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TOPOLOGICAL AND GEOMETRICAL PHASES DUE TO GRAVITATIONAL FIELD WITH CURVATURE AND TORSION

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Abstract

The gravitational phase factor, which may be used to determine the evolution of a wave function in the WKB approximation, is modified to incorporate the Fermi-Walker transport of the wave function when it is accelerated, for example by a wave guide. The modified phase factor is used to obtain geometrical and topological phases in the interference of two coherent beams around a cosmic string containing mass and intrinsic spin. An exact solution for the string from the Einstein-Cartan- Sciama-Kibble gravitational field equations is used to interpret the topological phases as being due to the fluxes of curvature and torsion inside the string.

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Two types of phases in quantum interference are of interest in physics. First type is a geometrical phase, by which is meant here a phase acquired by a wave function that is determined by geometric structures along a curve, such as a connection that is used to parallel transport along the curve. Second is a topological phase by which is meant a phase that is independent of the choice of curve in some region of interest, for example when the curvature of the connection or the classical forces vanish in that region.

Here I shall investigate these two type of phases due to the gravitational field of a cosmic string [1] that has mass and angular momentum. I shall show that if the cross-section of the string is infinitesimal, in the sense that its geometry is slowly varying and there are no surface energy-momentum or spin, then it must contain torsion. I shall obtain the phases for interference around the cosmic string and relate them to the curvature and torsion inside the string by means of an exact solution of the Einstein-Cartan-Sciama-Kibble (ECSK) theory [2], which is the simplest modification of general relativity to include torsion.

Both these types of phase may be illustrated by the phase shift due to the electromagnetic field in the interference of two coherent wave functions of a charged particle. In the Aharonov-Bohm (AB) effect [3], the field strength $F_{\mu\nu}$ is zero along the interfering beams. The phase factor that determines the shift in the interfering fringes is

$$\Phi_\gamma = \exp\left(-\frac{ie}{\hbar} \oint_\gamma A_\mu dx^\mu\right), \quad (1)$$

where A_μ is the electromagnetic 4-potential representing a $U(1)$ connection, and the closed curve γ passes through the two wave functions, and encloses a region in which the field strength is non zero*. Consequently, this phase shift is constant as the beams are varied in the outside region in which the field strength is zero, which makes this effect topological.

In the generalization of this effect to an arbitrary gauge field [3,4], the phase shift is

* Units in which the velocity of light $c = 1$ will be used throughout.

determined by the gauge field phase factor

$$\Phi_\gamma = P \exp\left(-\frac{ig}{\hbar} \oint_\gamma A_\mu{}^k T_k dx^\mu\right), \quad (2)$$

where T_k generate the Lie algebra of the gauge group, $A_\mu{}^k$ is the Yang-Mills gauge potential, P denotes path ordering, and γ is a closed curve through the interfering beams consisting of integral curves of p^μ . Hence, Φ_γ is an element of the gauge group. Clearly, (1) is a special case of (2) corresponding to the fact that the gauge group of electromagnetism is $U(1)$. When γ is an infinitesimal closed curve spanning a surface element represented by $d\sigma^{\mu\nu}$,

$$\Phi_\gamma = 1 - \frac{ig}{2\hbar} F_{\mu\nu}{}^k T_k d\sigma^{\mu\nu} \quad (3)$$

where the Yang-Mills field strength

$$F^k = dA^k - gC^k{}_{ij} A^i \wedge A^j. \quad (4)$$

More generally, the restriction that at each point in space-time either the field strength or the wave function should be zero may be removed. Then the phase shift is still determined by (1) or (2) with the restriction that γ should be made up of the unperturbed trajectories between successive mirrors (including the beam splitter) of the interferometer [4,5]. The relevant phase factor is still of the general form (1) or (2) and is geometrical because it is determined by the integral of the 1-form A_μ . Conversely, if this geometrical phase is given as an experimental fact then it may be inferred that electromagnetism is a gauge field. Formally, this may be shown by modifying Feynman path integral by this geometric phase which then gives Schrodinger's equation with A_μ minimally coupled.

The special case of the topological AB phase is also useful because it determines clearly which field variables properly describe of the field. This was recognized by Wu and Yang [6] who used the AB effect to argue that an *intrinsic and complete* description of the electromagnetic field is provided by (1), and not the field strength, in a multiply connected region. For non abelian gauge fields, the field strength does not determine

the gauge potential even in a simply connected region in general [6]. The AB effect and its generalization to a non abelian gauge field show that this indeterminacy has physical consequences in quantum theory.

