Classification of inflationary Einstein–scalar–field–models via catastrophe theory

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Abstract

Various scenarios of the initial inflation of the universe are distinguished by the choice of a scalar field potential $U(\phi)$ which simulates a temporarily non-vanishing cosmological term. Our new method, which involves a reparametrization in terms of the Hubble expansion parameter $H$, provides a classification of allowed inflationary potentials and of the stability of the critical points. It is broad enough to embody all known exact solutions involving one scalar field as special cases. Inflation corresponds to the evolution of critical points of some catastrophe manifold. The coalescence of its nondegenerate critical points with the creation of a degenerate critical point corresponds the reheating phase of the universe. This is illustrated by several examples.

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**Introduction.** In the 80’s, Guth [1] and Linde [2] have modelled an inflationary phase of the universe (cf. [3]). Scalar fields (Higgs, axion) are expected to generate, shortly after the big bang, an exponential increase of the universe. The so-called *graceful exit* to the Friedmann cosmos was partly solved in the *new inflationary universe* [4]. In this model, the scalar field is ruled by a slightly different self–interaction potential which possesses a slow–roll part (a plateau) of the potential (acting as a vacuum energy) which dominates the universe at the beginning. Power–law models were constructed which possess no exponential but an $a(t) \sim t^n$ increase of the expansion factor of the universe [5–7]. The *intermediate inflation* is merely a combination of exponential and power–law increase [8]. Further solutions were found in [9]. Recently, by using the Hubble expansion parameter $H$ as a new “time” coordinate, we [10] were able to derive the general Robertson–Walker metric for a *spatially flat* cosmos. Our formal solution for arbitrary $U(\phi)$ comprises all previous exact solutions.

For a rather general class of inflationary models the Lagrangian density reads

$$\mathcal{L} = \frac{1}{2 \kappa} \sqrt{|g|} \left( R + \kappa [g^{\mu \nu}(\partial_\mu \phi)(\partial_\nu \phi) - 2U(\phi)] \right),$$

(1)

where $\phi$ is the scalar field and $U(\phi)$ the self–interaction potential. We use natural units with $c = \hbar = 1$. A constant potential $U_0 = \Lambda / \kappa$ would simulate the cosmological constant $\Lambda$. We do not separately discuss non–minimally coupled Jordan–Brans–Dicke type models [11] since they can be reduced to (1) via the Wagoner–Bekenstein–Starobinsky transformation [12–15]. Let us concentrate on the *flat* Friedmann–Robertson–Walker cosmos

$$ds^2 = dt^2 - a^2(t) \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right],$$

(2)

where $a(t)$ is the time–dependent expansion factor with the dimension length. Flat space is anyhow favored in the inflationary scenario. The scalar field depends only on the time $t$, i.e. $\phi = \phi(t)$.

**The Friedmann evolution equations.** Let us assume that $a(t) \neq 0$, such that we can express our equations completely [10] in terms of the Hubble expansion rate $H := \dot{a}(t) / a(t)$. Only the diagonal components of the Einstein equation are non–vanishing. The $(0, 0)$ component involving the density $\rho$, reads
\[3H^2 = \kappa \rho = \kappa \left( \frac{1}{2} \dot{\phi}^2 + U \right). \tag{3}\]

It describes the conservation of the energy. The (1, 1), (2, 2), and (3, 3) components are given by

\[2\dot{\mathcal{H}} + 3H^2 = -\kappa p = -\kappa \left( \frac{1}{2} \dot{\phi}^2 - U \right), \tag{4}\]

where \(p\) is the pressure generated by the scalar field. The resulting Klein–Gordon equation is

\[\ddot{\phi} = -3H \dot{\phi} - U'(\phi), \tag{5}\]

which is, after multiplication by \(\dot{\phi}\),

\[\frac{1}{2}((\dot{\phi})^2) = -3H(\dot{\phi})^2 - \dot{U}. \tag{6}\]

By linear combination of (3) and (4) we find the autonomous nonlinear system

\[\dot{\mathcal{H}} = \kappa U(\phi) - 3H^2 =: V(H, \phi), \tag{7}\]
\[\dot{\phi} = \pm \sqrt{\frac{2}{\kappa}} \sqrt{3H^2 - \kappa U(\phi)} = \pm \sqrt{-\frac{2}{\kappa} V(H, \phi)}. \tag{8}\]

