Gravity in 2+1 dimensions

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Abstract
A review is given of some classical and quantum aspects of 2+1 dimensional gravity.

Two lectures given at the XI Italian Relativity Meeting, Trieste 26-30 September 1994.

Work supported in part by M.U.R.S.T.
1. Introduction

Gravity in $2+1$ dimensions [1] can be given two different meanings: one as the study of special solutions of $3+1$ dimensional gravity in presence of a space-like Killing vector; the other as a simplified model of gravity which can lend itself as playground to hopefully learn how to deal with problems in $3+1$ which up to now have defeated our comprehension. The first aspect is of interest in connection of cosmic strings.

One starts by counting the number of independent components of the Riemann tensor as a function of the space-time dimensions $D$.

In $D = 2$ due to the symmetries of the Riemann tensor we have only one independent component $R_{0101}$ and actually we can write

$$R^{\mu\nu}_{\lambda\rho} = (\delta^\mu_\lambda \delta^{\nu}_{\rho} - \delta^\mu_\rho \delta^{\nu}_{\lambda}) R/6. \quad (1.1)$$

In $D = 3$ due to the antisymmetry of $R_{\mu\nu\lambda\rho}$ in the pairs $(\mu\nu)$ and $(\lambda\rho)$ we have that each pair can assume only three values and due to the symmetry under the exchange of the two pairs we have that the Riemann tensor has only 6 independent values i.e. as many as the number of independent components of the Ricci tensor. We can also write

$$R^{\mu\nu}_{\lambda\rho} = \epsilon^{\mu\nu\kappa} \epsilon_{\lambda\rho\sigma} (R_\kappa - \delta_\kappa^\sigma R/2) = \epsilon^{\mu\nu\kappa} \epsilon_{\lambda\rho\sigma} G_\kappa^\sigma \quad (1.2)$$

being $G_\kappa^\sigma$ the Einstein tensor.

Einstein’s equations are written as

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R = \frac{8 \pi G}{c^4} T_{\mu\nu} \quad (1.3)$$

where $G$ is Newton’s constant whose dimension is $l^{D-3} m^{-1} c^2$. If we consider in $D = 4$ a geometry in which the $g_{\mu\nu}$ are independent of $x^3$ and $g_{\mu\nu}$ has the structure

$$g_{03} = g_{30} = 0, \quad g_{33} = 1 \quad (1.4)$$
then \( R_{0\mu} = R_{\mu 0} = 0 \) and for \( \mu, \nu = 0, 1, 2 \) we have the three dimensional Einstein’s equations, where \( R_{\mu\nu} \) is just computed from the components 0, 1, 2 of the four dimensional metric \( g_{\mu\nu} \). From eq.(1.3) we see that the components 3, \( \mu \) with \( \mu = 0, 1, 2 \) of the energy momentum tensor vanish, while we have

\[
\frac{8\pi G}{c^4} T_{33} = -\frac{g_{33}}{2} R
\]

which in general is not zero. The written formulas prove that to all solution in 2+1 dimensions there corresponds a solution in 3+1 with a space like Killing vector and an energy momentum tensor which is simply related to the one in 2+1; in 3+1 a tension develops along the third dimension.

The situation described above is interesting in connection of cosmic strings [2].

We note however that not all solutions of Einstein equations in 3+1 dimensions with an open space-like Killing vector give rise to solutions of 2+1 dimensional gravity. We can quote as an example the axial solutions given by Levi-Civita [3]

\[
ds^2 = \rho^{-2m} \left[ \rho^{2m^2} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] - \rho^{2m} dt^2.
\]

We see here that \( g_{33} = \rho^{-2m} \) and as such not constant.

In all dimensions \( T_{\mu\nu} = 0 \) implies \( R_{\mu\nu} = 0 \) but in 2+1 dimensions \( R_{\mu\nu} = 0 \) implies \( R_{\mu\nu\lambda\rho} = 0 \) and thus where matter is absent space-time is flat. This is the most important property of 2+1 dimensional gravity.

2. Energy, momentum and angular momentum in 2+1 dimensions

After these general comments we come to the definition of the fundamental quantities in the theory, i.e. energy-momentum and angular momentum.

The energy-momentum and angular momentum of a system in general relativity in \( D \) dimensions with \( D \geq 4 \) is defined by means of the flux of a pseudotensor through a
$D - 2$ dimensional space-like surface located at space infinity. For example for the energy-momentum we have in four dimensions [4]

$$\frac{16\pi G}{c^4} T_{\lambda} = -\frac{1}{2} \int \delta^{\alpha\beta}_{\mu\nu\lambda} \ g^{\nu\delta} \ \Gamma^\mu_{\delta\beta} \ dS_{\sigma\alpha}$$

(2.1)

where $dS_{\sigma\alpha}$ is the element of a space-like two dimensional surface at space infinity.

It is well known that such energy-momentum can be defined when the metric approaches at space infinity the minkowskian metric. Such a definition cannot be taken over directly to $2+1$ dimensional gravity; in fact it is true that the space outside the sources is exactly flat, but the global structure of the space time at space infinity is that of a cone with a non zero deficit angle. We shall see e.g. that for a stationary, static distribution of matter such a deficit angle equals the space integral of $T^0_0$ i.e. the total amount of mass of the matter present in the system. The result however does not hold in general. With regard to angular momentum the situation is more complicated. One can consider the rather artificial massless (zero deficit angle), and then one can carry over the usual procedure of $3+1$ dimensions and thus one is able in the stationary case to identify the angular momentum with the time jump which occurs in a synchronous reference system when one performs a closed trip around the source [5].

A general definition of energy-momentum and angular momentum in $2+1$ dimensions can be derived by computing the holonomies of the $ISO(2,1)$ group, which were introduced in connection to the formulation of pure $2+1$ dimensional gravity as a Chern-Simon theory.

Lorentz and more generally Poincaré holonomies will play a fundamental role in all what follows and thus we turn now to them.

