Particle Spectrum Created through Bubble Nucleation and Quantum Field Theory in the Milne Universe

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Using the multi-dimensional wave function formalism, we investigate the quantum state of a scalar field inside a true vacuum bubble nucleated through false vacuum decay in flat spacetime. We developed a formalism which allows us a mode-by-mode analysis. To demonstrate its advantage, we describe in detail the evolution of the quantum state during the tunneling process in terms of individual mode functions and interpret the result in the language of particle creation. The spectrum of the created particles is examined based on quantum field theory in the Milne universe.

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I. INTRODUCTION

Various phenomena associated with phase transitions at the early stage of the universe have been a subject of great interest in cosmology for two decades. Cosmological first order phase transitions are caused by the decay of a metastable (false) vacuum in field theory, which produces true vacuum bubbles in the sea of false vacuum. A particular interest was taken in the false vacuum decay during an inflationary stage of the universe [1-3].

Recently, several authors considered a possible inflationary universe scenario which realize $\Omega_0 \sim 0.1$ open universe [4-6], in contrast to the standard inflationary universe which predicts $\Omega_0 = 1$ flat universe [7]. In this scenario, the bubble nucleation process plays an important role. That is, one assumes a scalar field with a potential like in the new inflationary scenario but with a high potential barrier before the slow rolling inflationary phase. If the nucleation rate is small, nucleated bubbles do not collide, and a homogeneous and isotropic universe with negative curvature is realized in one bubble thanks to the $O(4)$-symmetry of the bounce solution. The second inflation starts inside a bubble due to the flat part of the potential. Then after the inflation, the universe gradually becomes curvature-dominated as it expands and the present universe with $\Omega_0 \sim 0.1$ can be explained. The important problem is the quantum state of the scalar field after bubble nucleation because its quantum fluctuations may be considered as the origin of the cosmic structure.

As is well-known, the false vacuum decay rate can be calculated by means of the Euclidean path integral dominated by an $O(4)$-symmetric bounce solution and the classical motion of the nucleated bubble is described by the analytic continuation of the bounce solution [8,9]. But studies of the quantum state after the false vacuum decay have not been done so much. Several pioneering works on this problem were done by Rubakov [10], Kandrup [11], Vachaspati and Vilenkin [12]. Recently, we developed a formalism to investigate the quantum state after false vacuum decay based on the WKB wave function in a multi-dimensional tunneling system [13-16], which was originally developed by Banks, Bender and Wu [17], and Gervais and Sakita [18].

For a simplified model of spatially homogeneous false vacuum decay discussed by Rubakov [10], we showed that our formalism leads to the same results [13]. As for the false vacuum decay due to an $O(4)$-symmetric vacuum bubble, we also applied our formalism to a simple toy model and evaluated the expectation value of the energy momentum tensor inside the vacuum bubble [14]. The result indicates that the field excitation occurs inside the vacuum bubble, i.e., particle creation occurs there and it resembles a thermal state.
In spite of these investigations, we have not yet obtained a clear understanding of the quantum state after bubble nucleation. Neither the spectrum of created particles, nor the relation between the efficiency of the particle creation and the shape of the field potential has not been made clear. A difficulty originates from the fact that the mass of the quantum field is spatially inhomogeneous and time-dependent due to the non-trivial background field configuration.

In this paper, we tackle the issue by looking at the evolution of mode functions of a scalar field in a systematic manner and by investigating the spectrum of created particles, to obtain a better understanding of the quantum state after bubble nucleation. Our strategy is as follows. The background field configuration is inhomogeneous and time-dependent in the conventional Minkowski space-time coordinates. However, it keeps the Lorentz group \((O(3,1))\) symmetry, which originates from the \(O(4)\)-symmetry of the bounce solution. This implies that the system is homogeneous on the hyperbolic time slicing inside the future light cone within one bubble. This time slicing of the Minkowski spacetime is called the Milne universe. Quantum field theory in the Milne universe has been well investigated as an example of the quantization of a field in curved spacetime [19-22], and the relation of the vacuum state between the Milne universe and the Minkowski spacetime is well understood for a massive field, at least for two-dimensional spacetime. Using this fact, we can define a natural vacuum state in the Milne universe that corresponds to the usual Minkowski vacuum. Then we can gain an insight into the quantum state of a field inside an expanding bubble in terms of the particle spectrum observed by the comoving observers in the Milne universe at late times.

Following this strategy, we first carefully investigate the evolution of the mode functions through the tunneling process in two-dimensional spacetime, based on the wave functional formalism recently developed by us [13]. We find that the resulting mode functions can be expressed in a very transparent manner and the created particle spectrum can be evaluated mode by mode. Then we extend the result to the case of four dimensional spacetime. This technical advantage will be very useful when we consider realistic cases of bubble nucleation.

The paper is organized as follows. In section 2, we review our formalism to investigate the quantum state after bubble nucleation. In section 3, we investigate the quantum state inside a vacuum bubble in two-dimensional spacetime on the basis of quantum field theory in the Milne universe, and derive the formula for the spectrum of created particles. Then the result is extended to the four-dimensional case in section 4. In section 5, we apply our result to a simple thin-wall model which is similar to the one discussed by Rubakov [10]. The results are found to be similar to those obtained by Rubakov in the case of a spatially homogeneous decay model. Section 6 is devoted to summary and discussions. We use the units \(\hbar = 1\) and \(c = 1\) and a bar (\(^\ast\)) to denote the complex conjugate.

II. REVIEW OF FORMALISM

First, let us start reviewing our formalism to investigate the quantum state after false vacuum decay [13]. It is based on the WKB wave function for a multi-dimensional tunneling system. For generality, we consider the \((n+1)\)-dimensional Minkowski spacetime. The conventional Minkowski spacetime coordinates are denoted by \((t,x)\) with \(x = (x_1, \cdots, x_n)\). We consider a system of two interacting real scalar fields \(\sigma\) and \(\phi\) with the Lagrangian [10],

\[
\mathcal{L} = \mathcal{L}_\sigma + \mathcal{L}_\phi,
\]

where

\[
\mathcal{L}_\sigma = - \int d^n x \left[ \frac{1}{2} (\partial_\mu \sigma)^2 + V(\sigma) \right],
\]

\[
\mathcal{L}_\phi = - \int d^n x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2(\sigma) \phi^2 \right],
\]

Here the \(\sigma\) field has a potential \(V(\sigma)\) as shown in Figure 1, and decays from the false vacuum at \(\sigma(x) = \sigma_F\) to the true vacuum at \(\sigma(x) = \sigma_T\) through bubble nucleation.