The function \(V(H, \phi)\) will turn to be the “height function” in Morse theory [16]. Observe that (8) is, in view of (3) and (4), a first integral of (5). Moreover, \(V \leq 0\) in order to avoid scalar ghosts. For the metric (2), the Lagrangian density (1) reduces

\[\mathcal{L} = -\frac{3}{\kappa} \dot{a}^2 a + \left[ \frac{1}{2} \dot{\phi}^2 - U(\phi) \right] a^3. \tag{9}\]

Since the shift function is normalized to one for the metric (2), the canonical momenta are given by \(p = \partial \mathcal{L} / \partial \dot{a} = -6Ha^2 / \kappa\) and \(\pi = \partial \mathcal{L} / \partial \dot{\phi} = a^3 \dot{\phi}\). [Incidentally, this suggest to take the volume \(a^3\) as a generalized coordinate and use \(P = \partial \mathcal{L} / \partial (a^3) = -2H/\kappa\) as new momentum.] The Hamiltonian or “energy function” is given by

\[E = \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{\kappa} V(H, \phi) \right] a^3. \tag{10}\]
and vanishes for all solutions.

**Catastrophe of Whitney manifold and the critical points of the evolution.**

In the phase space \([n = b_1, n = d]\), the equilibrium states of the system \((7)\) and \((8)\) are given by the constraint \(\{\dot{H}, \dot{\phi}\} = 0\). The critical or equilibrium points, respectively, of this system are determined by \(V(H, \phi_c) = 0\). This constraint is globally fulfilled by \(\kappa U(\phi) = 3H^2\), where the Hubble expansion rate is constant, i.e. \(H_\Lambda =: \sqrt{\Lambda/3}\). For \(\dot{\phi} = 0\) and \(\Lambda \neq 0\), we obtain the de Sitter inflation with \(a(t) = a_0 \exp(\sqrt{\Lambda/3} t)\).

The Jacobi matrix \(J\) of the system \((7)\) and \((8)\) is given by

\[
J = \begin{pmatrix}
-6H & \kappa U'' \\
\pm 6H(-2\kappa V)^{-1/2} & -\kappa U'(-2\kappa V)^{-1/2}
\end{pmatrix},
\]

where \(U' = dU/d\phi\). Since \(\det J = 0\), the system is degenerate. For the analysis of stability, it suffices therefore to consider only \((7)\) and later to reconstruct \(\phi\). Since \(H\) and \(\phi\) are independent variables, we can introduce the non–Morse potential \(W(H, \phi)\), which is defined via \(V := -\partial W/\partial H\) and analyse the system with the aid of catastrophe theory. From \((7)\) we obtain

\[
W(H, \phi) = H^3 - \kappa U(\phi)H + C(\phi),
\]

where \(C\) is an arbitrary function of \(\phi\). The function \(W\) is already in canonical form in \(H\)-space, and belongs to a Whitney surface or to the Arnold singularity class \(A_2\) (cf. [18]). That means that our catastrophe manifold here is the Whitney surface. This manifold has only one control parameter, which is here given explicitly as the potential \(U\). Thus, an evolution of critical points are determined via the values of the potential \(U\). Let us analyse the types of critical points at different fixed values of the control parameter \(U\).

If \(U_c := U(\phi_c) < 0\), the equation has no stable critical point due to the shape of the Whitney surface. However, if \(U_c > 0\), there are two critical points: stable at \(H_c = \sqrt{U_c/3}\) and unstable at \(H_c = -\sqrt{U_c/3}\). For \(H_c > 0\), Walliser [19] comes to the same conclusion.

The value \(U_c = 0\) is the bifurcation point. Provided this is also an extrema of \(V\), we necessarily have \(\partial V/\partial H |_c = -6H_c = 0\), \(\partial V/\partial \phi |_c = \kappa U'_c = 0\), and \(\dot{\phi}_c = 0\). Thus, also the
critical points of the Klein–Gordon equation are involved. Hence, the Hubble parameter has to vanish and $\phi_c$ is a double zero of the potential $U$. The Hessian of (7) takes the form

$$\text{Hess}(V) = \begin{pmatrix} -6 & 0 \\ 0 & \kappa U'' \end{pmatrix}. \quad (13)$$

The sub–determinant of the Hessian is $\Delta_0 = \partial^2 V/\partial H^2 = -6 < 0$ and $\Delta_1 = \det \text{Hess}(V) = (\partial^2 V/\partial H^2) (\partial^2 V/\partial \phi^2) - (\partial^2 V/\partial H \partial \phi)^2 = -6\kappa U''$. For a maximum of the potential $U$, i.e. $U' = 0$ and $U'' < 0$, the function $V$ possesses a maximum; for a minimum of the potential $U$ we find a saddle point for $V$.

