STRING QUANTIZATION IN CURVED SPACETIMES:
NULL STRING APPROACH

H. J. DE VEGA\(^{(a,b)}\), I. GIANNAKIS\(^{(c,d)}\) and A. NICOLAIDIS\(^{(e)}\)

\(^{(a)}\) LPTHE, Laboratoire Associé au CNRS UA 280, Université Paris VI
F-75252 Paris, France

\(^{(b)}\) Isaac Newton Institute, Cambridge, CB3 0EH, United Kingdom

\(^{(c)}\) Center for Theoretical Physics, Texas A\&M University
College Station, TX 77843-4242, USA

\(^{(d)}\) Astroparticle Physics Group, Houston Advanced Research Center (HARC)
The Mitchell Campus, Woodlands, TX 77381, USA

\(^{(e)}\) Theoretical Physics Department, University of Thessaloniki
Thessaloniki 54006, Greece

Abstract

We study quantum strings in strong gravitational fields. The relevant small parameter is \(g = R_c \sqrt{T_0}\), where \(R_c\) is the curvature of the spacetime and \(T_0\) is the string tension. Within our systematic expansion we obtain to zeroth order the null string (string with zero tension), while the first order correction incorporates the string dynamics. We apply our formalism to quantum null strings in de Sitter spacetime. After a reparametrization of the world-sheet coordinates, the equations of motion are simplified. The quantum algebra generated by the constraints is considered, ordering the momentum operators to the right of the coordinate operators. No critical dimension appears. It is anticipated however that the conformal anomaly will appear when the first order corrections proportional to \(T_0\), are introduced.
Classical and quantum string propagation in curved spacetimes is a very important subject. The investigations in this domain are relevant for the physics of quantum gravitation as well as for the understanding of the cosmic string models in cosmology.

Strings are characterized by an energy scale $\sqrt{T_\theta}$ ($T_\theta$ is the string tension). The frequencies of the string modes are proportional to $T_\theta$ and the length of the string scales with $\frac{1}{\sqrt{T_\theta}}$. The gravitational field provides another length scale, the curvature radius of the spacetime $R_c$. For a string moving in a gravitational field a useful parameter is the dimensionless constant $g = R_c \sqrt{T_\theta}$. Large values of $g$ imply weak gravitational fields, the metric does not change appreciably over distances of the order of the string length. We may reach large values of $g$ by letting $T_\theta \to \infty$. In this limit the string shrinks to a point and a suitable expansion (small string oscillations around the center of mass of the string) has been proposed in ref [1]. In the opposite limit, small values of $g$, we encounter strong gravitational fields and it is appropriate to consider $T_\theta \to 0$. In ref [2] a systematic expansion in terms of the string tension has been presented. To zeroth order we obtain the null string [3], every point of the string moves independently along a null geodesic. In this letter we study the quantization of the null string in curved spacetime, notably in a de Sitter geometry.

Let us summarize the main results of ref. [2]. The limit $T_\theta \to 0$ cannot be reached using the Nambu-Goto action for the strings. Following the analogous massless particle case we are led to a reformulated Lagrangian [2]

$$L = \frac{1}{4\lambda} [\dot{X}^\mu \dot{X}^\nu G_{\mu\nu}(X) - c^2 X^\mu X^\nu G_{\mu\nu}(X)]$$

(1)

where $c = 2\lambda T_\theta$ is the world-sheet speed of light (a dot and a prime denote respectively differentiation with respect to the world-sheet time and space variables, $\tau$ and $\sigma$). The string equations of motion read

$$\ddot{X}^\mu - c^2 X^\mu \Gamma_{\kappa\lambda}(\dot{X}^\kappa \dot{X}^\lambda) = 0$$

(2)

supplemented by the constraints

$$\dot{X}^\mu X^\nu G_{\mu\nu} = 0 \quad \ddot{X}^\mu \dot{X}^\nu G_{\mu\nu} + c^2 X^\mu X^\nu G_{\mu\nu} = 0$$

(3)

A systematic expansion in powers of $c$ is feasible now [2]. If we write

$$X^\mu(\sigma, \tau) = A^\mu(\sigma, \tau) + c^2 B^\mu(\sigma, \tau) + \cdots$$

(4)

where the dots indicate higher powers of $c^2$ and substitute this form of $X^\mu$ in (2) and (3) we find that to zeroth order in $c$ the string dynamics is given by the equations of motion

$$\ddot{A}^\mu + \Gamma^\mu_{\nu\rho} A^\nu A^\rho = 0$$

(5)
and the constraints
\[ \dot{A}^\mu \dot{A}^\nu G_{\mu\nu} = 0 \]
\[ \dot{A}^\mu A^\nu G_{\mu\nu} = 0 \]  
(6a)
(6b)