In the lowest WKB picture, the decay from the false vacuum can be described by a bounce solution [8]. The bounce solution satisfies the classical Euclidean field equation, and has the maximum symmetry to minimize the Euclidean action. In our case, it has the \(O(n+1)\)-symmetry, and depends only on \(\tau^2 + x^2\), where \(\tau (\equiv it)\) is the time coordinate in the Euclidean space. Thus we can write the bounce solution as \(\sigma_0(\tau^2 + x^2)\). The subsequent motion of the bubble after nucleation is given by the analytic continuation of the bounce solution to the Lorentzian time, which is given by \(\sigma_0(-\tau^2 + x^2)\).

For simplicity, we neglect the quantum fluctuations of the \(\sigma\) field and consider only the quantum state of the \(\phi\) field which will be affected through the coupling with \(\sigma\) as it decays from the false vacuum. Note that one may regard \(\phi\)
as the fluctuation of $\sigma$ itself, if desired [26]. The wave function that describes the quantum state in the Euclidean region is written as [13,14]

$$\Psi_E = \mathcal{N}(\tau)e^{-S_E[\sigma_0]}\psi_E[\tau, \phi(\cdot)],$$  \hspace{1cm} (2.4)

where

$$S_E[\sigma_0] = \int_{-\infty}^{\infty} dt' \int d^4x \left[ \frac{1}{2} \left( \frac{\partial \sigma_0}{\partial t'} \right)^2 + \frac{1}{2} \left( \frac{\partial \sigma_0}{\partial x} \right)^2 + V(\sigma_0) \right],$$ \hspace{1cm} (2.5)

$$\psi_E[\tau, \phi(\cdot)] = \exp \left[ -\frac{1}{2} \int d^4x d^4y \phi(x)\phi(y)\Omega_E(x, y; \tau)\phi(y) \right].$$ \hspace{1cm} (2.6)

In Eq.(2.4), $e^{-S_E[\sigma_0]}$ is the lowest WKB part and gives the classical picture of the tunneling described by the bounce solution $\phi$. The factor $\mathcal{N}(\tau)$ describes the next WKB order corrections to the normalization factor and is independent of $\phi$. The part $\psi_E$ is the wave function that describes the state of the field $\phi$ on the classical background of $\sigma_0$. Thus all the information of the quantum state is contained in the function $\Omega_E(x, y; \tau)$. It is expressed in terms of the Euclidean mode functions $g_k(\tau, x)$ as

$$\Omega_E(x, y; \tau) := \int d^4k \frac{\partial g_k(\tau, x)}{\partial \tau}g_k^{-1}(\tau, y),$$ \hspace{1cm} (2.7)

where $g_k(\tau, x)$ satisfies the Euclidean field equation,

$$\left[ \frac{\partial^2}{\partial \tau^2} + \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - m^2(\sigma_0(\tau^2 + x^2)) \right] g_k(\tau, x) = 0,$$ \hspace{1cm} (2.8)

with the boundary condition at $\tau = -\infty$ as

$$g_k \to e^{i\omega_k \tau}Y_k(x),$$ \hspace{1cm} (2.9)

where $\omega_k := \sqrt{k^2 + m^2(\sigma_F)}$, $\sigma_F = \sigma_0(\tau = -\infty)$, and $Y_k(x)$ is a spatial harmonic function. This boundary condition is fixed by requiring that the system is set initially in a ground state in the false vacuum. The inverse of $g_k(\tau, x)$, $g_k(\tau, x)^{-1}$ is defined by

$$\int d^4x g_k(\tau, x)g_k^{-1}(\tau, x) = \delta(k - k').$$ \hspace{1cm} (2.10)

The quantum state after false vacuum decay is described by the analytic continuation of the above wave functional into the Lorentzian region ($\tau \to it$) where the classical motion is allowed [13]:

$$\Psi_L \propto e^{iS[\sigma_0]}\psi_L[t, \phi(\cdot)],$$ \hspace{1cm} (2.11)

where

$$S[\sigma_0] = \int_0^t dt' \mathcal{L}[\sigma_0(-t'^2 + x^2)],$$ \hspace{1cm} (2.12)

$$\psi_L[t, \phi(\cdot)] = \exp \left[ -\frac{1}{2} \int d^4x d^4y \phi(x)\phi(y)\Omega(x, y; t)\phi(y) \right],$$ \hspace{1cm} (2.13)

and $\Omega(x, y; t)$ is expressed in terms of the mode functions $\overline{u_k(t, x)}$, which is the analytic continuation of $g_k(\tau, x)$ to the Lorentzian time, as

$$\Omega(x, y; t) := \frac{1}{i} \int d^4k \frac{\partial \overline{u_k(t, x)}}{\partial t}\overline{u_k^{-1}(t, y)}.$$ \hspace{1cm} (2.14)

Just as in the case of Eq.(2.4), the first part $e^{iS[\sigma_0]}$ in Eq.(2.11) describes the classical motion of the expanding bubble, and the second part $\psi_L[t, \phi(\cdot)]$ does the quantum state of the $\phi$ field on the classical background of the expanding bubble $\sigma_0(x^2 - t^2)$. 

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It has been shown that the quantum state of the $\phi$ field after false vacuum decay can be understood in the Heisenberg picture by using the mode function $u_k(t, x)$ that satisfies the field equation for $\phi$ on the classical background $\sigma_0$ [13,16]. As the boundary condition (2.9) does not fix the normalization of $g_k(\tau, x)$ and the function $\Omega(x, y; \tau)$ is independent of it, we may assume that the mode function $u_k(t, x)$ satisfies the Klein-Gordon normalization (if it is not the case, we can take a linear combination of them to satisfy the normalization, i.e., $v_p = \int d^3 k \lambda(p, k) u_k$),

$$-i \int d^3 x \left( u_k(t, x) \overline{\partial_x u_k(t, x)} - u_k(t, x) \overline{\partial_x u_k(t, x)} \right) = \delta(k - k'),$$

where a dot denotes the $t$-differentiation. Introducing the creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$, respectively, associated with these mode functions, the Heisenberg field operator $\hat{\phi}_H$ may be expanded as

$$\hat{\phi}_H = \int d^3 k \left( \hat{a}_k u_k(t, x) + \hat{a}_k^\dagger \overline{u_k(t, x)} \right).$$

Then the quantum state described by $\psi_k[T, \phi(\cdot)]$ is found to be a “vacuum” state, $|\psi\rangle$, defined by

$$\hat{a}_k |\psi\rangle = 0 \quad \text{for any } k,$$

in the Heisenberg picture. Note that this “vacuum” state is non-trivial in general. If there exists a natural choice of vacuum, say $|0\rangle$, the state $|\psi\rangle$ is a squeezed state over $|0\rangle$, described by a non-trivial Bogoliubov transformation. Then it is possible to discuss the particle creation by evaluating the Bogoliubov coefficients between the two states.

