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A PROPOSAL TO SOLVE THE LANDAU GHOST AND RENORMALON PROBLEMS

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Abstract : A systematic procedure to remove the ultraviolet renormalons from non asymptotically free field theories is presented, which imitates the corresponding procedure for infrared renormalons in infrared non trivial theories. It is argued the resulting continuum theory may have no Landau ghost, and still be consistent with the triviality bounds.

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1. Renormalizable ultraviolet (UV) non-asymptotically free field theories are plagued by two related UV problems :

- i) Renormalized coupling constants and amplitudes display unphysical singularities at finite space-like momenta : the Landau ghost [1] (and the associated “triviality” [1,2]) problem.
- ii) Internal UV integration in certain class of Feynman diagrams with vacuum polarization insertions produces a factorial growth of perturbation theory coefficients at large order [3-5]. After Borel summation, an unphysical and ambiguous imaginary part is generated : the renormalon [5] problem. In this note, I suggest a possible solution based on a formal analogy with the analogous infrared (IR) problems in IR non trivial theories. It consists in a systematic removal of the UV renormalons, through the addition of an (infinite) number of specific “non-perturbative” terms (which are however determined in principle from “all-order” perturbation theory). I shall further argue that this proposal may be consistent with the known “triviality bounds” [2], although putting them in a different light.

2. Consider for definiteness in QED with massless fermions the renormalization group (RG) invariant quantity : $R(Q) = -\frac{1}{\beta_1} \left(\frac{d\pi}{d \ln Q^2} - \frac{d\pi}{d \ln Q^2} \Big|_{\alpha=0} \right)$, where $\pi(Q)$ is the photon vacuum polarization, β_1 the 2-loop β function coefficient, and α the coupling constant. In perturbation theory, $R(Q)$ is given by the formal power series : $R(Q) = \sum_{n=0}^{\infty} r_n \alpha^{n+1}$ (with $r_0 = 1$), and one can define an “all-order” perturbative R_{PT} by the Borel sum :

$$R_{PT}(\alpha) = \int_0^{\infty} dz \exp(-z/\alpha) R_{PT}(z) \quad (1)$$

with

$$R_{PT}(z) = \sum_{n=0}^{\infty} \frac{r_n}{n!} z^n$$

The Borel transform $R_{PT}(z)$ is believed to have a finite convergence radius [6], with branch point singularities (the UV renormalons) on the positive z axis at $z_n \beta_0 = n$ ($n \geq 1$) ($\beta_0 > 0$ is the one-loop β function coefficient), which contribute $O(\exp(-z_n/\alpha))$ imaginary parts to $R_{PT}(\alpha)$. In a consistent theory, they must be cancelled by those coming from additional, “non-perturbative”, contributions.

A **minimal** way to do this is to define the “non-perturbative” $R(Q)$ by the following subtraction procedure :

$$R(Q) \equiv R_{PT}(\alpha) - \sum_{n=1}^{\infty} R_n(\alpha) \quad (2)$$

where :

$$R_n(\alpha) = \int_{z_n}^{\infty} dz \exp(-z/\alpha) R_n(z) \quad (3)$$

with

$$R_1(z) \equiv R_{PT,1}(z) \equiv \text{singular part of } R_{PT}(z) \text{ for } z \sim z_1$$

$$R_2(z) \equiv R_{PT,2}(z) - R_{1,2}(z) \equiv \text{singular part of } (R_{PT}(z) - R_1(z)) \text{ for } z \sim z_2, \text{ etc } \dots$$

I shall assume the singular parts have the form :

$$R_n(z) = K_n \left(\frac{Q^2}{\mu^2} \right)^{z_n \beta_0} \frac{1}{\left(1 - \frac{z}{z_n} \right)^{\gamma_n}} \left[1 + \sum_{p=1}^{\infty} r_{p,n} \left(1 - \frac{z}{z_n} \right)^p \right] \quad (4)$$

(for simplicity, I neglect “mixing”, i.e. the occurrence of several R_n ’s with different γ_n ’s for a given z_n), where μ is the renormalization point and K_n a real constant. The $(Q^2/\mu^2)^{z_n \beta_0}$ factor comes from RG invariance (see below), and $r_{p,n}$ are polynomials in $\ln Q^2/\mu^2$. The Borel integrals must be taken consistently above or below the positive z axis, so that $R(Q)$ in eq.(2) is real and free of ambiguity.