While these results show that the field strength is insufficient to describe the field, the fact that the gauge covariant operators (2) are sufficient to determine the field uniquely is shown by the following theorem [7]: Given the holonomy transformations (2) for piecewise differentiable curves which begin and end at a given point in space-time, the gauge potential $A_\mu{}^k$ may be reconstructed, and it is then unique up to gauge transformations. But since the set of such curves form an infinite dimensional manifold L , the corresponding operators (2) have a great deal of redundancy. Indeed, the gauge field in space-time may be reconstructed from a minimal set of these operators defined on a four dimensional submanifold of L [8]. This is mathematically equivalent to working with the gauge potential defined in a particular gauge on the four dimensional space-time. Therefore, once the redundancy in the loop space L has been removed, there is no advantage to using the holonomy transformation (2) as opposed to the gauge potential in a particular gauge.

It follows that in quantizing the electromagnetic or more general gauge fields, one must quantize the gauge potential instead of the field strength. This suggests that by studying topological phases for the gravitational field, we could similarly determine what variables properly describe this field, which must then be the variables that must be quantized.

Gravitational Phase Factor

The tension between general relativity and quantum theory arises fundamentally from the fact that the former requires a space- time geometry while the latter is formulated using the Hilbert space geometry. If we probe the gravitational field using classical systems such as classical clocks, light signals, or freely falling particles which define geodesics, the appropriate geometry obtained is the curved space-time geometry of general relativity. But, as seen earlier, the gauge field structure of a gauge field becomes evident when the field is probed using quantum systems. So, one way of determining whether gravity is a

gauge field or not may be by probing it with a quantum particle.

To do so, consider the gravitational phase factor introduced by the author [9]

$$\Phi_\gamma = \text{Pexp}[-i \int_\gamma (\epsilon_\mu{}^a P_a + \frac{1}{2} \omega_\mu{}^a{}_b M^b{}_a) dx^\mu], \quad (5)$$

which is associated with an arbitrary piece-wise differentiable curve γ which may be open or closed*. Here, P_a and $M^b{}_a$, $a, b = 0, 1, 2, 3$ are the translation and Lorentz transformation generators which generate the covering group of the Poincare group that acts on the Hilbert space. Then $e_\mu{}^a$ and $\omega_\mu{}^a{}_b$ have the interpretation of the gauge potentials of a Poincare gauge field. Indeed, (5) may be used to regard gravity as a Poincare gauge field in the sense of Yang's integral formulation of gauge field[10]. But to do so, it is necessary for us to see how the phase acquired by a wave function in a gravitational field can be obtained from (5).

To give physical meaning to (5), consider the invariant, irreducible subspaces of the Hilbert space under the action of the Poincare group. As noted by Bargmann and Wigner [11], each such subspace may be regarded as the Hilbert space of states of a "particle". The two Casimir operators of the Poincare group, which have constant values for each irreducible invariant subspace, determine the mass and spin of the corresponding particle. The actions of P_a and $M^b{}_a$ on this subspace provide the representations of energy-momentum and total angular momentum observables, respectively, for this particle.

Let us now see the physical meaning of $e_\mu{}^a$ in terms of the phase. Now $e_\mu{}^a$ depends on the observer: A Lorentz transformation of the observer results in $e_\mu{}^a$ transforming as a contravariant vector in the index a while P_a transforms as a covariant vector. Suppose that a particle is in a state $|\psi\rangle$ that is approximately an eigenstate of P_a with eigenvalues p_a .

* In (5), $e_\mu{}^a$ and $M^b{}_a$ are dimensionless, while P_a and $\omega_\mu{}^a{}_b$ have dimensions of inverse length. Alternatively, one may multiply P_a and $M^b{}_a$ by $\hbar$ which would make them have dimensions of momentum and angular momentum respectively, and divide the exponent in (5) by $\hbar$ as in (1) or (2).

