Signatures of Confinement in Axial Gauge QCD

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Abstract

A comparative dynamical study of axial gauge QED and QCD is presented. Elementary excitations associated with particular field configurations are investigated. Gluonic excitations analogous to linearly polarized photons are shown to acquire infinite energy. Suppression of this class of excitations in QCD results from quantization of the chromoelectric flux and is interpreted as a dual Meissner effect, i.e. as expulsion from the QCD vacuum of chromo-electric fields which are constant over significant distances. This interpretation is supported by a comparative evaluation of the interaction energy of static charges in the axial gauge representation of QED and QCD.

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1 Introduction

Confinement of colored objects is one of the fundamental properties of the strong interaction. Apart from the obvious importance for the spectrum of observed hadrons in the absence of free gluons and quarks, confinement is quite likely responsible for other more general properties such as the existence of Regge trajectories and the spontaneous breakdown of chiral symmetry. It is believed that color confinement is only one of the possible phases associated with the strong interaction which at finite temperature and possibly at finite baryon density undergoes a transition to an unconfined phase. Theoretical evidence based on lattice gauge calculations strongly suggests that QCD indeed exhibits the phenomenon of confinement and the transition to an unconfined phase at finite temperature. Significant experimental efforts are underway to establish the existence and to investigate the properties of the unconfined phase.

Despite many efforts, a generally accepted analytical explanation or qualitative description of confinement within the framework of QCD is still missing. The only theoretical model of a confining theory, which has a well defined derivation from QCD, is provided by the strong coupling limit of lattice QCD. Unfortunately, this strong coupling limit does not distinguish QED and QCD as far as confinement is concerned. These two theories supposedly develop their characteristic differences only in a phase transition, which as a function of the coupling constant is known to occur for QED and for which no evidence is found in lattice QCD calculations.

Canonical formulations of QCD constitute a complementary approach to the issue of confinement. We will show how, within this framework, non-abelian properties of QCD provide strong signatures of confinement. Distinguishing elements of QED and QCD remain manifest. A variety of gauge fixed formulations of QCD have already been studied (cf. [2, 4] – [8]) with the aim of clarifying the structure of QCD. The general strategy of all these efforts consists in explicitly resolving the constraint equations of QCD, such as the Gauss law in the Weyl-gauge, and thereby eliminating the redundant variables from the QCD Hamiltonian. After this elimination, the Hamiltonian is, by construction, formulated in terms of gauge invariant variables (leaving aside the issue of global, residual symmetries). The resulting gauge fixed formulation of the dynamics is accessible to approximate treatments which are fully compatible with the underlying local gauge symmetry. Without gauge fixing, approximations generally violate gauge invariance and are therefore not appropriate for investigations of issues like confinement, where exact local color conservation must be particularly important.

The paradigm of a gauge fixed canonical formalism is provided by the Coulomb-gauge representation of QED. Here, by explicitly resolving Gauss’s law the longitudinal components of the vector potential are eliminated. The resulting description in terms of transverse vector fields is explicitly gauge invariant irrespective of the approximations involved in application of the formalism to specific dynamical problems. This gauge choice has played a very important role also in the development of gauge fixed formulations of QCD [2, 4, 6, 7]. However while in QED the Coulomb-gauge is singled out as the gauge in which static charges do not radiate, gluons and color spin remain coupled in QCD in any gauge. Moreover, the choice of radiation gauge variables does
not conform with the angular momentum algebra of the QCD Gauss law operators, thereby causing significant technical disadvantages. As a consequence, the Hamiltonian of QCD in the Coulomb-gauge representation cannot be constructed explicitly; its calculation involves inversion of the Fadeev-Popov operator which can be performed only perturbatively. Therefore, within the canonical formalism, the Coulomb-gauge has been particularly useful in perturbative studies of, for example, the running coupling constant \([2, 9, 10, 11]\) or the small volume limit of QCD \([12]\). However dynamical calculations within the Coulomb-gauge are of great technical complexity \([13]\).

In the absence of compelling physics reasons, the choice of gauge has been dictated in most studies by formal and technical considerations. We are only at the beginning of a development where common properties of the resulting quite different formulations of QCD, such as the appearance of centrifugal barriers in the electric field energy \([14, 15, 16]\), are being recognized and their dynamical role is being understood. At this point it appears unlikely that the non-perturbative phenomena of QCD are most easily described in one unique gauge. Given the complexity of the Hamiltonian in all the gauge fixed formulations, it seems appropriate to attempt to identify in a first step those degrees of freedom whose dynamics gets particularly simplified in a special gauge and concentrate the studies on their dynamics. We present here the results of such an investigation in the framework of the axial gauge representation of QCD.

In the axial gauge representation, the “simple” degrees of freedom are linearly polarized plane waves. This simplification arises by identifying one of the coordinate axes (the 3-axis in the following) with the spatial direction of the electric field and furthermore, in exploiting local gauge invariance, by identifying the color 3-axis of SU(2) QCD with the color orientation of the electric field. It can be shown \([14]\) that, in implementing the Gauss law, all other field configurations with electric fields pointing in the spatial 3-direction have a longitudinal component and can therefore be eliminated. The simplification of the dynamics occurs for both QED and QCD. In order to control infrared difficulties the system is enclosed in a box with periodic boundary conditions.

The main part of our studies will be concerned with the elementary excitations of these “simple” degrees of freedom. In QED, these elementary excitations are nothing else than photons described in this particular gauge by a single field. In QCD, the corresponding elementary excitations will be seen to not exhibit the standard dispersion equation for massless particles; rather these excitations are found to be frozen in the infinite volume limit. We shall interpret the disappearance of this class of excitations from the spectrum as an indication of the dual Meissner effect: the QCD vacuum apparently does not support color-electric fields which remain constant in a certain direction over “large” distances. The parallel treatment of QED and QCD in analogous gauges is very important for identifying those elements of QCD which make it differ so significantly from QED. Unlike lattice theories, the canonical description is not, by assumption, formulated in terms of compact variables; a restriction in the range of certain dynamical variables occurs rather as a consequence of modifications in the electric field energy which are typical for non-abelian theories. As another application of the simplified dynamics of plane wave excitations we consider the static quark-antiquark interaction. Again, a proper choice of coordinates and variables allows us to simplify significantly the dynamics and, under additional assumptions, derive within the canonical gauge fixed formalism the strong coupling result of lattice gauge theories.
2 Formalism of axial gauge QCD

Our investigation starts with the QCD Hamiltonian in the axial gauge. For SU(2) color and in the presence of static charges, the Hamiltonian density is given by [14]

$$\mathcal{H} = \text{tr} \left[ \vec{E}_\perp^2 + \vec{B}_\perp^2 \right] + \frac{1}{2} \epsilon_3^2 + \frac{1}{2 L^2} \epsilon_3^2(\vec{x}_\perp) \epsilon_3(\vec{x}_\perp) + \frac{1}{2} \epsilon_3^2(\vec{x}_\perp). \quad (2.1)$$

