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Hamiltonian reduction and the R -matrix of the Calogero model.

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Abstract

The Calogero model can be formulated as the Hamiltonian reduction of the geodesic flow on a symmetric space. We show that this allows to obtain the R -matrix in an elegant and compact way.

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1 Introduction

Let us consider dynamical systems with a finite number of degrees of freedom, whose equations of motion can be encoded in Lax form [1]:

$$\dot{L} = [L, M] \quad (1.1)$$

This happens frequently for systems which can be naturally interpreted as motions on Lie algebras, such as top-like systems.

It is then obvious that the quantities:

$$I_n = \text{Tr } L^n \quad (1.2)$$

are conserved quantities under the evolution given by the equation (1.1). Moreover assuming that the Lax matrix is globally defined on the phase space of the dynamical system, the invariants I_n are also globally defined. Then the Liouville theorem, as refined by Arnold [2] allows to assert that the dynamical system is integrable if and only if the conserved quantities I_n are in involution under the Poisson bracket, i.e.

$$\{I_n, I_m\} = 0 \quad (1.3)$$

Note that such a condition is only significant when the I_n 's are globally defined since one can always complete the hamiltonian locally to a Poisson commuting set of quantities, due to the Darboux theorem.

The involutory character of the conserved quantities is equivalent to the existence of an R -matrix as shown in [3]. Here one speaks about a classical R -matrix, a concept whose connection with that of the quantum R -matrix is quite loose [4]. To be brief this means that one can write:

$$\{L_1 \otimes L_2\} = [R_{12}, L_1] - [R_{21}, L_2] \quad (1.4)$$

with $L_1 = L \otimes 1$, $L_2 = 1 \otimes L$ and so on. Hence the R -matrix encodes all the information on the Poisson brackets of the system, but may be any sort of object with values in the tensor product of two spaces in which L lives, in particular R may very well depend on dynamical variables. For the usual simple dynamical systems in which R is introduced e.g. the Toda chain, this happens not to be the case.

We shall now see that for an other famous integrable dynamical system, the Calogero model [5], the associated R -matrix suffers from all the diseases that are potentially present in the general case, that is, it is dynamical, non antisymmetric, non invertible. This means that the connection of such a classical R -matrix to a quantum counterpart is highly hypothetical, notwithstanding the fact that the Calogero model is also a perfectly respectable integrable quantum system.

This model consists of N points sitting on a line with a non-“local” interaction given by:

$$H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{1}{\sinh^2(q_i - q_j)} \quad (1.5)$$

The hyperbolic sine may be replaced by an ordinary sine or even by $(q_i - q_j)^2$ without any changes in the discussion below. It may even be replaced by some elliptic function but the present methods are not adequate to discuss this integrable situation. The associated equations of motion can be recast in Lax form with an L -matrix given by

$$L_{ii} = p_i, \quad L_{ij} = \frac{i}{\sinh(q_i - q_j)} \text{ if } i \neq j \quad (1.6)$$

so that $L^\dagger = L$ and $H = 1/2 \text{Tr } L^2$ and some appropriate M . There even exists a Lax pair with spectral parameter found by Krichever [6], and $L(\alpha)$ is obtained by adding to L $1/\alpha$ times the matrix with ones everywhere, which commutes to M .

This model is classically integrable because one can exhibit the R -matrix, obtained in [7] by brute force computation:

$$R_{12} = \sum_{k \neq l} \coth(q_k - q_l) E_{kl} \otimes E_{lk} + \frac{1}{2} \sum_{k \neq l} \frac{1}{\sinh(q_k - q_l)} E_{kk} \otimes (E_{kl} - E_{lk}) \quad (1.7)$$

The very unusual features alluded to above are easily observed:

- The first half of R is antisymmetric under exchange of k and l which is equivalent to the transposition in the tensor product but not the second half, hence R is non antisymmetric.
- R is non invertible means that the dualized form is not an invertible operator since diagonal terms E_{kk} appear on the left but not on the right.
- Finally R obviously depends on the dynamical variables q_j .

In the following we shall present a simplified version of the discussion of Olshanetsky and Perelomov [8] showing why the above seemingly complicated dynamical system may be obtained by Hamiltonian reduction from a very simple and obviously integrable one. We shall then benefit from this viewpoint to explain how easily one can obtain the R -matrix, following the analysis in [9].

