A Hyperbolic Scenario to Chaotic Scattering: Horseshoe Truncation by Potential Deformation

G. Troll

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A HYPERBOLIC SCENARIO TO CHAOTIC SCATTERING : HORSESHOE TRUNCATION BY POTENTIAL DEFORMATION

G. TROLL

Techn. Univ. Berlin, FB Mathematik, D-10623 Berlin, Germany,
FAX: 030-314 24413, *E-mail*: troll@math.tu-berlin.de
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ABSTRACT. In this paper we study a transition scenario of a class of 2D potential scattering systems from a regular to a chaotic dynamics. We consider potentials which are attractive, central, continuous and smooth on their non-overlapping compact supports. The dynamics is discretized by a Poincaré map describing a single scattering incident. The origin of chaos is trapping by multiscattering. We identify conditions under which it is possible to induce a transition to chaos by deforming the potentials almost smoothly in such a way that the associated parameterized family of Poincaré maps is generated from a single map by restriction on a shrinking support. As a consequence, all hereditary properties of the dynamics, e.g. hyperbolicity, are conserved under parameter variation. This is in stark contrast to commonly known scenarios of bounded systems. The price one has to pay for this remarkable property is a loss of differentiability on a finite set of points, essentially on the boundaries of the potentials supports. We construct the deformation family of potentials by truncating the associated deflection functions and using classical inverse scattering theory to reconstruct the corresponding potentials. A simple scattering system introduced by U. Smilansky and myself in [5] falls into this class, but the inversion conditions are weak enough to admit a broad range of other examples such as the truncated Coulomb potential and also hard core potentials.

Key words and phrases. chaotic scattering, transition to chaos, Hamiltonian dynamics, truncated horseshoe.

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1. INTRODUCTION

1.1. The problem. Suppose one has a set of attractive potentials in $\mathbb{R}^2$, which are central, of finite range (compact support) and non-overlapping. From a finite distance one injects mono-energetic test particles into this set and detects escaping particles at a screen also situated within a finite distance. Suppose this Hamiltonian multi-scattering dynamics is *chaotic* (irregular). By this we mean that there is a Cantor set of trapped trajectories whose points are Lyapunov unstable. Points on escaping trajectories are stable if the potentials are smooth, because for them the time needed to reach the detector is finite and represents a function continuous at initial conditions for which the trajectory is bound to escape.

Such a multi-scattering dynamics offers a natural discretization procedure by using one single scattering as a time-one step. We suppose that a time-one (Poincaré) map has been constructed in some low dimensional phase space of a few angles and angular momenta. A straightforward way to do this uses the deflection functions of the potentials and it will be the deflection functions which we are going to vary in order to study the following **transition problem**: Is there a way to deform the potentials (how smoothly ?) in such a way that the associated Poincaré map preserves as many features as possible (foremost hyperbolicity) but that nevertheless the scattering eventually becomes *regular* in the sense that the set of trapped trajectories becomes trivial (does not any more contain a Cantor set).

The motivation for this question is an attempt to reduce the transition to chaotic scattering to its pure bones which can be studied without additional problems of more complicated bifurcations. Besides this, however, the answer substantiates, in a more general context, earlier claims that truncating a horseshoe instead of smoothly deforming it is not only a formal procedure to simulate chaotic scattering but is also realizable as potential scattering.

1.2. Truncated horseshoes as formal scattering systems. A clue as to how this problem may be tackled (actually a motivation for me to pose this problem) is given by the formal multi-scattering systems of truncated horseshoes, which were introduced in reference [7]. Truncating a horseshoe (hyperbolic, in particular no tangencies) is – for the time being – a formal way to open up a bound dynamical system represented by a horseshoe. More specifically, one cuts off a part of the horseshoe map's T full compact domain Q from which the horseshoe is generated by forward and backward iteration as $\Lambda = \bigcap_{i \in \mathbb{Z}} T^i(Q)$. In the classical example of Smale's horseshoe this full fundamental domain Q is just the unit square $[0, 1]^2$. The part $C(Q)$ cut off the fundamental domain Q represents a window through which particles may escape and we will call it escape window here.

Actually, we are going to introduce a nested and with respect to this nesting monotonically increasing parametrized family $\{Q_\kappa\}$ of closed subsets Q_κ of the full fundamental domain Q . These nested subsets $\{Q_\kappa\}$ will be interpreted as fundamental domains of an associated parametrized family $\{T_\kappa\}$ of maps whose essential property is that two such maps coincide on the smaller one of their domains: $T_\kappa|_{Q_\kappa} = T|_{Q_\kappa}$. On their escape windows the maps are

discontinuously redefined to assure escape without return to their fundamental domain. Points which both under forward and backward iteration never enter the escape window but stay forever in its fundamental domain are interpreted as lying on bound or trapped orbits. These points form the it trapped set Λ_κ :

$$(1) \quad \Lambda_\kappa := \Lambda(T_\kappa, Q_\kappa) = \bigcap_{i \in \mathbb{Z}} T_\kappa^i(Q_\kappa) := \bigcap_{i \in \mathbb{Z}} T^i(Q_\kappa)$$

Those points which eventually enter the escape window are considered to belong to escaping (or scattered in the strong sense) trajectories.

What simplifies now the study of the evolution of the trapped sets is the fact that the associated dynamical systems on their trapped sets are just restrictions of the full horseshoe map. Therefore all properties of the full horseshoe which are *hereditary*¹ are common to the whole family of dynamical systems. Hyperbolicity is such a hereditary property. This justifies the name *truncated horseshoes* for the resulting dynamical systems.

It is important to note that although truncated horseshoes are hyperbolic they are not structurally stable in any case. This is due to a boundary effect: For those parameter values where some trapped trajectories have points on the boundary of the fundamental domain an arbitrarily small parameter change is enough to enlarge the escape window beyond these points thus changing the trapped set into a non-equivalent one.

1.3. The truncated horseshoe scenario. Now, although orbits which do not enter an enlarged escape window are not changed in the new dynamical system, the new horseshoe for an enlarged escape window is not just the intersection of the old horseshoe with the smaller fundamental domain. Once an orbit has even a single point within the enlarged escape window it is broken up and will escape at this point never to return. Therefore all points of this orbit including those points within the new and smaller fundamental domain do not belong to the new horseshoe of trapped orbits.

This fact together with the expansiveness of the horseshoe map leads to a fast (exponential) thinning out of the horseshoe when varying the family parameter in the direction of enlarging the escape window at the expense of the fundamental domain: Since arbitrarily close points on the horseshoe will eventually separate by at least some global constant depending on the specific horseshoe map one expects under moderate conditions that there is a finite parameter range where the horseshoe degenerates to a finite set or even to a single point.

Indeed, suppose the family parameterization is a continuous function of the diameter of the fundamental domain measured along the unstable manifolds. If this *unstable diameter* of the fundamental domain shrinks to zero one finds a regime of the horseshoe family in a parameter interval of positive length, where the horseshoes are degenerate, i.e. they are empty (empty regime) or they consist only of a finite number of points (periodic regime).

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¹We call a property **P** of a map $T : M \rightarrow M$ hereditary if each restriction of T to a closed invariant set has **P**.

We call the parameter set where the horseshoe is degenerate the *regular regime* because the trapped set of the dynamics is trivial.

The *critical value*, i.e. the upper limit of this interval, separates the regular and chaotic regimes. Calling any parameter value where the trapped set changes a bifurcation value, one can identify two bifurcation scenarios: Either the critical value is isolated in the bifurcation set from the left, or the bifurcation values accumulate towards it from the left. These scenarios were called the *abrupt* and the *cascade (truncation) scenario* in reference [7, 9].