For simplicity, “gauge transform” away $\omega_\mu{}^a{}_b$ along a curve γ . Then the phase difference along γ is obtained from $\Phi_\gamma|\psi\rangle = \exp(i\phi_\gamma)|\psi\rangle$, where $\phi_\gamma = \int_\gamma e_\mu{}^a p_a dx^\mu$. This phase may be observed for an open curve γ by physical experiments such as the Josephson effect or by an atomic clock [12]. This implies that the “wave vector” $p_\mu \equiv e_\mu{}^a p_a$ is observable. Requiring that the correspondence between p_μ and p_a is $(1-1)$ implies that $e_\mu{}^a$ is a non singular matrix. Therefore, it has an inverse $e^\mu{}_b$:

$$e_\mu{}^a e^\mu{}_b = \delta^a{}_b. \quad (6)$$

Hence, $p_a = e^\mu{}_a p_\mu$. As mentioned, the Casimir operator $\eta^{ab} P_a P_b$ of the Poincare group has a definite value, say m^2 , for the given particle. Therefore,

$$\eta^{ab} p_a p_b = g^{\mu\nu} p_\mu p_\nu = m^2, \quad (7)$$

where $g^{\mu\nu} \equiv \eta^{ab} e^\mu{}_a e^\nu{}_b$ is a non singular matrix. Its inverse $g_{\mu\nu}$ defines a pseudo-Riemannian metric of Lorentzian signature on space-time. On using (6),

$$g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b. \quad (8)$$

It is important to note here that it was possible to obtain a metric on space-time from the gravitational phase only because this phase is observable for an open curve. For other gauge fields usually there is a Casimir operator for the gauge group. But this does not give a metric on space-time because the phase is observable only for a closed curve. This is implied by the fact that the gauge potential $A_\mu{}^i$ can be gauge transformed away at a given space-time point. It follows that, $A_\mu{}^i$ cannot satisfy a relationship of the form (8) with η_{ab} replaced by the quadratic form of the Casimir operator of the gauge group. The above argument deriving the space-time metric shows that the gravitational phase factor (5) provides a link between the Hilbert space geometry and the space-time geometry. This derivation was only approximate because of the uncertainty principle: it could be supposed only that $|\psi\rangle$ was approximately an eigenstate of P_a in order to localize the state around

the curve γ . This could be a good approximation when the size of the wave packet is large compared to the Compton wave length, but small compared to the radius of curvature. But in the case of a gauge field the values of the wave function along γ may be regarded as precise eigenstates of the generators of the gauge group, which is a fundamental difference between gravity and gauge fields.

The field strength or curvature of this Poincare gauge field is obtained by evaluating (5) for an infinitesimal closed curve γ :

$$\Phi_\gamma = 1 - \frac{i}{2}(Q_{\mu\nu}{}^a P_a + \frac{1}{2}R_{\mu\nu}{}^a{}_b M^b{}_a)d\sigma^{\mu\nu}, \quad (9)$$

using the Poincare Lie algebra, where

$$Q^a = de^a + \omega^a{}_b \wedge e^b \quad (10)$$

is the torsion and

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad (11)$$

is the linear curvature. Eqs. (9), (10) and (11) are the analog of (3) and (4) for gravity.

The above general concepts will now be illustrated for a spin 1/2 particle in a general curved space-time. Its wave function is a Dirac spinor field, ψ , on space-time. Operationally, $\psi(x)$ is the value of this wave function as measured by the observer with frame $\{e^\mu{}_a\}$ at the space-time point x . This connects the space-time wave function $\psi(x)$ with the Hilbert space state vector $|\psi\rangle$ referred to above. In the WKB approximation, we may write $\psi = \alpha e^{i\phi}$, where ϕ is real, with $p_\mu \equiv -\hbar\partial_\mu\phi$ and α satisfying [4]

$$p^\mu p_\mu = m^2, \quad (12)$$

$$p^\mu \nabla_\mu \alpha = -\frac{1}{2}(\nabla_\mu p^\mu)\alpha. \quad (13)$$

Eq. (12) implies that the integral curves of p^μ are geodesics. According to (13), α gets parallelly transported along these integral curves except that its magnitude gets thinned out as these curves diverge or thickens as they converge, as expected intuitively.

If these geodesics are parallel at the beam splitter, they will not in general be parallel as ψ propagates in a general curved space-time. However, then $\nabla_\mu p^\mu \sim \frac{1}{R}p$ where R is the radius of curvature. Suppose that the wave function is contained in a region whose linear dimensions are of the order of $L \ll R$. Then $\nabla_\mu \alpha \sim \frac{1}{L}\alpha$ and therefore the right hand side of (13) is negligible. Since p^μ and α are both being parallel transported in the direction of p^μ , with $\nabla_\mu p^\mu = 0$, it follows that the wave function as a whole is being parallel transported along the integral curves of p^μ . The evolution of the wave function obtained by integrating (12) and (13), may then also be obtained by acting on the initial state by the operator (5), with γ being one of the integral curves of p^μ . Then $M^b_a = S^b_a + L^b_a$, where S is the spin angular momentum and L is the orbital angular momentum. To parallel transport α , S is sufficient, but to parallel transport p , L is needed also.