\( A^\mu(\sigma, \tau) \) represents a collection of massless particles moving independently along null geodesics. The only reminiscence from the string is the constraint, eq. (6b), which requires the velocity to be perpendicular to the string. The next order correction \( B^\mu(\sigma, \tau) \) obeys the following equation of motion
\[ \ddot{B}^\mu + \Gamma^\mu_{\kappa\lambda}(\dot{A}^\kappa \dot{B}^\lambda + \dot{A}^\lambda \dot{B}^\kappa) + \Gamma^\mu_{\kappa\lambda\rho} \dot{A}^\kappa \dot{A}^\lambda B^\rho = A^\mu_{\kappa\lambda} + \Gamma^\mu_{\kappa\lambda} A^\nu A^{\lambda\nu} \]  
(7)
supplemented with the constraints
\[ 2 \dot{A}^\mu \ddot{B}^\nu G_{\mu\nu} + \dot{A}^\mu \dot{A}^\nu G_{\mu\nu,\rho} + A^\mu_{\kappa\lambda} A^\nu_{\kappa\lambda} = 0 \]
\[ \ddot{B}^\mu A^\nu G_{\mu\nu} + \dot{A}^\mu B^\nu G_{\mu\nu} + A^\mu_{\kappa\lambda} B^\nu G_{\mu\nu,\rho} = 0 \]  
(8)
where \( \Gamma^\mu_{\kappa\lambda\rho} \) and \( G_{\mu\nu,\rho} \) indicate the derivatives with respect to \( A^\rho \).

We would like to study the quantization of the null string in de Sitter spacetime. The line element is defined as
\[ ds^2 = C^2(X_0)(dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2) \]  
(9)
with
\[ C(X_0) = \frac{R_0}{X_0} \]  
(10)
where \( X_0 \) is the conformal time and \( R_0 \) is the scale factor. The Lagrangian for a null string propagating in (9) takes the form
\[ L = C^2(A_0) \dot{A}_i = P_i(\sigma) \quad i = 1, 2, 3 \]  
(11)
The equations of motion, using also the constraints, provide
\[ C^2(A_0) \dot{A}_i = P_i(\sigma) \]  
(12a)
\[ C^2(A_0) \dot{A}_0 = P_0(\sigma) \]  
(12b)
Using eq. (10), we obtain from eq. (12b)
\[ A_0(\sigma, \tau) = \frac{\bar{A}_0 R_0^3}{R_0^3 - \bar{A}_0 P_0 \tau} \]  
(13)
with \( \bar{A}_0 = A_0(\sigma, \tau = 0) \). Eq. (12a) gives then
\[ A_i(\sigma, \tau) = \bar{A}_i = \frac{P_i \bar{A}_0^2 \tau}{R_0^3 - \bar{A}_0 P_0 \tau} \]  
(14)
String Quantization in Curved Spacetimes...

with \( \overline{A}_i = A_i(\sigma, \tau = 0) \). The constraints (6) tie the initial shape and momentum of the string

\[
\overline{A}_0 P_0 = \sum_{i=1}^{3} \overline{A}_i P_i \quad (15a)
\]

\[
P_0^2(\sigma) = \sum_{i=1}^{3} P_i^2(\sigma) \quad (15b)
\]

Notice that the relationship

\[
A_i - \overline{A}_i = \frac{P_i}{P_0}(A_0 - \overline{A}_0) \quad (16)
\]

holds, i.e. in terms of the cosmic variables we obtain straight lines for the individual particles of the null string. The constraints retain their form under the reparametrization \( \tau = f(\overline{\tau}, \overline{\sigma}) \) and \( \sigma = g(\overline{\sigma}) \) with \( f \) and \( g \) arbitrary functions. We may use this reparametrization freedom in order to simplify the equations of motion. Choosing

\[
\tau = \frac{R_0^2 \overline{\tau}}{P_0 \overline{\tau} + \overline{A}_0} \quad (17)
\]

we obtain (after dropping tildes)

\[
A_\mu = P_\mu \tau + \overline{A}_\mu \quad \mu = 0, 1, 2, 3 \quad (18a)
\]

\[
P_\mu = 0 \quad (18b)
\]

\[
\overline{A}_\mu P^\mu = 0 \quad (18c)
\]

Due to the periodicity in the \( \sigma \)-direction, since we deal with closed strings, we can expand \( P^\mu(\sigma) \) and \( \overline{A}^\mu(\sigma) \) in Fourier series as follows

\[
P^\mu(\sigma) = \sum_n p_n^\mu \exp(i n \sigma) \quad \overline{A}^\mu(\sigma) = \sum_n x_n^\mu \exp(i n \sigma) \quad (19)
\]

Similarly the constraints can be expanded in Fourier series

\[
P^\mu P_\mu = \sum_\kappa H_\kappa \exp(i k \sigma) \quad P^\mu \overline{A}_\mu = \sum_\kappa G_\kappa \exp(i k \sigma) \quad (20)
\]

By substituting the expressions for the \( P^\mu(\sigma) \) and \( \overline{A}(\sigma) \) in the above relations we find

\[
H_\kappa = \sum_n p_{k-n} p_n \quad G_\kappa = i \sum_n p_{k-n} x_n \quad (21)
\]