III. PARTICLE SPECTRUM

In this section we first briefly review quantum field theory in the two-dimensional Milne universe [19–22]. Then we present a method to investigate the quantum state of a field inside a true vacuum bubble. Here we focus on the case of two-dimensional spacetime. The extension to the four-dimensional case will be given in section 4.

The Milne universe is the hyperbolic time slicing of the Minkowski spacetime in the future light cone (see Fig.2). Introducing the coordinates $T$ and $\chi$ in the two-dimensional Milne universe as

$$t = T \cosh \chi, \quad x = T \sinh \chi, \quad (0 < T < \infty, \quad -\infty < \chi < \infty)$$

the line element is written as

$$ds^2 = -dT^2 + T^2 d\chi^2.$$ (3.2)

The field equation of a massive scalar field with mass $M$ is

$$\left[ \frac{\partial^2}{\partial T^2} + \frac{1}{T} \frac{\partial}{\partial T} - \frac{1}{T^2} \frac{\partial^2}{\partial \chi^2} + M^2 \right] \phi(T, \chi) = 0.$$ (3.3)

Taking the mode expansion,

$$\phi(T, \chi) = \int_{-\infty}^{\infty} dp \varphi_p(T) \frac{e^{-ip\chi}}{\sqrt{2\pi}},$$ (3.4)

we easily find the solution as

$$\varphi_p(T) = c_p^{(1)} \sqrt{\frac{\pi}{4}} e^{-\pi p/2} H_{i_p}^{(1)}(MT) + c_p^{(2)} \sqrt{\frac{\pi}{4}} e^{\pi p/2} H_{i_p}^{(2)}(MT),$$ (3.5)

where $c_p^{(1)}$ and $c_p^{(2)}$ are constants, and $H_{i_p}^{(1)}(z)$ and $H_{i_p}^{(2)}(z)$ are the Hankel functions of the first and second kinds, respectively. Following the prescription of the second quantization of a field, we obtain

$$\hat{\phi}_H(T, \chi) = \int_{-\infty}^{\infty} dp \left[ \varphi_p(T) \frac{e^{-ip\chi}}{\sqrt{2\pi}} b_p + \overline{\varphi_p(T) \frac{e^{ip\chi}}{\sqrt{2\pi}}} \right],$$ (3.6)
where $\hat{b}_p$ and $\hat{b}_p^\dagger$ are the annihilation and creation operators, respectively, and $|c_p^{(2)}|^2 - |c_p^{(1)}|^2 = 1$. The following choice of the coefficients have a special meaning.

$$c_p^{(1)} = 0 \quad c_p^{(2)} = 1. \quad (3.7)$$

Hereafter we regard $\hat{b}_p$ and $\hat{b}_p^\dagger$ as those defined for this choice of the coefficients. The calculation of the propagator for the vacuum annihilated by $\hat{b}_p$ leads to the same expression as that for the usual Minkowski vacuum [19,20],

$$\langle 0 | T \phi_H(x) \phi_H(x') | 0 \rangle = \frac{1}{4i} H_0^{(2)}(-M^2(x^\mu - x'^\mu)^2 - i\epsilon). \quad (3.8)$$

Thus $H_{ip}^{(2)}(MT)$ is the positive frequency mode function for the Minkowski vacuum in the Milne universe.

For later convenience, we show this fact by directly relating the mode functions in the Milne universe and the usual Minkowski mode functions. By using the integral representation for the Hankel function [23], we find

$$e^{-i\eta \cosh K + i\zeta \sinh K} = \frac{1}{2i} \int_{-\infty}^{\infty} dp e^{-iK_p} \left( \frac{\eta + \zeta}{\eta - \zeta} \right)^{i\eta/2} e^{\eta^2/2} H_{ip}^{(2)}((\eta^2 - \zeta^2)^{1/2}), \quad (3.9)$$

which holds under the condition,

$$Im(\eta \pm \zeta) < 0. \quad (3.10)$$

Setting $\zeta = Mx, \eta = Mt$ and $K = \text{arcsinh}(k/M)$ in the above expression yields

$$e^{-i\omega_k t + iKx} = \frac{1}{2i} \int_{-\infty}^{\infty} dp e^{-iK_p} \left( \frac{t + x}{t - x} \right)^{i\eta/2} e^{\eta^2/2} H_{ip}^{(2)}(M(t^2 - x^2)^{1/2}), \quad (3.11)$$

where $\omega_k = M \cosh K = \sqrt{k^2 + M^2}$. If we use the coordinates $(T, \chi)$ introduced in Eq.(3.1), we have

$$e^{-i\omega_k t + iKx} = \frac{1}{2i} \int_{-\infty}^{\infty} dp e^{-iK_p} e^{i\eta p} e^{-\eta^2/2} H_{ip}^{(2)}(MT). \quad (3.12)$$

The above formula means that $H_{ip}^{(2)}(MT)$ is indeed the positive frequency mode function for the Minkowski vacuum. Note that, because of the validity condition (3.10), the presence of a small negative imaginary part in $t$ is understood, which corresponds to the familiar prescription.

The negative frequency functions are also obtained by taking the complex conjugate of Eq.(3.12),

$$e^{i\omega_k t - iKx} = -\frac{1}{2i} \int_{-\infty}^{\infty} dp e^{iK_p} e^{-i\eta p} e^{-\eta^2/2} H_{ip}^{(1)}(MT), \quad (3.13)$$

where we have used the relation,

$$e^{-\eta^2/2} H_{ip}^{(2)}(MT) = e^{\eta^2/2} H_{ip}^{(1)}(MT). \quad (3.14)$$

In contrast to the positive frequency mode function, Eq.(3.13) is valid on the upper half complex $t$-plane.

Now we consider the mode functions describing the quantum state after bubble nucleation. Following the general discussion given in the previous section, we first need to solve the Euclidean field equation to find the Euclidean mode function $g_k(\tau, x)$. Introducing the $O(2)$-symmetric coordinates $\xi_E$ and $\theta$ in the Euclidean region as

$$x = \xi_E \sin \theta, \quad \tau = -\xi_E \cos \theta, \quad (-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \ 0 \leq \xi_E < \infty) \quad (3.15)$$

the equation for the Euclidean mode function is written as

$$\left[ \frac{\partial^2}{\partial \xi_E^2} + \frac{1}{\xi_E} \frac{\partial}{\partial \xi_E} + \frac{1}{\xi_E^2} \frac{\partial^2}{\partial \theta^2} - m^2(\sigma_0(\xi_E^2)) \right] g_k(\tau, x) = 0. \quad (3.16)$$

The boundary condition is $g_k(\tau, x) \to e^{\omega_k \tau - iKx}$ at $\tau \to -\infty$. 