Examples for these two scenarios are the truncated double horseshoe (Baker map) and the truncated sawshoe whose chaos and complexity transitions were also studied in references [7, 9]: The Baker map is the simplest example being a variant of the classical horseshoe with two bends in order to make the Jacobian positive definite everywhere. It is truncated by a family of horizontal lines and shows an abrupt transition to chaos. The sawshoe is the linearization, which happens to be also the asymptotics, of a simple model of actual potential scattering introduced formally in reference [5] and proven to derive from a potential in reference [9]. The truncation parameter is here the range of the potentials. The transition scenario is a cascade transition. The model will be briefly described in section 2.

1.4. Solution (overview). We are going to use the truncation scenario for the solution of the problem stated above. The question we have to address is: What besides a formal analogy have these truncated horseshoes to do with potential scattering? This question will be answered in two steps during which we will already introduce some basic notation for the following main theorem.

First, truncated horseshoes arise naturally in potential scattering when one uses the deflection functions of the scattering potentials for the construction of the Poincaré map. The details are worked out in the next section. Actually one meets in the more general context a slight generalization of the horseshoe construction of equation (1) where one allows in each iteration a different map (cf. equation (6) below). Suppose a set $\{V^{[p]}\}_{p \in P}$ of potentials $V^{[p]}$ of finite range $b_{\max}^{[p]}$ is given. The finite range of the potentials corresponds to the compactness of the supports of the deflection functions $\theta_{b_{\max}^{[p]}}$. The easiest way to truncate the resulting horseshoe of trapped trajectories is to truncate the deflection functions as follows: Shrink the supporting interval $[0, b_{\max}^{[p]}]$ of each deflection function $\theta_{b_{\max}^{[p]}}^{[p]}$ of the full horseshoe to an interval $[0, \beta^{[p]}]$, $0 < \beta^{[p]} \leq b_{\max}^{[p]}$, and define a new truncated deflection function $\theta_{\beta^{[p]}}^{[p]}$ as the restriction of the full deflection function $\theta_{b_{\max}^{[p]}}^{[p]}$ on the supporting interval $[0, \beta^{[p]}]$ and 0 outside. The *control parameters* are thus the ranges $\{\beta^{[p]}\}$ of the deformed potentials associated to the truncated deflection functions.

We postpone for a moment the natural question, how to get potentials $\{V_{\beta^{[p]}}^{[p]}\}_{p \in P}$ implementing these truncated deflection functions and take a look at the behaviour of trajectories in continuous time under change of the control parameter. It should be clear that as the corresponding discretized orbits are preserved, continuous time trajectories which are trapped and survive a decrease of the control parameter remain unchanged outside the range of the

potentials. They will of course change inside the potentials, but this is irrelevant for the transition problem we are interested in.

The second step now to be taken, which closes the circle back to potential scattering, requires the solution of the inverse scattering problem of finding potentials generating the truncated deflection functions. Following methods of [4, 3] we find this to be possible if one is satisfied with smoothness but for finitely many points² and if the potentials are arranged and deformed in a manner such that they satisfy a miss-one-miss-all condition. By this condition we mean that if a trapped trajectory misses a potential because of a change in the control parameter then the new trajectory must escape to infinity, that is it will not encounter another potential at the new value of the control parameter. Furthermore, we require that there are *open channels* between the potentials, i.e. for any fixed potential any two other potentials are either aligned shadowing each other completely or they are separated by an angle bounded away from zero through which particles escape.

Using the notation introduced in this subsection we can state the main theorem of this paper:

Theorem 1.1. *Given is a finite set of continuous central scattering potentials $\{V^{[p]}\}_{p \in P}$, $V^{[p]} :]0, \infty[\rightarrow \mathbb{R}$, and a particle energy $E_0 > 0$. Suppose each potential $V^{[p]}$ in this set satisfies the conditions:*

- (1) *each support $\text{supp } V^{[p]} \subset [0, b_{\max}^{[p]}]$, for some $0 < b_{\max}^{[p]} \leq b_{\max} < \infty$,*
- (2) *each restriction $V^{[p]}|_{]0, b_{\max}^{[p]}[}$ is smooth (C^2 suffices),*
- (3) *each deflection function $\theta^{[p]} : [0, b_{\max}^{[p]}] \rightarrow \mathbb{R}$ of $V^{[p]}$ is well-defined and bounded.*

and for $r^{[p]*} := \inf\{r \in]0, b_{\max}^{[p]}]; r^2(1 - V^{[p]}(r)/E_0) = b_{\max}^2\}$ and radius of closest approach $r_*^{[p]} := \sup\{r \in]0, b_{\max}^{[p]}]; V^{[p]}(r) = E_0\}$ either

- (4) *$r_*^{[p]} < r^{[p]*} < b_{\max}$ and $\theta^{[p]}'$ is monotone increasing*

or

- (4') *$r^{[p]*} = b_{\max}$ and $\theta^{[p]} > 0$*

then for each $p \in P$ there exists a nearly smooth (C^1) (see definition 3.1) deformation $\beta^{[p]} \mapsto V^{[p]}$ whose deflection function is just the truncated deflection function $\theta_{\beta^{[p]}}^{[p]}$.

If, furthermore, the multiscattering system, i.e. the discretized dynamical system, whose map describes a single scattering incident, consists of finitely many scatterers and satisfies the following conditions that

- (a) *the truncation family has the miss-one-miss-all property,*
- (b) *there are open channels of angular size bounded away from zero;*

and the following analytic conditions that

- (c) *the multiscattering system is hyperbolic,*

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²The non-differentiability of the potentials on the boundaries of the supports is unavoidable because if left and right derivatives coincided and thus equalled 0 on the boundary then each truncated deflection functions were also 0 on its boundary and consequently the full deflection function identically 0 and hence trivial.

(d) *and the unstable diameter of the fundamental domain is a continuous function of the control parameter $\kappa := \sup_p \{\beta^{[p]}\}$ tending to 0 for $\kappa \rightarrow 0$, then for $\kappa > 0$ small enough the multiscattering system becomes regular without losing hyperbolicity or any other hereditary property.*

Remark 1.1. With the sign convention chosen (cf. eq. (10)) conditions (4) and (4') exclude purely repulsive potentials ($V \geq 0$, $V' < 0$).

We split the proof of this theorem into several parts. Section 2 proves the second half of the theorem resulting from conditions (a) -(d). Section 3 proves the first half. There, suitable potential deformations are constructed in the proofs of propositions 3 and 4. In that section we shall give also a few examples, such as the truncated Coulomb potential, the sawshoe potential, and a hard-core potential.

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2. THE TRUNCATED HORSESHOE SCENARIO IN POTENTIAL SCATTERING

In this section we study how truncated horseshoes arise in potential scattering. The connection becomes apparent when one uses the deflection functions of the scattering potentials for the construction of the Poincaré map.

Given is a family $\{V^{[p]}\}_{p \in P}$ of central potentials $V^{[p]} : \mathbb{R}^2 \rightarrow \mathbb{R}$, all with compact, non-overlapping supports of finite diameters $0 < 2b_{\max}^{[p]} \leq 2b_{\max} < \infty$. We assume that there are *open channels* between the potentials, i.e. for any fixed potential any two other potentials are either aligned shadowing each other completely or they are separated by an angle which is bounded away from zero and through which particles escape. Furthermore, particles are injected from and detected at a detector/injector screen within a finite distance but outside all potential supports.