In the axial gauge representation of QCD, the degrees of freedom are the (three color projections of the) perpendicular components of the gauge fields $A_\perp$ and their conjugate electric fields $\tilde{E}_\perp$. (We use an unconventional sign for the electric field, so that $A$ and $\tilde{E}$ satisfy standard commutation relations of coordinates and momenta.) One of the cartesian components of the fields, the 3-component, has been eliminated. Naive gauge fixing gives rise to infrared problems; these can be avoided by enclosing the system in a box (length $L$) and imposing periodic boundary conditions. In this way, two-dimensional, i.e. $x_3$-independent, color neutral gauge fields $a_3(\vec{x}_\perp)$ and their conjugate electric fields $\epsilon_3(\vec{x}_\perp)$ remain as unconstrained degrees of freedom. In turn, the two-dimensional neutral, longitudinal components of $A_\perp$ and $\tilde{E}_\perp$ are identified as dependent degrees of freedom and have been eliminated, i.e.,

$$\text{div}_\perp \int_0^L d\vec{x}_3 \tilde{E}_\perp^3(\vec{x}) = 0. \quad (2.2)$$

(Note that the canonical commutation relations are correspondingly modified.) The relation between magnetic and gauge fields is the standard one,

$$B_3 = \partial_1 A_2 - \partial_2 A_1 - ig \left[ A_1, A_2 \right],$$
$$B_i = \epsilon_{ik3} \left[ \partial_1 a_3 r^3/2 - \partial_3 A_1 - ig \left[ A_k, a_3 r^3/2 \right] \right] \quad (i = 1, 2). \quad (2.3)$$

Except for the modifications by the two-dimensional fields, the contributions of the perpendicular degrees of freedom to electric and magnetic field energy are as usual; the second term in eq. (2.1) is just the contribution of the 3-component of the electric field

$$\frac{1}{2} \epsilon_3^2 = \frac{1}{L^2} \int_0^L d\vec{x}_3 \int_0^L d\vec{y}_3 \sum_{p, q, n} (1 - \delta_{p, q} \delta_{n, 0}) \frac{G_{p, q}^4(\vec{x}_\perp, \vec{y}_3) G_{p, q}(\vec{x}_\perp, z_3)_3}{\left[ \frac{2\pi n}{L} + g (p - q) a_3(\vec{x}_\perp) \right]^2} e^{i2\pi n(z_3 - y_3)/L},$$

which, by resolving Gauss’s law, is given in terms of the other degrees of freedom,

$$G_{\perp}(\vec{x}) = \vec{\nabla}_\perp \tilde{E}_\perp(\vec{x}) + g e^{a b c} r^a \frac{\vec{a}_\perp}{2} \left( E_\perp^c(\vec{x}) - \eta(\vec{x}_\perp) \delta_{c, 3} \right) + g \rho^m(\vec{x}). \quad (2.4)$$

We have omitted the quark contributions in the Hamiltonian but included for later application static color charges as described by the density $\rho^m(\vec{x})$. The Hamiltonian density of eq. (2.1) also contains the two-dimensional, neutral color-electric field $\eta$,

$$\eta(\vec{x}_\perp) = \frac{g}{L} \vec{\nabla}_\perp \int d^2 y \mathbf{d} (\vec{x}_\perp - \vec{y}_\perp) \left( e^{3d r} \bar{A}_\perp^4(\vec{x}) \bar{E}_\perp^4(\vec{x}) + \rho^m(\vec{x}) \right) \quad (2.5)$$

which is defined in terms of the two-dimensional Green function

$$d(\vec{z}_\perp) = -\frac{1}{L^2} \sum_{\vec{q}_n \neq 0} \frac{1}{q_n^2} e^{i \vec{q}_n \vec{z}_\perp}, \quad \vec{q}_n = \frac{2\pi}{L} (n_1, n_2). \quad (2.6)$$

(Note that the canonical commutation relations are correspondingly modified.) The relation between magnetic and gauge fields is the standard one,
This “electrostatic” field appears as a consequence of the elimination of the neutral, longitudinal components from $\tilde{E}_\perp$ and $A_\perp$ (cf. eq. (2.2)). Finally, as a remnant of the Gauss law constraint, periodicity of the fields requires the neutral component of the charge to vanish in the space of physical states $|\Phi\rangle$,

$$Q^3|\Phi\rangle = 2 \int d^3x (G_\perp)_{11}(x)|\Phi\rangle = 0 \ .$$

(2.8)

Further insight into the dynamics as described by the above QCD Hamiltonian can be gained from a comparison with the Hamiltonian of QED in the axial gauge

$$H = \frac{1}{2} \left[ \tilde{E}_\perp^2 + \tilde{B}^2 \right] + \frac{1}{2} \mathcal{E}_3^2 + \frac{1}{2L^2} \mathcal{E}_3^2(\tilde{x}_\perp) + \frac{1}{2} \tilde{\eta}^2(\tilde{x}_\perp) \ .$$

(2.9)

The same choice of the degrees of freedom as physical ($\tilde{E}_\perp$) or constrained ($E_3$) ones preserves the similarity of the structure of the underlying Weyl-gauge Hamiltonians of QED and QCD. Obvious differences are connected with the color structure, e.g. in the definition of the QED magnetic fields ($g = 0$ in eq. (2.3)), or in representing the 3-component of the QED electric field in terms of the unconstrained variables,

$$\frac{1}{2} \mathcal{E}_3^2 = \frac{1}{2L^2} \int_0^L dz_3 \int_0^L dy_3 \sum_n \left( 1 - \delta_{n,0} \right) \frac{G_\perp(\tilde{x}_\perp, z_3) G_\perp(\tilde{x}_\perp, y_3) e^{i\tilde{x}_\perp y_3(y_3 - z_3)/L}}{2\pi n^2} \ .$$

(2.10)

with

$$G_\perp(\tilde{x}) = \nabla_\perp \tilde{E}_\perp(\tilde{x}) + \epsilon \rho^\perp(\tilde{x}) \ .$$

(2.11)

In QED, the electrostatic two-dimensional field $\tilde{\eta}$ is generated by the density of the charges which again are assumed to be static,

$$\tilde{\eta}(\tilde{x}_\perp) = \frac{e}{L} \nabla_\perp \int d^3y \delta(\tilde{x}_\perp - \tilde{y}_\perp) \rho^\perp(\tilde{y}) \ .$$

(2.12)

As in QCD, two-dimensional fields appear as a consequence of implementing the Gauss law on a torus. Here, the physics reason is particularly transparent. The electric fields $e_3(\tilde{x}_\perp)$ are purely transverse fields and are therefore not part of the Gauss law constraint. These fields together with the conjugate variables $a_3(\tilde{x}_\perp)$ describe photons polarized in the 3-direction and propagating in the 1-2 plane. These degrees of freedom cannot be eliminated while their counterparts, the purely longitudinal components of $\tilde{E}_\perp$, are completely determined by the Gauss law and eliminated by the constraint (2.2).

### 3 Elementary excitations

The role of the two-dimensional degrees of freedom appearing in both the QCD and QED axial gauge Hamiltonians can be discussed from two quite different points of view. On the one hand, their presence can be understood purely formally to guarantee consistency of the formulation and provide proper infrared behavior of the energy density. We note, for example, the occurrence of $a_3(\tilde{x}_\perp)$ in the propagator defining $\mathcal{E}_3$ in eq. (2.4). Apart from this formal role one might expect physically these special
two-dimensional degrees of freedom to contribute in a thermodynamic sense negligibly to physical observables. On the other hand, these two-dimensional fields generate legitimate elementary excitations of the system. In QED, any linearly polarized photon can, after an appropriate choice of coordinates, be described by these lower dimensional fields. In this sense, the degrees of freedom associated with \( a_3(\vec{x}_\perp) \) and \( \epsilon_3(\vec{x}_\perp) \) are of rather general nature, and it is only the peculiar choice of coordinates which simplifies significantly their dynamics.