To find the solution, we first consider a solution of Eq.(3.16) in the form $F_p(\xi_E)e^{\gamma}$. Then $F_p(\xi_E)$ must obey

$$
\left[ \frac{\partial^2}{\partial \xi_E^2} + \frac{1}{\xi_E} \frac{\partial}{\partial \xi_E} + \frac{p^2}{\xi_E^2} - m^2((\sigma)(\xi_E^2)) \right] F_p(\xi_E) = 0. \tag{3.17}
$$

In the limit $\xi_E \to \infty$, we have $\sigma \to \sigma_F$. Let $M^2 = m^2(\sigma_F)$, then the asymptotic solution of $F_p(\xi_E)$ which approaches zero at $\xi_E \to \infty$ is

$$
F_p(\xi_E) = K_{ip} (M \xi_E), \tag{3.18}
$$

where $K_{ip}(z)$ is the Modified Bessel function. On the other hand, from Eq.(3.13) with $t \to -i\tau$ ($\tau < 0$), we find

$$
\epsilon^{\omega + i\tau - ik} = \frac{1}{\pi} \int_{-\infty}^{\infty} dp e^{ikp} e^{-\gamma} K_{ip} (M \xi_E). \tag{3.19}
$$

Thus the mode function $g_k(\tau, x)$ can be expressed in the form,

$$
g_k(\tau, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp e^{ikp} e^{-\gamma} F_p(\xi_E), \tag{3.20}
$$

where $F_p(\xi_E)$ is the solution to Eq.(3.17) with the condition $F_p(\xi_E) \to K_{ip} (M \xi_E)$ at $\xi_E \to \infty$.

Once we obtain the Euclidean mode function, we continue it to the Lorentzian region by $\tau \to it$. This procedure can be readily performed in the region outside the light cone ($x^2 - t^2 > 0$),

$$
u_k(t, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp e^{ikp} \left( \frac{t + x}{t - x} \right)^{-i\gamma/2} F_p(\xi), \tag{3.21}
$$

where $\xi = \sqrt{x^2 - t^2}$. In order to obtain the mode function inside the future light cone, a bit more consideration is necessary. To keep the regularity of the mode function, we must perform the analytic continuation at an infinitesimally small Euclidean time before $\tau = 0$, $\tau \to it - \epsilon (\epsilon > 0)$, or we must understand Eq.(3.21) with $t$ replaced by $t + i\epsilon$ [14]. In terms of $\xi$, this implies $\xi = e^{-i\gamma/2}T$ inside the future light cone. Then the mode function there is obtained as follows. In a sufficiently small neighborhood of $\xi = 0$, we have $m^2(\xi) \approx M^2 = \text{const.}$ Hence $F_p(\xi)$ can be expressed as

$$
F_p(\xi) = A_p K_{ip} (M \xi) + B_p (I_{ip} (M \xi) + I_{-ip} (M \xi))/2
\quad = \tilde{A}_p K_{ip} (M \xi) + B_p I_{ip} (M \xi), \tag{3.22}
$$

where $\tilde{A}_p = A_p + \frac{i}{\pi} \sin \tau p B_p$. Note that, from the reality of $F_p(\xi_E)$, the coefficients $A_p$ and $B_p$ are real. Following the above prescription, the Modified Bessel functions in the above equation become

$$
K_{ip} (M \xi) \to K_{ip} (M e^{-\tau p/2} (t^2 - x^2)^{1/2}) = \frac{\pi i}{2} e^{-\pi y/2} H_{ip}^{(1)} (MT), \tag{3.23}
$$

$$
I_{ip} (M \xi) \to I_{ip} (M e^{-\tau p/2} (t^2 - x^2)^{1/2}) = \frac{1}{2} \pi e^{\pi y/2} \left( H_{ip}^{(1)} (MT) + H_{ip}^{(2)} (MT) \right). \tag{3.24}
$$

Therefore we have the following expression for the mode function inside the future light cone at $T \simeq 0$,

$$
u_k(t, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp e^{ikp} e^{-ipx} \left[ c_p e^{-\pi y/2} H_{ip}^{(1)} (MT) + d_p e^{\pi y/2} H_{ip}^{(2)} (MT) \right], \tag{3.25}
$$

where

$$
c_p := \frac{\pi}{2T} \tilde{A}_p + \frac{\cosh \pi p}{2} B_p, \quad d_p := \frac{1}{2} B_p. \tag{3.26}
$$

Taking the complex conjugate of Eq.(3.25), we obtain

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where we have used Eq. (3.14).

If we take the limit of no $\sigma$-dependence of mass, i.e., $m^2(\sigma(\xi_E)) = M^2 = \text{constant}$ for all $\xi_E$, we have $c_p = \pi/2i$ and $d_p = 0$, and Eq.(3.27) reduces to

$$u_k(t, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp e^{-i\mathbf{k} \cdot \mathbf{p}} e^{i p x} \left[ c_p e^{\tau/2} H^{(2)}_{p} (\mathbf{M}T) + d_p e^{-\tau/2} H^{(1)}_{p} (\mathbf{M}T) \right],$$

where

Thus the particle creation does not occur in this limit, as expected.

In general, when the width of bubble wall cannot be neglected, the evolution of the classical background field inside the bubble becomes important. This effect leads to time variation of the mass of the field, and additional particle creation may occur. In such a case, Eq.(3.27) merely gives the (asymptotic) initial condition of $u_k(t, x)$ at $T \rightarrow 0$.

We now focus on the region inside the true vacuum bubble. Expanding the mode function as

$$u_k(t, x) \rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} dp e^{-i\mathbf{k} \cdot \mathbf{p}} e^{i p x} \varphi_p (T),$$

the field equation reduces to

$$\left[ \frac{\partial^2}{\partial T^2} + \frac{1}{T} \frac{\partial}{\partial T} + \frac{p^2}{T^2} + m^2(\sigma(T^2)) \right] \varphi_p (T) = 0,$$

thanks to the symmetry of the system. The initial condition for this equation is specified by Eq.(3.28). We assume that $m^2(\sigma(T^2)) = \mu^2 = \text{constant}$ at $T \rightarrow \infty$, that is, $\sigma(T^2)$ settles down to a finite value (presumably that corresponds to the true vacuum value $\sigma_T$). Then the asymptotic solution at $T \rightarrow \infty$ takes the form,

$$u_k(t, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp e^{-i\mathbf{k} \cdot \mathbf{p}} e^{i p x} \left[ \hat{c}_p e^{\tau/2} H^{(2)}_{p} (\mu T) + \hat{d}_p e^{-\tau/2} H^{(1)}_{p} (\mu T) \right],$$

where $\hat{c}_p$ and $\hat{d}_p$ depend on the evolutionary history of $m^2(\sigma(T^2))$.