Because of conservation of angular momentum one can uniquely describe each trajectory piece between but outside two potentials by a triple (p, x, b) . The label p denotes the approached potential $V^{[p]}$ or escape, x is the angle with respect to a distinguished global direction, and b the angular momentum or impact parameter with respect to the local potential $V^{[p]}$. We call the compact cylinder $Q_{b_{\max}^{[p]}} = S^1 \times [-b_{\max}, b_{\max}]$ of pairs (x, b) where one may have another scattering incident at potential $V^{[p]}$ before a possible escape the *local fundamental domain*. As the ranges of the potentials are bounded, we may imbed the local fundamental domains into a common fundamental domain, the compact cylinder $Q_{b_{\max}} := S^1 \times [-b_{\max}, b_{\max}]$ (where S^1 is the unit circle). A discretization $\tilde{T}$ is then given by mapping a triple (p, x, b) to the following one (p', x', b') . $\tilde{T}$ decomposes naturally into two maps: $\tilde{T} = \tilde{T}_2 \circ \tilde{T}_1$. The first map $\tilde{T}_1$ translates the angle by the value of the deflection function $\theta^{[p]}$ and preserves angular momentum, the second map $\tilde{T}_2$ preserves the angle w.r.t the global direction and transforms the coordinates to the coordinate system of the next scatterer.

Definition 2.1. The deflection discretization $\tilde{T}$ of the potential scattering system introduced above is given by

$$(2) \quad \tilde{T} : P \times Q_{b_{\max}} \rightarrow P \times Q_{\infty}, \quad (p, x, b) \mapsto (p', x', b')$$

where $\tilde{T}$ decomposes into two maps $\tilde{T} = \tilde{T}_2 \circ \tilde{T}_1$.

$$(3) \quad \tilde{T}_1 : P \times Q_{b_{\max}} \rightarrow P \times Q_{b_{\max}}, \quad (p, x, b) \mapsto (p, x + \theta^{[p]}(b), b)$$

$$(4) \quad \tilde{T}_2 : P \times Q_{b_{\max}} \rightarrow P \times Q_{\infty}, \quad (p, x, b) \mapsto (p', x, b + G^{[p]}(x, b))$$

Here $\theta^{[p]}$ is the deflection function of the potential $V^{[p]}$ and the maps $G^{[p]}$ depend only on the geometric arrangement of the potentials.

In a slight generalization of equation (1) the trapped set can then be written as

$$(5) \quad \tilde{\Lambda} := \tilde{\Lambda}_{b_{\max}} := \Lambda(\tilde{T}, Q_{b_{\max}}) := \bigcap_{i \in \mathbb{Z}} \tilde{T}^i(P \times Q_{b_{\max}})$$

When we try to reduce this case to a truncated horseshoe on phase space $S^1 \times \mathbb{R}$, we find that in the general case of non-identical potentials in an arbitrary arrangement one cannot project this map to a map on the cylinder $Q_{b_{\max}}$ because one might find $(p, x, b) \mapsto (p_1, x_1, b_1)$ and $(p', x, b) \mapsto (p_2, x_2, b_2)$ with $(x_1, b_1) \neq (x_2, b_2)$. One way to treat this difficulty is to decompose the trapped set into subsets labeled by sequences of potential labels $(n_i)_{i \in \mathbb{Z}}$ which determine the sequence of potentials a trajectory is visiting.

Definition 2.2. We call the sequence $(n_i)_{i \in \mathbb{Z}}$ of labels of visited potentials the *itinerary* of the trajectory. We denote by $I \subset P^{\mathbb{Z}}$ the set of possible itineraries. For an itinerary $(n_i) \in I$ we define a sequence of Poincaré maps $(T^{[n_i]})$ by $\tilde{T}(p, x, b) =: (p', T^{[p]}(x, b))$.

The projection of the trapped set (5) to the fundamental set $Q_{b_{\max}}$ can now be written as:

$$(6) \quad \Lambda := \Lambda_{b_{\max}} := \Lambda(\{T^p\}_{p \in P}, I, Q_{b_{\max}}) := \bigcup_{(n_i) \in I} \Lambda^{[(n_i)]}$$

$$\Lambda^{[(n_i)]} := \dots \cap (T^{[n_{-2}]})^{-1} ((T^{[n_{-1}]})^{-1} (Q_{b_{\max}})) \cap (T^{[n_{-1}]})^{-1} (Q_{b_{\max}}) \cap Q_{b_{\max}} \cap$$

$$T^{[n_1]}(Q_{b_{\max}}) \cap T^{[n_2]}(T^{[n_1]}(Q_{b_{\max}})) \cap \dots$$

$$= \bigcap_{i \in \mathbb{N}} \left(\prod_{j=1}^i (T^{[n_{-j}]}) (Q_{b_{\max}}) \cap Q_{b_{\max}} \cap \prod_{j=1}^i (T^{[n_j]}) (Q_{b_{\max}}) \right) \subset Q_{b_{\max}}$$

where $T^{-[p]} := (T^{[p]})^{(-1)}$. The trapped set Λ is closed because the complement $C(\Lambda)$ is obviously open: a point on an escaping trajectory will eventually miss a potential by $\delta b > 0$; as we assumed open channels and the dynamics restricted to the escape window to be continuous, $C(\Lambda)$ is open.

We assume that this trapped set Λ is a horseshoe. By this we mean that for all families $\{T^{[n_i]}\}_i$ the partial trapped set $\Lambda^{[(n_i)]}$ is *hyperbolic* (or empty), i.e. the width of both the vertical strip sequence $(\prod_{j=1}^i (T^{[n_{-j}]}) (Q_{b_{\max}}))_{i \in \mathbb{N}}$ and the

horizontal strip sequence $(\prod_{j=1}^i (T^{[n_j]})(Q_{b_{\max}}))_{i \in \mathbb{N}}$ are both uniformly contracted to 1 dimensional manifolds transversal to each other.

We assume that each potentials $V^{[p]}$ has a deflection function $\theta^{[p]}$ which is continuous when restricted to the potential's support or its complement (there it is zero). In the case of central potentials the deflection functions are independent of the impact angle. We truncate the horseshoes via the deflection function by introducing a family of cut-off parameters $\beta^{[p]} \leq b_{\max}^{[p]}$ whose variation amounts formally to decreasing the potential ranges $b_{\max}$:

$$(7) \quad \theta_{\beta^{[p]}}^{[p]}(b) := \begin{cases} \theta^{[p]}(b) & \text{if } |b| \leq \beta^{[p]} \\ 0 & \text{otherwise} \end{cases}$$

The associated Poincaré map $T_{\beta^{[p]}}^{[p]}$ is then given by restriction. Its precise form on the escape window is uninteresting for our purposes. The family of truncated horseshoes is then $(\Lambda_{\beta})_{\beta}$. It is also closed.

The simplest scenario arises if there are only bifurcations where points are truncated. One meets such a situation in the following case:

Definition 2.3. A truncated multiscattering system $\{V^{[n_i]}, T^{[n_i]}, \beta^{[n_i]}\}_{n_i \in I}$ with itinerary set $I \subset P^{\mathbb{Z}}$ of trapped trajectories has the *miss-one-miss-all property* if each itinerary $(n_i) \in I$ is constant as a function of $(\beta^{[n_i]})$ as long as the corresponding trajectory is trapped.

When studying the conditions under which one gets a regular regime by truncating the generalized horseshoe Λ one runs into the problem that although in the hyperbolic case the map $\tilde{T}$ is automatically expansive when restricted to the trapped set $\tilde{\lambda}$ this need not be true for the projection. Simple sufficient conditions are given by

Proposition 1. *If a finite set of scatterers satisfies as geometric conditions that*

- (1) *the truncation family has the miss-one-miss-all property and that*
- (2) *there are open channels of angular size bounded away from zero;*

and the following analytic conditions that

- (1) *the deflection functions restricted to their supports are continuous,*
- (2) *the associated multiscattering system is hyperbolic,*
- (3) *and that the unstable diameter of the fundamental domain is a continuous function of the control parameter $\kappa := \sup\{\beta^{[p]}\}$ tending to 0 for $\kappa \rightarrow 0$,*

then even in the projection there is a regular regime on an interval of the control parameter κ .

