Endomorphism Semigroups and Lightlike Translations

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Abstract

Certain criteria are demonstrated for a spatial derivation of a von Neumann algebra to generate a one-parameter semigroup of endomorphisms of that algebra. These are then used to establish a converse to recent results of Borchers and of Wiesbrock on certain one-parameter semigroups of endomorphisms of von Neumann algebras (specifically, Type III$_1$ factors) that appear as lightlike translations in the theory of algebras of local observables.

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I Introduction

The standard situation for a pair of complementary spacetime regions in the theory of algebras of local observables, under the assumption of duality in the vacuum sector, is just that termed standard in the theory of von Neumann algebras: we have a von Neumann algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ acting on a Hilbert space $\mathcal{H}$, with a common cyclic and separating unit vector $\Omega$, the vacuum vector. In the particular situation in which $\mathcal{M}$ and $\mathcal{M}'$ correspond to the observables for a pair of complementary wedge regions (for definiteness let us take them to be $W_R = \{x \mid x_1 > |t|\}$ and $W_L = \{x \mid x_1 < -|t|\}$ respectively) it is expected [1] that the modular automorphism group $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$ will correspond to the Lorentz velocity transformations $V_t(2\pi t)$ in the direction orthogonal to the common face $x_1 = t = 0$ of the two wedges, and that the modular conjugation $J$ will be a slight variant of the TCP operator $\Theta$ (so as to give a reflection about that face). In that case the lightlike translations $U(a) = T(a(x_0 + x_1))$ will be a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$, which should have the following four properties:

(a) By Lorentz covariance, $\Delta^{it} U(a) \Delta^{-it} = U(e^{-2\pi t} a)$ and $J U(a) J = U(-a)$;
(b) By the spectral condition, $U(a)$ should have a positive generator $H$;
(c) By isotony, for $a \geq 0$ the corresponding adjoint action $A \to U(a) A U(-a)$ should give a one-parameter semigroup of endomorphisms of $\mathcal{M}$ (and thus for $a \leq 0$ likewise of $\mathcal{M}'$); and, finally,
(d) The vacuum vector $\Omega$ should be fixed by all $U(a)$, and thus annihilated by $H$.

In this connection Borchers has shown [2] that these four conditions are not all independent: in particular, if the last three hold, then the Lorentz covariance conditions follow automatically. Wiesbrock then proved [9] conversely that if (a), (c), and (d) hold, $U(a)$ automatically has a positive generator. In this note we demonstrate that the results of Borchers and of Wiesbrock are part of a larger chain of converses, and in the process perhaps shed some further light on these remarkable theorems. Specifically, we show that if the generator $H$ gives a derivation $\delta$ of $\mathcal{M}$ satisfying certain additional conditions, then (a), (b), and (d) imply (c), and in fact any three of the conditions listed above for $U(a)$ together imply the fourth. Note that in the local algebra context, it can be shown [6] that $\mathcal{M}$ and $\mathcal{M}'$ must be Type III$_1$ factors, but this will not be used in the following; the results will simply be stated in terms of arbitrary von Neumann algebras.
The situation here is analogous to, but in some respects altogether different from, the case of spatial derivations that generate automorphism groups of von Neumann algebras, which has been extensively studied ([5], Section 3.2.5, and references therein). We will develop the analogy more specifically after stating Theorem 1, but the obvious relevant condition is that $U(a)$ should commute with $J$ and with all $\Delta^a$; then the key question is to determine precisely what additional conditions on the derivation $\delta$ suffice to show that it generates an automorphism group. The best result in this direction is that of [3], in which the only additional assumption is that the derivation has a domain $D(\delta)$ such that $D(\delta)\Omega$ is a core for $H$. However, the proof of this result is rather difficult, and does not generalize to the endomorphism case. We will make do with more restrictive conditions here, but it would be very interesting to determine precisely what conditions suffice to guarantee that $\delta$ generates an endomorphism semigroup. Note that the endomorphism semigroups studied here are non-pathological counterexamples to the conjecture of [4] (for which many counterexamples are known [3]).

II Endomorphism Semigroups

If we have a von Neumann algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ acting on a Hilbert space $\mathcal{H}$, with a common cyclic and separating vector $\Omega$, we may define real linear spaces $R = \overline{\mathcal{M}^{sa}}$ and $R' = \overline{\mathcal{M}'^{sa}}$. Then $\langle \psi \mid \phi \rangle$ is real for all $\psi \in R, \phi \in R'$, and furthermore $R'$ is precisely the set of all $\psi$ such that $\langle \psi \mid \phi \rangle$ is real for all $\phi \in R$. Also, $D(\Delta^{1/2}) = R + iR$ and $D(\Delta^{-1/2}) = R' + iR'$ are dense in $\mathcal{H}$, $R = \{ \psi \mid \psi \in D(\Delta^{1/2}), J\Delta^{1/2}\psi = \psi \}$, and $R' = \{ \psi \mid \psi \in D(\Delta^{-1/2}), J\Delta^{-1/2}\psi = \psi \}$.