Let us consider to start the Lorentz holonomy

$$M = \text{Pexp}( -i \int J_a \omega^a_{\mu} dx^\mu )$$

(2.2)
where
\[ \omega_\mu^a = \frac{1}{2} \epsilon^a_{\mu b} \omega^b_{\nu} \]  
and \( J_a \) are the generators of the \( SO(2, 1) \) group with commutation relations given by
\[ [J_a, J_b] = i \epsilon_{abc} J^c \]
and the traces given by
\[ \text{Tr}(J_a J_b) = -\frac{1}{2} \eta_{ab} \]
where \( \eta_{ab} \equiv \text{diag}(-1, 1, 1) \). Here we shall consider for definiteness the fundamental representation of \( SU(1,1) \). It is not difficult to show that an element of \( SU(1,1) \) can always be written in the form \( \pm e^{-i J_a \Theta^a} \) according to the following cases:

If \( \Theta^a \) is a light-like vector then in the fundamental representation we have
\[ M = \pm e^{-i J_a \Theta^a} = \pm (I - i \Theta^a J_a). \]

If \( \Theta^a \) is space-like
\[ M = \pm e^{-i J_a \Theta^a} = \pm (\cosh \frac{\sqrt{-\Theta^a \Theta_a}}{2} I - 2i \sinh \frac{\sqrt{-\Theta^a \Theta_a}/2}{\sqrt{-\Theta^a \Theta_a}} (J_a \Theta^a)). \]

If \( \Theta^a \) is time-like we have
\[ M = \cos \frac{\sqrt{-\Theta^a \Theta_a}}{2} I - 2i \sin \frac{\sqrt{-\Theta^a \Theta_a}/2}{\sqrt{-\Theta^a \Theta_a}} (J_a \Theta^a). \]

The \( \pm \) alternative is a necessary one for \( \Theta^a \) space-like or light-like.

In the light-like and space-like case by computing \( \text{tr} M \) and \( \text{tr}(MJ^a) \) one determines completely \( M \) and thus \( \Theta^a \) while in the time-like case \( \Theta^a \) is completely determined in the range \( 0 \leq \sqrt{-\Theta^a \Theta_a} \leq 2\pi \). In this way we are also able to establish whether \( \Theta^a \in V^+ \) or \( V^- \), where \( V^+ \) and \( V^- \) denote the set of future and past directed time-like vectors.
In addition to eq. (2.2) one can consider the Poincaré holonomy given by

\[ W = \text{Pexp}( -i \int (J_\alpha \omega^a_\mu + P_a \epsilon^a_\mu) dx^\mu ) = \pm e^{-iJ_\alpha \Theta^a - iP_a E^a} \] (2.9)

which can also be written as

\[ W = \text{Pexp}( -i \int P_a T^a_\mu \epsilon^\mu_\mu dx^\mu ) \times \text{Pexp}( -i \int J_\alpha \omega^a_\mu dx^\mu ) = \pm e^{-iP_a E^a} e^{-iJ_\alpha \Theta^a} \] (2.10)

where

\[ E^a = \int_1^0 (T^a_\mu(t) dt) E^\mu_\mu \]

with

\[ T^a_\mu(t) = [e^{-it^\mu J^\alpha} \Theta^a]_\mu \] (2.11)

in the adjoint representation. It is useful to represent \( W \) as [6]

\[ W = \begin{pmatrix} M & E \\ 0 & I \end{pmatrix} \] (2.12)

with multiplication rules

\[ \begin{pmatrix} M_1 & E_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} M_2 & E_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} M_1 M_2 & M_1 E_2 + E_1 \\ 0 & I \end{pmatrix} \] (2.13)

being \( \mathcal{M} \) the transformation \( M \) in the adjoint representation.

Let us now consider the Poincaré holonomy for a closed loop; we can perform a gauge transformation at the origin of the loop or equivalently move the point along the contour or move the whole loop in space provided it always moves without intersecting matter.

In all cases \( W \) is subject to a gauge transformation \( W' = TWT^{-1} \) with

\[ M' = A M A^{-1} \] (2.14)

where \( A \) is an element of \( SU(1,1) \) from which

\[ \Theta' = A \Theta \] (2.15)

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and

\[ E' = \mathcal{M} E + (I - \mathcal{M} \mathcal{A}^{-1}) S \]  \hspace{1cm} (2.16)

being \( \mathcal{M} \) and \( \mathcal{A} \) the transformations \( M \) and \( A \) in the adjoint representation and \( S \) the translation vector of \( T \). So while \( \Theta \) transforms like a Lorentz vector, \( E^a \) transforms inhomogeneously. We can envisage two invariants \( \Theta^a \Theta_a \) and \( \Theta^a E_a \). The occurrence of such invariants was noticed by Achucarro and Townsend \cite{7} and by Witten \cite{8} and exploited by them to relate \( 2 + 1 \) dimensional gravity to Chern-Simon theory.

Thus from the Lorentz holonomy one can extract a vector under local Lorentz transformations. Such a vector does not depend on deformations of the loop keeping the origin fixed provided the loop extends only outside the matter, and the square of the vector can be naturally identified with the square of the mass of the system. Change in the origin of the loop is equivalent to a Lorentz transformation on the vector.

The energy momentum of the system will be identified by \(( c = 1 \) from now on\)

\[ \mathcal{P}^a = \Theta^a / 8 \pi G \]  \hspace{1cm} (2.17)

when \( M \) is the Lorentz holonomy computed along a closed contour which encloses all matter.

A major result of classical general relativity is the positive energy theorem \cite{9} which states that the energy-momentum pseudovector as defined in eq. (2.1) (something which we recall is possible in \( D \) dimension with \( D \geq 4 \) and for spaces which are asymptotically flat) is a future directed time-like vector under transformations which are asymptotically global Lorentz transformations. The ingredients for the proof are the energy condition and the existence of a space-like “initial condition” \( D - 1 \)-dimensional hypersurface. A similar result can be obtained in \( 2 + 1 \) dimension \cite{10} with the definition of energy given above eq. (2.17) but as we shall see now there are some differences.
We shall assume the existence of an edgeless space-like initial data two dimensional surface \( \Sigma \) with the topology of \( R^2 \). Let us consider on \( \Sigma \) a family of closed contours \( x^\mu(s, \lambda) \) with \( 0 \leq \lambda \leq 1 \), \( x^\mu(s, 0) = x^\mu(s, 1) \), and such that for \( s_1 < s_2 \) the contour \( x^\mu(s_1, \lambda) \) is completely contained in \( x^\mu(s_2, \lambda) \); moreover \( x^\mu(s, \lambda) \) shrinks to a single point for \( s = 0 \).