Proof. Because of the open channels any two potentials which are both reachable from a given potential are separated by an angle of at least $\delta(x) > \epsilon > 0$. Together with the continuity of the deflection functions this excludes branching of itineraries (n_i) below a parameter κ_0 (i.e. 2 itineraries do not have coinciding segments). Below κ_0 , the finiteness of the number of scatterers implies that there are only finitely many itineraries (n_i) . Hyperbolicity within the partial

trapped set and the shrinking of the domain transversal to the unstable directions imply that each itinerary has only finitely many corresponding trajectories below some $\kappa_1 \leq \kappa_0$ and below some $\kappa_2 \leq \kappa_1$ the unstable diameter is so small that these trajectories will contain only finitely many points. $\square$

The reason why we restricted ourselves to central potentials and consequently to deflection functions independent of x , is that the inverse scattering problem we will have to solve is much more difficult for this general case.

In the simple scattering model of [5], where an infinite string of identical, central, non-overlapping, and attractive potentials is arranged at unit distances along a straight line, a single Poincaré map is enough to describe the full discretized dynamics. Indeed, one finds here a periodic regime in spite of the infinite number of potentials. This is due to the translational symmetry of this model. Its Poincaré map T^{SM} assumes the following form

$$(8) \quad T^{\text{SM}} : Q_{b_{\max}} \rightarrow Q_{\infty} \\ \begin{pmatrix} x \\ b \end{pmatrix} \mapsto \begin{pmatrix} x + \theta(b) \\ b + G(x + \theta(b)) \end{pmatrix}$$

where $G(x) = -\text{sgn}(\cos x) \sin x$, so that the expression (6) for the trapped set reduces to the simpler equation (1). In order to minimize the difficulty for an analytical treatment we chose in reference [5] as deflection function θ the sawtooth function Z_k , $k > 0$:

$$(9) \quad Z_k : \mathbb{R} \rightarrow \mathbb{R}, \quad b \mapsto \begin{cases} kb & \text{if } |b| \leq b_{\max} \\ 0 & \text{otherwise} \end{cases}$$

The transition to chaos of the piecewise linearized version of this model which happens to coincide with the asymptotics $b_{\max} \rightarrow 0, k \rightarrow \infty$ but $0 < kb_{\max} < \infty$ leads to a pruning problem of a (generalized) symbolic dynamics and this can be solved rigorously (cf. references [6, 8]). It exemplifies a cascade scenario: One finds a periodic regime whose bifurcation values accumulate toward a critical value, where a homoclinic orbit initiates chaos. Simultaneously, the linearized dynamics undergoes a transition in the structural complexity (Chomsky hierarchy) of the associated formal languages (cf. ref. [7, 9]).

Up to this point one could argue (cf. reference [2]) that this particular model and the whole truncation scenario just introduced is purely formal and unrealistic, because even the existence of a family of potentials associated to the deflection family is dubious. However, in reference [9] this question was settled. There, the inverse scattering problem for the simple model was solved resulting in a potential pot which is as well-behaved as one could wish but for a discontinuity of the derivative at the boundary of its support. Following the ideas developed for this particular model, our next step will be to solve the inverse scattering problem for the general case.

3. THE REALIZATION OF TRUNCATED HORSESHOES BY POTENTIAL DEFORMATION

3.1. The classical inverse scattering problem. Consider the Hamiltonian dynamics in a central potential V , whose support has radius $b_{\max} > 0$. The

forward scattering problem is: For a given asymptotic particle energy $E_0 > 0 =: \lim_{r \rightarrow \infty} V(r)$ and an impact parameter $b \geq 0$, find the total curvature of the trajectory, that is the historical angle $\theta_V(b)$. If the potential V allows orbiting or if it has singularities with orders in $[r^{-2}, r^{-4}[$ then the resulting deflection function θ_V will have singularities (cf. ref. [4]). Otherwise θ_V can be calculated from the potential V just by integrating the equations of motion to yield:

$$(10) \quad \begin{aligned} \theta_V(b) &= 2 \int_{r_{\min}(b)}^{\infty} \frac{b \, dr}{r \sqrt{r^2(1 - \frac{V(r)}{E_0}) - b^2}} - \pi \\ &= -2 \arccos\left(\frac{b}{b_{\max}}\right) + 2 \int_{r_{\min}(b)}^{b_{\max}} \frac{b \, dr}{r \sqrt{r^2(1 - \frac{V(r)}{E_0}) - b^2}} \end{aligned}$$

where $r_{\min}(b)$ is the classical turning point of the trajectory and is determined as the largest zero of the radicand $r^2(1 - \frac{V(r)}{E_0}) - b^2$. For the construction of the Poincaré maps it is convenient to extend for $b < 0$ the deflection function as $\theta_V(-b) := -\theta_V(b)$. Furthermore, we regularize $\theta_V(b_{\max}) := \lim_{b \rightarrow b_{\max}-0} \theta_V(b)$, i.e. assume θ_V to be left continuous.

The inverse scattering problem assumes a deflection function θ (possibly having singularities) and an asymptotic particle energy E_0 to be given and asks for the reconstruction of a potential $V = V_\theta$ whose deflection function $\theta_V = \theta$. Our task is to deform a given V as smoothly as possible in such a way that the deformed potentials V_β have the truncations θ_β (equation (7)) of the deflection function θ of V as their deflection functions.

3.2. Deforming potentials by truncating the deflection functions. First, we make the term "as smoothly as possible" more specific:

Definition 3.1. We call a continuous deformation $]0, b_{\max}] \ni \beta \mapsto V_\beta \in C^m([0, \beta]) \cap C([0, \infty[)$, where the supports $\text{supp}(V_\beta) \subset [0, \beta]$ are nested, *smooth* (C^n , $n \leq m$) *on the support but for finitely many points* or *nearly smooth* for short if for a finite complement subset $I_\beta \subset \text{supp } V_\beta$, V_β is smooth (C^n) and $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $\beta_2 - \beta_1 < \delta$, $\beta_2 > \beta_1$, then

$$(11) \quad \max_{i=0 \dots n} \left(\sup_{r \in I_{\beta_1}} (|D^{(i)} V_{\beta_2}(r) - D^{(i)} V_{\beta_1}(r)|) \right) < \epsilon$$

In this section we are going to prove the following sufficient conditions permitting such a deformation:

Theorem 3.1. Let $V :]0, \infty[\rightarrow \mathbb{R}$ be a continuous potential, $E_0 > 0$, $v(r) := 1 - V(r)/E_0$, r_* the radius of closest approach:

$$(12) \quad r_* := \sup\{r \in]0, b_{\max}]; V(r) = E_0\}$$

and r^* the parameter

$$(13) \quad r^* := \inf\{r \in]0, b_{\max}]; r^2 v(r) = b_{\max}^2\}$$

Assume that

- (1) the support $\text{supp } V \subset [0, b_{\max}]$, $0 < b_{\max} < \infty$,
- (2) the restriction $V|_{]0, b_{\max}[}$ is smooth (C^2 suffices),

(3) the deflection function $\theta|]0, b_{\max}[$ of V is bounded.

and either

(4) $r_* < r^* < b_{\max}$ and θ' monotone increasing

or

(4') $r^* = b_{\max}$ and $\theta > 0$

then there is a solution V_β of the inverse scattering problem for the truncation θ_β , $0 < \beta \leq b_{\max}$, such that the deformation $\beta \mapsto V_\beta$ is smooth (C^1) on the support but for finitely many points.

