Differential Renormalization, 
the Action Principle and Renormalization Group Calculations

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Abstract

General prescriptions of differential renormalization are presented. It is shown that renormalization group functions are straightforwardly expressed through some constants that naturally arise within this approach. The status of the action principle in the framework of differential renormalization is discussed.
1 Introduction

Differential renormalization [1, 2] was invented as an alternative renormalization scheme useful for calculations strictly in four dimensions [3, 4]. The basic idea of this renormalization is to represent products of propagators in coordinate space through derivatives of sufficient order acting on locally integrable functions. Here the product is over the lines of a given graph, \( \tau_{\pm}(l) \) are respectively beginning and the end of a line \( l \),

\[
\Pi_{\Gamma}(x_1, \ldots, x_N) = \prod_l G(x_{\tau_{+}(l)} - x_{\tau_{-}(l)}),
\]

through derivatives of sufficient order acting on locally integrable functions. Here the product is over the lines of a given graph \( \Gamma \), \( \tau_{\pm}(l) \) are respectively beginning and the end of a line \( l \),

\[
G(x) = P(\partial / \partial x, m) \frac{m}{4\pi^2 \sqrt{x^2}} K_1(m \sqrt{x^2})
\]

is a propagator, with \( P \) polynomial and \( K_1 \) a modified Bessel function. This procedure explicitly characterizes the \( R \)-operation (i.e. renormalization at diagrammatic level) as an extension of the functional \( \Pi_{\Gamma}(x_1, \ldots, x_n) \) from the subspace of test functions which vanish in a vicinity of points where the coordinates \( x_i \) coincide to the whole space \( D(\mathbb{R}^{4n}) \).

The first step within initial version of differential renormalization [1, 2] is to reduce the problem to the case of diagrams depending on one coordinate difference. To do this at low orders of perturbation theory, it suffices to exploit certain manipulations based on the Leibniz rule. At higher orders, the only way of performing such a reduction is to integrate over all coordinate differences except one. However it is then possible to run into infrared problems since this ‘naive’ integration generally induces infrared divergences. In [6] the original version of differential renormalization was supplied with simple prescriptions which enabled infrared troubles to be avoided so that differentially renormalized expressions could be found with no more difficulty than determining the corresponding counterterms in dimensional renormalization. It was also shown that in writing down differentially renormalized quantities it is very useful to apply calculational experience based on dimensional regularization.

The second step [1, 2] is performed with prescriptions of the following type:

\[
\begin{align*}
\frac{1}{x^4} &\rightarrow -\frac{1}{4} \ln \mu^2 x^2, \\
\ln \frac{\mu^2 x^2}{x^4} &\rightarrow -\frac{1}{8} \ln \mu^2 x^2 + 2 \ln \mu^2 x^2, \\
\frac{1}{x^6} &\rightarrow -\frac{1}{32} \ln \mu^2 x^2 \end{align*}
\]

\[\text{In [5] a renormalization prescription of differential style was much earlier formulated at the level of primitively divergent diagrams, using the language of the } \alpha \text{-representation.}
\]

\[\text{Euclidean and Minkowski spaces can be treated on the same footing. For simplicity in what follows Feynman amplitudes will be considered in four-dimensional Euclidean space-time.}
\]
etc., where $\partial_\alpha \partial_\alpha$ is the usual Laplacian, $x^4 = (x^2)^2$, $x^6 = (x^3)^3$, and $\mu, \mu'$ are massive parameters which play the role of subtraction points. For $x \neq 0$, the expressions in the left-hand side and the right-hand side of (3–5) are identical. By definition, the extension of functionals in the left-hand side from the subspace of test functions which vanish near $x = 0$ to the whole space is determined by the right-hand side. Note that all the derivatives involved are understood in the distributional sense, i.e. a derivative $D^\alpha f$ of a distribution $f$ acts on a test function $\phi$ as

$$ (D^\alpha f, \phi) = (-1)^{[\alpha]} (f, D^\alpha \phi), $$

where $\alpha$ is the order of the derivative.

In [7] a second version of differential renormalization was presented in the case of scalar massless logarithmically divergent diagrams. It was based on ‘pulling out’ another differential operator instead of the Laplacian. In particular, (3) is replaced by

$$ \frac{1}{x^4} \rightarrow \hat{S} \ln \frac{\mu^2 x^2}{x^4}, $$

where

$$ \hat{S} = \frac{1}{2} \frac{\partial}{\partial x_\alpha} x_\alpha. $$

Within this version, there is no necessity of reducing the problem of renormalization to propagator-type diagrams. (This reduction is as usual important in renormalization group calculations — see below.) Thus there is no asymmetry of treating vertices of the given graph.

The purpose of this paper is to present a general prescription of this version of differential renormalization which is applicable for arbitrary diagrams including massive ones. The status of the renormalized action principle within differential renormalization will be also discussed. Another task is to show that some constants that naturally arise within this approach [7] are straightforwardly related to the renormalization group coefficients. It will be proved that the beta function and anomalous dimensions are expressed through these constants by the same formulae that, in the case of the MS scheme, the RG coefficients are expressed through counterterms. Note that the differential renormalization happens to be a mass-independent scheme.

The plan of the paper is as follows. In the next section necessary differential operators similar to (8) will be presented and standard formulae for the $B$-operation will be listed. Then in Section 3 renormalization of massless lower order diagrams is characterized. In Section 4 an auxiliary technique necessary for renormalization in the massive case is introduced through examples of lower order graphs. In Section 5 the general prescriptions are formulated and justified. Section 6 is devoted to discussion of the action principle within differential renormalization. In Section 7 explicit formulae for RG coefficients will be derived. Finally, Section 8 contains discussion of the results obtained.
2 Notation

2.1 Differential operators

Let us define the following differential operator:

\[ \hat{S}_x = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial x_{io}} (x_{io} - \bar{x}_o), \]  

where \( x \equiv x_1, \ldots, x_n \) is a set of \( n \) four-dimensional variables, and \( \bar{x} = \frac{1}{n} \sum x_i \). If \( F(x) \) is a translationally invariant function, i.e. \( F(x + a) = F(x) \), then \( F(x) = f(\bar{u}) \) for \( u_i = x_i - x_{i0} \), \( i \neq i_0 \) and

\[ \hat{S}_x F(x) = \hat{S}_{\bar{u}} f(\bar{u}), \]  

where

\[ \hat{S}_{\bar{u}} = \frac{1}{2} \sum_{i \neq i_0} \frac{\partial}{\partial u_{io}} u_{io}. \]  

Since the Feynman amplitudes are translationally invariant we will use subsequently this form for the operator \( \hat{S} \).

In fact, the homogeneity properties of Feynman amplitudes play an essential role. If \( f(u_1, \ldots, u_{n-1}) \) is a homogeneous function or distribution of degree \( \lambda \) then

\[ \hat{S} f = \frac{1}{2} (\lambda + 4(n - 1)) f. \]  

Note that the operator \( \hat{S} \) involves a preliminary multiplication by variables \( u_i \) which vanish at points where initial amplitudes are singular. Correspondingly, these singularities are reduced. In case the ultraviolet divergence is logarithmic, it disappears if the subsequent differentiation is understood in the distributional sense — see (6).