2 The Calogero model as Hamiltonian reduction of the geodesic flow on a Lie group

We work in the phase space $M = T^*G$, that is the cotangent bundle of a semi-simple Lie group G with its canonical symplectic form, and consider the geodesic flow whose solutions are simply:

$$g(t) = g_0 e^{tX} \quad X \in \mathcal{G} = \text{Lie}(G)$$

As a matter of fact any point in M can be parametrized as:

$$\omega = (g, \xi) \quad g \in G, \quad \xi \in \mathcal{G}^* \quad (2.1)$$

so that on any tangent vector $v \in T_g(G)$ the one-form ω takes the value $\omega(v) = \xi(g^{-1}v)$. Then the Hamiltonian of the geodesic flow is $H = 1/2 \|\xi\|^2$ and the equations of motion are solved by ξ constant and g as above. This dynamical system has a very big symmetry group allowing to find interesting systems under Hamiltonian reduction.

More precisely let us consider a non compact Lie group G and a maximal compact subgroup H such that G/H is a non compact *symmetric space*. In the following we shall restrict ourselves to the case $G = SL(N, \mathbb{C})$ and $H = SU(N)$ although the general case may be treated along essentially the same lines, see [9]. Then $H_L \times H_R$ acts on the left and on the right on M and in the above coordinates this action reads:

$$(h_L, h_R) \cdot (g, \xi) = (h_L g h_R^{-1}, h_R \xi h_R^{-1}) \quad (2.2)$$

Note that we have written everything as a left action and that h_L does not move ξ due to its definition while h_R acts by the coadjoint action. This is obviously a symmetry of the geodesic flow since $\|\xi\|^2$ is invariant and of the symplectic structure by naturality.

From Noether's theorem there are associated conserved momenta ($\mathcal{H}$ is the Lie algebra of H):

$$\begin{aligned} H_{(X_L, X_R)}(\omega) &= \alpha((X_L, X_R) \cdot \omega) \leftarrow \text{infinitesimal action} \\ X_L, X_R \in \mathcal{H} &\quad \uparrow \text{canonical 1-form} \\ &= \langle g \xi g^{-1}, X_L \rangle - \langle \xi, X_R \rangle \end{aligned} \quad (2.3)$$

so that we can write the momentum $\mathcal{P}$ with values in the Lie algebra of $H_L \times H_R$ identified with its dual under the Killing form:

$$\mathcal{P}(\omega) = (P_{\mathcal{H}} g \xi g^{-1}, -P_{\mathcal{H}} \xi) \quad (2.4)$$

where $P_{\mathcal{H}}$ is the orthogonal projector on $\mathcal{H}$. Note that the symmetry group $H_L \times H_R$ acts on $\mathcal{P}$ by the adjoint action.

The Hamiltonian reduction consists in:

- Constraining ourselves to a surface of fixed momentum:

$$N_\mu = \{\omega \in M \mid \mathcal{P}(\omega) = \mu\} \quad (2.5)$$

for a given momentum μ .

- Eliminating the extra degrees of freedom due to the fact that the stabilizer of μ in $H_L \times H_R$ is a residual symmetry on N_μ .

In order to fix this reduction we must now recall some facts about symmetric spaces: the Lie algebra $\mathcal{G}$ decomposes into the Lie algebra $\mathcal{H}$ of H and its orthogonal complement $\mathcal{K}$. Here $\mathcal{H}$ is the algebra of traceless anti-hermitean matrices, and $\mathcal{K}$ the algebra of traceless hermitean matrices. Under exponentiation we have the Cartan decomposition $G \simeq HK$ and $K = \exp(\mathcal{K}) \simeq$ symmetric space (G/H) . Finally any element of K can be brought to an element of a so-called Cartan algebra of G/H , here the diagonal matrices of K , by conjugation with an element of H (Cartan conjugation theorem). The net result is that one can write:

$$\forall g \in G, \quad g = h_1 Q h_2, \quad h_1, h_2 \in H, \quad Q \in K \text{ diagonal} \quad (2.6)$$

(Of course this is directly obvious in the present case.) Moreover this decomposition is *unique* up to the redefinitions:

$$h_1 \rightarrow h_1 d \quad h_2 \rightarrow d^{-1} h_2 \quad d \in H \text{ diagonal} \quad (2.7)$$

i.e. d has phases on the diagonal. This diagonal ambiguity is unfortunately a major source of troubles in the following.