Remark 3.1.

ad eq. (12): The radius of closest approach $r_* = 0$ iff $E_0 > \sup V$.

ad (3): Condition (3) is satisfied for any purely repulsive potential ($V \geq 0$, $V' < 0$) and for those potentials which do not trap any particles of the given energy E_0 . As $E_0 > 0 = \lim_{r \rightarrow \infty} V(r)$ and $V|]0, b_{\max}[$ is smooth trapping cannot occur if we exclude *orbiting* (i.e. $\theta_V|]0, b_{\max}[$ must be finite) and a fall into the singularity ($\theta_V(0)$ must be finite). In other words condition (3) is equivalent to

(3') *Either V is purely repulsive or it must not show orbiting and it must not have a singularity of order in $[-r^{-2}, -r^{-4}[$ at $r = 0$.*

ad (4): Condition (4) cannot be satisfied by a purely repulsive potential V because $r^2 v(r)$ is then strictly increasing in the interval between $[r_*, b_{\max}]$ and can therefore not cross the line $z = b_{\max}^2$ before $r = b_{\max}$ where $b_{\max}^2 v(b_{\max}) = b_{\max}^2$.

ad (4'): Condition (4') requires $V \leq 0$. This can be seen from the following proposition where for this case $R = 0$ and $w \geq 1$. Furthermore, condition (4') implies that $\theta(b_{\max}) > 0$ and that $r_{\min}(b_{\max}) = b_{\max}$. This means that for an impact parameter $b_{\max}$ a circular orbit with infinite deflection arises. We regularize in this case $\theta(b_{\max}) := \lim_{b \rightarrow b_{\max}-0} \theta(b)$.

If condition (4') is weakened to

(4'') $r^* = b_{\max}$ and $\theta \geq 0$, $\theta(b_{\max}) = 0$

then the left derivative of v makes a jump at β : $\forall 0 < \beta < b_{\max}$: $v_\beta(\beta) = (-2/\beta)$.

We split the proof of this theorem into several propositions. In propositions 3 and 4 explicit deformations which do the job will be constructed. But first we discuss how to reconstruct the potential from its deflection function.

Proposition 2. *Assume that the conditions (1)-(3) of theorem 3.1 hold. Then the deflection function $\theta = \theta_V$ is smooth on its support $\text{supp}(\theta) \subset [0, b_{\max}]$ and the restriction $V|]r_*, r^*]$ of the potential V satisfies*

$$(14) \quad (r(s), V(r(s))) = (s w(s)^{-1/2}, E_0(1 - w(s))), \quad s \in]0, b_{\max}[$$

$$(15) \quad w(s) = c \exp \left(\frac{2}{\pi} \int_s^{b_{\max}} \frac{\theta(b) - R(b)}{\sqrt{b^2 - s^2}} db \right)$$

where $c = b_{\max}^2/r^{*2}$ if $r_* < r^* < b_{\max}$ and $c = 1$ otherwise and

$$(16) \quad R(b) = \begin{cases} 2 \int_{r_*}^{b_{\max}} \frac{b \, dr}{r \sqrt{r^2 v(r) - b^2}} & \text{if } r^* < b_{\max} \\ 0 & \text{if } r^* = b_{\max} \text{ or } r^* < r_*. \end{cases}$$

Observe that $r_* = r(s = 0)$ because $s(r) = r\sqrt{v(r)}$ and hence $s \in]0, b_{\max}] \Leftrightarrow r \in]r_*, r^*]$. For repulsive potentials, V is indeterminate below the radius r_* of closest approach, whereas for attractive potentials V is essentially arbitrary (but for matching conditions) in the range $r^* < r < b_{\max}$.

Before we are going to prove this proposition we state the following lemma:

Lemma 1. *The potential V does not show orbiting iff $r^2 v$ does not have a critical point $r_c \in [0, b_{\max}]$ with $r_c^2 v(r_c) < b_{\max}^2$ and if it has r^* of eq. (13) as a critical point then $r^* = b_{\max}$.*

Proof. The well-definedness of the deflection function in the absence of orbiting and a singularity of order between r^{-2} and r^{-4} is immediate (cf. [4]). The other direction can be seen as follows: The critical points satisfy:

$$(17) \quad 2r v(r) + r^2 v'(r) = 0$$

The existence of a critical point r_c is obviously equivalent to the existence of a constant $C > 0$ s.t. $v(r_c) = C/r_c^2$ and $v'(r_c) = -2C/r_c^3$, i.e. v is locally inverse quadratic at any critical point. Now suppose $0 \leq b_0 \leq b_{\max}$ is the impact parameter of a trajectory with classical turning point $0 < r_c < b_{\max}$, where the potential v is assumed to be locally inverse quadratic. Then the deflection function of v has a nonintegrable singularity at b_0 and is consequently unbounded: One way to see this is to set the Taylor expansion of v around r_c into (10). One finds a nonintegrable singularity $(r - r_c)^{-1}$. A more physical argument is given by the observation that criticality at r_c is equivalent to the equality of the potential's gradient at r_c and the centrifugal force for curvature radius r_c . Together with the turning point condition $\dot{r}(r_c) = 0$ and the uniqueness of the initial value problem of motion in a potential field, this yields a trajectory which ends as circular motion, winding infinitely often around the potential's center. Now, consider the turning point function

$$(18) \quad r_{\min} : [0, b_{\max}] \rightarrow [0, b_{\max}], \quad b \mapsto \max\{r; r^2 v(r) - b^2 = 0\}$$

Even if $r_{\min}(0) = 0$ and $r_{\min}(b_{\max}) = b_{\max}$, not every critical point r_c need to appear as a turning point for some impact parameter b , because the turning point function need not be continuous. However, since v was supposed to be C^2 in $]0, b_{\max}[$, the only way a discontinuity may arise is via a tangent bifurcation at a local extremum of

$$(19) \quad \tilde{v} :]0, b_{\max}[\rightarrow \mathbb{R}, \quad r \mapsto r^2 v(r)$$

Two cases of discontinuities of the turning point function $r_{\min}$ can be distinguished:

case 1: There is a critical point in $]0, b_{\max}[$ at which $\tilde{v}$ assumes a value $< b_{\max}^2$. As $\tilde{v}(b_{\max}^2) = b_{\max}^2$ there must be relative minima among the critical points. Since $\tilde{v}$ is positive and C^2 and must not have a singularity of order $-r^{-2}$ at 0 there is a lowest relative minimum at some $r_c \in]0, b_{\max}[$. By construction this lowest minimum is both a critical point and a turning point. Thus trajectories with infinite total curvature occur, in other words the deflection function would have to be unbounded.

borderline case: Suppose there is a critical point of $\tilde{v}$ on the line $b_{\max}^2$ that is the accumulation point of a sequence of turning points. This would again lead to a limit circle, so that this case is also excluded because θ_V would be unbounded.

case 2: $\tilde{v}$ assumes values $> b_{\max}^2$ and all critical points of $\tilde{v}$ in $]0, b_{\max}[$ assume values $\geq b_{\max}^2$. In this case there is a $0 < r^* < b_{\max}$ s.t. $r^2 v(r) \geq b_{\max}^2 \forall b_{\max} > r \geq r^*$.