In order to interpret the quantum state described by the mode functions obtained in Eq.(3.31) in the Heisenberg picture, we need to orthonormalize them. This is accomplished by taking the following linear combination of $u_k(t, x)$:

$$v_q(T, \chi) = \int_{-\infty}^{\infty} dk \lambda(q, k) u_k(t, x),$$

where

$$\lambda(q, k) = \frac{e^{i\chi q}}{M \cosh K} \frac{1}{4\sqrt{2(|\hat{c}_p|^2 - |\hat{d}_p|^2)}}.$$ 

Then we have

$$v_q(T, \chi) = \left[ \alpha_q \sqrt{\pi} e^{\pi/2} H^{(2)}_{q} (\mu T) + \beta_q \sqrt{\pi} e^{-\pi/2} H^{(1)}_{q} (\mu T) \right] e^{i\chi q} \sqrt{2\pi},$$

where

$$\alpha_q := \frac{\hat{c}_q}{\sqrt{|\hat{c}_q|^2 - |\hat{d}_q|^2}}, \quad \beta_q := \frac{\hat{d}_q}{\sqrt{|\hat{c}_q|^2 - |\hat{d}_q|^2}},$$

or

$$\varphi_q(T) = \frac{\sqrt{\pi}}{2\sqrt{|\hat{c}_q|^2 - |\hat{d}_q|^2}} \varphi_q(T).$$
One can easily check these mode functions satisfy the Klein-Gordon normalization on the $T = \text{constant}$ hypersurface in the Milne universe [24].

The coefficients $a_q$ and $\beta_q$ are the Bogoliubov coefficients with respect to the natural vacuum in the Milne universe which corresponds to the usual Minkowski vacuum. The spectrum of created particles with respect to this natural vacuum is

$$n_q = \left| \beta_q \right|^2 = \frac{1}{\left| \xi_q / \tilde{d}_q \right|^2 - 1}.$$  \hspace{1cm} (3.37)

When the particle creation after the bubble nucleation can be neglected, as is the case in the thin wall approximation, we have $\xi_q = c_q$ and $\tilde{d}_q = d_q$. Then the spectrum of the particle can be expressed as

$$n_q = \frac{1}{\left( \pi A_q / B_q \right)^2 + \sinh^2 \pi q} = \left| \frac{B_q}{\pi A_q} \right|^2.$$  \hspace{1cm} (3.38)

As clear from Eq.(3.36), once we have the expression for $\varphi_q(T)$ in the bubble, it is trivially easy to obtain the orthonormalized mode functions $v_q(T, \chi)$. In other words, the flat Minkowski coordinates play only an auxiliary role and we may focus on the evolution of a single mode function whose mode indices are specified with respect to $(\xi_E, \theta)$- or $(T, \chi)$-coordinates.

Let us summarize our result. First we solve Eq.(3.17) to find $F_p(\xi_E)$ with the boundary condition that it decreases exponentially as $\xi_E \to \infty$. Then we analytically continue it to the interior of the future light cone $F_p(\xi_E) \to \varphi_p(T)$ by $\xi_E \to e^{-\pi / 2 T}$ and take its complex conjugate. The resulting function $\varphi_p(T)$ at $T = 0$ gives the initial condition of the mode function which describes the quantum state inside the bubble. If the evolution of the classical background field $\sigma_q(T^2)$ can be neglected so that the mass of $\phi$ is constant (\(= \mathcal{M}\)) inside the bubble, as in the case of a thin-wall bubble, $\varphi_p(T)$ at $T = 0$ is the only information we need. By decomposing it into a linear combination of the asymptotic forms of $e^{\pi / 2 \mathcal{M}^2} H^{(2)}_{\nu_p}(\mathcal{M} T)$ and $e^{-\pi / 2 \mathcal{M}^2} H^{(1)}_{\nu_p}(\mathcal{M} T)$ at $T = 0$, the coefficients in front of them give the (unnormalized) Bogoliubov coefficients with respect to the natural vacuum of the Milne universe. The normalized Bogoliubov coefficients are then readily obtained as given in Eq.(3.35). If the evolution of the classical background cannot be neglected, as in the case of a thick-wall bubble, we must further solve the equation for the mode function with the initial condition specified by $\varphi_p(T)$ at $T = 0$. Then at $T \to \infty$ where the mass of $\phi$ settles down to a constant value (\(= \mu\)), we again decompose $\varphi_p(T)$ into the two independent solutions $e^{\pi / 2 \mathcal{M}^2} H^{(2)}_{\nu_p}(\mu T)$ and $e^{-\pi / 2 \mathcal{M}^2} H^{(1)}_{\nu_p}(\mu T)$ and read off the Bogoliubov coefficients with respect to the natural vacuum. We may then interpret the final quantum state in the language of particle creation due to bubble nucleation by considering the particle spectrum described by the Bogoliubov coefficients.

### IV. EXTENSION TO FOUR DIMENSION

The analysis given in the last section can be extended to a four-dimensional system. We introduce the coordinates $(T, \chi)$ in the four-dimensional Milne universe as

$$r = T \sinh \chi, \quad t = T \cosh \chi, \quad (0 < T < \infty, \quad 0 < \chi < \infty)$$  \hspace{1cm} (4.1)

where $r$ and $t$ are the radial and time coordinates in the Minkowski spacetime, respectively. The line element of the four-dimensional Milne universe is written as

$$ds^2 = -dT^2 + T^2 (d\chi^2 + \sinh^2 \chi d\Omega^2_{(2)}),$$  \hspace{1cm} (4.2)

where $d\Omega^2_{(2)}$ is the line element of the unit two-sphere. The field equation for a massive scalar field is

$$\left[ \frac{\partial^2}{\partial T^2} + \frac{3}{T} \frac{\partial}{\partial T} - \frac{1}{T^2} L^2 + m^2 \right] \phi(T, \chi, \Omega) = 0,$$  \hspace{1cm} (4.3)

where $L^2$ is the Laplacian operator on the unit three-dimensional spatial hyperboloid,

$$L^2 = \frac{1}{\sinh^2 \chi} \frac{\partial}{\partial \chi} \left( \sinh^2 \chi \frac{\partial}{\partial \chi} \right) + \frac{1}{\sinh^2 \chi} L^2_{\Omega},$$  \hspace{1cm} (4.4)
and $L^2_\Omega$ is the Laplacian on the unit two-sphere. The eigenfunctions $Y_{\nu m}$ of $L^2$ satisfy the equation [22],

$$L^2 Y_{\nu m}(\chi, \Omega) = -(1 + p^2)Y_{\nu m}(\chi, \Omega),$$

and is in the form,

$$Y_{\nu m}(\chi, \Omega) = f_{\nu}(\chi)Y_{\nu m}(\Omega),$$

where

$$f_{\nu}(\chi) = \frac{\Gamma(ip + l + 1)}{\Gamma(ip)} \frac{1}{\sqrt{\sinh \chi}} P_{ip-1/2}^{-l} \left( \cosh \chi \right),$$

and $Y_{\nu m}(\Omega)$ is the spherical harmonics on the unit sphere, $\Gamma(z)$ is the Gamma function and $P_{ip-1/2}^{-l} \left( z \right)$ is the associated Legendre function of the first kind. The eigenfunctions $Y_{\nu m}$ with $0 \leq p < \infty$ form a complete orthonormal set for square-integrable functions on the unit hyperboloid. Expanding the field operator in terms of the usual Minkowski vacuum, just the same as in the case of two dimensions.