To get the Calogero model one *chooses* a peculiar value of the momentum:

$$\mu_L = i \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 0 \end{pmatrix} = i(vv^+ - \mathbf{1}) \quad \mu_R = 0 \quad (2.8)$$

The rationale behind these choices is the following:

- $\mu_R = 0$ implies that the whole H_R stabilises μ , hence one works on the symmetric space G/H .
- A matrix U in H stabilises μ_L i.e. $U\mu_L U^{-1} = \mu_L$ if and only if $Uv = e^{i\theta} v$, where v entering the definition of μ_L is a vector with all components equal to 1. It is then obvious that the corresponding stabilizer is:

$$H_\mu \simeq SU(N-1) \times U(1)$$

and is a *maximal subgroup* of H . Hence the reduced dynamical system is the minimal non trivial one obtained by such a process.

Finally any element $\omega = (g, \xi)$ in N_μ can be brought to the form (Q, L) with Q in K diagonal, and L some element in $\mathcal{G}$ through the action of the residual symmetry group $H_{\mu,L} \times H_R$ and the decomposition $g = h_1 Q h_2$ on N_μ is *unique* since no diagonal element of H can fix v .

We can now take $\omega = (Q, L) \in N_\mu$ and write $\mathcal{P}(\omega) = \mu$. Since Q is the exponential of a diagonal traceless real matrix we write $Q = \text{Diag}(e^{q_i})$, with $\sum q_i = 0$. Then the condition on the right momentum says $P_H L = 0$ i.e. $L \in \mathcal{K}$, and the condition on the left momentum reads $P_H Q L Q^{-1} = \mu_L$, i.e.

$$\frac{1}{2}(e^{q_i} q_j L_{ij} - e^{q_j} q_i L_{ji}) = \begin{cases} 0 & i = j \\ i & i \neq j \end{cases}$$

Since $\tilde{L}_{ji} = L_{ij}$ one immediately finds that L has exactly the form of the Lax matrix (1.6) with unfixed real diagonal terms p_i 's.

It is easy to solve the equations of motion. In fact starting from the initial point (Q_0, L_0) the evolution according to the geodesic flow is obviously $g(t) = Q_0 e^{L_0 t}$, $\xi = L_0$, and it remains to decompose $g(t) = h_1(t) Q(t) h_2(t)$. But we have:

$$g g^\dagger = h_1 Q^2 h_1^{-1} = Q_0 e^{2L_0 t} Q_0$$

so one only needs to diagonalize this matrix. In particular for t small, the eigenvalues will be $e^{2q_i(t)}$, and one expands up to order t^2 , which leads to (dropping the subscript 0):

$$e^{2q_i}(1 + 2t\dot{q}_i + t^2(\ddot{q}_i + 2\dot{q}_i^2)) + O(t^3).$$

On the other hand we have to compute the eigenvalues of:

$$Q^2 + 2QLQt + 2QL^2Qt^2 + O(t^3)$$

which is readily done using the canonical orthonormal basis and standard quantum mechanical perturbation theory at first and second order. One gets exactly the equations of motion of the Calogero model at the initial point, but this is arbitrary. Finally this decomposition allows to express $L(t) = h_2 L_0 h_2^{-1}$ since ξ is constant, hence $L(t)$ evolves as in the Lax equation (1.1) with $M = \dot{h}_2(t) h_2^{-1}(t)$ and one can make choices so that $h_2 \in H_\mu$ instead of h_1 leading to $Mv = 0$, that is the sum of the elements of a line in M is 0. This allows to find easily M which happens to be equal to that introduced in [8].

3 Poisson brackets on the reduced phase space.

We first sketch the geometric situation to which we arrived:

Notice that a function F on Calogero phase space is equivalent to a function on the section of N_μ characterized by $g = Q$, itself equivalent to a function F on N_μ invariant under the action of the stabilizer $H_\mu \times H_R$. One then extends F arbitrarily to a function on the whole $M = T^*G$. Then the Hamiltonian reduction of the phase space is equivalent to using Dirac brackets on M of the form:

$$\{F, G\}_{\text{red}} = \{F, G\} + \frac{1}{2}((v_G, \omega) \cdot F - (v_F, \omega) \cdot G) \quad (3.1)$$

Here the element v_F of the Lie algebra $\mathcal{H}$ of the symmetry group is determined by the equation

$$\forall X \in \mathcal{H} \quad \mu([X, v_F]) = (X, \omega) \cdot F$$

It generates a vector field v_F, ω at the point $\omega \in M$ which itself acts on the function G .