It is easy to prove that the Wilson loop under the deformation induced by the parameter \( s \) changes according to the following equation [10]

\[
\frac{DM}{ds}(s, 1) \equiv \frac{dM}{ds}(s, 1) + i [J_\alpha \omega^{\alpha}_\mu(s, 1), \frac{dx^\mu}{ds}, M(s, 1)] = \\
iM(s, 1) \int_0^1 d\lambda M(s, \lambda)^{-1} R_{\mu\nu}(s, \lambda) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{ds} M(s, \lambda)
\tag{2.18}
\]

where \( R_{\mu\nu} \) is the curvature form in the fundamental representation. Eq.(2.18) is true in any dimension but in 2+1 dimensions the curvature 2-form is given directly by the energy-momentum tensor

\[
R_{\mu\nu} = 8\pi G \eta_{\mu\nu\rho} T^a_{\rho} J_a
\tag{2.19}
\]

with \( \eta_{\mu\nu\rho} \equiv \sqrt{-g} \epsilon_{\mu\nu\rho} \) and \( T^a_{\rho} = \epsilon^a_{\mu} T^\mu_{\rho} \). Taking into account that \( -\eta_{\mu\nu\rho} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{ds} \) is the area vector \( N_\rho \) corresponding to the 2-dimensional surface \( \Sigma \), and thus time-like and the dominant energy condition (DEC) [11] we have

\[
\eta^a(s, \lambda) \equiv T^a_{\rho} N_\rho \in V^+.
\tag{2.20}
\]

By substituting into eq.(2.18) we have

\[
\frac{DW}{ds}(s, 1) = -8\pi i G W(s, 1) J_a Q^a(s)
\tag{2.21}
\]

with \( Q^a \) time-like and future directed.

If we parameterize \( M \) in the fundamental representation as \( M(s, 1) = w(s) I - 2i J_a \theta^a(s) \) the evolution equation gives

\[
\frac{dw(s)}{ds} = 4\pi G Q^a(s) \theta^a(s)
\tag{2.22}
\]
Such a system of differential equations can be discussed rigorously [10] with the following results:

For $s = 0$ obviously $M(0, 1) = w(0, 1) = 1$ and $\Theta^\alpha = 0$. Then as the loop expands and starts including matter $\Theta^\alpha$ becomes a time-like future directed vector whose norm (i.e. the mass squared) increases monotonically as the loop embraces more and more matter and in the meantime $w(s)$ decreases monotonically from the initial value 1. The value $w(s) = -1$ can be reached in two ways, either with $\theta^\alpha = 0$ or with $\theta^\alpha$ light-like. In the first case the universe has a time-like momentum and $\sqrt{-\Theta^\alpha \Theta_\alpha}$ has reached the value $2\pi$; in absence of further matter the universe becomes a cylinder at large distances. Instead for $\theta^\alpha$ light-like we have a light-like universe. If by still expanding the loop we enclose more matter in case 1 the momentum stays time-like and the deficit angle becomes larger that $2\pi$ and the universe closes kinematically with the topology of a sphere; instead in case 2, $\Theta^\alpha$ becomes space-like. An example of such universe is the one envisaged by Gott [12], built up by two fast moving particles and which was subject to close scrutiny in the literature [13]. Again the rigorous discussion of the evolution equations [10] shows that it is not possible, by still adding matter to such space-like universe, to go back to an open time-like universe.

It is of interest to see what becomes of the energy for the stationary static case. Here the dreibeins can be chosen independent of time and of the form $e^\alpha_0 = N \delta^\alpha_0$, $e^\alpha_i = 0$ and we have $T^{00} = NT^{00}$. In this case the evolution equations can be solved explicitly [10] to get for the mass of the system

\[ \delta = 8\pi Gm = 8\pi G \int T^0_0 \sqrt{-\gamma} dx^1 dx^2 = 4\pi \int R^{(2)} \sqrt{-\gamma} dx^1 dx^2 \]  

(2.24)
where $T_0^0$ is the energy momentum tensor in the coordinate basis and $\gamma$ is the determinant of the space metric, in agreement with the result of [5]. The obtained result is expected from special relativity taking into account that in $2 + 1$ dimensions there are no gravitational forces, thus no gravitational potential energy and thus static masses combine additively.

Summing up in comparison to $D$ dimensions with $D \geq 4$ we have the following differences:

1. In $3 + 1$ dimensions we can consider open universes of arbitrary mass. In $2 + 1$ dimensions for small loops one always starts with an energy momentum vector which is time-like future directed and such vector remains time-like future directed putting together more matter until we reach the $2\pi$ limit for the deficit angle. Then adding more matter one goes over either to a time-like closed universe or with proper kinematical conditions one can reach a “space-like” universe in the sense that the Lorentz holonomy becomes of the type $-e^{-iJ_s \Theta^s}$ with $\Theta^s$ space-like.

2. If the energy-momentum of the matter enclosed by a loop is space-like in the sense mentioned above, then by enclosing more matter one cannot go back to a time-like open universe. Thus we have a generalization to all forms of matter satisfying the DEC of the theorem by Carrol, Fahri, Guth and Olum [14] which can be stated as follows: If a subsystem of the universe has space-like momentum then either the universe is closed or it has space-like momentum.

As we mentioned above $\Theta^s \Theta_s$ is not the only invariant of the Poincaré holonomy; we have also $\Theta^s E_s$. This is related to the angular momentum (which in $2 + 1$ dimensions has just one component) by

$$J = -\frac{\Theta^s E_s}{8\pi G \sqrt{-\Theta_s \Theta^s}} \quad (2.25)$$
A first check of the correctness of our definition comes from the value it assumes for the “Kerr” solution in $2 + 1$ dimensions

$$ds^2 = -(dt + 4GJ d\theta)^2 + (1 - 4Gm)^2 \rho^2 d\theta^2 + d\rho^2. \quad (2.26)$$

We find in fact that $J$ of eq.(2.25) coincides with the $J$ appearing in eq.(2.26) which is related to the time shift $\Delta t$ that appears in a synchronous coordinate system when one encircles once the source $\Delta t = 4GJ \times 2\pi$. The $J$ of eq.(2.25) also coincides (in the limit $m \to 0$) with the value obtained from the $3 + 1$ dimensional prescription in the case of a massless source with angular momentum, for which the angular deficit at infinity is zero.

For the Poincaré holonomy one can write down evolution equations [10] similar to the ones written and discussed for the Lorentz holonomy. These can be solved in the weak limit giving for $J$ the expression of the angular momentum in special relativity and thus providing further support to the identification (2.25). Thus formulas (2.17, 2.25) provide a good definition for energy-momentum and angular momentum in $2 + 1$ dimensions; a positive energy theorem holds with some characteristics which are intrinsic to $2 + 1$ dimensions.