Consequently :

$$R_n(\alpha) = -\tilde{C}_n \left(\frac{Q^2}{\mu^2} \right)^{z_n \beta_0} \exp\left(-\frac{z_n}{\alpha}\right) (-\alpha)^{\delta_n} \left[1 + \sum_{p=1}^{\infty} r_{p,n} \frac{\delta_n(1+\delta_n)\dots(p-1+\delta_n)}{(-z_n)^p} \alpha^p \right] \quad (5)$$

with

$$\begin{aligned} \delta_n &= 1 - \gamma_n \\ \tilde{C}_n &= K_n \Gamma(1 - \gamma_n) z_n^{\gamma_n} \end{aligned} \quad (6)$$

For $\alpha > 0$ and $|\delta_n| \neq \text{integer}$ the $R_n(\alpha)$ ’s are complex, and their imaginary parts cancell against the ones coming from $R_{PT}(\alpha)$, but the subtracted $R_n(\alpha)$ also have a real part, as required by simple analyticity properties in the α -plane. Actually, the structure of $R_n(\alpha)$ is very close to that of $\text{Im } R_n(\alpha)$ (obtained from eq.(5) by changing $\tilde{C}_n \rightarrow \pm \tilde{C}_n \sin \pi \delta_n$) and of $\text{Im } R_{PT}(\alpha)$, and in this sense the definition eq.(2) can be seen as a “minimal” extension of the notion of Borel summation (the meaning of the infinite sum in eq.(2) remains to be clarified, however (see section 3 for an example)). Also, although the $R_n(\alpha)$ ’s are “non-perturbative” $O(\exp(-z_n/\alpha))$ quantities, they are

determined from the singularities of $R_{PT}(z)$, i.e. from information contained in perturbation theory (to all orders !). Furthermore, the assumed RG invariance of $R_{PT}(\alpha)$ in eq.(1) implies¹ that of the R_n 's, so that $R(Q)$ in eq.(2) is also RG invariant. Indeed, the RG equation for $R_{PT}(\alpha)$:

$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha} \right] R_{PT} = 0$ becomes in Borel space [7] :

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta_0 z \right] R_{PT} + \int_0^z dy R_{PT}(y) b(z-y) = 0 \quad (7)$$

where $b(z)$ is the Borel transform of $\frac{\beta(\alpha)}{\alpha^2} - \beta_0 = \beta_1 \alpha + O(\alpha^2)$. On the other hand, RG invariance of the singular part $R_{PT,n}(\alpha)$ associated to the n^{th} renormalon is equivalent in Borel space to :

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta_0 z \right] R_{PT,n} + \int_{z_n}^z dy R_{PT,n}(y) b(z-y) = 0 \quad (8)$$

and one easily checks from eq.(7) that $R_{PT,n}(z)$ do satisfy eq.(8), order by order in an expansion around $z = z_n$. One can then show recursively that $R_n(z)$ also satisfies² eq.(8), hence $R_n(\alpha)$ is RG invariant. It further follows from eq.(5), since $\alpha(Q) \sim 1/\beta_0 \ln \Lambda^2/Q^2$ for $Q^2 \rightarrow 0$, and $z_n \beta_0 = n$, that $R_n(\alpha)$ is an $O\left(\left(\frac{Q^2}{\Lambda^2}\right)^n\right)$ quantity, and $R(Q)$ in eq.(2) is given by a **low-energy** expansion, as natural in an IR free theory.