□

Proof of Prop. 2. Suppose first that $r_* < r^* < b_{\max}$. By lemma 1 we know that $r_{\min} \in [0, r^*]$, so that one may decompose:

$$(20) \quad \theta(b) = 2 \int_{r_{\min}(b)}^{r^*} \frac{b dr}{r \sqrt{r^2 v(r) - b^2}} + 2R(b) - 2 \arccos\left(\frac{b}{b_{\max}}\right)$$

Now, we proceed similarly as in [4, 3].

$$(21) \quad x(r) := \frac{1}{r^2 v(r)}$$

defines a positive and monotone function on the interval $]0, r^*]$ because $x'(r) \neq 0$ from lemma 1. Furthermore, if $r^* < b_{\max}$ then $R(b_{\max}) = \lim_{b \rightarrow b_{\max}-0} \theta(b)$ because $r^2 v$ does not have a critical point at r^* . Therefore eq. (21) defines a permissible integral transformation. We define

$$(22) \quad \begin{aligned} g(x) &:= -\frac{\sqrt{x}}{r} \frac{dr(x)}{dx}, \quad y := b^{-2}, \quad y_0 := b_{\max}^{-2} \\ \hat{\theta}(y) &:= \theta(b), \quad \hat{R}(y) := R(b) \end{aligned}$$

and can write:

$$(23) \quad \hat{\theta}(y) = 2 \int_{y_0}^y \frac{g(x) dx}{\sqrt{y-x}} + \hat{R}(y) - 2 \arccos \sqrt{\frac{y_0}{y}}$$

Since $V[0, b_{\max}[$ is smooth all terms on the right side are continuously differentiable, and hence $\hat{\theta}$ and θ are so, too. Consequently, with $\hat{\theta}_R := \hat{\theta} - \hat{R}$, yielding an Abel integral equation:

$$(24) \quad \hat{\Theta}(y) := \hat{\theta}_R(y)/2 + \arccos \sqrt{\frac{y_0}{y}} = \int_{y_0}^y \frac{g(x) dx}{\sqrt{y-x}}$$

where $\hat{\Theta}$ is also continuously differentiable. Therefore (cf. for example [10]), eq. (24) has a unique solution $g \in C([y_0, \infty[) \cap L^1([y_0, \infty[)$. It has the form

$$(25) \quad g(x) = \frac{\sin(\pi/2)}{\pi} \left(\frac{\hat{\Theta}(y_0)}{\sqrt{x-y_0}} + \int_{y_0}^x \frac{\hat{\Theta}'(y) dy}{\sqrt{x-y}} \right)$$

$$(26) \quad = 1/(2\sqrt{x}) + \frac{1}{\pi} \int_{y_0}^x \frac{\hat{\theta}'_R(y) dy}{2\sqrt{x-y}}$$

because $\hat{\Theta}(y_0) = \hat{\theta}_R(y_0) = 0$ and $\hat{\Theta}'(y) = \hat{\theta}'_R(y) + y_0^{1/2}/(2y^{3/2}\sqrt{1-y_0/y})$. The change of order of differentiation and integration in (25) is allowed because

$$(27) \quad R'(b) = 2 \int_{r^*}^{b_{\max}} \frac{rv(r) dr}{(r^2v(r) - b^2)^{3/2}}$$

so that $\frac{\hat{\Theta}'(y_0)}{\sqrt{x-y_0}}$ has only poles of order $-1/2$ at $y = x$ and $y = y_0$ and is hence dominated by some $L^1([y_0, \infty[)$ -function. Setting $\hat{v}(x) := v(r)$ one receives

$$(28) \quad \frac{d \ln \hat{v}}{dx} = 2 \frac{g(x)}{\sqrt{x}} - \frac{1}{x}$$

so that together

$$(29) \quad \frac{d \ln \hat{v}}{dx} = \frac{1}{\pi} \int_{y_0}^x \frac{\hat{\theta}'_R(y) dy}{\sqrt{x(x-y)}}$$

Integration followed by partial integration using $\hat{v}(y_0) = v(r^*) = b_{\max}^2/r^{*2}$ yields

$$(30) \quad \ln \hat{v}(x) - \ln \frac{b_{\max}^2}{r^{*2}} = \frac{1}{\pi} \int_{y_0}^x \sqrt{\frac{x}{y(x-y)}} \hat{\theta}_R(y) dy$$

The reparameterization $s(r) := x(r)^{-1/2} = r\sqrt{v(r)}$, $w(s) := \hat{v}(x) = v(r)$ is monotone because $r \mapsto x(r)$ is monotone and yields eq. (15).

If $r^* = b_{\max}$ we receive equation (30) with $\hat{\theta}$ instead of $\hat{\theta}_R$ because even if $\hat{\theta}(y_0) \neq 0$ this term is absorbed by partial integration after eq. (29). $\square$

Next, we are going to study the conditions under which the integral transformation of eq. (21) defines a monotone function. Afterwards we are going to check that this condition remains satisfied when we truncate the deflection function.

Lemma 2. *Eq. (21) defines a strictly monotone increasing function iff*

$$(31) \quad \Phi_\beta(s) := \int_s^{b_{\max}} \frac{b \theta'_R(b)}{\sqrt{b^2 - s^2}} db - \frac{b_{\max} \theta_R(b_{\max})}{\sqrt{b_{\max}^2 - s^2}} < \pi \quad \forall 0 < s < b_{\max}$$

where $\theta_R = \theta - R$.

Observe that if $r^* < b_{\max}$ then $\theta_R(b_{\max}) = 0$ whereas for $r^* = b_{\max}$ we have $\theta_R = \theta$ and $\theta_R(b_{\max}) \neq 0$ is possible.

Proof. By partial integration one gets from (15)

$$(32) \quad w(s) = \frac{b_{\max}^2}{r^{*2}} \exp \left(\frac{2}{\pi} \left(\theta_R(b_{\max}) \operatorname{arcosh} \left(\frac{b_{\max}}{s} \right) - \int_s^{b_{\max}} \theta'_R(b) \operatorname{arcosh} \left(\frac{b}{s} \right) db \right) \right)$$

Hence

$$(33) \quad w'(s) = \frac{2 w(s)}{\pi s} \left(\int_s^{b_{\max}} \frac{b \theta'_R(b)}{\sqrt{b^2 - s^2}} db - \frac{b_{\max} \theta_R(b_{\max})}{\sqrt{b_{\max}^2 - s^2}} \right)$$

so that from $r(s) = s w(s)^{-1/2}$

$$(34) \quad r'(s) = w(s)^{-1/2} \left(1 - \frac{1}{\pi} \left(\int_s^{b_{\max}} \frac{b \theta'_R(b)}{\sqrt{b^2 - s^2}} db - \frac{b_{\max} \theta_R(b_{\max})}{\sqrt{b_{\max}^2 - s^2}} \right) \right)$$

As the distance of closest approach is smaller than r^* and $r(s = b_{\max} - 0) = r^*$ a monotone $r(s)$ must be increasing. This yields the assertion. $\square$

Now, we are going to examine how to deform the original potential as smoothly as possible so that the resulting deformed potentials have the truncations of the original deflection functions as their deflection functions. First, suppose $\tilde{v}$ has a critical point. Lemma 1 above shows that the graph of $\tilde{v}$ must cross the line $y = b_{\max}^2$ transversely at a $r^* < b_{\max}$. When we deform v both r^* and the part of v above r^* are essentially arbitrary because of the non-uniqueness of the solution of the inverse scattering problem.