3. Solving Einstein’s equations in $2+1$ dimensions

Gauges of geodesic type play a special role in $2 + 1$ dimensional gravity. It was already pointed out in [15] that in gaussian normal coordinates the evolution equations in $2 + 1$ dimensions are reduced to a system of ordinary differential equations. There is a variety of geodesic gauges [16, 17], the best known being the Fermi-Walker gauge [16]. In the first order formalism such a gauge is defined by

$$\sum_i \xi_i \omega^a_{bi} = 0 \quad \sum_i \xi_i e^a_i = \sum_i \xi_i \dot{e}^a_i. \quad (3.1)$$
These equations are solved by

\[ \omega^{a}_{\xi i}(\xi) = \xi^{j} \int_{0}^{1} R^{a}_{\xi ji}(\lambda \xi, t) \lambda d\lambda, \]

\[ \omega^{a}_{\xi 0}(\xi) = \omega^{a}_{\xi 0}(0, t) + \xi^{i} \int_{0}^{1} R^{a}_{\xi i 0}(\lambda \xi, t) d\lambda, \quad (3.2) \]

\[ \epsilon^{a}_{\xi}(\xi) = \delta^{a}_{\mu} + \xi^{j} \int_{0}^{1} \omega^{a}_{\xi j}(\lambda \xi, t) \lambda d\lambda + \xi^{j} \int_{0}^{1} S^{a}_{j i}(\lambda \xi, t) \lambda d\lambda. \]

\[ \epsilon^{a}_{\xi 0}(\xi) = \delta^{a}_{\mu} + \xi^{i} \int_{0}^{1} \omega^{a}_{\xi i 0}(\lambda \xi, t) d\lambda + \xi^{i} \int_{0}^{1} S^{a}_{i j}(\lambda \xi, t) d\lambda, \]

where \( R^{a}_{\xi ji} \) and \( S^{a}_{j i} \) are the Riemann tensor and the torsion. Here we shall work with zero torsion. These resolvent formulas hold in any dimensions, however in \( 2 + 1 \) dimensions they assume a particular meaning because the Riemann tensor appearing in eq.\((3.2)\) is given directly in terms of the energy-momentum. Thus eqs.\((3.2)\) give the metric in terms of the sources by a quadrature. However one has to keep in mind that the energy-momentum tensor \( T_{c} \) which appears in Einstein’s equation

\[ \varepsilon_{abc} R^{a b} = -16 \pi G T_{c}, \quad (3.3) \]

\[ R^{a b} = -8 \pi G \varepsilon^{a b c} T_{c} = -4 \pi G \varepsilon^{a b c} \varepsilon_{\rho \mu \nu} T_{c}^{\rho} dx^{\mu} \wedge dx^{\nu}, \quad (3.4) \]

is not completely arbitrary but is subject to the symmetry and covariant conservation constraints summarized by

\[ \mathcal{D} T^{a} = 0, \quad \varepsilon_{abc} T^{b} \wedge e^{c} = 0. \quad (3.5) \]

Thus the problem of solving Einstein’s equations is reduced to that of solving \( [18] \) the constraints \((3.5)\). The conservation constraint is automatically solved if we express the energy momentum tensor in terms of the connection, through eq.\((3.3)\). The symmetry constraint is more difficult. After introducing the cotangent vectors \( T_{\mu} = \frac{\partial \xi^{0}}{\partial \xi_{\mu}} \),
\[ P_\mu = \frac{\partial \rho}{\partial \xi^\mu} \] and \( \Theta_\mu = \rho \frac{\partial \theta}{\partial \xi^\mu} \) where \( \rho \) and \( \theta \) are the polar variables in the \( (\xi^1, \xi^2) \) plane and writing the most general radial connection in the form

\[ \omega^{ab}_\mu(\xi) = \varepsilon^{abc} \varepsilon_{\mu \rho \nu} P^\rho A^\nu_c(\xi) \] (3.6)

with

\[ A^\rho_c(\xi) = T_c \left[ \Theta^\rho \beta_1 + T^\rho \frac{(\beta_2 - 1)}{\rho} \right] + \Theta_c \left[ \Theta^\rho \alpha_1 + T^\rho \alpha_2 \rho \right] + P_c \left[ \Theta^\rho \gamma_1 + T^\rho \frac{\gamma_2}{\rho} \right] \] (3.7)

the metric becomes

\[ -ds^2 = (A_1^2 - B_1^2)dt^2 + 2(A_1 A_2 - B_1 B_2)dt d\theta + (A_2^2 - B_2^2)d\theta^2 - d\rho^2 \] (3.8)

where \( A_1 + 1, B_1, A_2, B_2 \) are the primitives in \( \rho \) of the functions \( \alpha_1, \beta_1, \alpha_2, \beta_2 \). In terms of such functions the symmetry constraint becomes equivalent to the system [18]

\[
\begin{align*}
A_1 \alpha'_2 - A_2 \alpha'_1 + B_2 \beta'_1 - B_1 \beta'_2 &= 0 \\
\alpha_2 \gamma_1 - \alpha_1 \gamma_2 + A_2 \gamma'_1 - A_1 \gamma'_2 + \frac{\partial \beta_1}{\partial \theta} - \frac{\partial \beta_2}{\partial t} &= 0 \\
\beta_2 \gamma_1 - \beta_1 \gamma_2 + B_2 \gamma'_1 - B_1 \gamma'_2 + \frac{\partial \alpha_1}{\partial \theta} - \frac{\partial \alpha_2}{\partial t} &= 0.
\end{align*}
\] (3.9)

Three of the functions \( \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \) can be given freely while the others are determined by eq.(3.9). We are interested in solutions in which the source \( T^{\mu \rho} \) has some bounded support in space. The condition for a bounded support is that the invariants of the holonomies become constant outside the sources. These conditions are very simple to express in presence of a Killing vector. For example if \( \frac{\partial}{\partial \theta} \) is a Killing vector they assume the form

\[
\beta_2^2 - \alpha_2^2 - \gamma_2^2 = \text{const} \quad \text{and} \quad \alpha_2 B_2 - \beta_2 A_2 = \text{const} \] (3.10)

while if \( \frac{\partial}{\partial t} \) is a Killing vector they assume the form [18]

\[
\beta_1^2 - \alpha_1^2 - \gamma_1^2 = \text{const} \quad \text{and} \quad \alpha_1 B_1 - \beta_1 A_1 = \text{const}. \] (3.11)
The first correspond to the conservation in time of the mass and angular momentum, while the second correspond to the independence outside the source of the Poincaré invariants of lines parallel to the Killing vector $\frac{\partial}{\partial t}$.