Consider a wave packet that can be expanded as a superposition of locally plane waves each of which obey the WKB approximation. I have shown that the evolution of each of these locally plane waves may be obtained by acting on the initial state by (5). Therefore, by linearity, the evolution of the wave packet may be obtained by the action of (5).

An advantage of the present approach that uses the operator (5) to study evolution of matter in a gravitational field is that it generalizes easily to the second quantization of the matter field. Then the operators P_a and M^b_a in (5) are formed from the quantized matter field using Noether's theorem. An alternative approximation is to treat the matter field as first quantized fields but the gravitational field as quantized. Then the phase difference between two points in the wave function becomes uncertain owing to the quantum fluctuation of e_μ^a and $\omega_\mu^a_b$. This leads to *decoherence* of the wave function. If gauge fields are quantized, then (1) and (2) imply uncertainty of the phase around a closed curve. In contrast, as mentioned earlier, the gravitational phase is meaningful along an open curve and therefore gravitational fluctuation causes decoherence between distinct space-time points in a manner fundamentally different from quantized gauge fields.

Effect of Mirrors - Quantum Fermi-Walker Transport

In the absence of gravity, (5) is independent of the path γ connecting two fixed space-time points. But the phase difference along a beam going between the two points depends on the path γ traveled. The resolution of this apparant conflict is that the operator (5) determines the evolution of the wave packet between two mirrors. But each mirror gives a Lorentz boost [13] to the wave function which does not, in general, commute with (5). If we work to first order in perturbation then the effect of Thomas precession may be separated from the effect of gravity. In the particular case of neutron interferometry [14], the two beams which interfere at each detector has undergone the same net Lorentz boosts and therefore the contributions due to Thomas precession cancel to first order.

Suppose that the wave function experiences continuous acceleration but does not experience any torque. E. g. the beam goes through a curved tube, or wave guide, or suffers a dense sequence of infinitesimal Lorentz boosts, or acted upon by some external (non gravitational) force such as due to an electric field on a charged wave function. Then the expectation values of all the observables with respect to the wave function are Fermi-Walker transported (see [15] for a clear exposition of classical Fermi-Walker transport). I shall incorporate this into the propagation of the wave function by introducing the notion of quantum Fermi-Walker transport. Modify (5) to

$$\Phi'_\gamma = P \exp[-i \int_\gamma (\epsilon_\mu{}^a P_a + \frac{1}{2} \Omega_\mu{}^a{}_b M^b{}_a) dx^\mu], \quad (14)$$

where $\Omega_\mu{}^a{}_b = \omega_\mu{}^a{}_b + \tau_\mu{}^a{}_b$, with

$$\tau_\mu{}^a{}_b = a^\nu (\epsilon_\nu{}^a \epsilon_{\mu b} - \epsilon_\mu{}^a \epsilon_{\nu b}). \quad (15)$$

The evolution of a wave packet through the wave guide is given by the action of (14). While (5) gives rise to quantum parallel transport, (14) gives rise to quantum Fermi-Walker transport. Here, γ is an integral curve of p^μ along the center of the wave packet, the “acceleration” $a^\nu = m^{-2} p^\mu \nabla_\mu p^\nu$ is evaluated on γ , and the latin index is lowered using η_{ab} .

The need for this modification is ultimately due to the following fundamental difference between gravity and gauge fields: $e^a{}_\mu$ plays the role of a “solder form” that identifies tangent space vectors with internal vectors, in addition to its role as a translational gauge field [16,17].