The string coordinate \( \overline{A}^\mu(\sigma) \) and the conjugate momentum \( P^\mu(\sigma) \) satisfy the following Poisson brackets

\[
\{ P^\mu(\sigma), \overline{A}_\nu(\sigma') \} = \delta(\sigma - \sigma') G^{\mu\nu} \quad (22)
\]
while for their Fourier modes $p_n^\mu$ and $x_n^\mu$ we obtain
\[ \{ p_n^\mu, x_m^\nu \} = G^{\mu\nu} \delta_{m+n} \]

Using the above relations we can calculate the Poisson brackets of the moments $H_\kappa$ and $G_\kappa$ of the constraints. With a redefinition $G_\kappa \rightarrow -G_\kappa$ we find the following algebra (the analogue of the Virx Vir in the tensionful string)
\[ \{ G_n, G_m \} = i(n-m)G_{n+m}, \quad \{ G_n, H_m \} = i(n-m)H_{n+m}, \quad \{ H_n, H_m \} = 0 \]
\[ \{ G_n, G_m \} = i(n-m)G_{n+m}, \quad \{ G_n, H_m \} = i(n-m)H_{n+m}, \quad \{ H_n, H_m \} = 0 \quad (23) \]

The constraints $G_n$ and $H_n$ generate reparametrizations in $\sigma$ and $\tau$, respectively. The algebra of the constraints of the null string arises as the Inonu-Wigner contraction \[4\] \[ T_\hbar \rightarrow 0 \] of the algebra of the constraints of the tensile string. This can be seen by writing the constraint algebra of the tensile string in terms of the modes $H_m, G_n$ of the constraints $PX' = P^2 + T_\hbar^2 X'^2 = 0$. The modes of the constraints of the tensile string $G_n, H_n$ are given in terms of the modes of the constraints of the null string $G_n, H_n$

\[ G_n = G_n, \quad H_n = H_n + T_\hbar \sum \delta_n x_{n-\kappa} x_n \quad (24) \]

The algebra then takes the form
\[ \{ G_n, G_m \} = i(n-m)G_{n+m}, \quad \{ G_n, H_m \} = i(n-m)H_{n+m} \]
\[ \{ H_n, H_m \} = 4i\hbar^2(n-m)G_{m+n} \quad (25) \]
which clearly in the limit $T_\hbar \rightarrow 0$ reduces to the null algebra.

We may now quantize the null string by replacing the Poisson bracket by the commutator
\[ [p_n^\mu, x_m^\nu] = -iG^{\mu\nu} \delta_{m+n} \quad (26) \]

The passage from the classical domain to the quantum one offers the possibility for the emergence of central terms in the algebra. The algebra will take the form
\[ [G_n, G_m] = (n-m)G_{n+m} + A(n)G_{n+m}, \quad [G_n, H_m] = (n-m)H_{n+m} + B(n)\delta_{n+m}, \]
\[ [H_n, H_m] = C(n)\delta_{n+m} \quad (27) \]

In order to specify the form of the anomaly we make use of the Jacobi identity for the algebra. We find that the most general form of the anomaly is given by
\[ A(n) = \alpha n^3 + \beta n, \quad B(n) = \gamma n^3 + \delta n, \quad C(n) = 0 \quad (28) \]

where $\alpha, \beta, \gamma$ and $\delta$ are constants which need to be determined. The actual calculation depends crucially upon the ordering of the operators \[5\], \[6\], \[7\]. Given our Lagrangian eq. (1) with $c = 0$, the vacuum $|0\rangle$ satisfies the condition
\[ p_n^\mu |0\rangle = 0 \quad (29) \]
We order then all the $p_n$ to the right of the $x_n$. We arrive at similar conclusions by looking at the complete tensile theory and considering the limit $c \to 0$. The tensile string is quantized using harmonic oscillator operators, representing right and left movers. The operators which annihilate the vacuum state in the limit $c \to 0$ are reduced to momentum operators [7]. With the adopted ordering, it is relatively easy to evaluate the quantum algebra generated by the constraints. The quantum algebra does not contain central terms, namely we find that $\alpha = \beta = \gamma = \delta = 0$ and therefore a quantum null string may exist in any dimension without anomalies. We anticipate however that a critical dimension will appear, as soon as the first order correction $B^\mu(\sigma, \tau)$, proportional to $T_0$, is introduced (work in progress).

Summarizing, we presented a consistent framework within which it is feasible to study string quantization in strong gravitational fields. We have shown that for the emergence of the conformal anomaly, responsible is the dimensionful string tension. Historical precedent (Higgs mechanism) may allow us in contemplation for a mechanism generating string tension and at the same time respecting the symmetries.

Acknowledgements.

We would like to acknowledge useful discussions with our colleagues, HdV with N. Sánchez and G. Veneziano, IG with M. Evans and A. Polychronakos and AN with J. Iliopoulos and K. Tamvakis.

References.