Both in presence of the Killing vector $\frac{\partial}{\partial \theta}$ and of the Killing vector $\frac{\partial}{\partial t}$ the constraint equations can be solved by means of quadratures. For example in the stationary case we have [18]

$$\alpha_2 = \frac{B_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \rho} \left( \frac{N}{B_1} \right) + 2\alpha_1 I$$

$$\beta_2 = \frac{A_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \rho} \left( \frac{N}{A_1} \right) + 2\beta_1 I$$

$$\gamma_2 = \frac{B_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \theta} \left( \frac{A_1}{B_1} \right) + 2\gamma_1 I,$$

where

$$I = \int_0^\rho d\rho' \frac{N(A_1^2 \beta_1^2 - B_1^2)}{(B_1^2 - A_1^2)^2}; \quad N = \frac{1}{2\gamma_1} \frac{\partial}{\partial \theta} (A_1^2 - B_1^2)$$

(3.12)

(3.13)

By using these formulas or in the simpler cases by using directly eqs.(3.9) one can write down with great ease all solution written in the literature [1] and also other solutions [18].

In addition to the standard exterior “Kerr” solution [1, 5]

$$ds^2 = -(dt + 4GJd\theta)^2 + (1 - 4Gm)^2(\rho - \rho_0)^2 d\theta^2 + d\rho^2$$

(3.14)

we can mention the exterior metric generated by a closed string with tension [1]

$$ds^2 = -k(\rho + c_1)^2(dt + 4GJd\theta)^2 + g_{\theta\theta}d\theta^2 + d\rho^2$$

(3.15)

with $k > 0$ and $g_{\theta\theta} = \text{const} > 0$ and the “linear” universes [1]

$$ds^2 = 2g_{t\theta}dt d\theta + k(\rho + c_1)d\theta^2 + d\rho^2$$

(3.16)

$$ds^2 = -k(\rho + c_1)dt^2 + 2g_{t\theta}dt d\theta^2 + d\rho^2$$

(3.17)
with \( g_{t\theta} = \text{const}. \) It is also possible [18] to write down time dependent solutions which satisfy the energy condition over all space-time.

**Causality**

The discovery by Gott [12] of a simple system in \( 2+1 \) dimensions which possesses closed time-like curves (CTC) has revived the issue of the consistency of general relativity with causality [19, 20]. The first concrete example of a solution of Einstein's equations which possesses CTC was given in 1949 by Goedel [19]. Actually Goedel's example, due to the trivial dependence in the \( z \)-coordinate, is one of the first solution of \( 2+1 \) dimensional gravity (in presence of a negative cosmological constant). In Goedel's universe the violation of causality occurs for very large trips. Also the \( 2+1 \) dimensional "Kerr" solution eq.(2.26) as pointed out in [5] and easily checked, shows CTC near the origin, while at large distances there are no CTC's. The problem of CTC's is in part related to the energy theorem in the following sense: According to the theorem [14,10] which we discussed at the end of the last section, in an open universe with time-like momentum there cannot be Gott pairs. This decreases the probability that an open time-like universe composed of point particles contains CTC's, but is not yet a proof of their absence.

For a closed universe composed of a finite number of point-like spinless particles 't Hooft [21] gave a general treatment which allows to prove that even though the formation of Gott's pairs is energetically permitted, the universe collapses before a traveler can complete a CTC.

With regard to the occurrence of CTC's in the "Kerr" solution eq.(2.26) such a problem was examined in [23] by employing the Fermi-Walker gauge described in the previous section. The physical nature of such a gauge allows to extract useful information from the energy condition which here is used in the weak form (WEC) stating that for any time-like \( u^\mu \) we have \( u^\mu T_{\mu\nu}u^\nu < 0 \). The following theorem holds [23]:

\[
15
\]
For a stationary open universe with axial symmetry if the matter sources satisfy the WEC and there are no CTC at space infinity, then there are no CTC at all. Thus the “singular source” related to (2.26) does not satisfy the WEC.

In fact from the WEC and the absence of CTC’s at infinity it is possible to prove that

\[ \frac{d}{d\rho} \left( \frac{g_{\theta\theta}}{\det(\epsilon)} \right) > 0. \]

As \( g_{\theta\theta} = 0 \) for \( \rho = 0 \) and \( \det(\epsilon) \) cannot vanish in open universes with axial symmetry [23], \( g_{\theta\theta} \) is always positive. Thus the absence of CTC’s is the result of a subtle interplay between the WEC and boundary conditions. The hypothesis of absence of CTC’s at infinity is a necessary one as one can construct examples of universes which satisfy the WEC but have CTC’s at space infinity [23].

With the same techniques the theorem on the absence of CTC’s can also be proved for all closed stationary universes with axial symmetry. With regard to the extension to non axially symmetric stationary universes at present the proof goes through provided \( \det(\epsilon) \) in the Fermi-Walker gauge never vanishes but one expects to be generally true for stationary universes.

Due to the simple structure of 2+1 dimensional gravity one would expect a very general simple statement about CTC’s but at present this is lacking.

Finally we remark that in a model with point-like particles, such particles have to be taken with zero angular momentum otherwise we have a violation causality or equivalently of the energy condition.

**Canonical quantization**

The most direct approach to quantizing 2+1 dimensional gravity is the canonical approach. One starts fixing a priory the topology of space-time. Compact 3-dimensional manifolds are to be avoided due to causality requirements [24]. Then the simplest choice
is $\mathcal{M} = \Sigma \times R$ with $\Sigma$ a compact orientable two-dimensional manifold. The classical ADM [25] Hamiltonian formulation of gravity in $n+1$ dimensions starts with the action

$$ A = \int \left( \pi^{ab} g_{ab,\tau} - N H - N^a \mathcal{H}_a \right) dx^2 dt $$

(4.1)

with

$$ H = \frac{1}{\sqrt{g}} [\pi^{ab} \pi_{ab} - (\pi^a_a)^2] - \sqrt{g} R $$

(4.2)

and with

$$ \mathcal{H}_a = -2 \nabla_b \pi^b_a $$

(4.3)

where $\nabla_b$ is the covariant derivative with respect to the space metric of the $n$ dimensional space-like manifold defined by $t = \text{const}$, $g$ is the determinant of the metric of the space slice and $R$ is the scalar curvature of the space slice. It is well known that the role of the lapse and shift variables $N$ and $N^a$ is that of Lagrange multipliers. We know that for $\Sigma \approx S^2$ the connections are trivial and thus the model is trivial. There are two ways to give the theory a non trivial content; either one introduces a non trivial space topology or one adds matter. The first alternative is the simplest. Thus one takes for $\Sigma$ an orientable two dimensional manifold of genus $g \geq 1$.