Proposition 3. *Let V be a potential of range $b_{\max}$ satisfying the conditions of proposition 2 and let θ be its deflection function. If $r_* < r^* < b_{\max}$ and θ' is monotone increasing then there is a solution V_β of the inverse scattering problem for the truncation θ_β , $0 < \beta \leq b_{\max}$, such that the deformation $\beta \mapsto V_\beta$ is nearly smooth (C^1).*

Proof. Suppose θ is the deflection function of v defined in eq. (10) and θ_β its truncation defined in eq. (7). Now in order to get for the inverse scattering problem of the given θ_β a solution v_β which is a continuous deformation of v one has to decide how to deform the free parameter r^* and the essentially arbitrary part $r > r^*$ of the potential. We choose here a simple homogeneous transformation accepting an arising non-differentiability at the new r_β^* : Let $\tilde{v}(r) := r^2 v(r)$. We claim that the equations of proposition 2

$$(35) \quad w_\beta(s) = \frac{\beta^2}{r_\beta^{*2}} \exp \left(\frac{2}{\pi} \int_s^\beta \frac{\theta(b) - R_\beta(b)}{\sqrt{b^2 - s^2}} db \right) \quad \text{for } 0 < s \leq \beta$$

$$(36) \quad R_\beta(b) = 2 \int_{r_\beta^*}^\beta \frac{b dr}{r \sqrt{\tilde{v}_\beta(r) - b^2}} \quad \text{for } 0 \leq b \leq \beta$$

$$(37) \quad \tilde{v}_\beta(r) = \frac{\beta^2}{b_{\max}^2} \tilde{v} \left(\frac{b_{\max}}{\beta} r \right) \quad \text{for } r_\beta^* < r \leq \beta$$

$$(38) \quad r_\beta^* = \frac{\beta}{b_{\max}} r^*$$

define a continuous potential with deflection function θ_β . To see this, it is enough to show that the condition of lemma 2

$$(39) \quad \Phi_\beta(s) = \int_s^\beta \frac{b \theta'_{R_\beta}(b)}{\sqrt{b^2 - s^2}} db < \pi \quad \forall 0 < s < \beta$$

is satisfied for each truncation parameter $0 < \beta < b_{\max}$:

By assumption

$$(40) \quad \Phi_{b_{\max}}(s) < \pi \quad \forall 0 \leq s \leq b_{\max}$$

Using $\theta_{R_\beta} = \theta_\beta - R_\beta$ we decompose $\Phi_\beta = \Phi_\beta^{(\theta_\beta)} + \Phi_\beta^{(R_\beta)}$ where $\Phi_\beta^{(\theta_\beta)} := \int_s^\beta \frac{b \theta'_\beta(b)}{\sqrt{b^2 - s^2}} db$ and $\Phi_\beta^{(R_\beta)} := - \int_s^\beta \frac{b R'_\beta(b)}{\sqrt{b^2 - s^2}} db$. We are going to show that under the conditions imposed above

$$(41) \quad \Phi_\beta(s) \leq \Phi_{b_{\max}}\left(\frac{b_{\max}}{\beta} s\right) \quad \forall 0 \leq s \leq \beta$$

so that the assertion follows from equation (40).

(a) First we treat $\Phi_\beta^{(R_\beta)}$. Substituting $\rho := (b_{\max}/\beta) r$ we get

$$(42) \quad R_\beta(b) = 2 \int_{r^*}^{b_{\max}} \frac{b d\rho}{\rho \sqrt{\frac{\beta^2}{b_{\max}^2} \tilde{v}(\rho) - b^2}}$$

and using equation (27) while substituting $\tilde{b} := (b_{\max}/\beta) b$ yields

$$(43) \quad \begin{aligned} \Phi_\beta^{(R_\beta)}(s) &= -2 \int_s^\beta \frac{b}{\sqrt{b^2 - s^2}} \left(\int_{r^*}^{b_{\max}} \frac{\frac{\beta^2}{b_{\max}^2} \tilde{v}(\rho) d\rho}{\rho (\frac{\beta^2}{b_{\max}^2} \tilde{v}(\rho) - b^2)^{3/2}} \right) db \\ &= -2 \int_{\frac{b_{\max}}{\beta} s}^{b_{\max}} \frac{\tilde{b}}{\sqrt{\tilde{b}^2 - (\frac{b_{\max}}{\beta} s)^2}} \left(\int_{r^*}^{b_{\max}} \frac{\tilde{v}(\rho) d\rho}{\rho (\tilde{v}(\rho) - \tilde{b}^2)^{3/2}} \right) d\tilde{b} \\ &= \Phi_{b_{\max}}^{(R)}\left(\frac{b_{\max}}{\beta} s\right) \quad \text{for } 0 \leq s \leq \beta \text{ and } 0 < \beta \leq b_{\max} \end{aligned}$$

(b) Next, we treat $\Phi_\beta^{(\theta_\beta)}$.

$$(44) \quad \begin{aligned} \Phi_\beta^{(\theta_\beta)}(s) &= \Phi_\beta^{(\theta)}(s) = \int_s^\beta \frac{b \theta'(b)}{\sqrt{b^2 - s^2}} db = \int_{\frac{b_{\max}}{\beta} s}^{b_{\max}} \frac{\tilde{b} \frac{d}{d\tilde{b}} \theta(\frac{\beta}{b_{\max}} \tilde{b})}{\sqrt{\tilde{b}^2 - (\frac{b_{\max}}{\beta} s)^2}} d\tilde{b} \\ &= \Phi_{b_{\max}}^{(\theta(\cdot \beta/b_{\max}))}\left(\frac{b_{\max}}{\beta} s\right) \leq \Phi_{b_{\max}}^{(\theta)}\left(\frac{b_{\max}}{\beta} s\right) \\ &\quad \text{if } \frac{d}{d\tilde{b}} \theta\left(\frac{\beta}{b_{\max}} \tilde{b}\right) \leq \theta'(\tilde{b}) \quad \forall 0 < \beta \leq b_{\max} \text{ and } \forall 0 \leq \tilde{b} \leq b_{\max} \\ &\quad \text{if } \theta'(\lambda x) \leq \lambda^{-1} \theta'(x) \quad \forall 0 \leq x \leq b_{\max} \text{ and } \forall 0 < \lambda \leq 1 \end{aligned}$$

The latter sub-homogeneity condition is satisfied for monotone increasing θ' , so that we have shown equation (41). Apparently the deformation is nearly smooth. Exceptional points are β and $\mathbb{R}_\beta^*$. The non-differentiability at β is unavoidable if θ is not identically zero, whereas the non-differentiability at $\mathbb{R}_\beta^*$ might be avoided by a more sophisticated deformation. $\square$

Again by lemma 1 one knows that the only remaining case is that $\tilde{v}$ has no critical point in $[0, b_{\max}]$, so that $\tilde{v}$ must lie below the line $y = b_{\max}^2$.

Proposition 4. *Let V be a potential of range $b_{\max}$ satisfying the conditions of proposition 2 and let θ be its deflection function. If $\theta \geq 0$ then there is a solution $V_\beta \leq 0$ of the inverse scattering problem for the truncation (eq. (10)) θ_β , $0 < \beta \leq b_{\max}$.*

- *If $r = b_{\max}$ is a critical point (i.e. $\theta(b_{\max}) > 0$) then this deformation can be chosen to be nearly smooth (C^1).*
- *If $r = b_{\max}$ is not a critical point (i.e. $\theta(b_{\max}) = 0$) then the left derivative of v makes a jump at β : $\forall 0 < \beta < b_{\max}$: $v_\beta(\beta) = (-2/\beta)$.*

Proof. Defining Φ_β again as the left side of the inequality (31) in Lemma 2 and observing $\theta_R = \theta$ in our case yields

$$(45) \quad \Phi_\beta(s) = \int_s^\beta \frac{b \theta'(b)}{\sqrt{b^2 - s^2}} db - \frac{\beta \theta(\beta)}{\sqrt{\beta^2 - s^2}}$$

From this we calculate

$$(46) \quad \frac{\partial}{\partial \beta} \Phi_\beta(s) = \frac{\theta(\beta)}{\sqrt{\beta^2 - s^2}} \left(\frac{\beta^2}{\beta^2 - s^2} - 1 \right) \geq 0 \quad \forall s \in [0, \beta] \quad \text{because } \theta \geq 0$$

Since $\Phi_{b_{\max}} < \pi$ this implies that also $\Phi_\beta < \pi \forall 0 < \beta \leq b_{\max}$, so that

$$(47) \quad w_\beta(s) = \frac{\beta^2}{r_\beta^{*2}} \exp \left(\frac{2}{\pi} \int_s^\beta \frac{\theta(b)}{\sqrt{b^2 - s^2}} db \right)$$

with $r_\beta^* = \beta$ being a critical point if $\theta(\beta) > 0$, defines via proposition 2 a potential with deflection function θ_β . $\square$

Remark 3.2. If V is strictly repulsive ($\theta < 0$, $\theta' > 0$) then one sees directly from equation (45) that the inversion condition of lemma (2) fails for each $\beta \in]0, b_{\max}[$.

