SOME ASPECTS OF $q$-AND $qp$-BOSON CALCULUS

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Abstract

A set of compatible formulas for the Clebsch-Gordan coefficients of the quantum algebra $U_q(\mathfrak{su}_2)$ is given in this paper. These formulas are $q$-deformations of known formulas, as for instance: Wigner, van der Waerden, and Racah formulas. They serve as starting points for deriving various realizations of the unit tensor of $U_q(\mathfrak{su}_2)$ in terms of $q$-boson operators. The passage from the one-parameter quantum algebra $U_q(\mathfrak{su}_2)$ to the two-parameter quantum algebra $U_{qp}(\mathfrak{u}_2)$ is discussed at the level of Clebsch-Gordan coefficients.

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1. PRELIMINARIES

The aim of the present paper is to continue the program of extending in the framework of $q$-deformations the main results of the work in ref. 1 on the $SU_2$ unit tensor or Wigner operator (the matrix elements of which are coupling coefficients or $3 - jm$ symbols). A first part of this program was published in the proceedings of Symmetries in Science VI (see ref. 2) where the $q$-deformed Schwinger algebra was defined and where an algorithm, based on the method of complementary $q$-deformed algebras, was given for obtaining three- and four-term recursion relations for the Clebsch-Gordan coefficients (CGc's) of $U_q(\mathfrak{su}_2)$ and $U_q(\mathfrak{su}_1,1)$. The algorithm was fully exploited in ref. 3 where the complementary of three quantum algebras in a $q$-deformation of the symplectic Lie algebra $\mathfrak{sp}(8)$ was used for producing 32 recursion relations.

This paper is organized as follows. In section 2, we derive 12 explicit forms for the CGc's of $U_q(\mathfrak{su}_2)$. They are $q$-deformations of the most usual formulas displayed in the literature. [In the following, we shall use the terminology $X$ form which means that the corresponding formula can be identified with the one originally derived by the author(s) $X$ ($X = \text{Wigner}$, $\text{van der Waerden}$, $\text{Racah}$, $\text{Majumdar}$, etc.) in the limiting case where $q = 1$]. From each of the 12 $X$ forms, it is possible to derive, as explained in section 3, a $q$-boson realization of the $U_q(\mathfrak{su}_2)$ unit tensor. Finally,
section 4 deals with the two-parameter Hopf algebra $U_{qp}(u_2)$ and it is sketched there how to transcribe to $U_{qp}(u_3)$ the results obtained in section 2 for $U_{q}(su_2)$.

Some words about the notation are in order. In section 4, we shall use the notation

$$[[x]]_{qp} := \frac{q^x - p^x}{q - p}$$

while in sections 2 and 3 we shall use the abbreviation $[x]$ to denote

$$[x] := [[x]]_{q_{pq}} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

where $x$ may stand for an operator or a (real) number. The other notations to be used concern the $q$- and $qp$-deformed factorials

$$[n]! := [1][2] \cdots [n]$$

$$[n]_{qp}! := [[1]]_{qp}[[2]]_{qp} \cdots [[n]]_{qp}$$

$$[0]! := 1$$

$$[0]_{qp}! := 1$$

where $n$ is a positive integer.

We shall take the commutation relations of the quantum algebra $U_{q}(su_2)$ in the usual (Kulish-Reshetikhin-Drinfeld-Jimbo) form, viz.,

$$[J_3, J_-] = -J_- \quad [J_3, J_+] = +J_+ \quad [J_+, J_-] = [2J_3]_{q}$$

The co-product of the Hopf algebra $U_{q}(su_2)$ corresponds to eq. (41) below with $p = q^{-1}$. The extension of eq. (4) to the quantum algebra $U_{qp}(u_2)$ is given in section 4.

For Hermitean conjugation requirements [more precisely, to insure that $(J_+)^\dagger = J_-$], the values of the parameters $q$ and $p$ must be restricted to the following domains: either (i) $q \in \mathbb{C}$ and $p \in \mathbb{C}$ or (ii) $q \in \mathbb{R}$ and $p \in \mathbb{R}$ with $p = q^*$ (the $*$ indicates complex conjugation). In the special case where $p = q^{-1}$, we may take either (i) $q \in \mathbb{R}$ or (ii) $q = e^{i\beta}$ with $0 \leq \beta < 2\pi$. Therefore, in all cases the product $qp$ is real.

### 2. ANALYTICAL EXPRESSIONS FOR $q$-DEFORMED CGC’S

#### 2.1. The Philosophy

In this section, our aim is to derive analytical expressions for the $U_{q}(su_2)$ CGC’s from a given (fully checked) formula. This can be done with various means including: (i) the resummation procedure, (ii) the use of ordinary symmetry properties (corresponding to the 12 simple ordinary symmetries of the Regge array) for the $U_{q}(su_2)$ CGC’s, (iii) the use of pure Regge symmetries (corresponding to the 72 $- 12 = 60$ nonordinary symmetries of the Regge array) for the $U_{q}(su_2)$ CGC’s, (iv) the use of the mirror reflection symmetry for the $U_{q}(su_2)$ CGC’s, and (v) the transition from the given formula to its expression in terms of the $q$-deformed hypergeometric function $3F_2$ and the use of symmetry properties of this function.

The resummation procedure (i) to be used below is an adaptation, in the framework of $q$-deformations, of the procedure described by Jucys and Bandzaitis and applied to the standard SU$_3$ CGC’s. It amounts to introduce

$$\delta(a, b) = \sum_s (-1)^{s-a} \frac{q^{s(a-b-1)+a}}{[a-s][s-b]}$$

(5)
in the sum occurring in a given formula for the \( U_q(SU_2) \) CGc’s. The resummation of the so-obtained expression may then be achieved by using some of the following summation identities (or \( q \)-factorial sums)

\[
\sum_{s} q^{-as} \frac{1}{[s]![b-s]![c-s]![a-b-c+s]!} = \frac{[a]!}{[b]![c]![a-b]![a-c]!} (6)
\]

\[
\sum_{s} (-1)^s q^{s(b+c-s-1)} \frac{[a-s]!}{[s]![b-s]![c-s]!} = (-1)^c q^{bc} \frac{[a-c]![b+c-a-1]!}{[b]![c]![b-a-1]!} \quad (7)
\]

\[
\sum_{s} (-1)^s q^{s(b+c-s-1)} \frac{[a-s]!}{[s]![b-s]![c-s]!} = q^{bc} \frac{[a-b]![a-c]!}{[b]![c]![a-b-c]!} \quad a \geq b, \ a \geq c \quad (8)
\]

\[
\sum_{s} q^{s(b+c-s+2)} \frac{[b-s]![c+s]!}{s![a-s]!} = q^{a(c+1)} \frac{[b-a]![c]![b+c+1]!}{[a]![b+c-a+1]!} \quad (9)
\]

where, as in eq. (5), the factorials are \( q \)-factorials. The identities (6)-(9) coincide with the well-known factorials sums given in ref. 8 in the \( q = 1 \) limit.