In the case of linear divergences multiplication by a monomial of the first degree in coordinates is not sufficient. A second order monomial is necessary so that it is natural to apply the following operator

\[ \hat{S}^{(1)} = \frac{1}{4} \sum_{i,j,i',j'} \frac{\partial}{\partial x_{io}} \frac{\partial}{\partial x_{j\beta}} (x_{io} - \bar{x}_o) (x_{j\beta} - \bar{x}_\beta). \]  

For massless graphs, it is sufficient to apply (11), (12) and their generalizations. If massive lines are present, we may use homogeneity of Feynman amplitudes with respect to coordinates and inverse masses. Then a natural analog of (11) is given by

\[ \hat{S} = \frac{1}{2} \sum_i \frac{\partial}{\partial u_i} u_i - \frac{1}{2} \sum_i m_i \frac{\partial}{\partial m_i}, \]  

since differentiation in masses also improves the ultraviolet behaviour.
In the general case of degree of divergence \( \omega \) let us apply the following differential operator:

\[
\hat{S}^{(\omega)} = N\{\hat{S}^0 \ldots \hat{S}^0\},
\]

where \( \omega \) is the degree of divergence, \( \hat{S}^0 \equiv \hat{S} \) is defined by (13), and the symbol of the \( N \)-product implies that all the derivatives \( \frac{\partial}{\partial u_i} \) are to the left of all \( u_i \) while all the derivatives \( \frac{\partial}{\partial m_j} \) are to the right of \( m_j \). It is not difficult to show that

\[
\hat{S}^{(\omega)} = \hat{S}(\hat{S} + 1/2) \ldots (\hat{S} + \omega/2).
\]

In particular

\[
\hat{S}^{(1)} = \hat{S}(\hat{S} + 1/2),
\]

\[
\hat{S}^{(2)} = \hat{S}(\hat{S} + 1/2)(\hat{S} + 1).
\]

The following commutation relation will be also of use:

\[
\ln^k \mu^2 x^2 = \frac{1}{k+1} \left( \hat{S} \ln^{k+1} \mu^2 x^2 - \ln^{k+1} \mu^2 x^2 \hat{S} \right).
\]

Its generalization for \( k = 0 \) in the case of the operator \( \hat{S}^{(\omega)} \) looks like

\[
1 = a_{\omega} \hat{S}^{(\omega)} \ln \mu^2 x^2 - (\ln \mu^2 x^2 - 4b_{\omega})(\hat{S} + \omega/2)
\]

and is understood in the sense that it acts on a quantity that vanishes after the action of the square of the operator \( \hat{S} + \omega/2 \) (in the second term of the right-hand side the second and higher powers of this operator are omitted). Here

\[
a_{\omega} = (-2)^{\omega}/\omega!,
\]

\[
b_{\omega} = 1 + 1/2 + \ldots + 1/\omega.
\]

Generalizations of (19) for arbitrary \( k \) can be also derived but we shall not write them explicitly.

Furthermore, we shall need the following commutation relation:

\[
\left( \hat{S}_\Gamma + \omega_\Gamma/2 \right) \Pi_\Gamma = \Pi_{\Gamma \setminus \gamma} \left( \hat{S}_\gamma + \omega_\gamma/2 \right) \Pi_\gamma,
\]

where \( \Gamma \setminus \gamma \) denotes the subgraph which consists of lines that do not belong to the subgraph \( \gamma \).

### 2.2 \( R \)-operation

Unrenormalized Feynman amplitudes are obtained from the products \( \Pi_\Gamma \) by integrating over coordinates associated with internal vertices:

\[
F_\Gamma(x_1, \ldots, x_n) = \int dx_{n+1} \ldots x_N \Pi_\Gamma(x_1, \ldots, x_N).
\]
The ultraviolet divergences manifest themselves through local non-integrability of the function $\Pi_\Gamma$. The $R$-operation transforms this function into a locally integrable function $R\Pi_\Gamma$ which therefore can be naturally regarded as a distribution. The integration at large $x_i$ does not influence the ultraviolet divergences so that when defining $R\Pi_\Gamma$ all the vertices can be treated as external.

As is well-known, the renormalization can be based either on subtractions for each complete subgraph (a subgraph is called complete if, in case it contains the endpoints of some line, it necessarily contains the line itself, i.e. from $\pi_\pm(l) \in \gamma$ it follows that $l \in \gamma$), or for all 1PI subgraphs. The first type of renormalization was used in many early works on renormalization — see, e.g. [8, 9] and is designed for theories with Lagrangians and composite operators with normal ordering. The corresponding $R$-operation acts on the Feynman amplitude $\Pi_\Gamma$ for the graph $\Gamma$ as

$$
R\Pi_\Gamma = \sum_{\nu = \nu_1 \ldots \nu_j} \Lambda(\nu_1) \ldots \Lambda(\nu_j) \Pi_\Gamma
\equiv R'\Pi_\Gamma + \Lambda(\Gamma) \Pi_\Gamma. 
$$

The sum is over all decompositions of the set of vertices $\nu$ of the graph $\Gamma$ into non-empty non-intersecting subsets $\nu_1, \ldots, \nu_j$. Moreover, $\Lambda(\nu_i)$ is the counterterm operation for the subgraph $\gamma(\nu_i)$ composed of vertices $\nu_i$ and all lines that are internal to these vertices. Remember that $\Lambda(\nu_i) = 1$ if $\gamma(\nu_i)$ is an isolated vertex, and $\Lambda(\nu_i) = 0$, if $\gamma(\nu_i)$ is not an 1PI divergent subgraph.

The operation $R'$ is called incomplete $R$-operation. This operation removes all subdivergences of the diagram but does not include the overall counterterm $\Lambda(\Gamma)$. This implies that the function $R'\Pi_\Gamma$ is locally integrable in the space of coordinates except at the point where all the coordinates coincide. Thus, the problem reduces to the extension of this function to a distribution defined on the whole space.

For many reasons, a second type of renormalization is commonly used.\footnote{For example, the scheme based on subtractions at zero momenta is in the first case the BPH renormalization [8, 9] while its analog of the second type is the BPHZ renormalization [10]. For dimensional renormalization, only the second type is used in practice. In contrast to the first type, it provides a mass independent renormalization scheme.} The corresponding $R$-operation looks like

$$
R\Pi_\Gamma = \sum_{\gamma_1, \ldots, \gamma_j} \Delta(\gamma_1) \ldots \Delta(\gamma_j) \Pi_\Gamma
\equiv R'\Pi_\Gamma + \Delta(\Gamma) \Pi_\Gamma. 
$$

where $\Delta(\gamma)$ is the corresponding counterterm operation, and the sum is over all sets $\{\gamma_1, \ldots, \gamma_j\}$ of disjoint divergent 1PI subgraphs, with $\Delta(\emptyset) = 1$.

Note that these two types of renormalization coincide in the massless case, due to zero values of massless vacuum diagrams.
3 Lower order examples in the massless case

In the case of graph of Fig. 1a for \( u \neq 0 \) we have

\[
\Pi_{1a}(u) = \frac{1}{16\pi^4} \frac{1}{u^4} = \frac{1}{16\pi^4} S \ln \frac{\mu^2 u^2}{u^4}. \tag{24}
\]

The left-hand side of (24) is ill-defined as a distribution because this function is non-integrable in the vicinity of the point \( u = 0 \). However the right-hand side is correctly defined as a distribution everywhere in \( \mathcal{R}^4 \), since the operator \( S \) involves preliminary multiplication by \( u_\alpha \), and the function \( u_\alpha/u^4 \) is already locally integrable. By definition, an extension of the functional in the left-hand side to the whole space \( \mathcal{R}^4 \) is determined with the help of the right-hand side. The arbitrariness of this extension is explicitly contained in the parameter \( \mu \) which plays the same role as the corresponding parameters in the frameworks of dimensional and analytic renormalizations. Let us thus define the ‘differentially renormalized’ Feynman amplitude for the graph 1a by

\[
R \Pi_{1a} = \frac{1}{16\pi^4} S \ln \frac{\mu^2 u^2}{u^4} \tag{25}
\]

so that, in accordance with (6), the action of this distribution on a test function \( \phi(u) \in \mathcal{D}(\mathcal{R}^4) \) is defined by

\[
(R \Pi_{1a}, \phi) = -\frac{1}{16\pi^4} \int d u \ln \frac{\mu^2 u^2}{u^4} u_\alpha \frac{\partial}{\partial u_\alpha} \phi(u). \tag{26}
\]

The counterterm operation \( \Delta(\Gamma) \) for the graph 1a can be ‘formally’ represented as

\[
\Delta \Pi_{1a} = \frac{1}{16\pi^4} \left( S \ln \frac{\mu^2 u^2}{u^4} - \frac{1}{u^4} \right). \tag{27}
\]

This quantity alone (as well as other counterterms and unrenormalized or partially renormalized Feynman amplitudes) does not make sense as a functional on the whole space of test functions. However one can combine the sum of contributions of counterterm operations into renormalized quantities in such a way that all the obtained combinations will be sensible under integration. It should be noted that the counterterm (27) vanishes for \( u \neq 0 \).