This simplification becomes explicit for QED by observing that after an integration by parts in the magnetic field energy of eq. (2.9) (cf. also eq. (2.3)), the two-dimensional degrees of freedom decouple from the remaining ones and are described by the Hamiltonian

\[
h = \int d^2x \left[ \frac{1}{2L} \epsilon_3^\dagger(\vec{x}_\perp) + \frac{L}{2} \left( \nabla_\perp a_3(\vec{x}_\perp) \right)^2 \right].
\]

We compare this Hamiltonian with the one for the corresponding elementary excitations in QCD. Here, the color dynamics couple the two-dimensional fields to the perpendicular degrees of freedom. In QED, the presence of charged matter would also induce such a coupling. In a first step, we shall neglect this coupling and keep – with the electric field energy – only the abelian contribution to the magnetic field energy. The resulting Hamiltonian for \( a_3(\vec{x}_\perp) \) of QCD is, by construction, identical in structure with the QED one of eq. (3.13) but for the missing hermiticity of the QCD electric field operator \( \epsilon_3(\vec{x}_\perp) \) (cf. eq. (2.1)). This difference has important consequences. The hermiticity defect arises since SU(2) group elements \( W \), actually loops around the torus along the 3-direction, have been parametrized in terms of elements of the algebra \( \{a_3\}, \)

\[
W = \exp \{ ig L a_3(\vec{x}_\perp) /2 \} = \cos (g L a_3 /2) + i a_3(\vec{x}_\perp) \sin (g L a_3 /2).
\]

This is analogous to the hermiticity defect of the radial momentum operator when using polar coordinates and also leads here to modifications in the kinetic energy. In the Schrödinger representation, expressing the electric fields in terms of functional derivatives, one finds

\[
e_3^\dagger(\vec{x}_\perp) e_3(\vec{x}_\perp) = - \frac{1}{J(a_3(\vec{x}_\perp))} \frac{\delta}{\delta a_3(\vec{x}_\perp)} J(a_3(\vec{x}_\perp)) \frac{\delta}{\delta a_3(\vec{x}_\perp)}.
\]

The Jacobian \( J(a_3) \) is the Haar measure of SU(2),

\[
J(a_3(\vec{x}_\perp)) = \sin^2 \left( \frac{1}{2} g L a_3(\vec{x}_\perp) \right)
\]

and thus also appears in the volume element when evaluating matrix elements in terms of wave functionals of \( a_3(\vec{x}_\perp) \). As the transformation to polar coordinates becomes singular when attempting to define a direction for a vector of vanishing length, singularities in the kinetic energy eq. (3.15) arise for \( g L a_3 = 2\pi n \) in the parametrization of eq. (3.14). It is convenient to transform the kinetic energy into the standard form by defining, in analogy to the Schrödinger equation in polar coordinates, appropriate “radial” wave functions. We introduce for this purpose a lattice in the 1-2 plane (lattice constant \( \ell \)) and define the (rescaled) variables at the lattice sites \( \vec{b} = \vec{x}_\perp /\ell \),

\[
\varphi_{\vec{b}} = \frac{1}{2} g L a_3(\vec{b}\ell).
\]
The radial wave functions $\Phi$ are defined in terms of the original wave functions, which are the projections of the physical states $|\Phi\rangle$ (cf. eq. (2.8)) onto the field eigenvectors $|\varphi\rangle$, in the standard way

$$\Phi[\varphi] = \Phi[\varphi] \prod_{\ell} |\sin(\varphi_{\ell})|.$$  

In this way formally identical “free” Hamiltonians of the two-dimensional gauge degrees of freedom $\varphi$ for QED and QCD are obtained

$$h = h^e + h^m,$$

with the electric and magnetic field energies given by

$$h^e = -\frac{g^2 L}{8\ell^2} \sum_{\ell} \frac{\partial^2}{\partial \varphi^2_{\ell}},$$

$$h^m = \frac{2}{g^2 L} \sum_{\ell,\ell'} \left( \varphi_{\ell+\ell'}^2 - \varphi_{\ell}^2 \right).$$  

(3.20)

The fundamental vectors of the lattice (1, 0) and (0, 1) are denoted by $\delta$. Although of the same structure, the Hamiltonian (3.19) acts on wave functions belonging to different spaces in QED and QCD. The space of wave functions is restricted in QCD by the constraint

$$\Phi[\varphi] = 0 \text{ whenever } \varphi_{\ell} = n\pi \text{ for some } \vec{b}.$$  

(3.21)

In the transformation of the Hamiltonian to “radial” variables no effective potential but an energy shift

$$E \to E - \frac{g^2 L^3}{8\ell^4}$$

appears, which has been suppressed in eq. (3.20).

### 3.1 Photons in the 1-2 plane

In order to display the non-trivial dynamics described by the Hamiltonian $h$ in conjunction with the constraint of eq. (3.21) we consider first the case of electrodynamics. As is well known, in this case eigenstates and energies are determined by discrete Fourier transformation

$$\varphi_{\ell} = \frac{1}{K} \sum_{k} e^{2\pi ik/K} \varphi_{k}.$$  

(3.23)

The number of degrees of freedom is $K^2$ with

$$K = \frac{L}{\ell},$$

(3.24)

and limits the sum in eq. (3.23) to $|k_1,2| \leq K - 1$. In this way, the dispersion relation for the (lattice) photons,

$$\omega^2_{k} = \frac{4}{\ell^2} \sum_{\ell} \sin^2 \left( \frac{\pi\delta_{k}^2}{K} \right).$$  

(3.25)
is obtained, with the continuum limit
\[
\omega_k^2 \to \left( \frac{2\pi k}{L} \right)^2 \quad \text{for} \quad |k| \ll K. \quad (3.26)
\]

Thus the two-dimensional field \( a_3 (\vec{x}_\perp) \) describes free photons propagating in the 1-2 plane with polarization in the 3-direction. We also note that the dependence of the Hamiltonian on the coupling constant is of no relevance, a simple rescaling
\[
\varphi_\delta \to gL^{1/2} \varphi_\delta \quad (3.27)
\]
eliminates all but the dependence on the lattice spacing \( \ell \).

In momentum space, the ground state wave function factorizes into the contributions of the normal modes
\[
\Phi[\varphi] = \prod_k \left( \frac{8L^2}{\pi g^2 L} \right)^{1/2} \exp \left( -\frac{4L^2}{g^2 L} \omega_k \varphi_\delta \hat{\varphi}_k \hat{\varphi}_{-\delta} \right) \quad (3.28)
\]
and thus describes highly correlated zero-point motions in configuration space.

### 3.2 Jacobian and structure of the Hilbert space

The nature of the elementary excitations as described by the Hamiltonian (3.19) is qualitatively altered by the presence of the constraint (3.21) on the wave function. This difference in dynamics is easily appreciated in a mechanical interpretation. While the Hamiltonian (3.19), (3.20) describes a two-dimensional system of mass points which are coupled harmonically to the respective nearest neighbors with the photons as the normal modes, the constraint (3.21) can be visualized as walls of an infinite square well which limit strictly the amplitude of oscillations of the individual mass points. Apparently, the character of the normal modes of this system depends on the relative importance of the harmonic nearest neighbor couplings and the constraining force. The relevant parameter which controls the dynamics is the ratio of constants multiplying electric and magnetic field energy,
\[
\kappa = g^2 \frac{L}{4\ell}. \quad (3.29)
\]

Unlike the case of electrodynamics, this ratio receives significance by the constraint on the wave function which prevents the kinetic energy from becoming arbitrarily small for certain field configurations.