Among the ordinary symmetry properties (ii), we shall use the following relations

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_2-j_1-m} q^{m} \sqrt{\frac{[2j+1]}{[2j_2+1]}} (jj_1, -mm_1 | j_2, -m_2)_q \quad (10)
\]

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_1-j+m_2} q^{-m} \sqrt{\frac{[2j+1]}{[2j_1+1]}} (jj_2 m, -m_2 j_1 m_1)_q \quad (11)
\]

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_1-j+m_2} q^{-m} \sqrt{\frac{[2j+1]}{[2j_1+1]}} (jj_2 m_2, -m | j_1 m_1)_q \quad (12)
\]

We shall also use

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_2+m_2} q^{-m_2} \sqrt{\frac{[2j+1]}{[2j_1+1]}} (j_2 j, -m_2 m_1 j_1 m_1)^{-1} \quad (13)
\]

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_1-m_1} q^{m_1} \sqrt{\frac{[2j+1]}{[2j_2+1]}} (j_1 j m_1, -m | j_2, -m_2)^{-1} \quad (14)
\]

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_2+m_2} q^{-m_2} \sqrt{\frac{[2j+1]}{[2j_1+1]}} (j_2 j, -m m_1 j_1 m_1)^{-1} \quad (15)
\]

where it should be observed that, in contradistinction with eqs. (10)-(12), the index of the CGc in the right-hand sides of eqs. (13)-(15) is \( q^{-1} = 1 / q \). In the limiting situation where \( q = 1 \), eqs. (10)-(15) reduce to 6 of the 12 ordinary symmetry properties for the CGc’s of SU\(_2\) in an SU\(_2\) ⊃ U\(_1\) basis.

In the following, we shall be mainly concerned with the points (i) and (ii). However, the points (iii) (i.e., the Regge symmetries) and (iv) (i.e., the mirror reflection symmetry: \( m_1, m_2, m \) unchanged; \( j_i \leftrightarrow -j_i - 1 \) for \( i = 1, 2 \) and \( j \leftrightarrow -j - 1 \)) were used to check certain forms given below.
2.2. The Expressions

The problem of finding analytical expressions for the $U_q(\mathfrak{su}_2)$ CCs’ and for the corresponding Wigner (unit tensor) operator was attacked by numerous authors (see refs. 9 to 14 for a nonexhaustive list of works). Generally speaking, the methods valid for the ordinary CCs’ (for which $q = 1$) can be extended to $q$-deformed CCs’.

In the limiting case where $q = 1$, a useful form for the CCs’ of the nondeformed chain $\mathfrak{su}_2 \supset \mathfrak{u}_1$ was derived by Shapiro\textsuperscript{15} by making use of the (Löwdin) method of projection operators (see also the work by Calais\textsuperscript{16}). Such a method was adapted and applied by Smirnov, Tolsotov, and Kharitonov\textsuperscript{13} to the $q$-deformed chain $U_q(\mathfrak{su}_2) > \mathfrak{u}_1$.

Our starting point is

The Shapiro (Smirnov-Tolstoi-Kharitonov) form (1st form):

\begin{equation}
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^j (j + j_2 - j_1 - j_2 - m_1 - m_2 - j_1 m_2 - j_2 m_1)
\times \left(\frac{[j_2 - j_1 + j_2 - j][j_1 + j_2 + j + 1][j_2 - m_1][j + m]}{[j_2 - z][j_1 + j_2 - m - z]}\right)\frac{1}{2}
\end{equation}

which is eq. (5.17) of ref. 13. In eq. (16), as well as in eqs. (17)-(27), it is assumed that $m = m_1 + m_2$. [If $m \neq m_1 + m_2$, the right-hand sides of (16)-(27) should be replaced by 0.] The substitution $z \mapsto j_1 + j_2 - j - z$ allows us to rewrite eq. (16) in an alternative form. We thus get

The Shapiro (Smirnov-Tolstoi-Kharitonov) form (2nd form):

\begin{equation}
(j_1 j_2 m_1 m_2 | j m)_q = q^{-\frac{1}{2}(j + j_2 - j_1 - j_2 - m_1 + m_2 + 1 + j_1 m_2 + j_2 m_1)}
\times \left(\frac{[j + m][j_2 - m_2][j_1 - j_2 + j][j_1 + j_2 + j + 1]}{[j - m][j_2 + m_2][j_1 - m_1][j_1 + m_1][j_1 + j_2]}\right)\frac{1}{2}
\end{equation}

From the latter form, we can derive a useful intermediate form by using the resummation procedure. Indeed, by introducing eq. (5) into the right-hand side of eq. (17) and then by using successively eq. (8) and eq. (7), a long but straightforward calculation leads to

The intermediate form:

\begin{equation}
(j_1 j_2 m_1 m_2 | j m)_q = q^{-\frac{1}{2}(j + j_2 - j_1 - j_2 + m_2 - 1) + j_1 m_2 - j_2 m_1}
\times \left(\frac{[j_2 - m_1][j_1 + m_1][j + j_1 - j_2][j_1 + j_2 - j_2 + j_1][j + j_1 - j_2]}{[j + j_1 - m_1][j + m_1][j - j_1 + j_2 + 1]}\right)\frac{1}{2}
\end{equation}

which constitutes in turn an initial point for spanning other analytical expressions of the $U_q(\mathfrak{su}_2)$ CCs’.