The functional \( S R \Pi_{1a} \) equals zero for \( u \neq 0 \) and therefore its support coincides with the point \( u = 0 \). It is easy to verify that its action on test functions that are zero at this point is zero. Hence

\[
S R \Pi_{1a} = c_{1a} \delta^{(4)}(u). \tag{28}
\]

To calculate the constant \( c_{1a} \) one may introduce analytic regularization, to write (25) through \( \frac{d}{d\lambda} S(x^2)^{\lambda-2} \) at \( \lambda = 0 \) and apply the expansion

\[
(x^2)^{\lambda-2} = \frac{\pi^2}{\lambda} \delta^{(4)}(x) + O(\lambda^0), \tag{29}
\]
with $\lambda$ in the neighbourhood of the origin of the complex plane. The result is

$$c_{1a} = \frac{1}{16\pi^2}. \quad (30)$$

The next example is the graph of Fig. 1b. The subdivergence is removed according to prescription for the graph 1a. The ‘incomplete’ $R$-operation (i.e. without the last subtraction), when applied to the Feynman amplitude under consideration, gives

$$R' \Pi_{1b} \equiv (1 + \Delta(\gamma_1)) \Pi_{1b} = \frac{1}{(4\pi^2)^{4}} \frac{1}{v^2(u - v)^2} \hat{S}_{u} \ln \frac{\mu^2 u^2}{u^4}, \quad (31)$$

where $\gamma_1$ is graph 1a as a subgraph in 1b.

Using (18) at $k = 0$ one observes that if not all the coordinates $u, v, 0$ of the graph 1b coincide the following equation is valid:

$$R' \Pi_{1b} = \frac{1}{(4\pi^2)^{4}} \left\{ \hat{S}_{u, v} \frac{\ln \mu^2 u^2}{v^2(u - v)^2} \hat{S}_{u} \ln \frac{\mu^2 u^2}{u^4} - \frac{\ln \mu^2 v^2}{v^2(u - v)^2} \hat{S}_{u} R \ln \frac{\mu^2 u^2}{u^4} \right\}. \quad (32)$$

Let us now use (28) and the equation

$$\frac{\ln \mu^2 v^2}{v^4} = \frac{1}{2} \hat{S}_{v} \frac{\ln^2 \mu^2 v^2}{v^4}, \quad v \neq 0, \quad (33)$$

which enables us to differentially renormalize graph 1a with an additional logarithm:

$$R \ln \frac{\mu^2 v^2}{v^4} \equiv R \ln \frac{\mu^2 v^2}{v^4} \Pi_{1b} = \frac{1}{2} \hat{S}_{v} \frac{\ln^2 \mu^2 v^2}{v^4}. \quad (34)$$

After that, as for Fig. 1a, the right-hand side of (32) turns out to be defined as a functional on the whole space $D(R^8)$. The differentially renormalized Feynman amplitude for the graph 1b is defined as the corresponding extension of functional (32) from the subspace of test functions vanishing in a vicinity of the point $u = v = 0$.

As a result we obtain

$$R \Pi_{1b} = \hat{S} \ln \mu^2 v^2 \ R' \Pi_{1b} - c_{1b} R \ln \mu^2 v^2 \Pi_{1b}/\gamma_1$$

$$= \frac{1}{(4\pi^2)^{4}} \left\{ \hat{S}_{u, v} \frac{\ln \mu^2 v^2}{v^2(u - v)^2} \hat{S}_{u} \ln \frac{\mu^2 u^2}{u^4} - \frac{1}{2} c_{1b} \hat{S}_{v} \frac{\ln^2 \mu^2 v^2}{v^4} \delta(u) \right\}, \quad (35)$$

with $\Gamma = 1b, \gamma = 1a$. The arbitrariness of the subtraction operation for the graph 1b itself appears explicitly in the parameter $\mu'$. It is possible, for example, to introduce a unique mass scale $\mu$ that determines an energy scale for perturbation theory and fix all $\mu$-parameters which may arise as $\mu' = \zeta \mu$, with some constant $\zeta$. With this prescription $\mu$ determines a one-parametrical subgroup of RG transformations and is quite analogous, in its character, to the 't Hooft mass $\mu$ in dimensional renormalization.
The action of the counterterm operation for the graph 1b is formally written as

$$\Delta \Pi_{1b} = R \Pi_{1b} - R' \Pi_{1b}.$$  \hfill (36)

The counterterm (36) vanishes everywhere outside the point $u = v = 0$. Note that for all other points

$$\hat{S} R \Pi_{1b} = c_{1b} \hat{S} \Pi_{\Gamma / \gamma_1}.$$  

Therefore the left-hand and right-hand sides of this equation differ by a functional with support localized at the point $u = v = 0$. Using the same arguments as in the first example we obtain

$$\hat{S} R \Pi_{1b} = c_{\Gamma} \delta^{(8)}(u, v) + c_1 \hat{S} \frac{\ln \mu^2 v^2}{u^4} \delta^{(4)}(u)$$

$$= c_\Gamma \delta^{(8)}(u, v) + c_\gamma_1 R \Pi_{\Gamma / \gamma_1},$$  \hfill (37)

with some constant $c_\Gamma \equiv c_{1b}$.

To calculate this constant let us integrate (37) over $u$:

$$c_\Gamma \delta^{(4)}(v) = \frac{1}{(4\pi^2)^4} \left\{ \hat{S}_c \frac{1}{v^2} \int du \frac{1}{(u - v)^2} \hat{S}_u \frac{\ln \mu^2 u^2}{u^4} - \pi^2 \hat{S}_c \frac{\ln \mu^2 v^2}{v^4} \right\}. \hfill (38)$$

The integral in the braces can be evaluated by introducing analytic regularization

$$\int du \frac{1}{(u - v)^2} \hat{S}_u \frac{\ln \mu^2 u^2}{u^4} = \frac{d}{d\lambda} \left[ (\mu^2)^\lambda \lambda \int du \frac{1}{(u - v)^2(u^2)^{2-\lambda}} \right]_{\lambda=0}$$  \hfill (39)

and applying one-loop massless formula in four dimensions

$$\int du \frac{1}{(u^2)^{\lambda_1}((u - v)^2)^{\lambda_2}} = \pi^2 G(\lambda_1, \lambda_2) \frac{1}{(v^2)^{\lambda_1 + \lambda_2 - 2}}, \hfill (40)$$

with four-dimensional $G$-function given by

$$G(\lambda_1, \lambda_2) = \frac{\Gamma(\lambda_1 + \lambda_2 - 2)\Gamma(2 - \lambda_1)\Gamma(2 - \lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4 - \lambda_1 - \lambda_2)}. \hfill (41)$$

Therefore the first term in the braces in (38) is rewritten as

$$\pi^2 \hat{S}_c \frac{1}{v^2} \frac{d}{d\lambda} \left[ (1 + \lambda)(\mu^2)^\lambda (u^2)^{\lambda-1} \right]_{\lambda=0} = \pi^2 \hat{S}_c \left[ \frac{1}{v^4} + \frac{\ln \mu^2 v^2}{v^4} \right]. \hfill (42)$$

As a result we obtain the value $c_{1b} = 1/(16\pi^2)^2$.