For any $\psi \in R$, there is a sequence $X_n \in \mathcal{M}^{sa}$ such that $X_n\Omega \to \psi$, but there need not be a bounded operator $X \in \mathcal{M}^{sa}$ such that $X\Omega = \psi$; in general there is only a closed symmetric operator $\tilde{X}$ affiliated with $\mathcal{M}$ such that $\tilde{X}\Omega = \psi$, to which the $X_n$ converge on the common core $\mathcal{M}'\Omega$, so that $\tilde{X}Y\Omega = Y\psi$ for every $Y \in \mathcal{M}'$. If $\tilde{X}$ is self-adjoint, then the $X_n$ will converge to $\tilde{X}$ in the strong resolvent sense ([7], Theorem VIII.25).

If we are to have $U(a)\mathcal{M}U(-a) \subset \mathcal{M}$ for all $a \geq 0$, then the generator $H$ of the unitary group $U(a)$ must give a derivation $\delta$ of $\mathcal{M}$ by $\delta(X) = i[\Delta, X]$; however, this derivation will be unbounded, hence defined only on a dense set, and the problem is to give sufficient conditions for $\delta$ to generate a semigroup of endomorphisms of $\mathcal{M}$. Let

$$\mathcal{M}_\epsilon = \{ X \mid U(a)XU(-a) \in \mathcal{M} \text{ for all } 0 \leq a \leq \epsilon \},$$

(1)
and let $R_{\epsilon} = \overline{M_{\epsilon}} \cap \Omega$; then $M_\epsilon \supset M_{\epsilon'}$ and $R_{\epsilon} \supset R_{\epsilon'}$ whenever $\epsilon' \geq \epsilon$. In addition, let

$$M_+ = \bigcup_{\epsilon > 0} M_{\epsilon} \quad \text{and} \quad R_+ = \bigcup_{\epsilon > 0} R_{\epsilon}. \tag{2}$$

Then $M_\epsilon$ contains those elements $X$ of $M$ for which the differential equation $X(t)' = \delta(X(t))$, $X(0) = X$ in the Banach space $M$ has a solution curve of length at least $\epsilon$; likewise, $M_+$ contains those for which there is a solution curve of any positive length.

Conditions on $M_\epsilon$ and $M_+$ can thus be regarded as local existence conditions for this differential equation, and it is criteria of this sort that we will use to control the behavior of the derivation $\delta$.

**Theorem 1:** Suppose that $U(a)\Omega = \Omega$ and $U(a)R \subset R$ for all $a \geq 0$, and that for some $\epsilon > 0$, $\Omega$ is cyclic for $M_\epsilon$, i.e., $R_{\epsilon} + iR_{\epsilon}$ is dense in $H$. Then $U(a)M U(-a) \subset M$ for all $a \geq 0$.

**Proof of Theorem 1:** It will suffice to show that $U(a)M'U(-a) \supset M'$ for all $a \geq 0$; we have from our assumptions that $U(a)R' \supset R'$ for all $a \geq 0$. Let us pick a such that $0 \leq a \leq \epsilon$, so that $U(a)M'U(-a) \subset M'_\epsilon$. Let $X$ be a self-adjoint element of $M'$; then $X\Omega \in R' \subset U(a)R'$, so that there is a sequence $Y_n$ of self-adjoint elements of $M'$ such that $U(a)Y_n \Omega \rightarrow X\Omega$. Now, $X$ and every $X_n = U(a)Y_n U(-a)$ are all in $M'_\epsilon$, and $X_n \Omega \rightarrow X\Omega$, so as above the $X_n$ tend to $X$ on the common core $M_\epsilon \Omega$. But the $X_n$ and $X$ are all self-adjoint, so the $X_n$ tend to $X$ in the strong resolvent sense. Since each $X_n \in U(a)M'U(-a)$, $X$ is affiliated with $U(a)M'U(-a)$, hence $X \in U(a)M'U(-a)$ and $U(a)M'U(-a) \supset M'$. This is so for all $0 \leq a \leq \epsilon$, hence by the semigroup property for all $a \geq 0$.