By starting from the intermediate form (18), it is possible to derive, still in the context of the resummation procedure, the $q$-analog of the van der Waerden\textsuperscript{5} (symmetrical) formula. As a net result, we have found
The van der Waerden form:

\[(j_1 j_2 m_1 m_2 | jm)_q = q^{\frac{1}{2}(j_1 + j_2 - j)(j_1 + j_2 + j + 1) + j_1 m_2 - j_2 m_1} \times \left( \frac{[j + j_1 + j_2]!![j + j - j][-1][j + j_1 - j_2]!![j + j_2 - j_1]!![j_1 + j_2 + j + 1]!![j + m]!![j - m]!!}{[j_2 + m_2 + j_1 + j_2 - j - z]!!} \right)^{\frac{1}{2}} \]

\[\times \sum_z (-1)^z q^{-z(j_1 + j_2 + j + 1)} \frac{1}{[z]!![j_1 - m_1 - z][j_2 + m_2 - z]!} \times (j_1 - m_1)!![j_1 + m_1][j_2 - m_2]!![j_2 + m_2]!![j - m]!![j + m]!! \times \frac{1}{[j_1 - j - z]!![j_1 + j_2 - j - z]!!}
\]

From the van der Waerden form (19), we can obtain the q-analog of the Racah\(^6\) formula by using the resummation procedure together with a repeated application of the summation identity (6). This yields

The Racah form (1st form):

\[(j_1 j_2 m_1 m_2 | jm)_q = (-1)^{j_1 - m_1} q^{-\frac{1}{2}j_1(j_1 + 1) - j_2(j_2 + 1) + j(j + 1) + j_1 j_2 + j_1 m_1 + m_1 j + m_1 m_1 + m_1} \times \left( \frac{[j + j_1 + j_2 + j_1]!![j + j_1]!![j + j_2]!![j + j_1 + j_2]!![j + j_1 + j_2 + j + 1]!![j_1 + m_1][j_2 + m_2]!![j + m]!![j - m]!!}{[j_2 - m_2]!![j_1 - m_1]!![j_1 - j - z]!![j_1 + j_2 - j - z]!!} \right)^{\frac{1}{2}} \]

\[\times \sum_z (-1)^z q^{z(j_1 - j_2 + j)} \frac{[z][j_1 - m_1 - z][j - m - z][j_1 + j_2 + j + 1 - z]!!}{[z]!![j_1 - m_1 - z][j - m - z][j_1 + j_2 - j - z]!!} \]

Then, from the Racah form (20), we can derive another useful form, viz., the q-analog of the formula (13.11) by Jucys and Bandzaitis,\(^8\) again through the resummation procedure. As a matter of fact, we have obtained

The Jacys-Bandzaitis form (2nd form):

\[(j_1 j_2 m_1 m_2 | jm)_q = (-1)^{j_1 - m_1} q^{-\frac{1}{2}j_1(j_1 + 1) - j_2(j_2 + 1) + j(j + 1) + j_1 j_2 + j_1 m_1 + m_1 j + m_1 m_1 + m_1} \times \left( \frac{[j + j_1 + j_2 + j_1]!![j + j_1 + j_2]!![j + j_1 - j_2]!![j + j_1 + j_2 + j + 1]!![j_1 + m_1][j_2 + m_2]!![j + m]!![j - m]!!}{[j_2 - m_2]!![j_1 - m_1]!![j_1 - j - z]!![j_1 + j_2 - j - z]!!} \right)^{\frac{1}{2}} \]

\[\times \sum_z (-1)^z q^{z(j_1 - j_2 + j)} \frac{[z][j_1 - m_1 - z][j - m - z][j_1 + j_2 + j + 1 - z]!!}{[z]!![j_1 - m_1 - z][j - m - z][j_1 + j_2 - j - z]!!} \]

Going back to the van der Waearden form, by making the substitution \(z \mapsto j_2 + m_2 - z\) in eq. (19) and by applying the resummation procedure to the so-obtained relation, we arrive at

The Majumdar form (1st form):

\[(j_1 j_2 m_1 m_2 | jm)_q = (-1)^{j_1 + m_2} q^{-\frac{1}{2}j_1(j_1 + 1) - j_2(j_2 + 1) + j(j + 1) + j_1 j_2 + m_1 j + j_1 m_2 - j_2 m_2} \times \left( \frac{[j - j_1 + j_2]!![j_1 - m_1][j_1 + m_1][j_2 - m_2][j_2 + m_2]!![j + m]!![j - m]!!}{[j_2 - m_2]!![j_1 - j_1 + j_2]!![j_1 - j_2 - j + 1]!![j_1 + j_2 + j + 1]!![j_2 + m_2]!![j - m]!!} \right)^{\frac{1}{2}} \]

\[\times \sum_z (-1)^z q^{z(j_1 - m_1 + 1)} \frac{[2j - z][j_1 + j_2 - j + z]!!}{[z][j + m - z][j_1 - j - m_2 + z][j - j_1 + j_2 - z]!!} \]
Finally, the substitution \( z \mapsto j + m - z \) in eq. (22) produces

The Majumdar form (2nd form):

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_2 + m_2} q^{\frac{1}{2}(j-j_1+j_2)(j-j_1-j_2+1)m_2-1-j_2 m} \times \left[ \frac{[j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2]}{[j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2]} \right]^{\frac{1}{2}}
\]

\[
\sum_z (-1)^z q^{-z(j_1 - m_1 + 1)} \frac{[j_j_1 - j_2] + m - z]! [j + m - z]! [j + m - z]! [j - j_1 + j_2 - z]!}{[j + m - z]! [j + m - z]! [j + m - z]! [j + m - z]!}
\]

Equation (22) is the \( q \)-analog of a formula originally derived by Majumdar while eq. (23) is a simple consequence of (22).