The presence of the constraint (3.21) on the wave function not only changes significantly the physical properties of the system but also requires, in comparison with the abelian case, a very different method of solution. As a consequence of the constraint, the probability current between regions of the wave functions separated by zeroes of the Jacobian vanishes. Therefore the system defined by eqs. (3.19) – (3.21) possesses an infinity of conserved charges
\[
Q^\alpha_\delta = \int_{\pi}^{(n+1)\pi} d\varphi \delta \left( \varphi - \varphi_\delta \right) \quad (3.30)
\]
(here we have made explicit the operator character of $\varphi_{\delta}^\prime$). These charges specify the probability of the system to live in one of these intervals. In general, stationary states do not exist for an arbitrary choice of these charges; they rather require each of these charges to be non-vanishing only in one of the intervals. For illustration we consider the case of two lattice sites.

In accordance with the constraint (3.21) we decompose the wave function

$$\psi(\varphi_1, \varphi_2) = \sum_{n_1, n_2 = -\infty}^{+\infty} \Theta_{n_1}(\varphi_1)\Theta_{n_2}(\varphi_2) \psi_{n_1, n_2}(\varphi_1, \varphi_2),$$

where $\Theta_n$ denotes the projection on the fundamental intervals

$$\Theta_n(\varphi) = \begin{cases} 1 & \text{for } n\pi \leq \varphi \leq (n + 1)\pi \\ 0 & \text{otherwise} \end{cases}.$$ (3.32)

Most important, the wave function constraint decouples the solution of the Schrödinger equation in the different intervals; consequently eigenvalues and eigenfunctions can be obtained by choosing the wave function to be non-vanishing in one of the intervals only, i.e.,

$$\psi_{n_1, n_2}(\varphi_1, \varphi_2) \neq 0 \quad \text{only if } n_1 = n_1^0, \ n_2 = n_2^0.$$ (3.33)

The complete spectrum is determined by variation of $n_1^0$ and $n_2^0$. In general, the potential energy and therefore the spectrum depends on the choice of the intervals and therefore the system cannot coexist in different intervals. When symmetries are present however, the wave function corresponding to identical eigenvalues in different intervals can be arbitrarily distributed over these intervals. For example, this happens as a consequence of the displacement symmetry of the Hamiltonian (3.19), (3.20); a global shift of the intervals

$$n_0^* \rightarrow n_0^* + m$$ (3.34)

does not change the spectrum and the wave functions can be linearly superimposed.

Further possibilities for combining wave functions arise, for instance, by the following more symmetric form of the discretized magnetic field energy,

$$h^m \rightarrow \frac{1}{g^2 L} \sum_{\delta, \delta'} \left( 1 - \cos 2(\varphi_{\delta+\delta'} - \varphi_{\delta}) \right).$$ (3.35)

The presence of symmetries makes the definition of the eigenstates ambiguous; however the theory contains no operators which would connect different intervals with each other and thereby be sensitive to these ambiguities. In the following we shall restrict our calculations to one definite (site-independent) choice of the fundamental intervals,

$$Q_\delta^* = \delta_{n,0}.$$ (3.36)

The site-independence guarantees that the state of lowest energy occurs in this sector of the Hilbert space (with the parametrization (3.35) of the discretized magnetic field energy, the spectrum is independent of the choice of the fundamental intervals).
3.3 Gluonic excitations in the 1-2 plane

The strong coupling limit ($\kappa \gg 1$) is physically most important. We note that this limit is determined by the strength of the dimensionless coupling constant $g$ relative to a negative power ($-1/6$) of the number of degrees of freedom in the system (cf. (3.29)). In the absence of a dimensionful quantity, the continuum ($\ell \to 0$) and “thermodynamic” ($L \to \infty$) limits are not distinguished. We therefore expect the dependence of the coupling constant $g$ on the number of degrees of freedom to be dictated by asymptotic freedom,

$$g^2(L/\ell) \propto 1/\ln (L/\ell),$$

(3.37)

and the thermodynamic limit to correspond to the strong coupling limit. In this limit, the electric field energy dominates and the stationary states are simply given by excitations of the degrees of freedom at the individual lattice sites. The wave function of a stationary state can be written as

$$\Psi[n] = \prod_i \left[ \frac{2}{\ell} \right]^{1/2} \sin \left( n_i \varphi_i \right),$$

(3.38)

with the energy eigenvalue

$$E_n = \frac{g^2 L}{8\ell^2} \sum_i n_i^2.$$  

(3.39)

In particular, in the ground state all the individual degrees of freedom are in the lowest $n = 1$ state, and the value

$$E_0 = \frac{g^2 L^3}{8\ell^4}$$

(3.40)

is obtained for the ground state energy. This is just the negative of the energy shift associated with the transformation to radial coordinates (cf. eq. (3.22)). Thus the total ground state energy is actually zero and the corresponding ground state wave function is an eigenfunction of the modulus of the chromo-electric field operator with vanishing eigenvalue (up to singular contributions arising from possible discontinuities at $\varphi = n\pi$). States of lowest excitation energy are obtained by exciting a degree of freedom at one particular site into its first excited state, with excitation energy

$$\Delta E = \frac{3g^2 L}{8\ell^2}.$$  

(3.41)

Corrections to the strong coupling limit can be calculated perturbatively. First order perturbation theory in $h^n$ (cf. eq. (3.20)) yields for the ground state energy

$$E_0 = \frac{g^2 L^3}{8\ell^4} + \frac{4L}{g^2\ell^4} \left( \frac{\pi^2}{6} - 1 \right).$$

(3.42)

The result justifies the perturbative treatment of the magnetic field energy for sufficiently large number of degrees of freedom $\kappa^2 = g^4 (L/4\ell)^2 \gg 1$.

Excited states of the system are highly degenerate in the strong coupling limit. This degeneracy is lifted due to the magnetic coupling of the degrees of freedom at different
lattice sites. We consider the energetically lowest excitation at the site $\tilde{b}$ described by the wave function

$$
\chi_{\tilde{b}}[\varphi] = \left(\frac{2}{\pi}\right)^{1/2} \sin\left(2\varphi_{\tilde{b}}\right) \prod_{\tilde{b}'(\neq \tilde{b})} \left(\frac{2}{\pi}\right)^{1/2} \sin \varphi_{\tilde{b}'} \right).
$$

(3.43)

In the subspace of these excitations, the nearest neighbor magnetic coupling becomes diagonal for the eigenstates of the translation operator $(\tilde{b} \rightarrow \tilde{b} + \tilde{e})$

$$
\hat{\chi}_{\tilde{b}}[\varphi] = \frac{1}{K} \sum_{\tilde{b}} e^{-2i\pi\tilde{b}\tilde{K}/K} \chi_{\tilde{b}}[\varphi].
$$

(3.44)

The calculation of the excitation energies is straightforward; we find

$$
\Delta E_{\tilde{K}} = \frac{3}{8} g^2 L + \frac{64}{9\pi^2} \frac{1}{g^2 L} \left[ \sum_{\tilde{b}} \sin^2 \left(\frac{\pi\delta\tilde{b}}{K}\right) \right] - 1
$$

(3.45)

corresponding to the state (3.44) labeled by the momentum $\tilde{K}$.