**Remarks:** The analogy between the automorphism and endomorphism cases is now evident: in the automorphism case, the relevant condition is that $U(a)R = R$ for all $a \in R$; this is equivalent to the commutation of $U(a)$ with $J$ and with all $\Delta^k$. The desired conclusion would then be that $U(a)M U(-a) = M$ for all $a \in R$. Although the situation here is in some respects similar, there are a number of significant differences. For example, if $H$ were positive in the automorphism case, then by the Borchers-Arveson theorem it would be affiliated with $M$, but since it annihilates the separating vector $\Omega$, it would have to vanish. By contrast, in the endomorphism case it is possible for $H$ to be positive without being affiliated with $M$. This will occur in the special case of Theorem 2, in which we are primarily interested, and about which we can say somewhat more.
Theorem 2: Suppose that $U(a)\Omega = \Omega$ and $U(a)R \subset R$ for all $a \geq 0$, that
$
\Delta^{it} U(a) \Delta^{-it} = U(e^{-2\pi it,a}), \text{ and that } \Omega \text{ is cyclic for } M_+, \text{ i.e., } R_+ + iR_+ \text{ is dense in } \mathcal{H}. \text{ Then } U(a)M U(-a) \subset M \text{ for all } a \geq 0.$

Proof of Theorem 2: Notice that $\Delta^{it} R_+ = R_+e^{-2\pi it}$, so that for any $\epsilon > 0$, $R_+ = \cup_{t\geq 0} \{\Delta^{it} R_+\}$. By assumption, for any $\psi \in \mathcal{H}$, there is some $\epsilon > 0$ and some $\phi \in R_+ + iR_+$ such that $\langle \psi| \phi \rangle \neq 0$. Thus given any particular $\epsilon > 0$, there is some $\phi \in R_+ + iR_+$ and some $t \geq 0$ such that $\langle \psi| \Delta^{it} \phi \rangle \neq 0$. But $\phi \in R + iR = D(\Delta^{1/2})$, so that $\phi$ is an analytic vector for $\Delta^{it}$ in the strip $-1/2 \leq Im t \leq 0$. Thus $\langle \psi| \Delta^{it} \phi \rangle$ is the boundary value of a function analytic in $t$ on that strip, and cannot vanish for all $t \leq 0$. It follows that $R_+ + iR_+ = \cup_{t\leq 0} \{\Delta^{it}(R_+ + iR_+)\}$ is dense in $\mathcal{H}$ already, and Theorem 1 applies.

The condition that $R_+ + iR_+$ be dense will be referred to as the local existence condition of Theorem 2; the condition of Theorem 1 is a uniform version of it. In specific cases, for example those involving perturbations of known endomorphism semigroups, we might expect to establish local existence conditions of these sorts by means of fixed point theorems and other standard methods for differential equations.

At this point, it seems worthwhile to present the motivating example for this discussion, in the simple form of a massive scalar free field in 1+1 spacetime dimensions. Let $h = \mathcal{L}^2(R)$ be the one-particle space, and let $\mathcal{H} = \text{exp}(h)$ be a symmetric Fock space constructed over it, whose $n$-particle subspace $\mathcal{H}^n$ is the $n$-fold symmetric tensor product of $h$ with itself. The vectors of $\mathcal{H}$ we will index by the exponential map for vectors
$$
\exp(f) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^n \quad \text{for } f \in h;
$$
then for any $f \in h$ we can define the unitary Weyl operator $w(f)$ by
$$
w(f) \exp(g) = e^{-\frac{1}{2} h[f,(f)g]} \exp(f + g).
$$
If $u$ is a unitary operator on $h$, then its multiplicative promotion $U$ given by $U \exp(f) = \exp(uf)$ will be a unitary operator on $\mathcal{H}$; if $a$ is a self-adjoint operator on $h$, then its additive promotion $A$, the generator of the multiplicative promotion of $u(t) = e^{it\alpha}$, will be a self-adjoint operator on $\mathcal{H}$. The additive promotion of the identity is an operator $N$, the number operator, which has the eigenvalue $n$ on $\mathcal{H}^n$. Then for any $f \in h$, $D_S = \cap_{n=1}^{\infty} D(N^n)$ will be a core for the generator $\phi(f)$ of $w(tf)$, such that $\phi(f)D_S \subset D_S$. 