Geometry of Cosmic String

To obtain a topological phase due to (14), consider a cosmic string whose metric exterior to the string is given in cylindrical coordinates as [18,19].

$$ds^2 = (dt + \beta d\phi)^2 - d\rho^2 - \alpha^2 \rho^2 d\phi^2 - dz^2, \quad (16)$$

where α and β are constants which will be determined here by matching this solution to a suitable interior solution. Then the metric $g_{\mu\nu}$ satisfies (8) for the following orthonormal co-frame field $\{e^a\}$ adapted to the above coordinate system:

$$e^0 = dt + \beta d\phi, e^1 = d\rho, e^2 = \alpha \rho d\phi, e^3 = dz. \quad (17)$$

The connection coefficients in this basis are $\omega_\mu{}^a{}_b \equiv e_\nu{}^a \nabla_\mu e^\nu{}_b = 0$, for all a, b, μ except for

$$\omega_\phi{}^1{}_2 = -\omega_\phi{}^2{}_1 = -\alpha. \quad (18)$$

It follows, on using (10) and (11), that $Q^a = 0, R^a{}_b = 0$ outside the string.

If γ is a closed curve around the cosmic string then from (5),

$$\Phi_\gamma = \exp\left(-i \oint_\gamma e_\mu{}^0 P_0 dx^\mu\right) P \exp\left[-i \oint_\gamma \left(\sum_{k=1}^3 e_\mu{}^k P_k + \omega_\mu{}^1{}_2 M^2{}_1\right) dx^\mu\right]. \quad (19)$$

Hence the gravitational phase factor is a product of time translation and a spatial Euclidean transformation. This factorization is possible because $e_\mu{}^0 P_0$ commutes with the other terms that occur in (19). Hence, if (9) is valid, then the time translation must correspond to torsion being non zero inside the string. Suppose that the surface of the string is given by $\rho = \rho_0$, where ρ_0 is a small constant. Then, from (9) and (19), the flux of torsion through a cross-section Σ of the string is

$$\int_\Sigma Q^0 = \oint_\gamma e^0 = 2\pi\beta. \quad (20)$$

Using the Gauss-Bonnet theorem, the flux of curvature is

$$\int_{\Sigma} R_{\rho\phi}{}^1{}_2 d\rho \wedge d\phi = \oint_{\gamma} \omega_{\phi}{}^1{}_2 d\phi + 2\pi = 2\pi(1 - \alpha). \quad (21)$$

In (21), 2π that is added to the line integral undoes the purely coordinate effect due to the rotation of e^2 by 2π as one goes around γ .

Eqs. (20) and (21) are independent of the particular geometry inside the string so long as Σ is “infinitesimal” so that (9) is valid. In the present situation, this means the following: a) the geometry inside the string is slowly varying in the sense that inside it

$$\rho_0^{n+1} \nabla^n Q^a \ll 1, \rho_0^{n+2} \nabla^n R^a{}_b \ll 1, n = 0, 1, 2, 3, \dots \quad (22)$$

b) There is no surface energy-momentum or spin which is ensured by the requirement that the geometry satisfies appropriate junction conditions [20] at the surface of the string. It is possible, however, for (a) or (b) to be violated by making the geometry as we go inside the string vary sufficiently rapidly. Then it would be possible to obtain a solution inside the string with torsion being zero. For example, the string may be a rotating hollow cylinder with β depending on the orbital angular momentum per unit length of this cylinder and there need be no torsion.

But if the string does contain torsion, then according to the ECSK theory [2] there must be intrinsic spin inside the string. The field equations of this theory of gravitation may be written as [16]

$$\frac{1}{2} \eta_{ijkl} e^l \wedge R^{jk} = -8\pi G t_i, \eta_{ijkl} e^l \wedge Q^k = 8\pi G s_{ij}, \quad (23)$$

where t_i and s_{ij} are 3-form fields representing the energy-momentum and spin densities. I shall present now an exact solution of these equations corresponding to a fluid at rest in the interior of the string with constant energy and spin densities, with the spin polarized along the length of the string. The details of the derivation were given elsewhere* [21].

* Note added: I was recently informed that this solution, which I had obtained independently, is contained in H. H. Soleng, Gen. Rel. Grav. 24 (1992) 111.

The only non zero componets of t_i and s_{ij} are, with respect to an orthonormal co-frame field $\{e^a\}$ are

$$t_0 = t_3 = \epsilon e^1 \wedge e^2 \wedge e^3, s_{12} = s e^1 \wedge e^2 \wedge e^3, \quad (24)$$

where the energy-density ϵ and the spin density s are constants. Then, on defining $\rho^* = (8\pi G\epsilon)^{-1/2}$, (23) are satisfied in the interior of the string by

$$e^0 = dt + 8\pi G s \rho^{*2} \left[1 - \cos\left(\frac{\rho}{\rho^*}\right) \right] d\phi, e^1 = d\rho, e^2 = \rho^* \sin\left(\frac{\rho}{\rho^*}\right) d\phi, e^3 = dz, \quad (25)$$