As shown in Appendix, the mode function

$$e^{\nu \tau/2} H^{(2)}_\nu(MT)/\nu$$

turns out to be the positive frequency function with respect to the usual Minkowski vacuum, just the same as in the case of two dimension.

Then we can repeat the procedure given in the previous section to find the mode function after bubble nucleation. Introducing the $O(4)$-symmetric coordinates $(\xi_E, \theta)$ in the Euclidean region,

$$r = \xi_E \sin \theta, \quad \tau = -\xi_E \cos \theta, \quad (0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \xi_E < \infty)$$

the equation for the Euclidean mode functions is written as

$$\left[ \frac{\partial^2}{\partial \xi_E^2} + \frac{3}{\xi_E} \frac{\partial}{\partial \xi_E} + \frac{1}{\xi_E^2} L^2_E - m^2(\sigma_\nu(\xi_E^2)) \right] g_k(\tau, x) = 0,$$

where

$$L^2_E = \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin^2 \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} L^2_\Omega.$$  

The boundary condition is $g_k(\tau, x) = e^{\nu \tau/2} j_\nu(kr)Y_{\nu m}(\Omega)$ at $\tau = -\infty$.

Using the formula (A9) in Appendix, $g_k(\tau, x)$ is expressed as

$$g_k(\tau, x) = (-1)^l \sqrt{\frac{2}{\pi}} \int_0^\infty dp \frac{P_{ip-1/2}^{-l} \left( \cosh K \right)}{\sqrt{\sinh K}} \times \sin^l \theta \left( \frac{d}{d \cos \theta} \right)^l \frac{\sinh p \theta F_\nu(\xi_E)}{\sin \theta M_{\xi_E}} Y_{\nu m}(\Omega),$$

where $F_\nu(\xi_E)$ satisfies the same equation and the boundary condition as in the two-dimensional case, Eqs.(3.17) and (3.18). This fact leads to the following expression for the mode function inside the future light cone,

$$u_k(t, x) = (-i)^{l+1} \int_0^\infty dp \frac{\Gamma(ip + l + 1)}{\Gamma(ip)} \frac{P_{ip-1/2}^{-l} \left( \cosh K \right) P_{ip-1/2}^{-l} \left( \cosh \chi \right)}{\sqrt{\sinh K} \sqrt{\sinh \chi}} \times \frac{1}{MT} \left[ c_p e^{\nu \tau/2} H^{(2)}_\nu(MT) + d_p e^{-\nu \tau/2} H^{(1)}_\nu(MT) \right] Y_{\nu m}(\Omega),$$

where $c_p$ and $d_p$ are the same as those defined in Eq.(3.26). Thus the argument goes completely parallel to the two-dimensional case.
If the background field evolves inside the bubble, we must further solve Eq.(3.30) for the mode function with the
initial condition at $T \rightarrow 0$ given by Eq.(4.13). The result will take the form,

$$u_k(t, x) = (-i)^{l+1} \int_0^\infty dp \frac{\Gamma(ip + l + 1)}{\sqrt{\sinh \chi}} \frac{P_{ip-1/2}^- (\cosh \chi)}{\sqrt{\sinh \chi}} \frac{P_{ip-1/2}^- (\cosh \chi)}{\sqrt{\sinh \chi}}$$

$$\times \frac{1}{MT} \left[ \hat{\epsilon}_p e^{\gamma_{q/2}} H_{ip}^{(2)} (\mu T) + \hat{\delta}_p e^{-\gamma_{q/2}} H_{ip}^{(1)} (\mu T) \right] Y_{\theta \phi} (\Omega),$$

(4.14)

where we have assumed $m^2 (\sigma_q (T^2)) \rightarrow \mu^2$ at $T \rightarrow \infty$.

To obtain the orthonormalized mode functions, instead of Eq.(3.33), we set $\lambda(q, k)$ as

$$\lambda(q, k) = \frac{\sqrt{\pi}}{2} \frac{i^{l+1}}{\sqrt{(K_q - K_q^*) \cosh \chi}} (\sinh \chi)^{3/2} \Gamma (iq + l + 1) \frac{P_{ip-1/2}^- (\cosh \chi)}{\Gamma (iq)} \frac{P_{ip-1/2}^- (\cosh \chi)}{\Gamma (iq)},$$

(4.15)

and take the linear combination,

$$v_q(T, \chi, \Omega) = \int_0^\infty dk \lambda(q, k) u_k(t, x).$$

(4.16)

Using the orthonormality condition of the harmonic functions,

$$\int_0^\infty d\chi \sinh \chi P_{ip-1/2}^- (\cosh \chi) P_{ip-1/2}^- (\cosh \chi) = \frac{[\Gamma (ip)]^2}{[\Gamma (ip + l + 1)]^2} \delta (p - p'),$$

(4.17)

we then find the mode functions which satisfy the Klein-Gordon normalization on the $T = constant$ hypersurface in
the Milne universe as

$$v_q(T, \chi, \Omega) = \left( \alpha_q \frac{\sqrt{\pi}}{2} e^{\gamma_{q/2}} \frac{H_{ip}^{(2)} (\mu T)}{T} + \beta_q \frac{\sqrt{\pi}}{2} e^{-\gamma_{q/2}} \frac{H_{ip}^{(1)} (\mu T)}{T} \right) f_{\theta \phi} (\chi) Y_{\theta \phi} (\Omega),$$

(4.18)

where the coefficients $\alpha_q$ and $\beta_q$ are the same as those given by Eq.(3.35). Thus the particle spectrum is also
the same form as in the two-dimensional case, Eq.(3.37).

V. SIMPLE MODEL

In this section, applying the technique developed in the previous sections, we consider a simple model of the particle
creation through the tunneling process of bubble nucleation. We assume a thin-wall bubble so that the mass of the
$\phi$ field is given by

$$m^2 (\sigma_q (\xi_E^2)) = \left\{ \begin{array}{ll}
M^2; & R < \xi_E < \infty, \\
\mu^2; & 0 < \xi_E < R,
\end{array} \right.$$ 

where $R$ is the radius of the bubble when it nucleates.