Moncrief [15] and Hosoya and Nakano [26] have shown that the classical system described by (4.1) in $2+1$ dimensions is equivalent to a Hamiltonian system of $6g - 6$ degrees of freedom for $g > 1$ and 2 degrees of freedom for $g = 1$ i.e. for the torus. The reductions to a finite number of degrees of freedom is not surprising if one takes into account that in $2+1$ dimensions there are no gravitons and thus all the dynamical role is played by the non trivial independent holonomies of $\Sigma$. The method is to introduce a foliation of space-time with space-like surfaces of constant exterior curvature $K$. Such a quantity plays the role of time. The metric description of the space-like surface of constant $K$ is provided by a metric $h_{ab}$ of constant scalar curvature $(-1$ for $g > 1$ and $0$ for $g = 1$)
and a conformal factor $e^{2\lambda}$

$$g_{ab} = e^{2\lambda} h_{ab}; \quad R(h) = -1 \quad \text{or} \quad 0 \quad \text{for} \quad g = 1 \quad (4.4)$$

plus the Teichmueller parameters $\tau_\alpha$ which are 2 for $g = 1$ and $6g - 6$ for $g > 1$.

The momentum constraints $\mathcal{H}_a = 0$ impose on the conjugate momenta the structure

$$\pi^{ab} = \pi^{a\overline{b}TT} + \frac{1}{2} K \sqrt{g} g^{ab} \text{ with } \pi^{TT} \text{ transverse and traceless, while the hamiltonian constraint } \mathcal{H} = 0 \text{ determines the conformal factor } \lambda \text{ as the solution of a non linear elliptic differential equation. There are [15] existence and uniqueness theorems for the solution of such differential equation. At this stage is not difficult to perform the reduction of the degrees of freedom. One substitutes in (4.1) } \mathcal{H}_a = 0, \mathcal{H} = 0 \text{ and defined}

$$p_\alpha = \int_{\Sigma} e^{2\lambda} \pi^{a\overline{b}TT}(x) \frac{\partial h_{ab}}{\partial \tau_\alpha}(x, \tau) dx^2 \quad (4.5)$$

one reaches apart from some boundary terms which are not relevant for the equations of motion the action

$$A = \int [p_\alpha \frac{d\tau_\alpha}{dt} - \frac{dK}{dt} \int_{\Sigma} \sqrt{g} \ dx^2] \ dt. \quad (4.6)$$

Thus the Hamiltonian is simply given by the area of the space-like slice $K = \text{const}$; however in the canonical formalism one has to express such an area in terms of the canonical variables. In the simple case of the torus this can be accomplished with the result [15, 28]

$$\int_{\Sigma} \sqrt{g} \ dx^2 = \frac{1}{K} [\tau_2^2 (p_1^2 + p_2^2)]^{1/2}. \quad (4.7)$$

The $K$ dependence in the Hamiltonian can be removed [15] by putting $K = \exp(t/(2\pi)^2)$. One can now proceed to the canonical quantization by posing $p_j = -i \frac{\partial}{\partial \tau_j}$ thus reaching the Schroedinger equation

$$i \frac{\partial \psi(\tau, t)}{\partial t} = [\frac{\tau_2^2 \nabla^2}{2}]^{1/2} \psi(\tau, t). \quad (4.8)$$
For higher genus the expression of the reduced Hamiltonian is given only implicitly through the solution of a non linear elliptic differential equation.

The eigenvalue equation associated to eq. (4.8) is of the Bessel type and thus appears to be easily soluble but this is not so, because one should impose on the solution the invariance under modular transformations of \( \tau = \tau_1 + i \tau_2 \)

\[
\tau \to \tau + 1, \quad \tau \to -\frac{1}{\tau}
\] (4.9)

which correspond to the large diffeomorphisms.

The ADM approach is the typical second order approach to gravity. One can however formulate gravity also in the first order approach with dreibeins and connections taken as independent variables and which shares a greater similarity with the usual gauge theories. The action in 2 + 1 dimensions is given by

\[
A = -\frac{1}{2} \int R_{ab} \wedge e^c \varepsilon_{abc}
\] (4.10)

which apart a divergence term has, in the notation employed in sect.3, the form

\[
A = \int dt \int d^2x^2 \epsilon^{ij} (-\omega_{aj} \epsilon_i^a + \epsilon_0^a \epsilon_{abc} R_{ij}^c + \omega_{a0} S_{ij}^a)
\] (4.11)

where \( R_{ab} \) is the curvature and \( S^a \) is the torsion. From eq.(4.11) one derives the canonical P.B.

\[
\{ \epsilon_i^a(x), \omega_j^a(y) \} = -\epsilon_{ij} \delta^a(x-y)
\] (4.12)

and two constraints \( R_{ab} \approx 0 \) and \( S^a \approx 0 \). Thus we have that the group curvature of the Poincaré connection is locally zero and the only non trivial quantities are the holonomies related to non contractible loops. It is of great interest to study the algebra of such holonomies or better of the invariant quantities which characterize them. We recall that there are two invariants for each holonomy i.e. the “angle” of the Lorentz transformation and the projection of the “displacement vector” on the “rotation axis”.

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An equivalent choice of invariants is given by \( q = \text{tr} \mathcal{M} \) and \( \nu = \mathcal{M}^a_{b'c} \varepsilon_a^b E^c \), where \( \mathcal{M} \) is the Lorentz transformation in the adjoint representation. Starting from the canonical P.B. eq.(4.12) Nelson and Regge [6, 27] have derived the P.B. of \( q \) and \( \nu \) relative to two loops \( u \) and \( v \) with a single intersection. They are [6]
\[
\{ q(u), \nu(v) \} = \frac{1}{2} (q(uv) - q(uv^{-1}))
\]
and
\[
\{ \nu(u), \nu(v) \} = \frac{1}{2} (\nu(uv) - \nu(uv^{-1}))
\]
all the other P.B. as well as those of non intersecting loops being zero. Using such relations one can compute recursively for example, all the invariants of the loops generated by two intersecting loops. We know however that there are only a finite number of invariant quantities (4 for the torus and \( 12g - 12 \) for \( g > 1 \)) and thus we have the problem of the reduction to a minimal set of invariant and that of giving the representation of this algebra at the quantum level.

For the torus we have only two generators for the group \( \pi_1 \) and in this simple case the program can be easily completed with the following result. By means of a global Poincaré transformation one can reduce both holonomies related to the two generators \( u \) and \( v \) to the form of a boost in the \( t, x \) plane and a translation in the \( y \) direction. If we call \( \lambda \) the boost parameter and with \( a \) the translation for the \( u \) loop and \( \mu \) and \( b \) those for the \( v \) loop we have from eqs.(4.13, 4.14) the following result
\[
\{ \lambda, 2b \} = \{ \mu, -2a \} = 1
\]
all the other P.B. being zero.