Alternatively, we can investigate the non–Morse potential (12) alone, e.g. for the chaotic inflationary model with $U(\phi) = \phi^2/\kappa$

$$W(H, \phi) = H^3 - \phi^2 H + C(\phi), \quad (14)$$

which has minimum and maximum at $H_c = \pm \phi_c/\sqrt{3}$ and a saddle point for $\phi_c = 0$. This corresponds to the end of inflation and a reheating of the universe [20].

In the case of power–law inflation, we have $U(\phi) = (\exp \phi)/\kappa$ and therefore obtain

$$W(H, \phi) = H^3 - e^\phi H + C(\phi), \quad (15)$$

which has the critical points $H_c = \pm e^{\phi_c/2}/\sqrt{3}$ and a saddle point for $\phi_c = -\infty$. The latter can be reached because of $\phi = \ln(1/t)$ for $t \to \infty$. In both models, the saddle point appears at that time of the reheating of the universe.

The last example here should be taken from the new inflationary model with $U(\phi) = [\phi^2(\phi^2 - A) + \delta]/\kappa$, where $A, \delta$ are two constants. The function

$$W(H, \phi) = H^3 - [\phi^2(\phi^2 - A) + \delta] H + C(\phi) \quad (16)$$

possesses minima and maxima at $H_c = \pm \sqrt{U(\phi_c)/3}$ and two saddle points at $\phi_c = \pm \sqrt{A/2}$, the zeroes of the potential, if $D = A^2/4$. That means that a shift of the potential is necessary, such that the zeroes of the potential are also its minima.

The reheating phase of the universe. Since $U(\phi_c)$ corresponds to the latent heat of the universe in this phase, we are now in the position to state the following:
Theorem: The critical points of the non–Morse potential \( W(H, \phi) \) determine the evolution in the inflationary phase. Along the minima and maxima \( H = \pm \sqrt{U(\phi)/3} \), the inflaton moves from the slow–roll to the hot regime. The saddle points of \( W \), i.e. more precisely, the minima of \( V \), determine the onset of reheating.

The critical points are related to some turning points of the evolutions. Since our potential \( U \) or the field \( \phi \), respectively, depend on time, we obtain the following scenario: The system starts from some initial conditions. In the first stage of the inflation, the stable critical points define the regime of the evolution, that is the system evolves towards such points. In the second stage, because our control parameters (here the potential \( U \) or the field \( \phi \), respectively) also depend on time, the evolution moves along the critical points. Consider, for example, the chaotic inflationary model in Eq. (14): The system being initially located in a minimum, evolves to the saddle point; physically, this means the end of the inflation and a reheating of the universe. In this way, we have derived a universality class for inflationary models. This picture is very general, as we are going to show in the next section by means of catastrophe theory.

Bifurcation and other catastrophes. For the analysis of critical points of multidimensional functions (see [18]) the Hessian of (12) is usually employed. It has the form:

\[
Hess(W) = \begin{pmatrix}
6H & -\kappa U' \\
-\kappa U' & -H \kappa U'' + C''
\end{pmatrix}.
\] (17)

Thus the classification of all critical points may be given by the Whitney theorem [18]. However, the explicit form of (12) allows to solve this problem without an analysis of the Hessian. The given form of \( W \) suggests that we have here umbilic catastrophes. There are only a few of them: elliptic \( D^e_4 \) or hyperbolic \( D^h_4 \), second elliptic or second hyperbolic \( (D^e_6 \text{ or } D^h_6 \), respectively), and symbolic umbilic \( E_6 \). With the knowledge of the form of these catastrophes, which are given for example in Ref. [18] one can completely predict the evolution of the system. For the example of the chaotic inflation, Eq. (14) describes an elliptic umbilic catastrophe which has, in this special form, only one critical point.