For Fig. 1c using relation (27) we have

$$R' \Pi_{1c} = [1 + \Delta(\gamma_1) + \Delta(\gamma_2)] \Pi_{1c}$$

$$= \frac{1}{(4\pi^2)^4} \left\{ \frac{1}{(u - v)^4} \hat{S} \frac{\ln \mu^2 u^2}{u^4} + \frac{1}{u^4} \hat{S} \frac{\ln \mu^2 (u - v)^2}{(u - v)^4} - \frac{1}{u^4(u - v)^4} \right\}, \hfill (43)$$
where \( \gamma_1 \) and \( \gamma_2 \) are respectively left and right simple loops 1a in 1c.

As in the case of Fig. 1b, using equations (28), (33), (18) and extending functional (43) from the space of test functions determined in \( \mathcal{R}^8 \) with ‘deleted’ point \( u = v = 0 \) to the whole space \( \mathcal{D}(\mathcal{R}^8) \) we come to the following result:

\[
R \Pi_{1d} = \hat{S} \ln \mu^2 v^2 R' \Pi_{\Gamma} - c_{\gamma_1} (R \ln \mu^2 v^2 \Pi_{\Gamma/\gamma_1} + R \ln \mu^2 v^2 \Pi_{\Gamma/\gamma_2}).
\]  

(44)

In the case of the ‘catseye’ diagram Fig. 1c with an additional logarithm \( \ln \mu^2 \) of (1d).

Let us consider the space \( \mathcal{R}^{12} \) with deleted origin \( u = v = w = 0 \). For each point in the vicinity of the origin we have at least one of the following two possibilities: (i) \( u \neq 0 \) or/and \( v \neq 0 \), (ii) \( u \neq w \) or/and \( v \neq w \). We consider first case (ii). Then the contribution from the counterterm of the subgraph \( \gamma_{22} \) disappears and \( R' \) takes the form

\[
R' \Pi_{1d} = \frac{1}{(4\pi^2)^6} \frac{1}{(u-w)^2(v-w)^2} \left\{ \hat{S} \ln \mu^2 w^2 \ln \mu^2 (u-w)^2 \right\} - \frac{1}{2} c_1 \hat{S} \ln \mu^2 \ln \mu^2 w^2 R' \Pi_{\Gamma/\gamma_1}. 
\]

(47)

and \( \Delta(\gamma_{22}) \Pi_{1d} \) is obtained by replacing \( u \) by \( w - u \) and \( v \) by \( w - v \). Here \( \gamma_1 \) is the central simple loop; \( \gamma_{21} \) and \( \gamma_{22} \) are respectively left and right graphs 1b as subgraphs of 1d.

Let us consider the space \( \mathcal{R}^{12} \) with deleted origin \( u = v = w = 0 \). For each point in the vicinity of the origin we have at least one of the following two possibilities: (i) \( u \neq 0 \) or/and \( v \neq 0 \), (ii) \( u \neq w \) or/and \( v \neq w \). We consider first case (ii). Then the contribution from the counterterm of the subgraph \( \gamma_{22} \) disappears and \( R' \) takes the form

\[
R' \Pi_{1d} = \frac{1}{(4\pi^2)^6} \frac{1}{(u-w)^2(v-w)^2} R' \Pi_{\gamma_{22}}.
\]

(48)

Using the procedure described above, in particular (18) at \( k = 0 \), (28) and (37), \( R' \Pi_{1d} \) can be represented in the form

\[
R' \Pi_{\Gamma} = \hat{S} \ln \mu^2 w^2 R' \Pi_{\Gamma} - c_{\gamma_1} \ln \mu^2 w^2 R' \Pi_{\Gamma/\gamma_1} - c_{\gamma_{22}} \ln \mu^2 w^2 R' \Pi_{\Gamma/\gamma_{22}}.
\]

(49)

The functional \( \hat{S} \ln \mu^2 w^2 R' \Pi_{\Gamma} \) is naturally extended to the whole \( \mathcal{R}^{12} \); the extension of simple loops (with ‘additional’ logarithms) \( \ln \mu^2 w^2 R' \Pi_{\Gamma/\gamma_{22}} \), \( i = 1,2 \) was described in the case of Fig. 1b. For the graph of Fig. 1c with an additional logarithm \( \ln \mu^2 w^2 R' \Pi_{\Gamma/\gamma_{1}} \), the procedure that was used for Fig. 1c itself and applies (18) at \( k = 2 \) can be straightforwardly generalized. A similar expression is obtained for the case (i).

As a result we obtain the following expression for the differentially renormalized diagram 1d which is valid in the whole space:

\[
R \Pi_{\Gamma} = \hat{S} \ln \mu^2 w^2 R' \Pi_{\Gamma} - c_{\gamma_{1a}} R \ln \mu^2 w^2 \Pi_{\Gamma/\gamma_{1a}} - c_{\gamma_{1b}} R \ln \mu^2 w^2 \Pi_{\Gamma/\gamma_{1b}} - c_{\gamma_{22}} R \ln \mu^2 \Pi_{\Gamma/\gamma_{22}}.
\]

(50)
4 Lower order examples in the massive case

Renormalization of the diagrams of Fig. 1 in the massive case is performed by formulae similar to the previous section, with $\tilde{S}$ given by (13), e.g. for graph 1a

$$R \Pi_{1a} \equiv R G(x)^2 = \tilde{S} \ln \mu^2 x^2 G(x)^2. \quad (51)$$

The constants $c_i$ involved here have the same values as in the massless case.

To see the additional problems that can arise let us consider other simple examples. New problems can appear for vacuum graphs or those with one external vertex, e.g. for tadpoles. Let us distinguish contributions (see Fig. 2a) which are formally given by $G(0)$ — the value of propagator at the origin where it is singular. They can exist separately or belong to other graphs. In the latter case, such contributions appear as independent factors. All the other tadpoles are products of propagators integrated over all coordinates but one. The methods of differential renormalization can be applied for these tadpoles without problem.

Note that even within dimensional regularization $G(0)$ is not defined without some further prescription. A standard way of handling this quantity is write it down as momentum space $d$-dimensional integral $\int d^d k / (k^2 + m^2)$, then calculate and renormalize it. In the limit $d \to 4$, one obtains $G(0) = \frac{1}{16\pi^2} m^2 \ln m^2 / \mu^2$ (up to finite renormalization).

At $d = 4$ it is possible to say that $G(0)$ is understood as the value at $x = 0$ of $G(x)$ from which the singular terms of the small $x$ expansion

$$G(x) = \frac{1}{4\pi^2} \left\{ \frac{1}{x^2} + \frac{m^2}{4} \left( \ln m^2 x^2 + 2\gamma_E - 1 \right) \right\} + o(x^2), \quad (52)$$

are subtracted ($\gamma_E$ the Euler constant). This procedure leads to the reasonable result $G(0) = \frac{1}{16\pi^2} m^2 (\ln m^2 / \mu^2 + 2\gamma_E - 1)$.

At first sight the strategy of extending functionals from some space of test functions to the whole space does not have anything to do with defining $G(0)$. Therefore the general recipes of differential renormalization seem to be of no use here. One possibility is to introduce additional prescriptions, and there can be at least two variants: to set $G(0) = 0$ or to take $G(0) = \frac{1}{16\pi^2} m^2 \ln m^2 / \mu^2$ (or some similar value for non-scalar propagators) — the former variant of course reduces to the latter one at $\mu = m$.

There is also a new problem of another type in the massive case. Consider the ‘setting sun’ diagram shown in Fig. 2b. In the case of renormalization (22) based on complete subgraphs the only subtraction involved is for the overall graph. However the operator (17) is not sufficient to remove all the ultraviolet divergences because differentiation in the mass removes only the most singular part of $G(x)^3$. If one tries to perform the second type of renormalization (23) in the differential style one will observe that the insertion of counterterms for three (overlapping) simple loops does
not change the diagram at $x \neq 0$. Furthermore, in the whole space, insertion of these counterterms does not make sense. It seems that we do not have ‘enough space’ to perform consecutive extensions of the initial functional in two steps that correspond respectively to renormalization of above three subgraphs and the graph itself.