The above analysis of the strong coupling limit is the central part of our investigations. We now present the physics implications of the results. As already emphasized, the ground state of the system is, in the strong coupling limit, an eigenstate of the electric field operator with vanishing eigenvalue. This possibility for a ground state with vanishing $x_3$-independent electric fields arises due to the Jacobian in the kinetic energy. In QED such states are not normalizable and would entail infinitely large fluctuations in the magnetic field energy. Thus the structure of the vacuum concerning these $x_3$-independent fields is very different in the abelian and non-abelian theory. The virial theorem applies to QED: magnetic and electric fields contribute equally to each normal mode. Hence, the expectation value $\langle \vec{E}^2 - \vec{B}^2 \rangle$ vanishes. In QCD the Jacobian invalidates equipartition. Chromoelectric $x_3$-independent fields are absent; in turn, the fluctuations in the magnetic field at different lattice sites are not correlated and the ground state energy is due exclusively to these uncorrelated magnetic field fluctuations. Similarly, the “gluon condensate” is dominated by the magnetic field contribution,

$$
\langle \vec{E}^2 - \vec{B}^2 \rangle = -\langle \vec{B}^2 \rangle < 0.
$$

(3.46)

It is interesting that even in the crude approximation of keeping only one particular kind of gluonic degrees of freedom, this model displays features which are reminiscent of the phenomenology of the “magnetic QCD vacuum”.

Concomitant with these qualitative differences in the structure of the ground states is the very different nature of the elementary excitations. Built on the highly correlated QED ground state the photons appear as collective excitations with excitation energies vanishing in the long-wavelength limit. In QCD, the elementary excitations are localized in configuration space and are due to formation of non-vanishing electric flux. Intuitively, the expression (3.41) for the excitation energy can be seen as a result of a quantization of the chromoelectric flux. In the absence of couplings to the other degrees of freedom, the chromoelectric fields formed in the elementary excitations with lowest excitation energy are located on just one transverse lattice site. Thus the flux
tubes are infinitely thin (i.e., the area is $\propto \ell^2$) and extend over the whole system in the 3-direction. The chromoelectric flux $\Phi_E$ associated with each component of the standing waves (eqs. (3.43), (3.44)) is quantized,

$$\Phi_E = ng \ .$$

(3.47)

(Note that since the standing waves are not eigenstates of the electric field operator, one has to decompose them into travelling wave components in order to “see” this flux quantization.) Beyond the strong coupling limit, waves propagate through the medium and transport the electric flux across a transverse plane. The spectrum acquires a band structure with a band width $\propto 1/(g^2 L)$ (cf. eq. (3.45)). Threshold energy and mass of the associated “particles” are $\propto g^2 L/\ell^2$ and tend to infinity with increasing size of the system. Irrespective of the value the transverse momentum, these elementary excitations never approach the free gluon limit.

4 Gluonic couplings

Here we continue our investigation of the elementary excitations of QCD as described by the two-dimensional fields $\varphi$ (cf. eq. (3.17)). Unlike in QED, where these degrees of freedom describe non-interacting photons in the absence of charged matter, in QCD such a decoupling does not occur. Here we study – within perturbation theory – the effect of the other degrees of freedom of QCD on the dynamics of the $\varphi$. We shall show that the effect of the infinity of other degrees of freedom is not sufficient to overcome the dominant role of the kinetic energy of these particular two-dimensional fields. The starting point of our investigation is the Hamiltonian of eq. (2.1) applied to SU(2) QCD. With the coupling constant $g$ treated as a small parameter, the neutral gluons as described by the conjugate pair of vector fields $A^3$, $E^3$ decouple from the other degrees of freedom and will not be considered further. At this point it is convenient to define “charged” gluon fields

$$\vec{\Phi}_\perp = \frac{1}{\sqrt{2}} \left( \vec{A}_\perp + i \vec{E}_\perp \right) \ , \quad \vec{\Pi}_\perp = \frac{1}{\sqrt{2}} \left( \vec{E}_\perp + i \vec{A}_\perp \right).$$

(4.48)

The Hamiltonian to be considered in the following,

$$h = h^e + h^m + h'[\varphi],$$

(4.49)

contains apart from the already discussed electric and abelian magnetic field energy the Hamiltonian $h'[\varphi]$ of the charged gluons coupled to the two-dimensional degrees of freedom $\varphi$,

$$h'[\varphi] = \int d^2x \left( \vec{\Pi}_\perp (\vec{x}) \vec{\Pi}_\perp (\vec{x}) + \vec{\beta}^1(\vec{x}) \vec{\beta}(\vec{x}) \right) + \frac{L}{4} \int d^2x_1 \int_0^L dz \int_0^L dz' \sum_n \vec{\nabla}_\perp \vec{\Pi}_\perp (\vec{x}_1, z) \frac{e^{i2\pi n(z-z')/L}}{(\pi n - \varphi(\vec{x}_1))^2} \vec{\nabla}_\perp \vec{\Pi}_\perp (\vec{x}_1, z').$$

(4.50)

The kinetic energy of the charged gluons receives contributions from the first and last term of the r.h.s. of the above equation. The last term also acts as a centrifugal
barrier on the two-dimensional degrees of freedom $\varphi(\hat{x}_\perp)$ (here, we have adopted a continuum notation for these variables, since most of the calculations can be carried through without resorting to the lattice in the perpendicular direction). The magnetic field $\beta$ is defined as

$$\beta_3 = \partial_1 \Phi_2 - \partial_2 \Phi_1, \quad \beta_{2,1} = \pm \left( \partial_3 - 2i\varphi(\hat{x}_\perp)/L \right) \Phi_{1,2}. \quad (4.51)$$

As the centrifugal term, the magnetic field energy contains a coupling between charged and neutral two-dimensional gluons via the covariant derivative $(\partial_3 - 2i\varphi/L)$.

The singular coupling (cf. eq. (4.50)) between the two-dimensional fields $\varphi(\hat{x}_\perp)$ and the charged gluons prevents straightforward application of perturbation theory. Expansion of the centrifugal term in eq. (4.50) in terms of the variables $\varphi(\hat{x}_\perp)$ yields non-integrable, infrared singularities. Thus the following considerations are more generally required for a systematic expansion in $g$ when these two-dimensional fields with their explicit non-perturbative dynamics are involved.