We want to compute for any two $X, Y \in \mathcal{K}$ the reduced Poisson bracket $\{L(X), L(Y)\}$ which is equivalent to $\{L_1 \otimes L_2\}$ on the Calogero phase space under the well-known dualization operation. At the generic point of N_μ : $(g, \xi) = (h_1(g)Qh_2(g), h_2^{-1}Lh_2)$ the invariant extension of $L(x)$ is obviously:

$$F_X(g, \xi) = \xi(h_2^{-1}Xh_2)$$

Out of N_μ the functions $h_1(g), h_2(g)$ are no longer uniquely defined due to the diagonal ambiguity, and one has in principle to choose arbitrary extensions and compute the Dirac bracket. However it is possible to show that infinitesimally around N_μ one can fix the choices of h_1, h_2 in such a way as to kill the Dirac correction term in equation (3.1). This amounts to putting all the diagonal variations into h_2 , and leaving them outside of h_1 . This happens because one can decompose $\mathcal{H}$ into a direct sum of: $\mathcal{H}_\mu$, the algebra of diagonal matrices, and a third subspace which can be nicely characterized as the complementary maximal isotropic subspace of the diagonal matrices under the quadratic form:

$$(X, Y) \rightarrow \mu([X, Y])$$

(which has kernel $\mathcal{H}_\mu$). For details we refer to [9].

It is then easy to compute our Poisson bracket using the canonical symplectic structure on T^*G which reads in the coordinates (g, ξ) :

$$\begin{aligned} \{\xi(X), \xi(Y)\} &= -\xi([X, Y]) \\ \{\xi(X), g\} &= -gX \\ \{g, g\} &= 0 \end{aligned} \quad (3.2)$$

Notice that the first of equations (3.2) is simply the Kirillov bracket on $\mathcal{G}^*$, and the second means that $\xi(X)$ is the momentum of right translations.

In the vicinity of the above defined section, for $X, Y \in \mathcal{K}$, we have using equations (3.2):

$$\begin{aligned} \{L(X), L(Y)\} &= \{F_X, F_Y\}_{\text{red}} \\ &= \{\xi(h_2^{-1}Xh_2), \xi(h_2^{-1}Yh_2)\}_{|h_2=1, \xi=L} \\ &= -L([X, Y]) + L([X, \nabla_Q h_2(Y)]) + [\nabla_Q h_2(X), Y] \end{aligned}$$

where $\nabla_Q h(X) = \frac{d}{dt} h(g e^{tX})|_{t=0}$. But since G/H is a symmetric space we have $[X, Y] \in \mathcal{H}$ hence $L([X, Y]) = 0$, and we recognize that the Poisson bracket is given by an R -matrix structure (the dualization of equation (1.4):

$$\{L(X), L(Y)\} = L([X, RY]) + [RX, Y] \quad (3.3)$$

with $R(X) = (\nabla_Q h_2)(X)$.

It only remains to compute this variation taking into account the above mentioned choices on the variations of h_1 and h_2 in the diagonal directions. As a matter of fact since any g can be written $g = h_1 Q h_2$, given the tangent vector v at Q one can decompose it as:

$$v = X_L \cdot Q + Q \cdot T + Q \cdot X_R$$

where $X_L, X_R \in \mathcal{H}, T \in \mathcal{K}$ diagonal and X_R is the sought-after $\nabla_Q h_2(Q^{-1}v)$. This decomposition becomes unique when X_L does not contain diagonal

elements. Here we have to compute X_R when $v = Q \cdot X$ with $X \in \mathcal{K}$, i.e. we solve:

$$X = Q^{-1} X_L Q + T + X_R$$

In the present case X_L, X_R are antihermitean while X, T are hermitean so this equation immediately determines the non-diagonal elements of X_L, X_R , while to fix the phases on the diagonal it is necessary to use the above hypothesis. Finally after dualization one exactly recovers the R -matrix given in equation (1.7).

The benefits of this approach to the computation of the R -matrix are:

- The existence of the R -matrix is almost obvious, stemming from a similar formula in T^*G .
- The above analysis can be extended directly to other symmetric spaces and allows to find the R -matrix for some fancy-looking Calogero models which are untractable by the direct computational method.
- A similar method has already been introduced in [10] to study the phase space of the Toda model. In this case the Iwasawa decomposition is used, a unique decomposition which greatly simplifies the discussion [9].

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