The same program has been carried through by Nelson and Regge explicitly for \( g = 2 \) and implicitly for any \( g \) [27]. It is of interest to relate this approach which contains only constants of motion to the ADM-type of approach described previously. Clearly
some space-like foliation has to be performed to introduce the concept of time. This has been accomplished for the torus by Carlip [28, 29] by introducing the foliation \( t = K^{-1} \cosh(\xi) \) and \( x = K^{-1} \sinh(\xi) \), \( K \) constant and \( \xi \) and \( y \) variable, where \( K \) is just the extrinsic curvature of the two dimensional surface embedded in Minkowski space and plays the role of time as in the Moncrief approach.

It is simple to compute the complex modulus of the space-slice

\[
\tau = (a + iK^{-1} \lambda)^{-1}(b + iK^{-1} \mu)
\] (4.16)

while the area and thus the Hamiltonian (see eq.(4.6)) becomes [28]

\[
H = \frac{a \mu - \lambda b}{K}.
\] (4.17)

Again the wave function will be required to be invariant under modular transformations expressed in the new \( \lambda, \mu \) coordinates. One can then relate the two approaches i.e. change from the \( \lambda, \mu \) representation to the \( \tau_1, \tau_2 \) representation through the following steps [29]: a) Compute the \( p_1, p_2 \) conjugate to \( \tau_1, \tau_2 \) as a function of the variables \( \lambda, a. \)

b) Compute the eigenfunctions \( \psi_\tau(\lambda, \mu) \) of \( \tau \). Being \( \tau \) eq.(4.16) a normal operator the eigenfunctions of \( \tau \) are simultaneously eigenfunctions of \( \tau_1 \) and \( \tau_2 \). c) Compute the \( p_1, p_2 \) in terms of \( \tau_1, \tau_2 \). The result is [28]

\[
p_1 = -i \frac{\partial}{\partial \tau_1}; \quad p_2 = -i \frac{\partial}{\partial \tau_2} + \frac{i}{\tau_2}
\] (4.18)

and substituting into eq.(4.17) one yields for the Hamiltonian

\[
H^2 = K^{-2}(\tau_2^2((\frac{\partial}{\partial \tau_1})^2 + (\frac{\partial}{\partial \tau_2})^2) + i \tau_2 \frac{\partial}{\partial \tau_1} - \frac{1}{4}).
\] (4.19)

The request of invariance under modular transformations in the \( \lambda, \mu \) space imposes the eigenfunction of eq.(4.19) to transform like modular form of weight \( 1/2 \). Thus while the two approaches are equivalent at the classical level they are not at the quantum level.
In a series of papers to which we refer for full details, Carlip [29] and Carlip and Nelson [30] have analyzed the source of this difference. The use of one or the other approach leads to a natural choice of fundamental variables which differ in the two cases. The translation of the classical theory to the quantum theory, as is well known, is subject to the problem of the ordering of the operators, which at least in part is related to the choice of the metric defining the scalar product in Hilbert space.

5. Tessellation and polygonal approaches

Usually the lattice is employed in field theory as a non perturbative regulator. In 2 + 1 dimensional gravity due to the local flatness of space-time it is possible to set up lattice-like schemes which are exact.

First we describe the tessellation approach of 't Hooft [21,22]. The main idea is to give a foliation of space-time in terms of two dimensional space-like surfaces which are piecewise flat and thus are built up by gluing together polygons. To have the time variable to flow at the same rate on two adjacent polygons one must choose the Lorentz frame on each polygon in such a way that the seam between the two polygons moves with the same (and “opposite”) speed with respect to the two Lorentz frames. The geometry of a given time slice is described by the length of the sides $L_i$ and the angles $a_i$ of the polygons and by the magnitude $2\eta_i$ of the Lorentz boost across the side $i$. Actually it is possible to compute the angles of the polygons from the boosts $\eta_i$ by imposing the compatibility of the Lorentz transformations across the sides which converge to a single vertex. Thus the $\eta_i$ and $L_i$ can be chosen as the fundamental variables in this approach.

There are constraints on the fundamental variables $L_i$ and $\eta_i$ because to close each polygon we must have

$$\sum_i (\pi - a_i) = 2\pi \quad (5.1)$$
and

\[ \sum_i e^{i\alpha_i} L_i = 0 \]  

(5.2)

where the sum extends to each single polygon.

Particles can be introduced by removing from a polygon a sector whose opening is given by \( \delta = 8\pi G m \) if the particle is at rest with respect to the Lorentz frame of the polygon. If the particle moves with respect to such frame a proper boost has to be applied. Due to the presence of the boosts and the motion of the particles, the lengths \( L_i \) of the sides change in time and can also vanish. Other sides can also originate for the same reason and these processes are encoded in nine transition rules which are described in detail in [22]. Such a deterministic system has been employed by 't Hooft to prove the absence of formation of CTC's in a closed universe containing point-like spinless particles and to study by means of numerical calculations simple three dimensional cosmological models. To proceed to quantization one has first to set up an hamiltonian description of the system. There are two elegant features in the hamiltonian formalism: the first are the remarkably simple P.B.

\[ \{2\eta_i, L_j\} = \delta_{ij} \]  

(5.3)

and the second the fact that the Hamiltonian is given by all the deficit angles

\[ H = \frac{1}{8\pi G} \sum_V (2\pi - \sum_i \alpha_i(V)). \]  

(5.4)

These deficits are both due to the presence of a particle and to the fact that \( \sum_i \alpha_i(V) \) extended to the angles converging to a single vertex \( V \) is not \( 2\pi \) due to the presence of the boosts. Recalling that the \( \alpha_i(V) \) can be computed from the boosts we have that the Hamiltonian is only function of the boosts. One has however keep in mind that in addition to the constraints eq.(5.1, 5.2) there are other constraints of the type of triangular inequalities among the boosts which have to be taken into account [22].
Quantization is achieved by replacing the P.B. with commutators and the classical transition rules mentioned above now play the role of boundary conditions on the wave function. A remarkable consequence of the form (5.4) of the Hamiltonian is that as the \( \alpha_i \) are determined in terms of the \( \eta_i \) by inverting trigonometric relations, the \( \alpha_i \) and as a consequence \( 8\pi GH \) is determined only modulo \( 2\pi \). This means that the evolution operator \( \exp(-iHt) \) is well defined only for discrete values of the time parameter \( t = 8\pi G n \). It would be nice to apply this quantization scheme to some simple instances.