To define an appropriately decoupled zeroth order Hamiltonian we introduce an $x_\perp$-independent external, neutral field $\chi$ which represents the average equilibrium position of the $\varphi(\hat{x}_\perp)$. In this way a well defined, though parameter dependent interaction Hamiltonian is introduced. Eventually, the external field $\chi$ will be treated as a variational quantity, used to minimize the ground state energy. Formally, we write

$$h'[\varphi] = h'[\chi] + (h'[\varphi] - h'[\chi]) \quad (4.52)$$

and define the $\varphi$-potential energy resulting from the coupling to the charged gluons as

$$h_{int} = \langle \Phi^{0}_\chi | (h'[\varphi] - h'[\chi]) | \Phi^{0}_\chi \rangle. \quad (4.53)$$

The state vector $|\Phi^{0}_\chi\rangle$ denotes the $\chi$-dependent ground state of the non-interacting charged gluons,

$$h'[\chi]|\Phi^{0}_\chi\rangle = E_0[\Phi]|\Phi^{0}_\chi\rangle. \quad (4.54)$$

In the first step towards the determination of $h_{int}$, we diagonalize the free charged gluon Hamiltonian. Since $h'[\chi]$ is quadratic in the charged gluon field operators, this diagonalization is achieved by a Bogoliubov transformation and yields the result

$$h'[\chi] = \sum_{\lambda,\bar{n}} |\bar{\alpha}^{\dagger}(\bar{n})\beta^{\dagger}(\bar{n})\rangle (\beta^{\dagger}(\bar{n})\beta^{\dagger}(\bar{n}) + \alpha^{\dagger}(\bar{n})\alpha^{\dagger}(\bar{n}) + 1) \quad (4.55)$$

The operators $\alpha^{\dagger}(\bar{n})$, $\beta^{\dagger}(\bar{n})$ create the two (SU(2)) charged massless gluon states with polarization specified by $\lambda$ and momentum given by $\vec{k}(\bar{n})$,

$$\vec{k}(\bar{n}) = \frac{2\pi}{L} (n_1, n_2, n_3), \quad (4.56)$$

while their energies are determined by the vector

$$\vec{p}(\bar{n}) = \frac{2\pi}{L} (n_1, n_2, n_3 - \chi/\pi) \quad (4.57)$$

associated with the covariant derivative. The coupling to the external, spatially constant neutral gluon field $\chi$ induces an asymmetry in the energies of the two charge.
states. Furthermore, the independence of the energies \(|\vec{p}(\vec{n})|\) of the polarization \(\lambda\) requires a consistent treatment of both magnetic and electric coupling of \(\chi\) to the charged gluons. For vanishing \(\chi\) and with the \(n=0\) term in the centrifugal term (cf. eq. (4.50)) disregarded, \(h[\chi]\) represents (apart from the corresponding two-dimensional fields) the free Hamiltonian for two types of photons in the axial gauge representation. By the Bogoliubov transformation this Hamiltonian gets transformed into the Coulomb-gauge representation and thereby the non-locality in the electric field energy is eliminated [17]. Having determined the Bogoliubov transformed ground state it is straightforward to calculate the separate contributions to the ground state energy. The color-electric contribution \(h^e_{int}\) to the potential energy \(h_{int}\) (cf. eq. (4.53)) arises from the centrifugal barrier term in eq. (4.50), the color-magnetic one \(h^m_{int}\) from the perpendicular components of the magnetic field,

\[
h_{int} = h^e_{int} + h^m_{int} .
\]  

To evaluate these matrix-elements we replace at this point the corresponding integrals over the perpendicular coordinates by sums and obtain

\[
h^e_{int} = \frac{\ell^2}{2L^2} \sum_{\ell,n} \left\{ \frac{k_{n3}^2 - 2\varphi / L}{\left( k_{n3} - 2\varphi / L \right)^2 - 1} \right\} \frac{k_{n\perp}^2}{\sqrt{p_{n3}^2 + k_{n\perp}^2}} ,
\]

\[
h^m_{int} = \frac{\ell^2}{2L^2} \sum_{\ell,n} \left\{ \frac{k_{n3}^2 - 2\varphi / L}{\left( k_{n3} - 2\varphi / L \right)^2 - 1} \right\} \frac{2p_{n3}^2 + k_{n\perp}^2}{\sqrt{p_{n3}^2 + k_{n\perp}^2}} .
\]  

(Here, we have used the shorthand notation \(k_{n3} = k_3(\vec{n})\), \(k_{n\perp} = k_{\perp}(\vec{n})\), etc.) As could be expected, these energies are divergent despite the subtraction of the most singular \(\chi\) and \(\varphi_\|\)-independent terms in the definition (4.53) of \(h_{int}\). Using a heat kernel regulator of the form

\[
e^{-\lambda\sqrt{p_{n3}^2 + k_{n\perp}^2}}
\]

and identifying the inverse momentum cut-off with the lattice spacing

\[
\lambda = \gamma \ell ,
\]

we obtain to leading order in \(\ell / L\)

\[
h^e_{int} = \frac{1}{2\pi \gamma^3 \ell} \sum_\ell \left\{ 2(\chi - \varphi_\|) \cot \varphi_\| + (\chi - \varphi_\|)^2 \frac{1}{\sin^2 \varphi_\|} \right\} ,
\]

\[
h^m_{int} = \frac{1}{2\pi \gamma^3 \ell} \sum_\ell \left\{ 2(\varphi_\| - \chi) \cot \chi + (\varphi_\| - \chi)^2 \frac{1}{\sin^2 \chi} \right\} .
\]  

Determination of the value of the external variable \(\chi\) requires computation of the expectation value of \(h_{int}\) in the unperturbed ground state (\(n_\ell = 1\) in eq. (3.38)). It is easily seen that the interaction energy becomes minimal at

\[
\chi = \pi / 2 ,
\]  

i.e. at the midpoint of the intervals in which the \(\varphi_\|\) degrees of freedom move. This choice also minimizes the total energy; the zero point energy in \(h[\chi]\) (eq. (4.55)),

\[
\begin{align*}
13
\end{align*}
\]
though $\chi$-dependent, does not contribute to leading order in $\ell/L$. Thus our final result for $h_{\text{int}}$ reads

$$
h_{\text{int}} = \frac{1}{2\pi^2}\sum_{\mathbf{b}} \left\{ \left( \frac{\pi}{2} - \varphi_\mathbf{b}(\varphi) \right)^2 \left( 1 + \left( \sin \varphi_\mathbf{b}(\varphi) \right)^{-2} \right) + 2 \left( \frac{\pi}{2} - \varphi_\mathbf{b}(\varphi) \right) \cot \varphi_\mathbf{b}(\varphi) \right\} . \quad (4.64)
$$

The properties of the potential energy $h_{\text{int}}$ are directly correlated with the structure of the centrifugal term and the non-abelian magnetic field energy in eq. (4.50). This potential energy is local, i.e., it does not connect degrees of freedom at different lattice sites with each other. It therefore does not significantly change the formation of waves across the transverse plane nor their suppression in the strong coupling limit ($\kappa \gg 1$). With the centrifugal term, $h_{\text{int}}$ is singular at $\varphi_\mathbf{b}(\varphi) = 0, \pi$. Close to the origin the kinetic energy of the “radial” motion (cf. eq. (3.20)) is thus supplemented by a centrifugal barrier

$$
- \frac{\partial^2}{\partial \varphi_\mathbf{b}(\varphi)^2} \rightarrow - \frac{\partial^2}{\partial \varphi_\mathbf{b}(\varphi)^2} + \frac{\pi \ell}{\gamma^3 g^2 L} \frac{1}{\varphi_\mathbf{b}(\varphi)} . \quad (4.65)
$$

The presence of the centrifugal barrier forces the full wavefunction ($\Phi$ in eq. (3.18)) to vanish at the origin. In addition to the centrifugal barrier, the potential energy (4.64) exhibits a repulsive Coulomb-like singularity. The modification of the ground state energy due to $h_{\text{int}}$ is given by

$$
\Delta E_0 = \frac{L^2}{4\pi (\gamma \ell)^3} \left( 1 + \frac{\pi^2}{2} \right) . \quad (4.66)
$$

Thus coupling to the charged gluons gives rise to a non-vanishing and, in the continuum limit, divergent energy density (energy per unit area). This is the leading contribution in the continuum limit (taking into account the redefinition of the energy in eq. (3.22)). Nevertheless, the dominant operator of the Hamiltonian (4.49) remains the electric field energy $h^e$ (cf. eq. (3.20)).