Other analytical formulas for the CGc's of \( U_q(\mathfrak{su}_2) \) may be spanned from the just obtained forms owing to the symmetry properties (10-15). For example, by applying the symmetry property (13) to the intermediate form (18) we obtain

The Wigner form:

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_2 + m_2} q^{\frac{1}{2}(j-j_1+j_2)(j-j_1-j_2+1)m_2-1-j_2 m} \times \left[ \frac{[j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2]}{[j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2]} \right]^{\frac{1}{2}}
\]

\[
\sum_z (-1)^z q^{-z(j_1 - j_2 + 1)} \frac{[j_j_1 - j_2] + m - z]! [j + m - z]! [j + m - z]! [j - j_1 + j_2 - z]!}{[j + m - z]! [j + m - z]! [j + m - z]! [j + m - z]!}
\]

[In other words, by introducing the intermediate form (18) for the CGc in the right-hand side of (13), then the Wigner form (24) is obtained as the left-hand side of (13).]

Similarly, the introduction of the intermediate form (18) in the right-hand side of the symmetry property (11) generates

The Zukauskas-Mauza form:

\[
(j_1 j_2 m_1 m_2 | j m)_q = (-1)^{j_2 + m_2} q^{\frac{1}{2}(j-j_1+j_2)(j-j_1-j_2+1)m_2-1-j_2 m} \times \left[ \frac{[j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2]}{[j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2]} \right]^{\frac{1}{2}}
\]

\[
\sum_z (-1)^z q^{-z(j_1 - j_2 + 1)} \frac{[j_j_1 - j_2] + m - z]! [j + m - z]! [j + m - z]! [j - j_1 + j_2 - z]!}{[j + m - z]! [j + m - z]! [j + m - z]! [j + m - z]!}
\]

The \( q \)-analog of the Racah form used by Kihler and Grenet in ref. 1 [at the level of their eq. (92)] can be deduced from the Racah form (20); it is sufficient to introduce (20) into the symmetry property (14); this leads to

The Racah form (2nd form):

\[
(j_1 j_2 m_1 m_2 | j m)_q = q^{\frac{1}{2}(j_1 + j_2 + j_3 + j_4 + j_5 + j_6 + j_7 + j_8)} \times \left[ \frac{[j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2]}{[j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2] [j_j_1 - j_2]} \right]^{\frac{1}{2}}
\]

\[
\sum_z (-1)^z q^{-z(j_1 - j_2 + 1)} \frac{[j_j_1 - j_2] + m - z]! [j + m - z]! [j + m - z]! [j - j_1 + j_2 - z]!}{[j + m - z]! [j + m - z]! [j + m - z]! [j + m - z]!}
\]

Finally, by putting the Shapiro form (1st form) [eq. (16)] in the right-hand side of the symmetry property (12), we obtain
The Jacys-Bandzaitis form (1st form):

\[(j_1j_2m_1m_2|jm)_q = (-1)^{j_2+m_2} q^{\frac{1}{2}(j_1-j_2-j)(j_1+j_2+j+1)-m_2-m_2-j-m_2} \times \left( \frac{[j_1-m_1]![j+m]!}{[j_1+j_2]!} \right)^{\frac{1}{2}} \times \left( \frac{[2j+1]}{[j_1+m_1]![j_2-m_2]![j_2+m_2]![j-m]!} \right)^{\frac{1}{2}} \times \sum_z (-1)^z q^{z(j_2+m_2)} \frac{[2j-z]![j+j_2+m_1-z]!}{[z]![j+m-z]![j+j_2-j_1-z]![j_1+j_2+j+1-z]!} \]

which is the \(q\)-analog of the relation (13.1B) obtained by Jacys and Bandzaitis and listed in ref. 8.

3. TOWARDS \(q\)-BOSON REALIZATIONS OF THE \(U_q(\text{su}_2)\) CGC'S

3.1. The Philosophy

Following the approach of ref. 1, we are now in a position to find \(q\)-boson realizations of the \(U_q(\text{su}_2)\) unit tensor. The components \(t[q : k \rho \Delta] \) of such a tensor operator are defined by\(^1,2\)

\[ \langle j'm'|t[q : k \rho \Delta]|jm \rangle := \delta(j', j + \Delta) \delta(m', m + \rho)(-1)^{2k} ([2j' + 1]_q)^{-\frac{1}{2}} (jkm\rho|j'm')_q \]

(28)

where \((jkm\rho|j'm')_q\) is a CGc for \(U_q(\text{su}_2)\). The operator \(t[q : k \rho \Delta]\) constitutes a \(q\)-deformation of the operator \(t_{k \rho a} \equiv t[1 : k \rho a]\) worked out by Kibler and Grenet.\(^1\)

It was shown in ref. 1 that the operator \(t[1 : k \rho a]\) can be expressed in the enveloping algebra of the Schwinger algebra (isomorphic with \(so_2,3\)). The extension to the case where \(q \neq 1\) is trivial so that it is possible to find a realization of the operator \(t[q : k \rho \Delta]\) in terms of \(q\)-boson operators defined in a two-particle Fock space (with two sets \(\{a_+, a_+^\dagger\}\) and \(\{a_-, a_-^\dagger\}\) of \(q\)-bosons). As in ref. 2, we take Macfarlane\(^1\) and Biedenharn\(^2\) \(q\)-bosons corresponding to

\[
\begin{align*}
a_+ |n_1n_2\rangle &= \sqrt{[n_1]_q} |n_1-1, n_2\rangle \\
a_+^\dagger |n_1n_2\rangle &= \sqrt{[n_1+1]_q} |n_1+1, n_2\rangle \\
a_- |n_1n_2\rangle &= \sqrt{[n_2]_q} |n_1, n_2-1\rangle \\
a_-^\dagger |n_1n_2\rangle &= \sqrt{[n_2+1]_q} |n_1, n_2+1\rangle \\
N_i |n_1n_2\rangle &= n_i |n_1n_2\rangle \quad (i = 1, 2)
\end{align*}
\]

(29)

from which it is clear that \(a_+^\dagger = (a_+)^\dagger\) and \(a_-^\dagger = (a_-)^\dagger\) when \(q \in \) or \(q \in S^3\). Then, from the transformation

\[ |jm\rangle \equiv |n_1n_2\rangle \quad j := \frac{1}{2}(n_1 + n_2) \quad m := \frac{1}{2}(n_1 - n_2) \]