3. The previous prescription is analogous to a similar subtraction procedure for **IR** renormalons in UV asymptotically free theories ($\beta_0 < 0$), which are in this case on the positive z axis at $z_n \beta_0 = -n$ ($n \geq 2$). In these theories the full, non perturbative $R(Q)$ is given [8] by the operator product expansion (OPE) (a high energy expansion this time) :

$$R(Q) = R_{PT}(\alpha) + \sum_n (G_n + \tilde{G}_n) \quad (9)$$

where $(G_n + \tilde{G}_n)$ represents the contribution of operators of dimension d_n , i.e. the vacuum condensates. Neglecting for simplicity operator mixing, the RG yields the α -dependance :

$$G_n(\alpha) = C_n \left(\frac{\mu^2}{Q^2} \right)^{d_n/2} \exp \left(\frac{d_n}{2\beta_0 \alpha} \right) \alpha^{\delta_n} [1 + O(\alpha)]$$

¹ A simple argument is that $R_{PT,n}(\alpha)$ has the same RG invariance property as $\text{Im } R_{PT,n}(\alpha)$ and $\text{Im } R_{PT}(\alpha)$.

² The $\left(\frac{Q^2}{\mu^2}\right)^{z_n \beta_0}$ factor in eq.(4) follows because the integral in eq.(8) can be neglected in leading order for $z \rightarrow z_n$.

(10)

$$\tilde{G}_n(\alpha) = \tilde{C}_n \left(\frac{\mu^2}{Q^2} \right)^{d_n/2} \exp \left(\frac{d_n}{2\beta_0\alpha} \right) (-\alpha)^{\delta_n} [1 + O(\alpha)]$$

where C_n , $\tilde{C}_n$ are real constants and the α -dependant terms in $G_n(\alpha)$ and $\tilde{G}_n(\alpha)$ are identical, except for the minus sign [9] in $(-\alpha)^{\delta_n}$ ($\delta_n = \frac{d_n}{2} \frac{\beta_1}{\beta_0^2} - \frac{\gamma_{0,n}}{\beta_0}$, where $\gamma_{0,n}$ is the one-loop anomalous dimension of the corresponding operator). The condensate contribution has therefore a “two-component” structure. The $\tilde{G}_n(\alpha)$ component is complex³ for $\alpha > 0$ (in the generic case where $|\delta_n|$ is not integer), and cancels [9-13] the imaginary part coming from IR renormalons in the Borel summed $R_{PT}(\alpha)$: it is the analogue of $-R_n(\alpha)$ in eq.(2) and (5) (in particular, cancellation requires $d_n/2 = -z_n\beta_0$ [10]), and is therefore determined from “all order” perturbation theory and represents the contribution of IR renormalons. On the other hand, the $G_n(\alpha)$ component is real and unrelated to renormalons (in this sense, it is genuinely “non-perturbative”). Although both components are expected on a general ground, in some cases only one component may be present. For instance, in QCD the quark condensate, being the order parameter of chiral symmetry breaking, is not expected to be related to renormalons ([11],[14]), and consequently only the $G_n(\alpha)$ component should occur for this peculiar condensate⁴. On the other hand, in simpler models such as the 2-dimensional non-linear $O(N)$ σ model [11,15], it may be that only the $\tilde{G}_n(\alpha)$ components are present, in which case all “non-perturbative” terms in the OPE would in fact be determined from “all-order” perturbation theory [9]. The latter kind of models would then be the UV free analogues of the “non-perturbative” ansatz for QED of section 2. However, in the QED case, there is no known a priori low energy expansion, and one has to start from the singularities of the Borel transform (the physical interpretation of the non-perturbative terms is also very different, see section 6).

Furthermore, UV free theories usually have an **IR** Landau ghost problem. If one assumes the

³ The complex component is related to the presence of UV renormalons and power UV divergences, generating mixing with lower dimensional operators in the corresponding matrix element [11,12].

⁴ The occurrence of condensate components unrelated to renormalons is also apparent on phenomenological grounds in QCD, when one compares [8] the vacuum correlators of (vector, vector) and (axial, axial) currents, which have identical perturbative expansions, but very different non-perturbative time-like spectra.