Proposition 1 together with theorem 3.1 completes the proof of the main theorem 1.1.

3.3. Examples. Of the following examples only the truncated horseshoe has been actually proved to represent a chaotic (and even hyperbolic) scattering system in a suitable parameter range. Repeating the standard construction of the trapped set of trajectories for the other two systems indicates heuristically the presence of chaos for suitable parameters (and arrangements). For the non-truncated attractive Coulomb potential (and variants) chaoticity was shown by Klein and Knauf in [1] for any arrangement of at least 3 potentials and high enough particle energy E_0 .

3.3.1. *The truncated sawshoe.* First, we come back to the truncated sawshoe. This example has already been treated in [9]. Since $\rho :]0, b_{\max}[\rightarrow]0, b_{\max}[$, $s \mapsto s \exp((-k/\pi)\sqrt{b_{\max}^2 - s^2})$ is strictly increasing for any $k > 0$, $b_{\max} \in]0, 0.5[$, the family of potentials parametrized by $0 < \beta < b_{\max} < 0.5$, and defined by

$$(48) \quad \left\{ \left(s e^{-k/\pi \sqrt{\beta^2 - s^2}}, E_0(1 - e^{2k/\pi \sqrt{\beta^2 - s^2}}) \right); s \in [0, \beta] \right\}$$

together with $V_{Z_k}(r) := 0$ if $r > \beta$ (see figure 1), realizes via proposition 4 the truncated sawshoe. These potentials have a critical point (of $\tilde{v}$) at $r = \beta$ indicating orbiting. However, as the deflection function $\theta = Z_k$ is not only non-negative but also has non-negative second derivative $\theta'' = 0$, proposition 3 shows that it is also possible to construct an associated potential family which does not show orbiting at $b = \beta$.

3.3.2. *The truncated Coulomb potential.*

$$(49) \quad V(r) = \begin{cases} -1/r + 1/b_{\max} & \text{for } 0 < r < b_{\max} \\ 0 & \text{for } r \geq b_{\max} \end{cases}$$

realizes proposition 3 in the case of $2 b_{\max} E_0 < 1$, because then one finds both for the parameter r^* :

$$(50) \quad 0 < r^* = \frac{b_{\max}^2 E_0}{1 - b_{\max} E_0} < b_{\max}$$

and for the deflection function θ , $b \in [0, b_{\max}]$:

$$(51) \quad \theta(b) = \pi - 2 \arccos\left(\frac{b}{b_{\max}}\right) + 2 \arctan\left(\frac{\sqrt{b_{\max}^2 - b^2} (b_{\max} - 2 b^2 E_0)}{2 E_0 b (b_{\max}^2 - b^2)}\right)$$

$$(52) \quad \theta''(b) = \frac{2 b b_{\max} (1 - 2 b_{\max} E_0) (b_{\max} - 12 b^2 E_0 + 8 b_{\max}^2 E_0 + 12 b^2 b_{\max} E_0^2 - 8 b_{\max}^3 E_0^2)}{(b_{\max}^2 - b^2)^{\frac{3}{2}} (b_{\max} - 4 b^2 E_0 + 4 b^2 b_{\max} E_0^2)^2}$$

one sees the monotonicity of θ' as follows: the second parenthesis term of the numerator of eq. (52) obeys

$$(53) \quad b_{\max} + 12 b^2 E_0 (-1 + b_{\max} E_0) + 8 b_{\max}^2 E_0 - 8 b_{\max}^3 E_0 \geq b_{\max} (1 - 4 b_{\max} E_0 + 4 b_{\max}^2 E_0^2) \geq 0$$

because the function $[0, 1] \ni x \mapsto 1 - 2x + x^2 \in [0, 1]$ is monotone decreasing from 1 to 0, so that $\theta'' \geq 0$.

The function R_β has the form:

$$(54) \quad R_\beta(b) = -2 \arctan\left(\frac{2b\sqrt{\beta^2 - b^2} b_{\max} E_0}{\beta^2 - 2b^2 b_{\max} E_0}\right) \\ + 2 \arctan\left(\frac{2b\sqrt{\beta^2 - b^2} (1 - b_{\max} E_0)}{\beta^2 - 2b^2 (1 - b_{\max} E_0)}\right) \\ + 2\pi \text{ if the denominator of the second arctan is negative}$$

A few members of this parameter family are shown in figure 2. The other case $2b_{\max} E_0 \geq 0$ is taken care of by proposition 4.

3.3.3. A hard core potential. The potential formulae of proposition 2 are analytically computable for polynomial deflection functions. A linear deflection function yields the sawshoe model of the first example. For a cubic deflection function

$$(55) \quad \theta(b) = \begin{cases} k b^3 & \text{if } 0 \leq b \leq b_{\max} \\ 0 & \text{if } b > b_{\max} \end{cases}$$

where $k > 0$ and $b_{\max} \in]0, 0.5[$, it is also easy to show for $r^* = b_{\max}$ when the reparameterization (21) is strictly monotone. Since

$$(56) \quad \Phi(s) = \frac{k s^2 (b_{\max}^2 - 2s^2)}{\sqrt{b_{\max}^2 - s^2}}$$

is bounded by π iff

$$(57) \quad k b_{\max}^3 < \frac{4\pi\sqrt{27}\sqrt{3+\sqrt{33}}}{(\sqrt{33}-3)(9-\sqrt{33})}$$

lemma 2 assures that in this case the equations of proposition 2 give solutions of the inverse scattering problem. As the deflection function is non-negative, proposition 4 establishes the existence of a truncation family of non-monotone potentials V_β defined by:

$$(58) \quad w(s) = \exp\left(\frac{2k\sqrt{\beta^2 - s^2}(\beta^2 + 2s^2)}{3\pi}\right)$$

$$(59) \quad (r(s), V(s)) = (s w(s)^{-1/2}, E_0(1 - w(s)))$$

for $s \in [0, \beta]$, $\beta \in]0, b_{\max}]$, $E_0 > 0$. Figure 3 shows a few members of this family.

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FIGURE CAPTIONS

Fig. 1 The sawshoe potentials V (equation (48)) as a function of the radial distance r for $b_{\max} = 0.2$, $\beta = 0.05, 0.1, 0.15, 0.2$. The inset shows the corresponding truncated deflection functions Z_k as functions of the impact parameter b for $k = 20$, $E_0 = 1$.

Fig. 2 The truncated Coulomb potentials V as a function of the radial distance r for $b_{\max} = 0.2$, $E_0 = 2.3$ realizing the case $2 E_0 b_{\max} < 1$. The fat dots mark non-differentiable points. The inset shows the corresponding truncated deflection functions as functions of the impact parameter b .

Fig. 3 The truncation family of a hard-core potential with cubic deflection function (equation (55)) for $E_0 = 1$, $k = 1000$. The inset again shows the corresponding truncated deflection functions.

TECHNISCHE UNIVERSITÄT BERLIN, FB MATHEMATIK MA 7-2, STR. D. 17.
JUNI 136, D-10623 BERLIN, GERMANY
E-mail address: troll@math.tu-berlin.de

Fig. 1

Fig. 2

Fig. 3