To overcome these problems let us exploit the following trick that was used in \cite{[15]}. Instead of propagators in coordinate space (2), let us consider its Fourier transform in respect to two additional components, $m_1$ and $m_2$ by considering the square of mass $m^2$ in the propagator as the square of this two-dimensional vector $m = (m_1, m_2)$. Denoting the corresponding coordinate-space variable by $y = (y_1, y_2)$ we obtain the massless propagator in six-dimensional space

$$G(x, y) = \frac{1}{4\pi^3 \left(x^2 + y^2\right)^2} \quad (53)$$

which satisfies

$$(x + y)G(x, y) = -\delta^{(4)}(x)\delta^{(2)}(y). \quad (54)$$

For general Feynman diagram let us introduce Fourier transformation for each massive line. As a result $\Pi_F$ is expressed as a product of propagators $G(x, y)$ depending on $4N$ usual coordinate-space variables and $2N_m$ additional ones, $N_m$ being the number of massive lines.

Remember that Feynman amplitudes must be well-defined distributions in order that Fourier transforms to momentum space (where physical quantities are calculated) are possible. Therefore it is necessary to perform integration over coordinates; at this step locally non-integrable singularities manifest themselves as the source of the ultraviolet divergences. Now, for the same reason, it is natural to consider the product of propagators $\Pi_F(z, y)$, with $y = (y_1, \ldots, y_{N_m})$, as a distribution not only in $x_i$ but also in $y_i$. Indeed, in the end it is necessary to perform inverse Fourier transformation and come back to masses (and put them equal to each other if there was initially only one mass). In this approach the ultraviolet divergences manifest themselves in integrations over small coordinate differences and also the variables $y_i$.

Note that it is possible to introduce Fourier transformation in masses even for massless lines. In this case, it is necessary, in the end of calculation, to integrate over the corresponding $y$-variables. Therefore, as products of propagators, we now have

$$\Pi_F(x, y) = \prod_i \frac{1}{4\pi^3 \left((x_{+i} - x_{-i})^2 + y_i^2\right)^2}. \quad (55)$$

One can show that usual power counting turns out to be the same and is governed by the same degree of divergence $\omega$. It is now possible to apply above formulae of differential renormalization (for the massless case) using the operator

$$\hat{S}_{x, y} = \frac{1}{2} \left( \sum_i \frac{\partial}{\partial x_i} x_i + \sum_i \frac{\partial}{\partial y_i} y_i \right). \quad (56)$$
We can now write the value of the propagator at \( x = 0 \)
\[
G(x, y) \bigg|_{x=0} = \frac{1}{4 \pi^3} \frac{1}{(y^2)^2},
\]
without running into division by zero. To make sense of it for all \( y \), i.e. as distribution in \( y \), let us apply the operator (since the divergence is quadratic) \( \delta_y^{(2)} \) given by (17):
\[
RG(0, y) = \frac{1}{4 \pi^3} 2 \delta_y^{(2)} \ln \mu^2 y^2 \quad \frac{y^2}{(y^2)^2}.
\]
Its inverse Fourier transformation in \( y \) gives
\[
RG(0; m) = \frac{1}{16 \pi^2} m^2 \left( \ln m^2/4 \mu^2 + 1 + 2 \gamma_E \right).
\]
The renormalized simple loop (51) is rewritten in the language of \( y \)-variables as
\[
R \Pi_{1a} = \frac{1}{(4 \pi^3)^2} \delta_{x,y_1,y_2} \frac{\ln \mu^2 V}{(x^2 + y_1^2)(x^2 + y_2^2)^2},
\]
with the corresponding counterterm represented as
\[
\Delta \Pi_{1a} = \frac{1}{(4 \pi^3)^2} \delta_{x,y_1,y_2} \left[ \frac{\ln \mu^2 V}{(x^2 + y_1^2)(x^2 + y_2^2)^2} - \frac{1}{(x^2 + y_1^2)(x^2 + y_2^2)^2} \right].
\]

One can use different arguments of the logarithm involved, for instance, (a) \( V = x^2 \); (b) \( V = y_1^2 \) or \( V = y_2^2 \), (c) \( V = x^2 + y_1^2 \) or \( V = x^2 + y_2^2 \); (d) \( V = x^2 + y_1^2 + y_2^2 \). It is possible to show that in cases (a) and (b) the limit \( m \to 0 \) exactly reproduces the ‘pure massless’ prescription (25).

Let us now return to the ‘setting-sun’ diagram. In the language of \( y \)-variables the product of corresponding propagators is
\[
\Pi_{2b} = \prod_{i=1,2,3} \left( \frac{1}{4 \pi^3 (x^2 + y_i^2)^2} \right).
\]
Inserting counterterms for three subgraphs 1a gives
\[
R' \Pi_{2b} = \Pi_{2b} + G(0, y_3) \left[ \delta_{x,y_1,y_2} \ln \mu^2 x^2 G(x, y_1) G(x, y_2) - G(x, y_1) G(x, y_2) \right] + \ldots,
\]
where the dots stand for permutations. This quantity is meaningful as a functional everywhere except the origin with respect to the variables \( (x, y_1, y_2, y_3) \). Let us now apply (19) at \( \omega = 2 \) and use the values of one-loop counterterms to write down the incompletely renormalized diagram as
\[
R' \Pi_{2b} = 2 \delta^{(2)} \ln \mu^2 V R' \Pi_{2b} - \frac{1}{4 \pi^3} \left( \ln \mu^2 y_3^2 - 6 \right) \frac{1}{(y_3)^2} \frac{1}{16 \pi^2} \delta^{(4)}(x) \delta^{(2)}(y_1) \delta^{(2)}(y_2) - \ldots.
\]
with $V$ chosen as $V = x^2 + y_1^2 + y_2^2 + y_3^2$. Now we extend this functional to the whole space of variables $(x, y_1, y_2, y_3)$ with the help of the renormalization of the $G(0)$ (see (58)) and its analog with an additional logarithm:

$$R \ln \mu^2 y^2 = \hat{S}^{(2)} \frac{\ln^2 \mu^2 y^2 + 6 \ln \mu^2 y^2}{y^4}.$$  

We obtain

$$R \Pi_{2b} = 2 \hat{S}^{(2)} \ln \mu^2 V R' \Pi_{2b}$$

$$- \frac{1}{4 \pi^2 \pi^2 y_1} \hat{S}^{(2)} \frac{\ln^2 \mu^2 y_1^2 - 6 \ln \mu^2 y_1^2 \ln \mu^2}{y_1^4} - \frac{1}{16 \pi^2} \delta^{(4)}(x) \delta^{(2)}(y_2) \delta^{(2)}(y_3) - \ldots,$$  

where we did not use a freedom to introduce a new parameter $\mu$ associated with the overall graph.

Let us now calculate the constants $c_\gamma$ which enter the following formula:

$$(\hat{S} + 1) R \Pi_{2b} = (c_{xx} + \sum_i c_{\gamma_i}) \delta^{(4)}(x) \prod_i \delta^{(2)}(y_i)$$

$$+ R \ln \mu^2 y_1^2 G(0, y_3) c_1 \delta^{(4)}(x) \delta^{(2)}(y_2) \delta^{(2)}(y_3) + \ldots.$$  

To calculate $c_\gamma$ we multiply (67) by $x^2$ and integrate it over $x$ and $y_i$. To calculate $c_y$ we integrate (67) over $x, y_2, y_3$ and perform inverse Fourier transformation in $y_1$. The results are

$$c_x = \frac{1}{2 (16 \pi^2)^2}, c_{y_i} = \frac{1}{(16 \pi^2)^2}.$$  

5 General prescriptions

By generalizing procedure described in the above examples let us define a renormalization procedure which will be naturally called as differential renormalization. To present quite general prescription let us apply the language of $y$-variables described in the previous section even in the case when some of the lines are massless. Let us consider renormalization of products (55) multiplied by powers of logarithms: $\ln^k (\mu^2 V \Gamma (x, y)) \Pi \Gamma$. Here we imply two possible variants: $V \Gamma = u_i^2 + \sum_i y_i^2$ and $V \Gamma = \sum_i y_i^2$, where $u_i = x_i - x_i'$ is any difference variable of the considered Feynman amplitude.