ghost is still present in the Borel-summed $R_{PT}(\alpha)$, it can only be removed by the “non-perturbative” contributions in the OPE. This cancellation may be checked in the $N = \infty$ limit of the two dimensional $O(N)$ σ model for the “ α -propagator” $D(Q)$, which describes σ - σ scattering. Defining a RG invariant dimensionless effective charge $\bar{R}(Q) = D(Q)/Q^2$, one gets [11,15] the **exact** “non-perturbative” $N = \infty$ result :

$$\bar{R}(Q)|_{N=\infty} = \frac{1}{Q^2} \frac{\sqrt{Q^2(Q^2 + 4m^2)}}{\ln \left(\frac{\sqrt{Q^2 + 4m^2} + \sqrt{Q^2}}{\sqrt{Q^2 + 4m^2} - \sqrt{Q^2}} \right)} \quad (11)$$

where m is the “non-perturbative” mass of the σ particles. Expanding eq.(11) for large Q^2 one recovers the OPE [11,15] :

$$\bar{R}(Q)|_{N=\infty} \underset{Q^2 \gg m^2}{=} \bar{\alpha} + \frac{2m^2}{Q^2} \bar{\alpha} (1 - \bar{\alpha}) - \frac{2m^4}{Q^4} \bar{\alpha} (1 + \bar{\alpha}/2 - 2\bar{\alpha}^2) + \dots \quad (12)$$

with $\bar{\alpha} = 1/\ln Q^2/m^2$. Eq.(12) shows that $\bar{R}_{PT}|_{N=\infty} \equiv \bar{\alpha}$ is just the one-loop β -function coupling, which indeed has an IR Landau ghost at $Q^2 = m^2$, whereas eq.(11) gives for $Q^2 \rightarrow 0$ $\bar{R}(Q)|_{N=\infty} \sim \frac{2m^2}{Q^2} \rightarrow \infty$, corresponding to an (infinite) IR fixed point in the “non-perturbative”

$\bar{R}(Q)$: the Landau ghost in $\bar{R}_{PT}$ (which is also present in all individual terms of the OPE eq.(12)) has been removed, as expected, by the contribution of the “non-perturbative” terms in eq.(12), i.e. (presumably) by the $\tilde{G}_{n|N=\infty}$ terms of this model. The summation of the **infinite** number of terms in the OPE eq.(12) is required to cure the Landau ghost problem. The following observation clarifies the nature of this summation : define R by :

$$\frac{d\bar{R}}{d \ln Q^2} = -\bar{R}^2 (1 + R) \quad (13)$$

(i.e., R represents all higher order terms beyond the leading one in the effective charge β function).

Then one finds from eq.(11) that :

$$R(Q) = a(x) \ln \frac{Q^2}{m^2} - b(x) \quad (14)$$

where $a(x)$ and $b(x)$ ($x \equiv m^2/Q^2$) are analytic around $x = 0$ with finite convergence radii (which fixes the summation procedure), and given by :

$$a(x) = \frac{2x}{(1 + 4x)^{3/2}} \quad (15)$$

$$b(x) = \frac{4x}{1 + 4x} \left[1 - \frac{\ln \left(\frac{1}{2} \sqrt{1 + 4x} + 1 \right)}{\sqrt{1 + 4x}} \right]$$

As another example of UV asymptotically free theory, consider QED with $\alpha < 0$. One can construct along similar lines as in section 2 a “non-perturbative” $\alpha < 0$ QED, hopefully free of IR Landau ghost, by subtracting out from $R_{PT}(\alpha)$ the contribution of the **IR** renormalons. In this case, one should use the analytic contribution of $R_{PT}(\alpha)$ in eq.(1) to $\alpha < 0$, which I assume to be given by Borel transformation over the **negative** z axis :

$$R_{PT}(\alpha) = - \int_{-\infty}^0 dz \exp(-z/\alpha) R_{PT}(z) \quad (16)$$

and define the “non-perturbative” $R(Q)$ by eq.(2), where $R_n(\alpha)$ now represents the contribution of the **IR** renormalons :

$$R_n(\alpha) = - \int_{-\infty}^{z_n} dz \exp(-z/\alpha) R_n(z) \quad (17)$$

with $z_n = -n/\beta_0$, $n \geq 2$, and $R_n(z)$ the singular parts of the IR renormalons, defined similarly to the procedure in section 2. The so defined $\alpha < 0$ QED is **different** from the $\alpha > 0$ QED constructed in section 2, and the two theories are **not** analytic continuation of each other. Furthermore, QED perturbation theory is asymptotic to the two different “non-perturbative” $R(Q)$ ’s in **different** sectors of the complex α plane ; in particular, as $\alpha \rightarrow 0^-$ one gets $R(Q) \sim \alpha$ only for the $\alpha < 0$ QED.