The solution to Eq.(3.17) can be easily found by matching the solutions in both regions,

$$F_p(\xi_E) = \left\{ \begin{array}{ll}
K_{ip} (M \xi_E); & R < \xi_E < \infty, \\
\hat{A}_p K_{ip} (M \xi_E) + B_p I_{ip} (M \xi_E); & 0 < \xi_E < R,
\end{array} \right.$$ 

(5.1)

where

$$\hat{A}_p = \mu R I_{ip} (\mu R) K_{ip} (MR) - M R I_{ip} (\mu R) K_{ip} (MR),$$

$$B_p = -\mu R K_{ip} (\mu R) I_{ip} (MR) + M R K_{ip} (\mu R) K_{ip} (MR),$$

(5.2)

and a prime denotes differentiation with respect to the argument.

The particle spectrum is given by $n_p = \left| B_p / \pi \hat{A}_p \right|^2$. Roughly, the Modified Bessel functions
asymptotically behave as $K_{ip}(z) \propto e^{-z/2}$ and $I_{ip}(z) \propto e^{z/2}$ for $p \rightarrow \infty$. Hence the particle spectrum always decreases exponentially for
large $p$ as $n_p \propto e^{-2\pi p}$. This suppression of the spectrum at high momentum limit resembles that of a thermal state. The same feature has been found in a spatially homogeneous model of false vacuum decay, first discussed by Rubakov [10] and subsequently by us [13]. Note that the mode index $p$ corresponds to a comoving wave number in the Milne universe and the scale $p = 1$ corresponds to the physical curvature scale on the hyperboloid. In what follows, we evaluate $n_p$ for several limiting cases.

(1) $M \ll R^{-1}$ and $\mu \ll R^{-1}$:

In this case, we can use the following asymptotic formulae for the Modified Bessel functions at $z \ll 1$,

$$I_{ip}(z) \simeq \frac{(z/2)^{ip}}{\Gamma(ip + 1)},$$

$$K_{ip}(z) \simeq \frac{\pi}{2 \sinh \pi p} \left( \frac{(z/2)^{-ip}}{\Gamma(-ip + 1)} - \frac{(z/2)^{ip}}{\Gamma(ip + 1)} \right).$$

(5.3)

After some algebra, we find the following expression for the particle spectrum,

$$n_p \simeq \frac{\sin^2(p \log(\mu/M))}{\sinh^2 \pi p}.$$  

(5.4)

We see that $n_p$ decreases exponentially at large $p$, in accordance with the general discussion in the above. Due to the sinusoidal dependence, it has zeros in a period of $\pi/\log(\mu/M)$. In the limit $p \to 0$, $n_p$ approaches a finite value, $n_p \to \left| \log(\mu/M)/\pi \right|^2$. Figure 3 shows the spectrum for the case $MR = 0.02$ and $\mu R = 0.01$.

(2) $M \gg R^{-1}$ and $\mu \ll R^{-1}$:

In this case, if we focus on the region $p \ll 1$, the asymptotic formulae for the Modified Bessel functions at $z \gg 1$ can be used,

$$I_{ip}(z) \simeq \frac{1}{\sqrt{2\pi z}} e^{z}, \quad K_{ip}(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z}, \quad \text{for } z \gg 1.$$  

(5.5)

Then $n_p$ is evaluated as

$$n_p = \frac{1}{4\sinh^2 \pi p} \left| \frac{\Gamma(ip)}{\Gamma(-ip)} \left( \frac{\mu R}{2} \right)^{-2ip} + 1 \right|^2.$$  

(5.6)

In the limit $p \to 0$, we find $n_p \to \left| (\gamma + \log(\mu R/2))/\pi \right|^2$, where $\gamma = 0.577 \ldots$ is the Euler constant. Figure 4 shows $n_p$ as a function of $p$ for the case $MR = 20$ and $\mu R = 0.01$.

(3) $M \gg R^{-1}$ and $\mu \gg R^{-1}$:

Using the asymptotic formulae (5.5), we easily find $n_p$ in the region $p \ll 1$,

$$n_p \simeq \left( \frac{\mu - M}{\mu + M} \right)^2 e^{-4\mu R}.$$  

(5.7)

(4) $M \ll R^{-1}$ and $\mu \gg R^{-1}$:

In this case, using Eqs. (5.5) and (5.3), we obtain $n_p$ at $p \ll 1$ as

$$n_p \simeq e^{-4\mu R}.$$  

(5.8)

The first two cases for which $\mu R \ll 1$, the spectrum shows a similar feature, that is, it is constant of order unity at $p \ll 1$, and decreases exponentially at $p \gg 1$. On the other hand, in the latter two cases when $\mu R \gg 1$, the particle creation is always suppressed exponentially by a factor of $O(e^{-4\mu R})$. This feature is the same as those obtained previously in the spatially homogeneous decay model of the false vacuum if we identify the Euclidean tunneling “time” in the homogeneous model with the radius of the bubble wall [10,11,13,25].
VI. DISCUSSION

In this paper, we have investigated the quantum state of a scalar field inside a vacuum bubble nucleated in the process of false vacuum decay, focusing on the aspect of particle creation. We have used the wave functional formalism [13] to investigate the quantum state. First, we have carefully analyzed the case of two-dimensional spacetime and presented a prescription to obtain the mode functions inside a future light cone of a bubble and an interpretation of the resulting quantum state in the particle creation picture. Because of the existence of a natural vacuum in the Milne universe, which corresponds to the conventional Minkowski vacuum, we have been able to introduce the concept of “particle” in a natural way. The spectrum of created particles follows immediately by reading off the Bogoliubov coefficients relative to the natural vacuum from the final form of the mode functions. We have shown that the Bogoliubov coefficients are diagonal in the mode indices so that a mode-by-mode analysis of the quantum state is possible. Then we have extended our method to four-dimensional spacetime and shown that it applies in essentially the same manner as in the case of two dimension. This technical advantage will be very useful when we consider realistic models, such as the case of a thick-wall bubble or the case of quantum fluctuations of the tunneling field itself [26].

We have applied the technique developed here to a simple thin-wall model and found the following. (1) The spectrum of created particles is always suppressed exponentially at large wavenumber with the characteristic scale of the spatial curvature on the hyperbolic time slicing inside the light cone (i.e., the cosmic time slicing in the Milne universe). (2) If the field becomes massive inside the true vacuum compared to the inverse of the wall radius, i.e., \( \mu \gg R^{-1} \), the particle creation is always suppressed by an exponential factor of \( O(e^{-\lambda_R}) \).

In this paper, we have assumed the spacetime to be flat and only considered a massive field inside the true vacuum bubble. However, it is of great interest to develop a formalism by relaxing these restrictions. In the case of a massive field, the quantum state corresponding to the Minkowski vacuum state can be written as a pure state in the Milne universe. But it will not be the case for a massless field [27]. This is because the massless modes propagate always along null directions and a half of them escape to future null infinity without intersecting the boundary light cone of the Milne universe. Thus the massless Minkowski mode functions can never be expressed in terms of those in the Milne universe alone. Apparently a more careful analysis is necessary for the massless case.