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We will present \( h \) in terms of functions \( f(\kappa) \) of the real variable \( \kappa \), or alternatively in terms of their Fourier transforms \( \hat{f}(\nu) \). Then we will let \( \mathcal{M} \) be the von Neumann algebra generated by \( w(f) \) for all \( f \) in the real linear space

\[
    r = \left\{ g(\kappa) + e^{-\pi\kappa} g^\ast(-\kappa) \mid g \in D(e^{\pi\kappa}) \right\}.
\]

It can be shown [8] that \( \mathcal{M}' \) is the von Neumann algebra generated by \( w(f) \) for all \( f \in r' = \{ g(\kappa) + e^{\pi\kappa} g^\ast(-\kappa) \mid g \in D(e^{\pi\kappa}) \} \), that \( J \) is the multiplicative promotion of \( j \) where \( (j g)(\kappa) = f^\ast(-\kappa) \), and that \( \Delta^\# \) is the multiplicative promotion of \( e^{2\pi i \kappa} \), so that \( \Delta^\# f(\nu) = f(\nu - 2\pi t) \). Then the unbounded operators \( \phi(f) \) for \( f \in r \) will be self-adjoint and affiliated with \( \mathcal{M} \), and in fact will generate \( \mathcal{M} \).

Clearly \( h_{\lambda, \rho} = \lambda e^\rho + \rho e^{-\rho} \) is an unbounded self-adjoint operator on \( h \) for each \( (\lambda, \rho) \in \mathbb{R}^2 \setminus (0, 0) \); we may then define the self-adjoint operator \( H_{\lambda, \rho} \) as the additive promotion of \( h_{\lambda, \rho} \), or alternatively by \( i[H_{\lambda, \rho}, \phi(f)] = \phi(i h_{\lambda, \rho} f) \). Then let

\[
    r_1 = \left\{ f(\kappa) = (\hat{g}(\sinh \nu) + i \cosh \nu \hat{h}(\sinh \nu))^{-1} \mid g, h \text{ real and supported in } [1, \infty) \right\}.
\]

It can be shown that for every \( (\lambda, \rho) \in \mathbb{R}^2 \), there is some \( \epsilon \) such that for every \( f \in r_1 \), \( \phi(f) \) is affiliated with \( \mathcal{M}_\epsilon \) with respect to \( H_{\lambda, \rho} \). Furthermore \( r_1 + ir_1 \) is dense in \( h \). It follows that for every \( H_{\lambda, \rho} \), the local existence condition of Theorem 2 and the uniform local existence condition of Theorem 1 both hold.

Then for \( (\lambda, \rho) \in \mathbb{R}^2 \setminus (0, 0) \), we have the following:

(i) \( H_{\lambda, \rho} \) is positive if and only if \( \lambda \) and \( \rho \) are both non-negative;
(ii) \( \Delta^\# H_{\lambda, \rho} \Delta^{-\#} = H_{e^{-2\pi i \lambda} e^{2\pi i \rho}} \), and \( JH_{\lambda, \rho} J = H_{-\lambda, -\rho} \);
(iii) \( H_{\lambda, \rho} \) generates a one-parameter semigroup of endomorphisms of \( \mathcal{M} \) if and only if \( \lambda \) and \( -\rho \) are both non-negative; and
(iv) \( H_{\lambda, \rho} \) generates a one-parameter semigroup of endomorphisms of \( \mathcal{M}' \) if and only if \( -\lambda \) and \( \rho \) are both non-negative.

Of course, this is a very simple example, in which it is easy to compute the effects of the \( U(a) \). In more complicated cases, Theorems 1 and 2 could perhaps be applied to greater effect. However, their conditions may well be more restrictive than is necessary; one might conjecture that the local existence conditions could be replaced by conditions purely on the domain \( D(\delta) \) of \( \delta \)—for example, as in [3], by the condition that \( D(\delta) \) be a core for \( H \).
III Lightlike Translations

Let us return to the situation described in the introduction, and consider again the conditions (a)–(d). We know already that (a) and (b) each follow from the remaining three conditions; we have now to consider (c) and (d). One branch is available immediately: suppose that (a) is satisfied, but $U(a)\Omega$ is not known. Then

$$
\langle \Omega \mid U(a)\Omega \rangle = \langle \Omega \mid \Delta^{it}U(a)\Delta^{-it}\Omega \rangle = \langle \Omega \mid U(e^{-2\pi t}a)\Omega \rangle
$$

is independent of $t$, and hence must be a constant for all $a > 0$ and for all $a < 0$. Taking the limit as $t \to \infty$, these constants must both be 1; but since $U(a)\Omega$ is a unit vector, it must therefore equal $\Omega$ for all $a$. Thus (a) alone implies (d). With this out of the way, we proceed to our main result:

**Theorem 3:** If $H$, the generator of $U(a)$, is positive and annihilates the vacuum, and if the local existence condition of Theorem 2 holds, then $U(a)MU(-a) \subset \mathcal{M}$ for all $a \geq 0$ (and thus $U(a)M'U(-a) \subset \mathcal{M}'$ for all $a \leq 0$) if and only if the Lorentz covariance relations hold in the form

$$
\Delta^{it}U(a)\Delta^{-it} = U(e^{-2\pi t}a) \quad \text{and} \quad JU(a)J = U(-a).
$$

**Proof of Theorem 3:** Theorem 2 allows us to reduce this to a question about the relations between $U(a)$ and $R$: it will suffice to show that $U(a)R \subset R$ for all $a \geq 0$ if and only if (8) holds. The result of Borchers [2] is essentially just that (8) holds whenever $H$ is positive and $U(a)R \subset R$ for all $a \geq 0$. Conversely, let us assume that (8) holds. Since $H$ is positive, $U(a)$ can be analytically continued to the upper half-plane, and in particular we have

$$
\Delta^{it}U(a) = U(a \cos(2\pi t) + ia \sin(2\pi t))\Delta^{i t}
$$

over the region $a \geq 0$ and $0 \leq t \leq 1/2$, upon which $a \sin(2\pi t) \geq 0$. It follows that $U(a)D(\Delta^{1/2}) \subset D(\Delta^{1/2})$ for all $a \geq 0$, and $\Delta^{1/2}U(a) = U(-a)\Delta^{1/2}$; furthermore $JU(a) = U(-a)J$, so that $J\Delta^{1/2}U(a) = U(a)J\Delta^{1/2}$. But from the Tomita-Takesaki modular theory, $R = \{ \psi \mid \psi \in D(\Delta^{1/2}), J\Delta^{1/2}\psi = \psi \}$, so we have that $U(a)R \subset R$ for all $a \geq 0$.

Corresponding results for the backwards lightlike translations $W(a) = T(a(\hat{x}_1 - \hat{x}_0))$ can be derived by exchanging $\mathcal{M}$ and $\mathcal{M}'$, and replacing $a$ by $-a$ in the above.
$W(a)$ should have a negative generator, and should satisfy Lorentz covariance in the form $\Delta^{it} W(a) \Delta^{-it} = W(e^{2\pi t} a)$ and $J W(a) J = W(-a)$. With these substitutions, the corresponding theorem obtains. The situation for the intermediate case, the spacelike translations $T(a \hat{x})$ taking $W_\mathbb{R}$ into itself, is somewhat more complicated, although not essentially different: Theorem 1 still holds, but now the relations between the generator (which in some frame of reference is the momentum component $P_1$) and the modular operators are no longer so simple. We must just show that $T(a \hat{x}) D(\Delta^{1/2}) \subset D(\Delta^{1/2})$ for all $a \geq 0$, and that $J \Delta^{1/2} T(a \hat{x}) = T(a \hat{x}) J \Delta^{1/2}$. For example, if $U(a)$ satisfies the conditions of Theorem 3, and $W(a)$ the corresponding requirements for a backwards lightlike translation, and if $U(a)$ and $W(b)$ commute for all $a, b \in \mathbb{R}$, then $U(\lambda a) W(\rho a)$ gives an endomorphism semigroup of this intermediate type for any $\lambda, \rho > 0$. This is just the situation described in the example at the end of the previous section.

If we combine the results of this note with those of [9], we have the following omnibus theorem, as advertised:

**Theorem 4**: Let $M$ be a von Neumann algebra acting on a Hilbert space $H$, which together with its commutant $M'$ has a separating and cyclic vector $\Omega$. Given a strongly continuous one-parameter group $U(a)$ of unitary operators on $H$, for which the local existence condition of Theorem 2 holds, then any three of the following four conditions imply the fourth:

(a) $\Delta^{it} U(a) \Delta^{-it} = U(e^{-2\pi t} a)$ and $J U(a) J = U(-a)$;

(b) the generator $H$ of the $U(a)$ is positive;

(c) $U(a) M U(-a) \subset M$ for all $a \geq 0$;

(d) $U(a) \Omega = \Omega$ for all $a$.

Likewise, any three of the following four conditions imply the fourth:

(a') $\Delta^{it} U(a) \Delta^{-it} = U(e^{2\pi t} a)$ and $J U(a) J = U(-a)$;

(b') the generator $H$ of the $U(a)$ is negative;

(c) $U(a) M U(-a) \subset M$ for all $a \geq 0$;

(d) $U(a) \Omega = \Omega$ for all $a$.

In addition, either (a) or (a') implies (d), so that if (a) holds, then (b) and (c) are equivalent, and if (d) holds, then (b') and (c) are equivalent; otherwise, no two of these conditions imply any other.
References