and

$$\omega^1{}_2 = -\cos\left(\frac{\rho}{\rho^*}\right) d\phi = \omega^2{}_1, \quad (26)$$

with all other $\omega^a{}_b$ being zero. The only non zero components of torsion and curvature are

$$Q^0 = 8\pi G s \rho^{*2} \sin\left(\frac{\rho}{\rho^*}\right) d\rho \wedge d\phi, R^1{}_2 = \frac{1}{\rho^*} \sin\left(\frac{\rho}{\rho^*}\right) d\rho \wedge d\phi. \quad (27)$$

This interior solution and the exterior solution (16) satisfy the junction conditions [20] for the ECSK theory at the boundary provided the mass per unit length

$$\mu \equiv \int_{\Sigma} \epsilon e^1 \wedge e^2 = \frac{1}{4G} (1 - \alpha) \quad (28)$$

and the angular momentum per unit length

$$J \equiv \int_{\Sigma} s e^1 \wedge e^2 = \frac{1}{4G} \beta, \quad (29)$$

where Σ is a surface at constant t and z . This determines α and β in the exterior solution (16).

The above solution differs significantly from the torsion strings of Prasanna and Tsoubelis [22]. In the special case when the spin is zero the above solution reduces to the one found by Gott [23] and Hiscock [24]. After the completion of the present work, I became aware that conical singularities with torsion, as opposed to finite torsion in an extended region studied in the present letter, have been treated in a purely classical context by Tod [25]. The present approach has the advantages that not only that it gives

physical meaning to the affine holonomy used in the latter paper and refs. [9,17] through (5), it shows that the string must necessarily contain torsion under the ‘infinitesimality’ conditions (a) and (b) stated above.

Phase Shifts due to the String

For simplicity, consider a circular interferometer with constant radius $\rho = r$ in a plane normal to the string, enclosing the string with its center on the axis of the string. It may be a superconducting interferometer, e. g. a superconducting ring interrupted by a Josephson junction. Or it may be an electron interferometer, or a wave guide, such as an optical fiber, at one point of which is the beam splitter that splits a beam into two which travel in opposite senses and interfere at a mirror that is at another point in the interferometer. The interferometer does not rotate relative to the distant stars, which may be ensured by requiring that telescopes rigidly attached to this interferometer are focused on the distant stars.

The phase shift may be obtained by letting (14), with γ along integral curves of p^μ which lie on a 2 dimensional submanifold σ with constant z and $\rho = r$, act on the state vector $|\psi\rangle$. Now, $p^\mu \Omega_\mu{}^a{}_b L^b{}_a$ when acting on $|\psi\rangle$, that is being quantum Fermi-Walker transported, determines the rate at which the spatial part of its wave function is precessing relative to the chosen basis. But in the present situation, assuming that $|\psi\rangle$ satisfies the WKB approximation, its approximate eigenvalues of energy-momentum p^a in the chosen basis do not change, as the wave is being guided around the interferometer. Therefore the spatial part of $|\psi\rangle$ also should not change, in this basis, apart from acquiring a phase factor. Hence, $p^\mu \Omega_\mu{}^a{}_b L^b{}_a |\psi\rangle = 0$. This may be formally proved using the fact that the Fermi-derivative of p^a is zero, because it is Fermi- Walker transported, and its components in the above basis e_a do not change. Therefore, using (14),

$$\Phi'_\gamma |\psi\rangle = \exp(-i \int_\gamma e_{\dot{\phi}}{}^0 P_0 d\phi) \exp(-i \int_\gamma e_{\dot{\phi}}{}^2 P_2 d\phi) \exp[-i \int_\gamma (\omega_{\dot{\phi}}{}^1{}_2 + \tau_{\dot{\phi}}{}^1{}_2) S^2{}_1 d\phi] |\psi\rangle. \quad (30)$$

Also, outside the string the curvature and torsion vanish, and $d\tau^a{}_b$ (treating a^a in the

1-form field (15) to be constants) have no $dt \wedge d\phi$ terms. Hence, the closed curve in σ around which the phase shift is being computed may be deformed within σ into a circle around σ with constant t , without altering the phase shift.