Definition. Let a renormalization $R$ of the graph $\Gamma$ be given by the following recursive formulae:

$$R \left. \ln \right|^{k+1} (\mu^2 V \Gamma) \Pi \Gamma = \frac{1}{k + 1} \hat{S}^{(2)} \ln^{k+1} (\mu^2 V \Gamma) R' \Pi \Gamma - \frac{1}{k + 1} \sum_{\gamma \subset \Gamma} R \ln^{k+1} (\mu^2 V \Gamma) \hat{C}_\gamma \Pi \Gamma,$$  

for $\omega = 0$ and integer $k \geq 0$;

$$R \Pi \Gamma = a_{\omega} \hat{S}^{(\omega)} \ln (\mu^2 V \Gamma) R' \Pi \Gamma - \frac{1}{k + 1} \sum_{\gamma \subset \Gamma} R (\ln (\mu^2 V \Gamma) - 4 b_\omega)(\hat{S} + \omega / 2) \hat{C}_\gamma \Pi \Gamma,$$  

(70)
for \( \omega > 0 \) and integer \( k = 0 \), as well by other relations for arbitrary \( \omega \) and \( k \) which follow from the corresponding generalizations of (19). Here \( R' \) is incomplete \( R \)-operation (22). The sum in the second term of the right-hand side of (69) and (70) runs over all 1PI proper subgraphs \( \gamma \) of \( \Gamma \). Furthermore,

\[
\Delta(\Gamma) \Pi_{\Gamma} = RÎPi_{\Gamma} - R' \Pi_{\Gamma}.
\]

Finally, the operations \( C_{\gamma} \) are determined from equations

\[
\left( \dot{S} + \omega/2 \right) R \Pi_{\Gamma} = \sum_{\gamma \subseteq \Gamma} C_{\gamma} R \Pi_{\Gamma} \equiv \sum_{\gamma \subseteq \Gamma} R C_{\gamma} \Pi_{\Gamma} + C_{\Gamma} \Pi_{\Gamma},
\]

where the operation \( C_{\gamma} \) inserts a polynomial \( \mathcal{P}_{\gamma} \) of degree \( \omega(\gamma) \) in masses of \( \gamma \) and its external momenta into the reduced vertex of the graph \( \Gamma/\gamma \). Symbolically we write

\[
C_{\gamma} \Pi_{\Gamma} = \Pi_{\Gamma/\gamma} \circ \mathcal{P}_{\gamma},
\]

where \( \circ \) denotes the insertion operation. In the language of coordinate space,

\[
C_{\Gamma} \Pi_{\Gamma}(x, y) = \mathcal{P}_{\Gamma}(\partial/\partial x_i, \partial/\partial y_{i'}) \prod_{i \in \Gamma} \delta^{(4)}(x_i - x_1) \prod_{l \in \Gamma} \delta^{(2)}(y_{i'}). 
\]

Indeed relations (69)–(72) and their generalizations for arbitrary \( \omega \) and \( k \) enable us to obtain \( R\ln^k(p^2 V_{\Gamma}) \) \( \Pi_{\Gamma} \), \( \Delta(\Gamma) \) and \( C_{\Gamma} \) provided we know the corresponding quantities for all proper subgraphs of \( \Gamma \) and its reduced graphs: in particular, \( \mathcal{P}_{\Gamma} \) is expressed from (72).

The following proposition is valid.

**Proposition.** The procedure \( R \) defined by relations (69)–(72) is a correct \( R \)-operation.

**Proof.** To prove this proposition we should show that (i) the expression (69) is finite, (ii) (69) is obtained from \( R' \) \( \Pi_{\Gamma} \) as an extension from the space from deleted origin of the whole space, i.e. the corresponding counterterm (71) is local, (iii) the quantity \( C_{\Gamma} \Pi_{\Gamma} \) found from (72) is local.

Let us consider, in the space of coordinates, an arbitrary point \( \mathbf{x}^0 \equiv \{x_1^0, \ldots, x_n^0\} \) in which at least one of the difference variables (e.g. \( u_1^0 = x_1^0 - x_n^0 \)) is non-zero: \( u_1^0 \neq 0 \). In respect to the point \( \mathbf{x}^0 \) all the set of vertices \( \mathcal{V} \) of the graph \( \Gamma \) is naturally decomposed over non-intersecting subsets \( \mathcal{V}_r \), with \( \mathcal{V} = \bigcup_r \mathcal{V}_r, x_i^0 - x_{i'}^0 = 0, \forall i, i' \in \mathcal{V}_r, r \neq r' \). In accordance with the assumption about the point \( \mathbf{x}^0 \), the number of subsets \( \mathcal{V}_r \) is not less than two.

Let \( \Gamma_r \) be subgraphs constructed with help of vertex sets \( \mathcal{V}_r \); by definition, each \( \Gamma_r \) contains any line that connects a pair of vertices from \( \mathcal{V}_r \). Note that the subgraphs \( \Gamma_r \) can be one-particle-reducible or disconnected. Let us denote by \( \mathcal{W} \) the set of maximal 1PI subgraphs of the graph \( \Gamma^0 = \bigcup_r \Gamma_r \). Furthermore, let \( \overline{\mathcal{W}} \) be the set of all divergent subgraphs of the graph \( \Gamma^0 \).
Let \( U^0 \) be a sufficiently small vicinity of this point so that for all \( x \in U^0 \) the same properties hold, \( x^0_i - x^0_i = 0, \forall i, i' \in \mathcal{V}_r \), and \( x^0_i - x^0_i \neq 0, \forall i' \in \mathcal{V}_r, \forall r \neq r' \). In the domain \( U^0 \), the incomplete \( R \)-operation \( R' \) does not include counterterms contributed by subgraphs containing vertices from different subsets \( \mathcal{V}_r \). Therefore, in \( U^0 \), we have the equation
\[
\hat{S} R' \Pi_{\Gamma} = \hat{S} \left\{ \sum_{\gamma_1} \Delta(\gamma_1) \ldots \Delta(\gamma_j) \right\} \Pi_{\Gamma},
\]
where the sum is over decompositions \( \mathcal{V} = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_j \) such that any \( \gamma_i \) belongs to \( \mathcal{W} \). In other words, the sum in (75) can be represented as analogous sum over decompositions in which any \( \gamma_i \) happens to be an element from the set \( \mathcal{W} \). After that each of the factors \( \Delta(\gamma_i) \) involved transforms into the \( R \)-operation \( R(\gamma_i) \) that acts on the Feynman amplitude for the subgraph \( \gamma_i \). Thus, (commutation relations (20)) are used
\[
(\hat{S} + \omega/2) R' \Pi_{\Gamma} = \hat{S} \left\{ \prod_{\gamma \in \mathcal{W}} \left( R \Pi_{\gamma} \right) \right\} \Pi_{\Gamma} \Pi_0 - \sum_{\gamma \in \mathcal{W}} \left( (\hat{S} + \omega/2) R \Pi_{\gamma} \right) \prod_{\gamma \neq \gamma} \left( R \Pi_{\gamma} \right) \Pi_{\Gamma} \Pi_0.
\]
(76)