4. Coming back to UV non asymptotically free theories, the next question is whether the “non-perturbative” $R(Q)$ of eq.(2) has no UV Landau ghost. This can perhaps be checked in the large N limit, where N is the number of fermion flavors (at finite N , the calculation of UV renormalons (i.e. of $R_{PT,n}(z)$) seems [16] beyond control⁵). However, I have to make the additional assumption that :

$$R_n(z)_{|N=\infty} = R_{PT,n}(z)_{|N=\infty} + \text{terms regular at } z = z_n \quad (18)$$

where $R_n(z)_{|N=\infty}$ means the $N \rightarrow \infty$ limit of eq.(4) (and a similar definition for $R_{PT,n}|_{N=\infty}$). In general, since the singularities of $R_{PT}(z)$ at $N = \infty$ are usually integer poles instead of cuts, $R_{PT,n}(z)_{|N=\infty}$ and $R_n(z)_{|N=\infty}$ contain both a singular pole part and an infinite number of terms regular at $z = z_n$. Assumption eq.(18) means that $R_{PT,n}(z)$ has no z_i renormalon for $i > n$ in the

⁵ A similar statement can be made for the calculation of the residues K_n of the IR renormalons [17].

$N = \infty$ limit ; this may occur if any singularity of $R_{PT,n}(z)$ occuring for finite N at $z = z_i$ ($i > n$) turns into a regular term at $N = \infty$. One can then make use of the result [14,18] :

$$R_{PT,n}(z)_{|N=\infty} = a_n \left(\frac{Q^2}{\mu^2} \right)^{z_n \beta_0} \left[\frac{1}{\left(1 - \frac{z}{z_n}\right)^2} + \frac{b_n}{1 - \frac{z}{z_n}} \right] + \text{terms regular at } z = z_n \quad (19)$$

with $a_n = \frac{8}{3} \frac{(-1)^{n+1}}{n^2(n+1)(n+2)}$, $b_n = -n \left(\ln \frac{Q^2}{\mu^2} + c_n \right)$ and $c_n = -\left(\frac{1}{n+1} + \frac{1}{n+2} \right)$, where the terms regular at $z = z_n$ in eq.(18) and (19) are also assumed to be regular for all $z > z_n$ (equivalently, the series in the numerator of eq.(4) are assumed to be regular for $z > z_n$ at $N = \infty$). Comparing with eq.(4), eq.(18) and (19) imply that $K_{n|N=\infty} = a_n$ and $\gamma_n = 2 - \frac{\lambda_n}{N} + \dots$ (where λ_n is not yet known, see below), hence (eq.(6)) $\delta_n = -1 + \frac{\lambda_n}{N} + \dots$ and $\tilde{C}_n \sim -\frac{a_n z_n^2}{\lambda_n} \times N = O(N)$ (the latter result is a consequence of the fact that $\gamma_{n|N=\infty} = 2$, a **positive** integer, and the integral in eq. (3) diverges at $z = z_n$). Furthermore, given that $r_{p,n}$ in eq.(4) are $O(N^0)$ (in particular, $r_{1,n|N=\infty} = b_n$), only the $p = 1$ term in eq.(5) survives at $N = \infty$, owing to the effect of the $1 + \delta_n \sim \frac{\lambda_n}{N}$ factor, present for $p > 1$, and one gets :

$$\left. \frac{R_n(\alpha)}{N} \right|_{N=\infty} = \frac{a_n z_n^2}{\lambda_n} \left(\frac{Q^2}{\mu^2} \right)^{z_n \beta_0} \exp\left(-\frac{z_n}{\alpha}\right) \left(-\frac{1}{\alpha}\right) \left(1 + \frac{b_n}{z_n} \alpha\right) \quad (20)$$