Finally we discuss the result (4.63) concerning the average field which, in view of the singularities at 0, $\pi$, is plausible. This result actually reflects a symmetry of the exact Hamiltonian which guarantees quite generally that the potential energy of the $\varphi_\mathbf{b}$ degrees of freedom must be stationary at $\pi/2$. As shown in [14], the Hamiltonian (2.1) is invariant under “displacements”, “central conjugations”, and reversal of the color 3-axis. Following arguments developed in the context of 1+1 dimensional QCD [18] we can combine these transformations to a relevant symmetry transformation which changes the variables in the following way

$$
\begin{align*}
\varphi_\mathbf{b} & \rightarrow - \varphi_\mathbf{b} + \pi \\
\hat{\phi}_\perp(x) & \rightarrow e^{2i\pi x/L} \hat{\phi}_\perp(x) , \quad \hat{\Pi}_\perp(x) \rightarrow e^{2i\pi x/L} \hat{\Pi}_\perp(x) \\
\hat{A}_\perp^3(x) & \rightarrow - \hat{A}_\perp^3(x) , \quad \hat{B}_\perp^3(x) \rightarrow - \hat{B}_\perp^3(x) .
\end{align*}
\quad (4.67)
$$

The invariance of the exact (eq. (2.1)) as well as of the approximate (eq. (4.49)) Hamiltonian under this transformation is easily verified. After integrating out the charged gluons, the only remnant of the above symmetry is the transformation of the variables $\varphi_\mathbf{b}$. The symmetry therefore reduces to the invariance of $h_{\text{int}}$ under a common reflection of the variables $\varphi_\mathbf{b}$ at $\pi/2$. This choice in turn implies a charge conjugation symmetric Hamiltonian (cf. eqs. (4.55), (4.56)). As the net effect of the coupling to the two-dimensional fields, the spatially periodic charged gluon fields acquire energies $|\vec{p}(\vec{n})|$ which actually correspond to anti-periodic boundary conditions.
5 Physics in axial gauge QCD

On the basis of the results derived in the preceding sections, we shall develop in this concluding section the physics picture of QCD as it presents itself in the axial gauge. The essential result of our investigations is the suppression of plane wave type excitations, i.e., excitations where the electric field is constant along a particular spatial direction. In our choice of coordinates and gauge, we have identified the spatial 3-axis with the direction in which the field does not vary and have used the local gauge invariance to have the field pointed in the color 3-direction everywhere in space. With this choice of coordinates, these particular degrees of freedom — neutral, linearly polarized gluons — are described by two-dimensional fields resulting in a significantly simplified description of the dynamics. Such a simplification is not too surprising considering the fact that in QED any linearly polarized photon, with the corresponding choice of coordinates and in the absence of matter, is described by a non-interacting two-dimensional scalar field. In QCD the decoupling from the other degrees of freedom is not complete, as happens in QED when coupled to charged matter. A characteristic difference to QED is that the gluon self interactions give rise to a significant modification of the kinetic energy (i.e. electric field energy) of these particular degrees of freedom. This modification is a direct consequence of the projection onto physical states as obtained in the implementation of the Gauss law. It is analogous to the modification of the kinetic energy of a quantum mechanical particle when projected onto states with definite angular momentum. As a result of this change, eigenstates to this modified kinetic energy operator of QCD become normalizable, i.e., the electric flux associated with these normalizable states gets quantized. Excitation energies of states with non-vanishing electric flux diverge linearly with the size of the system. Up to this point, the size of the system has been treated as a formal parameter introduced to properly define the theory in the infrared. Whenever $L$ is large compared with the relevant intrinsic length scale $1/\Lambda$ after renormalization or, phenomenologically, the size of hadrons — finite size effects should be negligible. Success of present lattice calculations actually indicates that the size of the system may not have to be much larger than hadronic sizes. In this sense, our results suggest that the QCD vacuum does not support color-electric fields which are constant over distances larger than hadronic sizes. Excitation energies associated with such fields grow with the distance over which such fields are constant. This behavior is independent of their variation in transverse directions, i.e., independent of the associated total momentum. Thus finite excitation energies can result only for structures radically different from plane waves. In summary, color-electric fields constant over large distances are expelled from the QCD vacuum, i.e. the QCD vacuum exhibits the dual Meissner effect. This in turn implies that magnetic fields cannot be correlated at points in space separated by distances over which constant electric fields cannot extend.

As an application of this picture, we consider the interaction between two static color charges in comparison with the electrostatic interaction in the Maxwell theory. In the axial gauge representation of QED, the Coulomb-interaction between static charges is not manifest, rather a linearly rising potential appears in the expression (2.10) for the field energy. This contribution is however only part of the electrostatic interaction. In the axial gauge representation of QED (as in any but the Coulomb-gauge) static
charges couple to the radiation field as described by the interference between $\mathbf{\nabla}_\perp \mathbf{E}_\perp$ and $\rho^m$ in eq. (2.10). It is easy to decouple the radiation field from the charges by shifting the electric field

$$\mathbf{E}_\perp \rightarrow \mathbf{E}_\perp + e \mathbf{\nabla}_\perp \frac{1}{\Delta} \rho^m.$$  (5.68)

Thus by introducing a pair of static charges (with net charge zero at a distance $d$ from each other) into the QED vacuum, the equilibrium position of the electric field changes. The radiation field oscillates around the electrostatic field generated by the static charges. For harmonic oscillations, such a shift in equilibrium position is irrelevant apart from a “$\mathcal{C}$-number” change in the zero point energy, which is nothing else than the electrostatic Coulomb-energy

$$U = \frac{e^2}{4\pi d}.$$  (5.69)

In QCD a corresponding shift can also be performed; it decouples however “static” color charges and radiation field only perturbatively. Obviously, the dynamics of the color spin of the static quarks necessarily remains coupled to the gluons. On the other hand our above discussion suggests that the QCD vacuum resists such a shift in the color-electric radiation field. We have seen that the vacuum does not support electric fields which have small variations over large distances. Such fields are introduced however in the shift (5.68). In order to follow more closely how a linearly dependent potential between static quark and antiquark arises in axial gauge QCD we assume – beyond our above considerations – that the QCD vacuum is void of any color-electric field. Introducing a static quark-antiquark pair requires adjustment of the vacuum to account for the correspondingly modified Gauss law. In a gauge fixed formalism, this readjustment is enforced by the “color-electrostatic” field energy appearing in the Hamiltonian in the process of implementing the Gauss law. In the axial gauge, the energy density of the electrostatic field is given by the operator $\frac{1}{4}(\mathbf{E}^2 + \mathbf{\tilde{E}}^2)$ in the Hamiltonian of eq. (2.1). With an appropriate choice of the coordinates, it is possible to account for the necessary readjustment of the vacuum by changes in one of the degrees of freedom only. For this purpose, the coordinates have to be chosen such that quark and antiquark are located on the $3$-axis. In this case, we may assume that the color-electric fields $\mathbf{E}_\perp(x)$ remain undisturbed and essentially zero. Also, due to the neutrality condition which the static charges have to satisfy, the two-dimensional neutral, color-electric fields $\tilde{\eta}(x_\perp)$ remain zero. Thus introduction of a static quark-antiquark system affects only the neutral two-dimensional gluons $\varphi_\perp$ in the Hamiltonian (2.1). Furthermore, as we have seen, degrees of freedom at different lattice sites are decoupled in the strong coupling limit. Only the neutral degree of freedom $\varphi_0$ at the site $\tilde{b}_0$, corresponding to the transverse coordinate of the quark-antiquark position, is coupled to the static charges. The Hamiltonian describing this system of quantum mechanical degrees of freedom reads

$$\delta \mathcal{H} = -\frac{g^2 L}{8L^2} \frac{\partial^2}{\partial \varphi_0^2} + u.$$  (5.70)