(30)

we have

\[
\begin{align*}
a_\pm |jm\rangle &= \sqrt{[j \pm m]_q} |j - \frac{1}{2}, m \mp \frac{1}{2}\rangle \\
a_\mp^\dagger |jm\rangle &= \sqrt{[j \pm m + 1]_q} |j + \frac{1}{2}, m \pm \frac{1}{2}\rangle 
\end{align*}
\]

(31)
which is at the root of the (Jordan-Schwinger) realization of $U_q(\text{su}_2)$. A simple iteration of (31) yields

$$(a_+)^n \lfloor jm \rfloor = \left( \frac{[j + m]_q^!}{[j + m - n]_q^!} \right)^{1/2} \lfloor j + n - \frac{n}{2}, m - \frac{n}{2} \rfloor$$

$$(a_-)^n \lfloor jm \rfloor = \left( \frac{[j - m]_q^!}{[j - m - n]_q^!} \right)^{1/2} \lfloor j - n + \frac{n}{2}, m + \frac{n}{2} \rfloor$$

$$(a_+^\dagger)^n \lfloor jm \rfloor = \left( \frac{[j + m]_q^!}{[j + m - n]_q^!} \right)^{1/2} \lfloor j + n + \frac{n}{2}, m + \frac{n}{2} \rfloor$$

$$(a_-^\dagger)^n \lfloor jm \rfloor = \left( \frac{[j - m]_q^!}{[j - m - n]_q^!} \right)^{1/2} \lfloor j - n - \frac{n}{2}, m - \frac{n}{2} \rfloor$$

for $n \in \mathbb{Z}$. Thus, it is possible to understand why $t[q : k\rho\Delta]$ can be developed in the polynomial form

$$t[q : k\rho\Delta] = \sum_{\alpha, \beta, \gamma, \delta} C_{\alpha, \beta, \gamma, \delta}(q, k, \rho, \Delta) (a_+^\dagger)^\alpha (a_+)^\beta (a_-^\dagger)^\gamma (a_-)^\delta$$

where the sum on $\alpha, \beta, \gamma, \delta$ is in fact a sum on a single variable $z$ [likewise in eqs. (16)-(27)].

### 3.2. Some $q$-Bosonized Expressions

As an example, the $q$-boson realization of the operator $t[q : k\rho\Delta]$ corresponding to eq. (18) gives

**The intermediate realization**:

$$t[q : k\rho\Delta] = (-1)^{2k} q^{\frac{k\Delta(2j + \Delta - k - 1 - 2\rho + j - k) + j\rho - k} \times \left( \frac{[k - \Delta]_q [k + \Delta]_q [k - \rho]_q [k + \rho]_q [2j + \Delta - k]}{[2j + \Delta + k + 1]_q} \right)^{1/2} \times \sum_z (-1)^z q^{-z(2j + \Delta - k + 1)} (a_+^\dagger)^{k - \Delta - z} (a_+^\dagger)^{k + \rho - z} (a_-^\dagger)^{k - \Delta - z} (a_-)^{\rho + z}$$

$$= \frac{[k - \Delta]_q [k + \Delta]_q [k - \rho]_q [k + \rho]_q [2j + \Delta - k]}{[2j + \Delta + k + 1]_q} \times \sum_z (-1)^z q^{-z(2j + \Delta - k + 1)} (a_+^\dagger)^{k - \Delta - z} (a_+^\dagger)^{k + \rho - z} (a_-^\dagger)^{k - \Delta - z} (a_-)^{\rho + z}$$

where, as in section 2, the abbreviation $[x]_q$ stands for $[x]_q$. The correctness of eq. (31) can be verified by taking the $j'm'j$ matrix element of the right-hand side of (31); then, by using (32) and (28), it can be checked that we obtain the CGC $(jkm \rho|j'm')_q$ in the intermediate form (18). It is to be observed that, in equations of type (34), the $q$-factorials in the denominators of the sum over $z$ give a guaranty that the powers of all $q$-boson operators are nonnegative integers.

In a similar way, we have obtained the $q$-analog of eqs. (89)-(92) of ref. 1. They are given by

**The van der Waerden realization**:

$$t[q : k\rho\Delta] = (-1)^{k + \Delta} q^{\frac{k\Delta(2j + \Delta + k + 1) + j\rho - k} \times \left( \frac{[k + \rho]_q [k - \rho]_q [k + \Delta]_q [k - \Delta]_q [2j + \Delta - k]}{[2j + \Delta + k + 1]_q} \right)^{1/2} \times \sum_z (-1)^z q^{-z(2j + \Delta + k + 1)} (a_+^\dagger)^{k - \Delta - z} (a_-^\dagger)^{k - \Delta - z} (a_-^\dagger)^{k + \rho + z} (a_+^\dagger)^{\rho + z}$$

$$= (-1)^{k + \Delta} q^{\frac{k\Delta(2j + \Delta + k + 1) + j\rho - k} \times \left( \frac{[k + \rho]_q [k - \rho]_q [k + \Delta]_q [k - \Delta]_q [2j + \Delta - k]}{[2j + \Delta + k + 1]_q} \right)^{1/2} \times \sum_z (-1)^z q^{-z(2j + \Delta + k + 1)} (a_+^\dagger)^{k - \Delta - z} (a_-^\dagger)^{k - \Delta - z} (a_-^\dagger)^{k + \rho + z} (a_+^\dagger)^{\rho + z}$$

(35)
The Zukauskas-Manzuré realization:

\[
\begin{align*}
t[q: k\rho\Delta] &= (-1)^{\rho+\Delta} q^{\frac{1}{2}(k+\Delta)(2j+\Delta-k+1+2\rho)-(j+\Delta+k+1)\rho-km} \\
& \times \left( \frac{[k+\rho]![k-\rho]![k+\Delta]![k-\Delta]![2j+\Delta-k]!}{[2j+\Delta+k+1]!} \right)^{1/2} \\
& \times \sum_z (-1)^z q^{-z(2j+\Delta-k+1)} \frac{(a_+^{\pm})^k\rho^{\pm}(a_+^{\pm})^{k+\Delta-z}(a_-^{\pm})^{k+\Delta-z}(a_-^{\pm})^{k+\rho}(a_-^{\pm})^{k+\Delta}}{[k+\Delta-z]! [z]!}
\end{align*}
\] (36)