Hence, since eq.(19) is obtained in a scheme where $\alpha(\mu)_{|N=\infty} \equiv 1/\beta_0 \ln \Lambda^2/\mu^2$ (Λ is the corresponding Landau ghost), and using $z_n \beta_0 = n$, one deduces :

$$\left. \frac{R_n(\alpha)}{N} \right|_{N=\infty} = \frac{n a_n z_n^2}{\lambda_n} \left(\frac{Q^2}{\Lambda^2} \right)^n \left(\ln \frac{Q^2}{\Lambda^2} + c_n \right) \quad (21)$$

where all parameters, **except** λ_n , are known. The latter require a calculation of $R_{PT,n}(z)$ at next to leading order in $1/N$, and is related to one-loop anomalous dimensions of operators of dimension $d_n = 2n + 4$ (section 6). It is therefore **not** sufficient to perform a leading $1/N$ calculation to get the correct “non-perturbative” terms at $N = \infty$, but one has also to know how the $N = \infty$ limit is reached, starting from the finite N ansatz eq. (4). Note also $R_n(\alpha)$ is $O(N)$ and real at $N = \infty$, whereas $R_{PT}(\alpha)$ is $O(N^0)$. On the other hand, $\text{Im } R_n(\alpha)$ is $O(N^0)$, as it must to cancell $\text{Im } R_{PT}(\alpha)$.

For instance, assuming for simplicity $R_n(\alpha) = -\tilde{C}_n \left(\frac{Q^2}{\mu^2} \right)^n \exp\left(-\frac{z_n}{\alpha}\right) (-\alpha)^{\delta_n}$, one gets for $N \rightarrow \infty$, putting $\tilde{C}_n = \tilde{C}_n^\infty \times N$:

$$R_n(\alpha) \sim N \tilde{C}_n^\infty \left(\frac{Q^2}{\mu^2} \right)^n \exp\left(-\frac{z_n}{\alpha}\right) \frac{1}{\alpha} \left[1 + \frac{\lambda_n}{N} \ln(-\alpha) + O\left(\frac{1}{N^2}\right) \right] \quad (22)$$

hence $\text{Im } R_n(\alpha)_{|N=\infty} = \pm \pi \tilde{C}_n^\infty \lambda_n \left(\frac{Q^2}{\mu^2} \right)^n \exp\left(-\frac{z_n}{\alpha}\right) \frac{1}{\alpha}$, an $O(N^\circ)$ quantity (with $\tilde{C}_n^\infty \lambda_n = -a_n z_n^2$ known).

Since $R(Q)_{|N=\infty} = -\sum_n R_n(\alpha)_{|N=\infty}$ is purely “non-perturbative”, one first has to check that the latter sum does not contain any Landau ghost or unphysical space-like ($Q^2 > 0$) singularity (each individual $R_n(\alpha)$ is O.K. in this respect). Eq. (21) shows that $R(Q)_{|N=\infty}$ is given in term of two power series in Q^2/Λ^2 , which may well have finite convergence radii around $Q^2 = 0$ (similarly to the functions in eq.(15)). On the other hand, a time-like singularity ($Q^2 < 0$) would be acceptable, and would signal new non-perturbative phenomena generated by the strong UV coupling.

A further check is provided by looking at the Gell-Mann-Low effective charge : $\alpha_{\text{eff}}(Q) = \frac{\alpha}{1 + \alpha \pi(Q)}$, which is related to R by :

$$\frac{d\alpha_{\text{eff}}}{d \ln Q^2} = \beta_0 \alpha_{\text{eff}}^2 \left(1 + \frac{\beta_1}{\beta_0} R \right) \equiv \beta_{\text{eff}} \quad (23)$$

(a relation similar to eq.(13)). One should check that $\alpha_{\text{eff}}(Q)$ has no Landau ghost either if the “non-perturbative” definition of R , eq.(2), is used. This is made plausible by the fact that $\beta_1/\beta_0 = O(1/N)$, hence, since ⁶ $R = O(N)$, $\frac{\beta_1}{\beta_0} R = O(N^\circ)$, and cannot be neglected at $N = \infty$, i.e. the “non-perturbative” α_{eff} does **not** satisfy the one-loop RG equation (which would yield a Landau ghost !) at $N = \infty$ (contrary to its perturbative counterpart, where R is replaced by R_{PT})⁷.