The coupling potential is the electrostatic field energy $\mathcal{E}_3^2/2$ (cf. eq. (2.4)),

$$u = \frac{g^2 L}{4} \int_0^L \frac{dz_3 \int_0^L \frac{dy_3}{L} \sum_{n, m} \left(1 - \delta_{p, q} \delta_{n, 0}\right) \rho^m_{pp} \rho^m_{pp} \left(\frac{\hat{b}_0, z_3}{\pi n + (p - q) \varphi(\hat{b}_0)^2} \right)^2 e^{i2\pi n(z_3 - y_3)/L},}$$  (5.71)
which is given in terms of the static color charge density

\[ \rho_{pq}^m(\vec{b}, z) = \frac{g}{2\ell^2} \sum_{\alpha = q, \bar{q}} \left[ \psi_{\alpha q}^\dagger(\vec{b}, z) \psi_{\alpha p}^\dagger(\vec{b}, z) - \frac{\delta_{pq}}{2} \sum_{r = 1, 2} \psi_{\alpha r}^\dagger(\vec{b}, z) \psi_{\alpha r}^\dagger(\vec{b}, z) \right]. \] (5.72)

Here we have represented the color degrees of freedom of the static quarks in second quantized form. The operator \( \psi_{\alpha q}^\dagger(\vec{b}, z) \) creates a quark (\( \alpha = q \)) or antiquark (\( \alpha = \bar{q} \)) at the position \((\vec{b}, z)\). With our sequence of approximations we have reduced the calculation of the static quark interaction energy to a problem of coupled quantum mechanical degrees of freedom. The above Hamiltonian \( \delta h \) is, after redefinition of the coupling constant, identical with the Hamiltonian describing the interaction of static charges in SU(2) QCD in one space dimension. The spectrum of this Hamiltonian has been determined in [19] and yields for the interaction energy of static color charges

\[ U = \frac{3g^2 d}{8 \ell^2}. \] (5.73)

This result agrees with the strong coupling limit of the SU(2) static quark-antiquark interaction in Hamiltonian lattice gauge theory [20]. It is quite satisfactory that approximations with similar physics content, the common assumption of vanishing color-electric fields, yield in the two quite different formal approaches the same physical results. This is particularly remarkable in view of the different ways in which the gauge symmetry is treated.

For our derivation, the proper choice of coordinates has been instrumental. It is instructive to consider the problem with a choice of the 3-axis which does not coincide with the direction of the quark-antiquark dipole moment. In this case, the color charges of quark and antiquark contribute to the \( y_3, z_3 \) integrations for different values of the perpendicular coordinates \( \vec{b} \). As a consequence, in the corresponding \( n = 0 \) terms, quark and antiquark contributions do not cancel. They rather give rise to a centrifugal barrier with strength \( \propto g^2 L/\ell^2 \). In the continuum limit, an infinite interaction energy results. Thus, with this choice of coordinates, the assumption of vanishing color-electric fields \( \vec{E}_\perp \) is unattainable. The system has to respond to the non-vanishing net charges arising in the \( x^3 \) integrations and compensate these charges by color-electric fields. Thereby a flux tube between quark and antiquark is formed which, with such a choice of coordinates, involves the complicated dynamics of the coupled \( \vec{E}_\perp \) fields rather than the simple dynamics of the single degree of freedom \( \varphi_0 \).

Comparison of the result (5.73) with the expression (3.41) for the energy associated with the lowest excitation of the two-dimensional degrees of freedom \( \varphi_\vec{b} \) supports the arguments concerning the quantization of electric flux (cf. eq. (3.47)). We observe that unlike the lattice formulation, the axial gauge representation does not introduce compact variables which by construction lead to such a quantization of the electric flux. In axial gauge QED no such phenomenon occurs. It is the appearance of the Jacobian which forces the relevant QCD degrees of freedom not to be periodic but rather to be constrained to a compact interval which is defined by consecutive zeros of the Jacobian. Thus most crucial properties of the two-dimensional degrees of freedom pertinent to the issues of confinement and the dual Meissner effect are traced back to a fundamental difference between QED and QCD which becomes manifest in the
gauge fixed formulation. In the lattice formulation of QCD, a similar dynamical role is played by the Haar measure in the definition of the partition function. Our findings are reminiscent of the disappearance of confinement when the Haar measure is replaced by a constant [21].

In our discussion of the interaction energy of static quarks, a series of approximations has been necessary whose validity might be difficult to assess quantitatively. On the other hand, our treatment of the dynamics of the two-dimensional degrees of freedom $\varphi(x_\perp)$ has been quite straightforward and therefore provides a novel access to the strong coupling limit of QCD. As the fundamental approximation in this approach, the coupling of the neutral $\varphi(x_\perp)$ to the charged gluon fields has been neglected. With this coupling taken into account, the flux tubes are expected to acquire finite transverse extension and thereby the strong coupling result for the excitation energy (3.41) to be determined by the string tension

$$\frac{3g^2}{\ell^2} \to \sigma.$$  \hfill (5.74)

We have performed a first step beyond the “strong coupling limit.” The singular nature of the coupling to the charged gluons prevents straightforward application of perturbation theory and we have determined in a variational calculation the average effect of the charged gluons on the $\varphi(x_\perp)$ ground state. This improved ground state can serve as starting point of a systematic expansion. Calculation of this lowest order correction reveals the presence of two different formal, small parameters involved in such an expansion. On the one hand, ordinary perturbative treatment of the non-linearities of QCD in both the non-abelian part of the magnetic field energy (cf. eq. (2.3)) as well as in the non-abelian contribution to the perpendicular Gauss law operator $G_\perp$ (2.5) naturally requires

$$g \ll 1.$$  \hfill (5.75)

On the other hand, as the comparison of the correction (4.66) with the lowest order ground state result (3.40) shows, such an expansion involves as a small parameter

$$\frac{\ell}{g^2 L} \ll 1.$$  \hfill (5.76)

Obviously these two expansions can be made compatible only if the coupling constant $g$ is considered a function of the number of degrees of freedom ($L/l$). With the dependence

$$g^2 \propto \frac{1}{\ln (L/\ell)}$$  \hfill (5.77)

expected to hold in the continuum limit, the two requirements (5.75), (5.76) are indeed compatible. It is encouraging that the non-perturbative effects indicating confinement require a behaviour of the coupling constant in the thermodynamic limit which is compatible with asymptotic freedom. It is tempting to speculate that absence of confinement in Higgs-gauge theories (e.g. SU(2) with a scalar doublet) is in turn related to a behaviour of $g$ in the continuum limit of such models [22] which, in contradistinction to (5.77), is controlled by a cutoff rather than by the number of degrees of freedom.
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