The Wigner realization:

\[
\begin{align*}
t[q: k\rho\Delta] &= (-1)^{k-\rho} q^{\frac{1}{2}(k+\Delta)(\Delta+k+1-2m-2\rho)-(j+\Delta+k+1)\rho-km} \\
& \times \left( \frac{[k+\Delta]![k-\Delta]![2j+\Delta-k]!}{[k+\rho]![k-\rho]![2j+\Delta+k+1]!} \right)^{1/2} \\
& \times \sum_z (-1)^z q^{z(k-\Delta+1)} \frac{(a_+^{\pm})^k\rho^{\pm}(a_+^{\pm})^{k+\Delta-z}(a_-^{\pm})^{k+\Delta-z}(a_-^{\pm})^{k+\rho}(a_-^{\pm})^{k+\Delta}}{[k+\Delta-z]! [z]!}
\end{align*}
\] (37)

The Racah (2nd) realization:

\[
\begin{align*}
t[q: k\rho\Delta] &= (-1)^{2k} q^{\frac{1}{2}(k+1-\Delta(2j+\Delta+1)+m\rho-1)} \\
& \times \left( \frac{[k+\Delta]![k-\Delta]![2j+\Delta-k]!}{[k+\rho]![k-\rho]![2j+\Delta+k+1]!} \right)^{1/2} \\
& \times \sum_z (-1)^z q^{-z(k-\rho+1)} \frac{(a_+^{\pm})^{k+\rho-z}(a_+^{\pm})^{k-\Delta}(a_-^{\pm})^{k-\Delta-z}(a_-^{\pm})^{k+\rho}(a_-^{\pm})^{k+\Delta}}{[k+\Delta-z]! [z]!}
\end{align*}
\] (38)

Note that in eqs. (34)-(38), it may be appropriate to replace the eigenvalues \( j \) and \( m \) in terms of the operators \((1/2)(N_1 + N_2)\) and \((1/2)(N_1 - N_2)\), respectively, in order to have basis independent operators. The remaining forms of section 2 can be \( q \)-bosonized too. The detailed results shall be given elsewhere. Remark that the Hermitean conjugation property

\[
t[q: k\rho\Delta] = (-1)^{\rho-j} t[q: k - \rho - \Delta]
\] (39)

[that is connected with the permutation of \( j \) and \( j' \) in eq. (28)] may be exploited to pass from one realization to another or to produce other \( q \)-boson realizations.

4. EXTENSION TO THE QUANTUM ALGEBRA \( U_{q\rho}(su_2) \)

The question of extending the one-parameter algebra \( U_{q}(su_2) \) to a two-parameter algebra was addressed by several authors.\(^{18-30}\) Indeed, the \( qp \)-quantized universal enveloping algebra \( U_{q\rho}(su_2) \) can be seen to be amenable to the one-parameter algebra \( U_{q}(su_2) \). To get a truly two-parameter algebra, it is necessary to \( qp \)-deform \( su_2 \) rather than \( su_2 \). We follow here the presentation of ref. 27 (see also ref. 2): the two-parameter quantum algebra \( U_{q\rho}(su_2) \) is spanned by the four generators \( J_\alpha \) (with \( \alpha = 0, 3, +, - \)) which satisfy the following commutation relations

\[
[J_3, J_\pm] = \pm J_\pm \quad [J_+, J_-] = (qp)^{J_0-J_3} [[2J_3]]_{q\rho} \quad [J_0, J_\alpha] = 0, \quad \alpha = 0, 3, +, -
\] (40)
In order to endow $U_{q,p}(u_2)$ with a Hopf algebraic structure, it is necessary to introduce a co-product $\Delta_{qp}$. It is defined by the application $\Delta_{qp} : U_{q,p}(u_2) \otimes U_{q,p}(u_2) \rightarrow U_{q,p}(u_2)$ such that

$$
\Delta_{qp}(J_0) := J_0 \otimes 1 + 1 \otimes J_0
$$
$$
\Delta_{qp}(J_3) := J_3 \otimes 1 + 1 \otimes J_3
$$
$$
\Delta_{qp}(J_\pm) := J_\pm \otimes (qp)^{1\over 2} (q^{-1})^{1\over 2} J_0 + (qp)^{1\over 2} J_0 (qp^{-1})^{1\over 2} \otimes J_\pm
$$

(41)

Note that with the constraint $p = q^*$, the co-product satisfies the Hermitian conjugation property $(\Delta_{qp}(J_{\pm}))^* = \Delta_{qp}(J_{\mp})$ and is compatible with the commutation relations for the four operators $\Delta_{qp}(J_\alpha)$ with $\alpha = 0, 3, +, -$. The universal $R$-matrix associated to the co-product $\Delta_{qp}$ reads

$$
R_{pq} = \begin{pmatrix}
  p & 0 & 0 & 0 \\
  0 & \sqrt{pq} & 0 & 0 \\
  0 & p - q & \sqrt{pq} & 0 \\
  0 & 0 & 0 & p
\end{pmatrix}
$$

(42)

and it can be proved that $R_{pq}$ verifies the so-called Yang-Baxter equation. The counit and antipode required for the Hopf algebraic structure of $U_{q,p}(u_2)$ are given in ref. 27.

The operator defined by

$$
C_2(U_{q,p}(u_2)) := \frac{1}{2}(J_+ J_- + J_- J_+) + \frac{1}{2} [[2]]_{qp} (qp)^{J_0 - J_2} [[J_3]]_{qp}^2
$$

(43)

is an invariant of the quantum algebra $U_{q,p}(u_2)$. The latter invariant gives back the well-known invariant of the quantum algebra $U_\hbar(u_2)$ when $p = q^{-1}$.