5. Comment on “triviality” : the above proposal may be consistent with “triviality”, or “zero charge” problem [1,2], which can be seen as a property of the **bare** coupling constant β function $\beta(\alpha)$ having no non-trivial UV fixed point, since one can deduce heuristically [20] the triviality bounds under the only assumption that the bare coupling $\alpha = \alpha(M)$ becomes infinite for a **finite** value Λ (the Landau ghost of the **bare** coupling) of the cut-off M . This assumption may be compatible with the **renormalized** coupling β function (for instance, the β_{eff} function in eq.(23))

⁶ The necessity that $R = O(N)$ is a signal one should not try to remove the singular renormalon parts sitting only at $N = \infty$: this procedure would be ambiguous, and would yield anyway an $O(N^\circ)$ “non-perturbative” R .

⁷ However, the absence of Landau ghost requires that R in eq.(23) becomes **negative** for sufficiently large Q^2 , despite the fact that at finite N , $R \sim \alpha_{\text{eff}} > 0$ for $Q^2 \rightarrow 0$.

having a non-trivial UV fixed point, owing to the effect of the “non-perturbative” terms in eq.(2), not necessarily present in $\beta(\alpha)$ (which presumably also has no UV renormalons, similarly to an analogous conjecture (E. Brezin, as quoted in ref. [19]) for the MS scheme β function associated to dimensional regularization). The “triviality bounds” [1,2] would then be essentially regularization scheme dependant results.

As a simple example, consider the QED analogue of the “Richardson potential” [21], namely define a renormalized coupling constant $\bar{R}(Q)$ by :

$$\bar{R}(Q) = \frac{1}{\beta_0 \ln \left(1 + \frac{\Lambda^2}{Q^2} \right)} \quad (24)$$

(where the 1 inside the logarithm removes the Landau pole at $Q^2 = \Lambda^2$ in the corresponding one-loop coupling). Eq.(24) implies the $\bar{\beta}(\bar{R})$ function⁸ :

$$\frac{d\bar{R}}{d \ln Q^2} = \beta_0 \bar{R}^2 \left[1 - \exp \left(-\frac{1}{\beta_0 \bar{R}} \right) \right] \equiv \bar{\beta}(\bar{R}) \quad (25)$$

Since $\bar{\beta}(\bar{R}) \sim \bar{R}$ for $\bar{R} \rightarrow +\infty$, $\bar{\beta}(\bar{R})$ has an (infinite) UV fixed point. Assuming further the “bare” coupling constant has the one-loop form : $\alpha(M) \equiv 1/\beta_0 \ln \Lambda^2/M^2$, with $\beta(\alpha) \equiv \beta_0 \alpha^2$ one gets from eq.(24), substituting Λ in term of $\alpha \equiv \alpha(M)$:

$$\bar{R}(Q) = \bar{R}(Q/M, \alpha) = \frac{\alpha}{1 + \alpha \beta_0 \ln \frac{M^2}{Q^2} + \alpha \beta_0 \ln \left[1 + \frac{Q^2}{M^2} \exp \left(-\frac{1}{\beta_0 \alpha} \right) \right]} \quad (26)$$

Since $\bar{R}(Q/M, \alpha)$ is a monotonously increasing function of α , one obtains, letting $\alpha \rightarrow +\infty$, the “triviality bound” (valid for $0 \leq \alpha \leq +\infty$) :

$$\bar{R}(Q/M, \alpha) < \bar{R}_{\max}(Q/M) = \frac{1}{\beta_0 \ln \left(1 + \frac{M^2}{Q^2} \right)} \stackrel{M^2 \gg Q^2}{\sim} \frac{1}{\beta_0 \ln \frac{M^2}{Q^2}} \quad (27)$$

where $\bar{R}_{\max}(Q/M) \equiv \bar{R}(Q/M, \alpha = +\infty)$, and, for $M^2 \gg Q^2$, one recovers the zero charge problem.