Extension to the case including gravity is most interesting, especially in connection with the open universe inflation [5,6]. As pointed out in the Introduction, if we consider a one-bubble inflationary universe model to realize a negative curvature universe, the spacetime will be a de Sitter universe and the quantum fluctuations of the inflation field inside a nucleated bubble will give rise to the cosmological density perturbations. Hence, if it is possible to include the effect of gravity in our approach, we will be able to determine the spectrum of the density perturbations in the open universe inflationary scenario. This is now under study [5].

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APPENDIX

In this Appendix, we derive the formulae which relate the mode functions in the Milne universe and those of the usual Minkowski spacetime in four-dimensional spacetime. As we use the spherical coordinates in both cases, we omit the spherical harmonics from the mode functions. We start from the following formula which may be derived from Eq.(3.11) or (3.12),

\[
 k e^{-i\omega_l t} j_l(kr) = - \int_0^\infty dp \sin \kappa p \frac{\sin p\chi}{\sin \chi} e^{\tau p/2} \frac{H_\nu^2(MT)}{T}. 
\]

This gives the relation between the \( l = 0 \) Milne and Minkowski mode functions.

We extend the above formula to the case of \( l \geq 1 \). By mathematical induction we can show that

\[
 \sinh^l \chi \left( \frac{d}{d \cosh \chi} \right)^l (ke^{-i\omega_l t} j_l(kr)) = \frac{1}{i^l \partial^\ell \kappa} (k^{\ell+1} e^{-i\omega_l t} j_l(kr)).
\]

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Using this fact, we operate \((\sinh \chi)^{(l/d\cosh \chi)^{l}}\) on both sides of Eq.(A1) and integrates both sides of the equation with respect to \(\omega_k\), we find

\[
k^{l+1} e^{-i\omega_k t} j_l(kr) = -i \int_0^\infty dp \, a_{pk} \sinh^l \chi \left( \frac{d}{d\cosh \chi} \right)^l \sin p \chi \frac{e^{\pi p/2}}{\sinh \chi} \frac{H^{(2)}_{ip} (MT)}{T},
\]

where

\[
a_{pk} = \int_M d\omega_1 \int_M d\omega_2 \int_M d\omega_3 \cdots \int_M d\omega_l \sin p \mathcal{K} (\omega_l) \\
= \int_M d\omega_2 (\omega_k - \omega_3) \int_M d\omega_3 \cdots \int_M d\omega_l \sin p \mathcal{K} (\omega_l) \\
= \int_M d\omega_1 (\omega_k - \omega_1)^{l-1} \frac{1}{(l-1)!} \sin p \mathcal{K} (\omega_1).
\]

Noting that \(\omega_k = M \cosh \mathcal{K}\), if we change the integration variable \(\omega_1\) to \(\arccosh (\omega_1/M)\), we see that this is just an integral representation of the associated Legendre function [23],

\[
a_{pk} = M^l \int_0^{\pi/2} (\sinh \mathcal{K})^{(l+1/2)} P_{ip}^{-l-1/2} (\cosh \mathcal{K}).
\]

Hence we have

\[
e^{-i\omega_k t} j_l(kr) = -i \int_0^\infty dp \sqrt{\frac{\pi}{2}} \frac{P_{ip}^{-l-1/2} (\cosh \mathcal{K})}{\sinh \mathcal{K}} \\
\times \sinh^l \chi \left( \frac{d}{d\cosh \chi} \right)^l \sin p \chi \frac{e^{\pi p/2}}{\sinh \chi} \frac{H^{(2)}_{ip} (MT)}{MT}.
\]

Using the formula,

\[
\sinh^l \chi \left( \frac{d}{d\cosh \chi} \right)^l \sin p \chi = \left( -1 \right)^l \frac{\pi}{p} \sqrt{\frac{\pi}{2}} \frac{\Gamma (ip + l + 1)}{\Gamma (ip)} \frac{P_{ip}^{-l-1/2} (\cosh \mathcal{K})}{\sqrt{\sinh \mathcal{K}}},
\]

which can be derived by using the differential relation for the associated Legendre functions, we find

\[
e^{-i\omega_k t} j_l(kr) = -(-i)^l \frac{\pi}{2} \int_0^\infty dp \left[ \frac{\Gamma (ip + l + 1)}{\Gamma (ip)} \right]^2 \frac{P_{ip}^{-l-1/2} (\cosh \mathcal{K})}{\sqrt{\sinh \mathcal{K}}} \\
\times \frac{P_{ip}^{-l-1/2} (\cosh \mathcal{K})}{\sqrt{\sinh \mathcal{K}}} \frac{H^{(2)}_{ip} (MT)}{MT}.
\]

This result shows that \(e^{\pi p/2} H^{(2)}_{ip} (MT)/T\) is indeed the positive frequency function in the Milne universe which describes the Minkowski vacuum, as in the two-dimensional case.

By taking the complex conjugate of Eq.(A6), and continuing it to the Euclidean time \(t \rightarrow -i \tau \) \((\tau < 0)\), we obtain

\[
e^{i\tau} j_l(kr) = (-1)^l \sqrt{\frac{\pi}{2}} \int_0^\infty dp \frac{P_{ip}^{-l-1/2} (\cosh \mathcal{K})}{\sqrt{\sinh \mathcal{K}}} \\
\times \sinh^l \chi \left( \frac{d}{d\cosh \chi} \right)^l \sin p \theta \frac{K_{ip} (m_q \xi)}{m_q E}.
\]

This equation corresponds to Eq.(3.19) in the two-dimensional case and gives the expression for the Euclidean mode function in the four-dimensional case.
Here we are implicitly assuming that the functional space describing the Cauchy data given on a $T = \text{constant}$ hypersurface in the Milne universe and those on a $t = \text{constant}$ hypersurface in the Minkowski spacetime are identical. For a field whose mass square is always positive, this is supported by the fact that no physical information can escape to the future null infinity. However, this will be violated for a field whose mass square becomes zero or negative in a certain (Lorentz invariant) region of spacetime. A simplest example is the case of a massless field. We defer detailed discussion of such a case to forthcoming papers [5,26].


FIGURE CAPTION

Fig.1. An illustration of a potential for the tunneling field $\sigma$.

Fig.2. A sketch of the coordinates describing the Milne universe.

Fig.3. The number spectrum of created particles as a function of the wave number $p$ for the case $MR = 0.02$ and $\mu R = 0.01$.

Fig.4. The same as Figure 3, but for $MR = 20$ and $\mu R = 0.01$. 