In the case where neither $q$ nor $p$ are roots of unity, the representation theory of $U_{q,p}(u_2)$ easily follows from the one of the Lie algebra $u_2$. An irreducible representation of the quantum algebra $U_{q,p}(u_2)$ is characterized by a doublet $(j_0, j)$ where $j_0 \in \mathbb{N}$ and $2j \in \mathbb{Z}$. Such a representation is associated to a subspace $\mathcal{E}(j_0, j) = \{|j_0, j, m\rangle : m = -j, -j + 1, \ldots, j\}$. The generic basis vector $|j_0, j, m\rangle$ of $\mathcal{E}(j_0, j)$ can be obtained from the highest weight vector $|j_0, j\rangle$ owing to

$$
|j_0, j, m\rangle = (qp)^{-j(j_0 - j)(j - m)} \sqrt{[[j + m]]_{qp} [[j - m]]_{qp} [[2j]]_{qp}} (J_-)^{j - m} |j_0, j\rangle
$$

(44)

Then, the action of the generators $J_\alpha$ (with $\alpha = 0, 3, +, -$) on the subspace $\mathcal{E}(j_0, j)$ is given by

$$
J_0 |j_0, j, m\rangle = j_0 |j_0, j, m\rangle
$$
$$
J_3 |j_0, j, m\rangle = m |j_0, j, m\rangle
$$
$$
J_+ |j_0, j, m\rangle = (qp)^{1\over 2} |j_0, j - 1\rangle \sqrt{[[j - m]]_{qp} [[j + m + 1]]_{qp} |j_0, j, m + 1\rangle}
$$
$$
J_- |j_0, j, m\rangle = (qp)^{-1\over 2} |j_0, j - 1\rangle \sqrt{[[j + m]]_{qp} [[j - m + 1]]_{qp} |j_0, j, m - 1\rangle}
$$

(45)

The eigenvalues of the invariant operator $C_2(U_{q,p}(u_2))$ on $\mathcal{E}(j_0, j)$ read

$$
\frac{(q^{j_0 + j_0 + 1} - q^{j_0 + 1} p^{j_0} - q^{j_0} p^{j_0 + 1} + q^{j_0 - j} p^{j_0} + 1)}{(q - p)^2} = (qp)^{(j_0 - j)} [[j]]_{qp} [[j + 1]]_{qp}
$$

(46)
in the general case where \( j_0 \neq j \). In the particular case where \( j_0 = j \), the eigenvalues (46) are equal to \([j]_{qp} \left[ [j + 1]_{qp} \right]_q\), a fact that was used as a basic ingredient for the qp-rotor model developed in refs. 31 and 32.

The quantum algebra \( U_{qp}(u_2) \) clearly depends on the two parameters \( q \) and \( p \). It is however interesting to show its relation with the well-known one-parameter algebra \( U_q(\text{su}_2) \). In this respect, eq. (41) suggests the following change of parameters

\[
Q := (qp^{-1})^{\frac{1}{2}} \quad P := (qp)^{\frac{1}{2}}
\]

In terms of the parameters \( Q \) and \( P \), we have

\[
[[x]]_{qp} = P_{x^{-1}} [x]_Q \quad [[x]]_{qp!} = P_{x(x^{-1})} [x]_Q!
\]

Then, by introducing the generators \( A_{\alpha} \) (with \( \alpha = 0, 3, +, - \))

\[
A_0 := J_0 \quad A_3 := J_3 \quad A_{\pm} := (qp)^{-\frac{1}{4} (j_0 - j_0^{\mp})} J_{\pm}
\]

it can be shown that the two-parameter quantum algebra \( U_{qp}(u_2) \) is isomorphic to the central extension

\[
U_{qp}(u_2) = u_1 \otimes U_Q(\text{su}_2)
\]

where \( u_1 \) is spanned by the operator \( A_0 \) and \( U_Q(\text{su}_2) \) by the set \{ \( A_3, A_+, A_- \) \}. The \( Q \)-deformation \( U_Q(\text{su}_2) \) of the Lie algebra \( \text{su}_2 \) corresponds to the usual commutation relations [cf. eq. (4)]

\[
[A_3, A_{\pm}] = \pm A_{\pm} \quad [A_+, A_-] = [2A_3]_Q
\]

(Of course, we have \([A_\alpha, A_\alpha] = 0 \) for \( \alpha = 0, 3, +, - \)). Furthermore, the co-product relations (41) leads to

\[
\Delta_{qp}(J_{\pm}) = \left( P_{A_0^{-\frac{1}{2}}} \otimes P_{A_0} \right) \Delta_{Q}(A_{\pm})
\]

\[
= P^{\Delta_{Q}(A_0^{-\frac{1}{2}})} \Delta_{Q}(A_{\pm})
\]

where the co-product \( \Delta_{Q} : U_Q(\text{su}_2) \otimes U_Q(\text{su}_2) \to U_Q(\text{su}_2) \) is given via

\[
\Delta_{Q}(A_0) := A_0 \otimes 1 + 1 \otimes A_0 \\
\Delta_{Q}(A_3) := A_3 \otimes 1 + 1 \otimes A_3 \\
\Delta_{Q}(A_{\pm}) := A_{\pm} \otimes Q^{\pm A_3} + Q^{-A_3} \otimes A_{\pm}
\]

The invariant \( C_2(U_{qp}(u_2)) \) may be transcribed in terms of the two parameters \( Q \) and \( P \). In fact, eq. (43) can be rewritten as

\[
C_2(U_{qp}(u_2)) = P^{2A_0^{-1}} C_2(U_Q(\text{su}_2))
\]

where

\[
C_2(U_Q(\text{su}_2)) := \frac{1}{2} \left( A_+ A_- + A_- A_+ \right) + \frac{1}{2} [2]_Q \left( [A_3]_Q \right)^2
\]

so that the eigenvalues (46) of \( C_2(U_{qp}(u_2)) \) on the subspace \( \mathcal{E}(j_0, j) \) can be rewritten as \( P^{2j_0^{-1}} [j]_Q [j + 1]_Q \).