Waelbroeck’s polygonal approach [31] is strictly related to the first order formulation of gravity and to the algebra of observables of Nelson and Regge of which it can be considered as a geometrical realization. The space-like slice is described by a polygon in flat Minkowski space in which pairs of sides are identified. Thus we have as variables the three vectors representing the sides and for each side an element of the Lorentz group which represents the Lorentz holonomy which takes from the given side to its image. One denotes the sides by \( E(\mu) \) with \( \mu = 1, \ldots, 2g \) with \( g \) the genus of the surface and the Lorentz holonomies which relate the side \( \mu \) with its image by \( M(\mu) \). Thus the Lorentz component of the holonomy related to a closed path which joins \( E(\mu) \) with its image is simply given by \( M(\mu) \) and it is also possible to reconstruct the translation part \( E_{\sigma} \) of the Poincaré holonomy related to a closed loop \( \sigma \), in terms of the \( M(\mu) \) and the \( E(\mu) \) [31].

There are two constraints: the first is that the polygon must close i.e.

\[
\sum_{\mu=1}^{2g} (I - M^{-1}(\mu)) E(\mu) = 0
\]  

(5.5)

and the other, like in the algebra of observables, tells us that the Lorentz relator equals the identity

\[
W = M(1) M^{-1}(2) M(1)^{-1} M(2) \ldots M(2g-1) M^{-1}(2g) M(2g-1)^{-1} M(2g) = I.
\]  

(5.6)
Spinless point particles can be introduced by inserting along the polygon two consecutive sides, one given by $E(\mu)$, $\mu = 2g + 1 \ldots 2g + N$, $N$ being the number of particles, and the other by its image $M^{-1}(\mu)E(\mu)$ under the time-like Lorentz transformation $M(\mu)$ subject to the constraint that its rotation angle is related to the mass of the particle by $\delta(\mu) = 8\pi G m(\mu)$ or equivalently $\text{tr} M(\mu) = 2 \cos^2 \delta(\mu) + 1$. The closure condition of the polygon has to be accordingly modified to include these additional couples of sides. As in the ADM approach the Hamiltonian is provided by a combination of the constraints with lapse and shift functions

$$H = \sum_a \frac{1}{2} N_a \varepsilon^a_{\,\,b} \, W^b_c + \sum_{\mu=2g+1}^{2g+N} N(\mu) \, \text{tr} M(\mu), \quad (5.7)$$

the second sum representing the contribution of the particles. The three-vector $N_a$ is a time-like vector which can be represented by a vector placed e.g. at the origin of the side $E(1)$ and by its images obtained by parallel transport along the polygon. The $N(\mu)$ are in principle arbitrary constants. The correct equations of motion are generated by the following simplectic structure

$$\{E^a(\mu), E^b(\mu)\} = \varepsilon^a_{\,\,b} E^c(\mu) \quad (5.8)$$

$$\{E^a(\mu), M^b_c(\mu)\} = \varepsilon^a_{\,\,b} dM^d_c(\mu) \quad (5.9)$$

all the others P.B. being 0, combined with the Hamiltonian $H$, and they are

$$\frac{dM(\mu)}{dt} = 0 \quad (5.10)$$

$$\frac{d^2 E(\mu)}{dt^2} = 0 \quad (5.11)$$

which simply express that the Lorentz holonomies are constants of motion and that the sides $E(\mu)$ (with this choice of time) vary linearly in time. The constraints (5.5,
5.6) are first order constraints. In order to proceed to quantization one has to give a representation of the eqs. (5.8, 5.9) in terms of coordinates and conjugate momenta. Defining the vector

\[ P_\alpha(\mu) = \frac{1}{2} \varepsilon^{b}_{\alpha} c M^c_{b}(\mu) \]  

(5.12)

the conjugate vector to it is [31]

\[ X(\mu) = \frac{1}{P^2(\mu)} (P(\mu) \wedge J(\mu) - \frac{2(E(\mu) \cdot P(\mu)) P(\mu)}{\text{tr} M(\mu) - 1}) \]  

(5.13)

with \( J(\mu) \equiv (1 - M^{-1}(\mu)) E(\mu) \). Thus one can pose

\[ [X^a(\mu), P_b(\mu)] = i \delta^a_b \]  

(5.14)

and then represent the momentum as \( P_b(\mu) = -i \partial / \partial X^b(\mu) \). The relation between the fundamental variables \( M^a_{b}(\mu) \) and \( E^a(\mu) \) and the \( X^a(\mu) \) and \( P_a(\mu) \) are algebraic and even though invertible they contain square roots in non simple combinations. The simplest quantum mechanical application is the writing of the equations corresponding to the constraints and to the Wheeler-de Witt equations. The Lorentz constraint equations are

\[ J^a \psi(X) = 0 \]  

(5.15)

where \( J^a \) in the new variables \( X \) and \( P \) takes the simple form

\[ J = \sum_{\mu} X(\mu) \wedge P(\mu) \]  

(5.16)

and the Wheeler de Witt equations are

\[ P^a \psi(X) = 0. \]  

(5.17)

Eq. (5.15) simply tells us that \( \psi \) has to be a scalar in the \( X(\mu) \) variables. We recall that \( P_a = \varepsilon^c_{a} b W^c \gamma / 2 \) and \( W \) is a function of the \( P_a(\mu) \) containing square roots of \( P^2(\mu) \pm 1 \).
and thus is a non rational function of $\frac{-i\partial}{\partial X^\mu(\mu)}$. This is similar to what is found in the ADM and first order formulation of quantum gravity already in the simple instance of the torus in absence of particles.

An alternative to writing the Wheeler de Witt equation is to introduce an internal time, like the length of a side of the polygon and by using the constraints solve the conjugate momentum to that variable in terms of the remaining variables. That gives the Hamiltonian relative to the given choice of internal time and one can proceed to write down the Schrödinger equation. Such an approach however appears to be still inequivalent to the previously described approaches [31].

**Conclusions**

We saw that $2 + 1$ dimensional gravity is well understood at the classical level even if there are still some problems to be solved. At the quantum level on the other hand we have a variety of approaches all of which appear to lead to inequivalent formulations. In addition only the simplest situations, e.g. absence of particles, and the simplest topologies have been thoroughly examined. One has to sort out which approach is the correct one and possibly examine more general situations.

**Acknowledgments**

I am grateful to the organizers of the conference for the kind invitation and to E. Guadagnini, R. Loll and especially D. Seminara for useful discussions.
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