Suppose ω is the frequency of the wave function which is assumed to be constant in time at the beam splitter. Then it is constant everywhere. Therefore, in this WKB approximation, the magnitude of the momentum $p = (E^2 - m^2)^{1/2}$ is also a constant along the beam, where $E = \hbar\omega$ is the energy of the wave function. The spin is assumed to be polarized in the z -direction, i. e. $|\psi\rangle$ is an eigenvector of S^2_1 with eigenvalue $S/\hbar$. The three factors in (30) commute with one another and their actions on $|\psi\rangle$ give rise to the following phase shifts: (i) The general relativistic Sagnac phase shift [26] is obtained from the first factor, on using (20) and (29), to be

$$\Delta\phi_E = - \oint_{\gamma} \frac{E}{\hbar} \epsilon_{\dot{\phi}}^0 d\phi = - \int_{\Sigma} \frac{E}{\hbar} Q_{\dot{\rho}\dot{\phi}}^0 d\rho \wedge d\phi = - \frac{8\pi G}{\hbar} E J, \quad (31)$$

where Σ is a 2-surface spanned by γ . If the interferometer is a parallelogram, then this topological phase becomes a special case of the phase due to torsion found in the appendix of [4] due to non closure of infinitesimal parallelogram in the presence of torsion, with the affine holonomy being now a time translation [27]. This is also a gravitational analog of the AB phase (see also [19,27]). (ii) The phase shift due to the coupling of spin to curvature [4] which is obtained by the third factor in (30) is

$$\Delta\phi_S = - \frac{S}{\hbar} \left\{ \oint_{\gamma} \omega_{\dot{\phi}}^1{}_2 d\phi + 2\pi \right\} = - \frac{S}{\hbar} \int_{\Sigma} R_{\dot{\rho}\dot{\phi}}^1{}_2 d\rho \wedge d\phi = - \frac{8\pi G}{\hbar} S \mu, \quad (32)$$

using (21) and (28). The phase shifts (31) and (32) may be evaluated using the line integral outside the string using (17) and (18), or the surface integral inside the string using (27). In (32), the 2π added to the line integral, to represent the singularity of $e^\mu{}_a$ along the z -axis, is significant only for fermions for which it gives an additional phase that is an odd multiple of π to compensate a purely coordinate effect. (iii) The phase shift due to the Thomas precession, also from the third exponential in (30), is

$$\Delta\phi_{\tau} = - \frac{S}{\hbar} \oint_{\gamma} \tau_{\dot{\phi}}^1{}_2 d\phi = \frac{S}{\hbar} \frac{\pi(1 - 4G\mu)p^2}{m^2}. \quad (33)$$

(iv) The phase shift due to the path difference between the two interfering beams from the second factor of (30) is

$$\Delta\phi_p = \frac{(1 - 4G\mu)rp\theta}{\hbar}, \quad (34)$$

where θ is the difference between the angular distances of the beams from the beam splitter and the point of interference.

The phase shifts (31) and (32) are expressed as flux integrals of torsion and curvature because they could have also been obtained from (5) which depends only on the affine connection. (The torsion flux contained in (31) is the same as (20).) It follows that these phase shifts are independent of the shape of the interferometer enclosing the string and therefore may be called topological. These results could therefore be applied to gravitational lensing of light produced by cosmic strings [1,23] if they do exist. But (33) and (34) depend on the shape of the interferometer because they contain metrical information which was incorporated into the modified gravitational phase factor (14). However, (34) may be regarded as a geometrical phase.

The topological phase shift (32) is obviously independent of J and exists in particular when $J = 0$. It is the gravitational analog of the phase shift experienced by a magnetic moment due to the electric field [28] when it interferes around a line charge which was discussed by Aharonov and Casher [29]. To show the duality [29,21] between (31) and (32), divide the string in (i) into segments of length $\delta\ell$ with spin $S = J\delta\ell$. Then replace each segment with the interfering particle with energy E and the particle by any segment, so as to obtain the situation corresponding to (ii). The resulting string has mass per unit length $\mu = \frac{E}{\delta\ell}$. Hence, the new phase shift given by (32) is equal to the original phase shift (31).

In conclusion, although gravitational phases have been obtained as geometrical phases previously [4], the topological phases obtained in the present letter suggest that what gives a complete and intrinsic description of the gravitational field are the variables e^a and $\omega^a{}_b$, analogous to the lesson we learned for the electromagnetic and other gauge fields. This

suggests that the gravitational field should be quantized by quantizing these variables [9,17,30], and it would seem necessary to include torsion.

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