From the point of view of the representation theory, eqs. (47)-(55) strongly suggest that the coupling (i.e., CGc's and \( 3 - jm \) symbols) and recoupling (i.e., \( 6 - j \) and
of $U_{qp}(u_2)$ are easily seen to satisfy the three-term recursion relations
\[
\sqrt{[j \mp m]_Q} [j \pm m + 1]_Q (m_1 m_2 | m \pm 1)_q = Q^+^{m_2} \sqrt{[j_1 \mp m_1]_Q [j_1 \pm m_1 + 1]_Q (m_1 \mp 1, m_2 | m)_q + Q^-{m_1} \sqrt{[j_2 \mp m_2]_Q [j_2 \pm m_2 + 1]_Q (m_1, m_2 \mp 1 | m)_q}
\]
which are identical to the ones satisfied by the CGc's $(j_1 j_2 m_1 m_2 | j m)_q$ of the quantum algebra $U_q(\text{su}_2)$ (see ref. 3). Therefore, there exists a proportionality relation between the $qp$-CGc's and the $Q$-CGc's. Reality and normalization conditions can be used to justify that the proportionality constant is equal to 1. Indeed, this may be checked by direct calculation: by adapting to the $qp$-deformation $U_{qp}(u_2)$ the method of projection operators used for $\text{su}_2$ (in ref. 15) and for $U_q(\text{su}_2)$ (in ref. 13), we can show that we have the connecting formula
\[
(j_{01} j_{02} j_1 j_2 m_1 m_2 | j_0 j m)_q = \delta(j_0, j_{01} + j_{02}) (j_1 j_2 m_1 m_2 | j m)_Q
\]
a result to be compared with the ones in refs. 28 and 29 for $U_{qp}(\text{su}_2)$. Therefore, all formulas of subsection 2.2 may be adapted to the case of the two-parameter quantum algebra $U_{qp}(u_2)$.

5. CLOSING REMARKS

We obtained in this work various analytical forms for the $U_q(\text{su}_2)$ CGc's. A special effort was put on the compatibility between the different forms discussed. Indeed, all the obtained forms were deduced from one single form, namely, the so-called Shapiro form, either by applying the resummation procedure or by using ordinary symmetry properties of the CGc's. Some other checks (not reported in the present paper) were also achieved by means of Regge symmetries and of the mirror reflection symmetry extended to the $U_q(\text{su}_2) > u_1$ chain. In addition, the forms were compared to existing formulas when possible.

In the classical limit where $q = 1$, the various $q$-dependent forms reduce to well-known expressions (like the formulas derived by Wigner, van der Waerden, and Racah in the early days of what is referred to as Wigner-Racah algebra of the rotation group) and to less-known expressions for the CGc's of the group $\text{SU}_2$ in an $\text{SU}_2 \supset \text{U}_1$ basis. For a given form, the passage from $q = 1$ to $q \neq 1$ manifests itself by: (i) the replacement of $2j + 1$ by $[2j + 1]_q$ and of ordinary factorials by $q$-factorials, (ii) the introduction of an internal $q$-factor which depends on a summation index $z$, and (iii) the introduction of an external $q$-factor ($z$-independent).

It is remarkable that the internal $q$-factor (which is $z$-dependent) assumes the form $q^{\pm \binom{N-D+1}{2}}$, where $N$ and $D$ stand respectively for the numerator and the denominator of the fraction involved in the sum over $z$. The occurrence of this heuristic rule might be rationalized on the basis of some properties of the $q$-deformation of the hypergeometric function $_2F_1$ in term of which it is possible to express the $q$-deformed CGc's of $\text{SU}_2$ [see ref. 33 for the use of $_2F_1(abc, dc; 1)$ in connection with $\text{SU}_2$ CGc's].
The various forms of the $U_q(\mathfrak{su}_2)$ CGc’s fall in three families: the Wigner, van der Waerden, and Racah families. Each family is characterized by a given distribution of the $q$-factorials $[j_1 \pm m_1]_q$, $[j_2 \pm m_2]_q$, and $[j \pm m]_q$ in the numerator of the factor in front of the sum over $z$. The van der Waerden family has the six $q$-factorials in the numerator. The Wigner family [including the Shapiro forms, the intermediate form, the Zukauskas-Mauza form, and the Jucys-Bandzaitis (1st) form] presents two $q$-factorials in the numerator while the Racah family [including the Majumdar forms and the the Jucys-Bandzaitis (2nd) form] has four $q$-factorials in the numerator. In principle, the parents in a family may be obtained from any member of the family by applying ordinary symmetry properties of the CGc’s.

Some preliminary results on the $q$-bosonization of the CGc’s of $U_q(\mathfrak{su}_2)$ have been reported. Various expressions have been given for the unit tensor operator $t[q : k \rho \Delta]$. Not all the possible forms of the operator $t[q : k \rho \Delta]$ have been described. In particular, the Majumdar realization of $t[q : k \rho \Delta]$ has been omitted since it presents some peculiarities as in the $q = 1$ case. A more complete listing of $q$-boson realizations of the operator $t[q : k \rho \Delta]$ shall be published elsewhere.

Finally, we have extended to a two-parameter deformation of $\mathfrak{su}_2$ the various expressions for the $q$-deformed CGc’s of the chain $SU_2 \supset SU_1$. The derivation of a true (nontrivial) two-parameter deformation of $\mathfrak{su}_2$ has been questioned by various authors. Although, the CGc’s for $U_{qp}(\mathfrak{su}_2)$ may be obtained from the ones for $U_Q(\mathfrak{su}_2)$ by means of a formula where the two parameters $q$ and $p$ are unified in a single one [i.e., $Q = (qp^{-1})^{\frac{1}{2}}$], the two parameters $q$ and $p$ [or, equivalently, $Q = (qp)^{\frac{1}{2}}$] really appear in the Casimir(s) of the Hopf algebra $U_{qp}(\mathfrak{su}_2)$. For the practitioner interested in putting some numbers on the (real) world, the two parameters $q$ and $p$ may have some interest when dealing with a comparison between theory and experiment (via fitting procedures for example). The latter point was applied to the derivation of a $qp$-rotor model for describing rotational bands of (superdeformed) nuclei.

To complete this work, it would be interesting to study the $U_q(\mathfrak{su}_2)$ CGc’s in terms of the $q$-deformed hypergeometric function $F_3$ and to examine the contraction of $F_3$ into a $F_1$ operator-valued function when going from the CGc to the Wigner operator. Also, to find $q$-boson realizations for the Racah operator (the matrix elements of which are $q$-deformed recoupling coefficients or $6 - j$ symbols) is an interesting problem. In addition, a treatment via symbolic programming languages (like Maple or Mathematica) of the formulas in this paper is presently under study. [A programme is obtainable for calculating any $U_q(\mathfrak{su}_2)$ CGc.] We hope to return on these matters in a future work.

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