Since in \( U^0 \) the propagators of lines which connect different elements of \( \mathcal{W} \) are not singular we may apply the Leibniz rule encoded in (18), (19) and their generalizations. For example, in the case \( \omega = 0 \) we have
\[
\ln(k^2(\mu^2 \mathcal{V}_r)) R' \Pi_{\Gamma} = \frac{1}{k+1} \hat{S} \ln(k+1)(\mu^2 \mathcal{V}_r) R' \Pi_{\Gamma} - \frac{1}{k+1} \ln(k+1)(\mu^2 \mathcal{V}_r) \hat{S} R' \Pi_{\Gamma}.
\]
(77)

Let us now apply relation (72) for subgraphs and turn from summation over elements \( \gamma \in \mathcal{W} \) and subgraphs of each \( \gamma \) to summation of subgraphs of \( \Gamma^0 \). After that we obtain the following expression for the second term in the right-hand side of (77):
\[
\frac{1}{k+1} \ln(k+1)(\mu^2 \mathcal{V}_r) \sum_{\gamma \in \mathcal{W}} R' C_{\gamma} \Pi_{\Gamma}.
\]
(78)

Here only subgraphs and reduced graphs with a smaller number of loops are involved. Therefore we know how to renormalize these quantities \( \ln(k+1)(\mu^2 \mathcal{V}_r) R' C_{\gamma} \Pi_{\Gamma} \) by extending them as functionals to the whole space and arriving at \( R \ln(k+1)(\mu^2 \mathcal{V}_r) C_{\gamma} \Pi_{\Gamma} \). As to the first term in (77) it does not have divergences because all subdivergences are removed by \( R' \) and the overall divergence is removed by the operator \( \hat{S}^{(\omega)} \). Thus we arrive at a finite differentially renormalized quantity (69) which is obtained by extension of the functional \( R' \Pi_{\Gamma} \) to the whole space (i.e. by adding a local counterterm). Note that in (69) the summation is all divergent 1PI subgraphs of \( \Gamma \); in each vicinity \( \Gamma^0 \) this summation reduces to the corresponding set \( \mathcal{W} \).

To prove (\( \hat{u} \hat{v} \)) it is sufficient to repeat the same manipulations as for (\( \hat{u} \)) starting from \( (\hat{S} + \omega/2) R' \Pi_{\Gamma} \) instead of \( \hat{S} \ln(k+1)(\mu^2 \mathcal{V}_r) R' \Pi_{\Gamma} \).

Comments. (\( \hat{u} \)) For renormalizable theories in the pure massless case there is no need to Fourier transform to \( y \)-variables. One can choose \( V_r = u^2_r \) where \( u_r = x_i - x'_i \).
is any difference variable of the considered Feynman amplitude such that the vertices 
\(i\) and \(j\) do not belong to the same divergent subgraph — see examples in Section 3.

(b) The renormalization prescriptions for Feynman amplitudes (21) are obtained from the above prescriptions for the products \(\Pi_{R}(x, y)\) by 1) integrating over coordinates associated with internal vertices 2) Fourier transforming in \(y\), and putting the corresponding \(m_i^2\) equal to squares of masses, e.g. \(m_i = 0\). It is then natural to consider the insertion polynomials \(\mathcal{P}_F\) dependent only on derivatives in coordinates that correspond to external vertices of the given graph.

6 Commutativity of \(R\)-operation with differentiation and the action principle

It is natural to check whether differential renormalization is in agreement with the action principle which expresses basic properties of quantum field theories such as equations of motion and the gauge invariance. Within dimensional renormalization the strategy for proving the renormalized action principle [12] is as follows: equations of motion, Ward identities etc. are proved for unrenormalized quantities, then for regularized quantities and finally (by more or less obvious commutativity of differentiation in coordinates with renormalization) for renormalized quantities. The most non-trivial point is to justify the relevant symmetries for regularized quantities.

In the context of differential renormalization there will be no such intermediate steps following this program because this is essentially a renormalization without regularization. To see what problems arise in reducing the problem to the case of unrenormalized quantities (via commutation of differentiation with respect to \(R\)-operation) let us consider the simplest example of Fig. 1a:

\[
RG(x)^2 = \hat{S} \ln \mu^2 x^2 G(x)^2. \tag{79}
\]

Let us try to see what is
\[
\partial_\alpha RG(x)^2,
\]
with \(\partial_\alpha = \partial/\partial x_\alpha\). First, remember that both \(\partial_\alpha\) and \(\hat{S}\) are understood in the distributional sense. We have
\[
\partial_\alpha \hat{S} = (\hat{S} + 1/2) \partial_\alpha,
\tag{80}
\]
and hence
\[
(\hat{S} + 1/2) \partial_\alpha \ln \mu^2 x^2 G(x)^2.
\]

Proceeding naively, for the moment, we continue to apply \(\partial_\alpha\) using the Leibniz rule. After this operation we obtain expressions that are ultraviolet divergent. However in a distributional sense the derivative is not defined because it acts on unrenormalized expression that is defined only at \(x \neq 0\). This is a manifestation of an important difference with respect to dimensional renormalization: we do not have a regularization associated with differential renormalization.
A different prescription is necessary in order to apply $\partial_a$. The point is that our differential $R$-operation should take account of the degree of divergence. For $G(x)^2$ it is equal to zero while formally it is $\omega = 1$ for $\partial_a G(x)^2$. According to our prescriptions when the degree of divergence $\omega$ is equal to one it is possible to use the operator (16) and the corresponding differentially renormalized expression is

$$R \partial_a G(x)^2 = -2(\hat{S} + 1/2) \hat{S} \ln \mu^2 x^2 \partial_a G(x)^2.$$  \hfill (81)

Let us use the following equation:

$$\hat{S} = -2\hat{S}(\hat{S} - 1/2) + 2\hat{S}^2,$$  \hfill (82)

with (28), (30) to show that

$$(\hat{S} + 1/2) \hat{S} \partial_a G(x)^2 = c_1 \partial_a \delta(x).$$  \hfill (83)

With the help of the auxiliary analytic regularization it is easy to find $-2c_1 = c_0 \equiv c_{1,a} = 1/16\pi^2$.

We have

$$\partial_a \hat{S}(\hat{S} - 1/2) = (\hat{S} + 1/2) \hat{S} \partial_a = \hat{S}^{(1)} \partial_a.$$  

Hence we may now apply $\partial_a$ to $\ln \mu^2 x^2 / x^4$ naively, in the sense of ordinary functions rather than in a distributional sense, because the operator $\hat{S}^{(1)}$ ensures that the overall expression is well defined. As a result we see that commutativity breaks down:

$$\partial_a \hat{S} \ln \mu^2 x^2 G(x)^2 = -2(\hat{S} + 1/2) \hat{S} \ln \mu^2 x^2 \partial_a G(x)^2 + \frac{3}{2} c_0 \partial_a \delta(x).$$  \hfill (84)

One can however define the renormalization of $\partial_a G(x)^2$ with another $\mu$-parameter, $\mu'$. This results in

$$\partial_a \hat{S} \ln \mu'^2 x^2 G(x)^2 = -2(\hat{S} + 1/2) \hat{S} \ln \mu'^2 x^2 \partial_a G(x)^2 + (3/2 - \ln(\mu'^2 / \mu^2)) c_0 \partial_a \delta(x).$$  \hfill (85)

and so we may recover commutativity when $\ln(\mu'^2 / \mu^2) = 3/2$.

This simple example shows that the commutativity of differentiation in coordinates with the $R$-operation is not satisfied automatically in differential renormalization. It is necessary to adjust renormalization parameters to provide it. Another possibility is to use desired commutation relations as definitions for renormalization of diagrams that are obtained as derivatives of some other diagrams [4]. In the above example, this amounts to applying $\partial_a R G(x)^2$ (rather than the right-hand side of (81)) as a definition of $R \partial_a G(x)^2$.