Furthermore, for $Q^2 \rightarrow \infty$, one gets :

$$\bar{R}(Q/M, \alpha) \sim \frac{1}{\beta_0} \frac{Q^2}{\Lambda^2} \equiv \frac{1}{\beta_0} \frac{Q^2}{M^2} \exp \left(-\frac{1}{\beta_0 \alpha} \right) \quad (28)$$

a behavior dictated by the infinite UV fixed point of $\bar{\beta}$, whereas for $Q^2 \rightarrow 0$ one obtains the low

⁸ One can also write $\bar{\beta}(\bar{R}) = \beta_0 \bar{R}^2 \left(1 - \frac{Q^2/\Lambda^2}{1 + Q^2/\Lambda^2} \right)$, where the second term is the analogue of $\beta_1/\beta_0 R$ in

eq.(23), and represents the simplest possible “non-perturbative” term, with a time-like singularity at $Q^2 = -\Lambda^2$ given by a simple pole !

energy expansion :

$$\begin{aligned}\bar{R}(Q/M, \alpha) &= \frac{\alpha}{1 + \alpha \beta_0 \ln \frac{M^2}{Q^2}} - \frac{Q^2}{M^2} \left(\frac{\beta_0 \alpha}{1 + \alpha \beta_0 \ln \frac{M^2}{Q^2}} \right)^2 \exp \left(-\frac{1}{\beta_0 \alpha} \right) + O(Q^4/M^4) \\ &\equiv \frac{1}{\beta_0 \ln \frac{\Lambda^2}{Q^2}} - \frac{Q^2}{\Lambda^2} \frac{1}{\beta_0 \ln^2 \frac{\Lambda^2}{Q^2}} + O(Q^4/\Lambda^4)\end{aligned}\quad (29)$$

Eq.(29) shows that in this example $\bar{R}_{PT}(Q/M, \alpha) \equiv \frac{\alpha}{1 + \alpha \beta_0 \ln \frac{M^2}{Q^2}} \equiv \frac{1}{\beta_0 \ln \frac{\Lambda^2}{Q^2}}$ is just the one-loop

coupling with Landau ghost at $Q = \Lambda$. The non-trivial UV fixed point is therefore generated by the “non-perturbative” terms in eq.(29), whereas **both** the bare coupling $\alpha(M)$ and $\bar{R}_{PT}(Q)$ have a Landau ghost. This remark suggests that in real life QED, the Landau ghost in the bare coupling, and the ensuing “triviality”, only reflects the one presumably still present in the Borel summed $R_{PT}(Q)$, while the full, “non-perturbative” $R(Q)$ may be free of Landau ghost. Furthermore, the “non-perturbative” terms in eq.(29) appear as power-suppressed terms in a large cut-off expansion, which suggests a similar interpretation in the general case (section (6)).

6 . It has been suggested by Parisi [7] that the imaginary parts associated to the UV renormalons at $z_n = n/\beta_0$ are proportionnal to zero momentum insertions of local composite operators of dimensions $d_n = 2n + 4$. This conjecture has been checked for the first UV renormalon in QED in ref.[16] and [22] (see also ref.[23]). An effective lagrangian description in term of an infinite number of these operators of arbitrary high dimensions, with exponentially small counterterms, has been also suggested in ref.[7] and checked in ref.[22] for the first UV renormalon. The present work suggests that not only the imaginary parts, but the full renormalon contributions $R_n(\alpha)$ (eq.(3)), may be represented by such an effective lagrangian.

One may further wonder about the physical origin of the “non-perturbative” terms $R_n(\alpha)$, so far added by hand. A clue is given by the work of ref.[19], which relates UV renormalons to cut-off dependant terms in the large cut-off expansion of the regularized theory :

$$R(Q/M, \alpha) = R_{PT}(Q/M, \alpha) + \sum_n \left(\frac{Q^2}{M^2} \right)^n \Delta R_n(Q/M, \alpha) \quad (30)$$

where the ΔR_n 's are power series in the bare coupling α with logarithmically cut-off dependant

coefficients, and can be described in Symanzik “local effective lagrangian” approach [24]. Moreover, according to the “improvement program” [25], it is possible to redefine the bare action by adding the above mentioned local irrelevant operators, in such a way that $\Delta R_n = 0$ order by order in perturbation theory. It is then plausible that the R_n ’s might remain as the exponentially suppressed relics of the ΔR_n ’s, after implementation of the improvement program.

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