Bearing in mind this conclusion let us consider, for example, the following equations in the $\phi^4$ theory:

$$m^2 \frac{\partial}{\partial m^2} R G^{(n)} = -RD_m G^{(n)},$$  \hfill (86)

$$g \frac{\partial}{\partial g} R G^{(n)} = -RD_4 G^{(n)},$$  \hfill (87)

$$R \left( D_m - D_2 + 2D_4 + \frac{n}{2} \right) G^{(n)} = 0,$$  \hfill (88)
where $D_m G^{(m)}$, $D_2 G^{(n)}$, $D_4 G^{(n)}$ denote, respectively, $n$-point Green functions with insertions of the following operators:

$$
\int dx \, m^2 \phi^2(x)/2, \quad \int dx \, (\partial \phi)^2(x)/2, \quad \int dx \, g^4(x)/4!,
$$

In dimensional renormalization these equations directly follow from the renormalized action principle \[12\] (in particular, \(88\) is the equation of motion).

However the renormalized action principle is not automatically guaranteed for differential renormalization. Nevertheless, it is possible to adjust finite arbitrariness (at a diagrammatic level) in renormalization of diagrams that contribute to the first two operators in \(89\) in such a way that equations \(86-88\) hold. In fact, it is sufficient to define renormalization of diagrams with one insertion of operators $m^2 \phi^2/2$ or $(\partial \phi)^2/2$ to satisfy

$$
R \int dx \, m^2 G(x - x_1)G(x - x_2) \Pi(x_1, x_2, \ldots)
= -m^2 \frac{\partial}{\partial m^2} RG(x_1 - x_2; m^p)\Pi(x_1, x_2, \ldots)|_{m^p = m}, \quad (90)
$$

$$
R \int dx \, \left[(m^2 - x)G(x - x_1)\right] G(x - x_2)\Pi(x_1, x_2, \ldots)
= RG(x_1 - x_2)\Pi(x_1, x_2, \ldots), \quad (91)
$$

where $\Pi$ is the rest of the product of the propagators, and in \(90\) the mass derivative acts only on the first propagator $G(x_1 - x_2)$. Since we certainly have possibility to adjust coefficients of proportionality $\zeta$ of the ‘overall’ $\mu$-parameter $\mu^\gamma = \zeta \mu$ to satisfy \(90,91\), it is sensible to use Eqs. \(90,91\) as definitions of the left-hand side.

### 7 Renormalization group coefficients

In dimensional renormalization the renormalization group equation

$$
\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g) \frac{\partial}{\partial g} - \gamma_m(g)m^2 \frac{\partial}{\partial m^2} + \frac{n}{2}\right) RG^{(n)} = 0 \quad (92)
$$

can be derived \[13\] from the so-called diagrammatic RG equation

$$
- \mu^2 \frac{\partial}{\partial \mu^2} RF_T + \epsilon h_TRF_T = \sum_{\gamma \subseteq T} h_\gamma R \left(\Delta^{(\gamma)}(\gamma) F_T\right), \quad (93)
$$

where $h_\gamma$ is the loop number, the operation $\Delta^{(\gamma)}(\gamma) \equiv \epsilon K^{(\gamma)}_\epsilon \Delta(\gamma)$ is obtained from the counterterm operation $\Delta^{MS}(\gamma)$ of the MS-scheme as the residue of the simple pole in $\epsilon$. 

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Then one uses relations (86–88) and obtains the well-known formulae for the RG coefficients:

\[
\beta(g) = g \frac{\partial Z_g^{(1)}}{\partial g} = g(2\gamma_4(g) - \gamma_4(g)),
\]

\[
\gamma_{m^2}(g) = g \frac{\partial Z_{m^2}^{(1)}}{\partial g} = g_2(g) - \gamma_{m^2}(g),
\]

\[
\gamma_i(g) = -g \frac{\partial Z_i^{(1)}}{\partial g}, \quad i = 2, \phi^2, 4,
\]

where \(Z_i^{(1)}\) are contributions of simple poles to the counterterms

\[
Z_2 = 1 + \frac{\partial}{\partial p^2} \tilde{K}_\epsilon R'G^{(2)}(p^2, m^2; \epsilon),
\]

\[
Z_{\phi^2} = 1 + \frac{\partial}{\partial m^2} \tilde{K}_\epsilon R'G^{(2)}(p^2, m^2; \epsilon),
\]

\[
Z_4 = 1 + \tilde{K}_\epsilon R'G^{(4)}(p_1, \ldots, p_4, m^2; \epsilon).
\]

In differential renormalization we have an equation similar to (93):

\[
-\mu^2 \frac{\partial}{\partial \mu^2} R_{F\Gamma} = \sum_{\gamma \in \Gamma'} R(C_{\gamma} F_{\Gamma'}),
\]

where \(C_{\gamma}\) is the operation (given by (73) and (74)) that inserts a finite polynomial of degree \(\omega\) in external momenta into the reduced graph \(\Gamma/\gamma\). To prove (100) it is sufficient to repeat arguments applied in proving (72). Moreover, (100) is in agreement with homogeneity of renormalized Feynman amplitudes in coordinates, inverse masses and \(1/\mu\).

Under these conditions, for evaluation of RG coefficients in differential renormalization, we can apply the following formulae that are quite similar to (94–96):

\[
\beta(g) = c_2 = g(2\gamma_4(g) - \gamma_4(g)),
\]

\[
\gamma_{m^2}(g) = c_{m^2} = g_2(g) - \gamma_{m^2}(g),
\]

\[
\gamma_i(g) = -c_i, \quad i = 2, \phi^2, 4,
\]

with

\[
(c_{m^2} m^2 - c_{2x}) \delta(x) = C R'G^{(2)}(x),
\]

\[
c_4 \prod_{i=2,3,4} \delta(x_i - x_1) = C R'G^{(4)}(x_1, \ldots, x_4).
\]

The constants \(c_i\) are calculated as sums of diagrammatic contributions \(C_{F\Gamma}\). Note that there is no factor \(g \frac{\partial}{\partial g}\) in (101–103) because, in contrast to (93), Eq. (100) does not involve the loop number \(h_{\gamma}\).

Thus, within differential renormalization, the constants \(c_i\) play the same role as the residues of the simple poles in dimensional renormalization counterterms.
8 Conclusion

We have seen that the action principle in differential renormalization is not satisfied automatically. It is necessary to adjust renormalization parameters to satisfy equations of motion etc. Moreover it is clearly natural and very useful to employ equations of motion etc. as definitions for renormalization of derivatives of the quantities involved (e.g. Green functions with insertions of composite operators) whenever possible \([4]\).

Based on subtractions on all divergent 1PI subgraphs differential renormalization turns out to be a mass-independent scheme — this property corresponds to locality of counterterms in the auxiliary parameters \(y_i\) as described in Sections 4 and 5. Renormalization group calculations in this version of differential renormalization are rather simple. Since the problem reduces to calculations of constants \(c_\gamma\) then one can use the method of infrared rearrangement \([14]\) which is based on possibility to put to zero masses and external momenta (\(\equiv\) integration over some coordinates, from coordinate-space point of view) — see examples of such calculations in Sections 3 and 4.

An alternative approach to renormalization group calculations within another version of differential renormalization \([4]\) is based, in the massive case, on the short distance expansion of propagator in coordinate space. This also results in a mass-independence property of the renormalization. Up to two-loop order, such scheme is successfully applied in various situations. However, for higher orders, the relevant short-distance expansion of the propagator should involve many terms which can essentially complicate the situation. It seems that the language of the auxiliary \(y\)-variables is a necessary price that must be paid in order to have a practical strictly four-dimensional scheme for multiloop calculations.

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References


Figures

Figure 1: Lower-order vertex diagrams from \( \phi^4 \)-theory.

Figure 2: Lower-order self-energy diagrams from \( \phi^4 \)-theory.