For \( \kappa U(\phi) \neq 3H^2 \), we find \( \{\dot{H}, \dot{\phi}\} \neq 0 \), which implies that the solutions \( \phi = \phi(t) \) and
$H = H(t)$ are invertible, i.e. $t = t(H) = \int \frac{dH}{\kappa \sqrt{-3H}}$. In contrast to the construction of Lidsey [22] in which the scalar field itself is employed as a new time variable, our approach is also valid for the end of inflation. Then we can write the potential in (7) and (8) in the reparametrized form [10]

$$U(\phi) = U(\phi(t)) = U(\phi(t(H))) = \tilde{U}(H).$$

(18)

The reduced problem is given by the one-dimensional equation

$$\dot{H} = \kappa \tilde{U}(H) - 3H^2 = g(H),$$

(19)

where $g(H)$ is the “graceful exit function” of Ref. [10].

All solutions of this equation may also be classified with the aid of the catastrophe theory [18]. Eq. (19) has critical points which are determined via the function $\tilde{W}(H)$, defined, analogously to $W$, by $g := -d\tilde{W}/dH$. The simplest case arises when $\tilde{W}(H)$ is exactly the Morse potential, i.e. $\tilde{W}(H) = \lambda H^2/2$, i.e. its critical points are nondegenerate. In this simple case the solution depends on the sign of $\lambda$. If $\lambda > 0$, the only possible stable state, to which the system evolves, is $H = 0$.

For $\lambda < 0$, we obtain $H \sim C \exp \lambda t$ and the system has no stable critical points. In that case $a \sim C_1 \exp(c \exp(\lambda t))$. Then, the universe would be expanding too fast.

The regimes of evolutions, or critical points, depend on the shape of the potential $\tilde{W}(H)$. With the aid of the theory of singularities (in that particular case with the aid of elementary catastrophe theory [18]) we may classify the inflation regimes. If the potential $\tilde{W}(H)$ is a smooth function it belongs to the one of the Arnold classes $A_n$, where $n \geq 2$.

The canonical form of the $n$-th class is $\tilde{W}(H) = \lambda H^{n+1}$. The co-dimension of that potential is equal to $n - 1$, that is, the number of control parameters is equal to $(n - 1)$.

The critical points, which are structurally stable correspond to the minima and maxima of the polynomial

$$\tilde{W}(H)_{\text{def}} = \lambda(H^{n+1} + \lambda_{n-1} H^{n-1} + \lambda_{n-2} H^{n-2} + \ldots + \lambda_0).$$

(20)
The system described by (19) make an evolution to the minima of $W(H)_{\text{deform}}$. In each of the minima, we have $\dot{H} = 0$ and $H = \text{const}$. The maxima of the potential (20) correspond to unstable points for which the instability transforms the system to its minima. With the change of the control parameters, some minima coalesce with maxima or vice versa. As a result, the number of minima changes. Each minimum is associated with a distinct constant value of the Hubble expansion rate.

That is, each of the critical points corresponds to a different de Sitter type inflation. However, as in the considered case of the Whitney catastrophe the coalescence of these nondegenerate critical points into degenerate one mean the reheating phase of the universe. That is, locally, on the considered complicated manifold, we have a Whitney sub–manifold. The inflationary evolution and the picture of the creation of the reheating phase is the same as it is described above.

However, there are a few exceptions related to the instabilities. One of these instabilities for the non–Morse potential we have already discussed above. Asymptotically, for a large time scale, the instabilities are defined by the leading term of the polynomial $W_{\text{deform}}(H)$.

Asymptotically, if $\lambda < 0$ and $n \geq 0$, we may write for the $n$–th Arnold class

$$g(H) = -(n + 1)\lambda H^n,$$

such that (19) yields the solution

$$H = \left[ C + (n^2 - 1)\lambda t \right]^{1/(1-n)}$$

where $C$ is some positive constant. These instabilities have no physical meaning.

Among the explicit models which have been analyzed in [10], we consider as an instructive example $g(H) = -2H^2/A^2 + 2A^2\lambda^2$ (in the notation of [9]). This corresponds again to the Whitney catastrophe or to the second Arnold class $A_2$. Therefore, we may expect the described above picture. Let us show it explicitly. This ansatz leads to

$$\dot{\phi}(t) = A \ln[\tanh(\lambda t)],$$

$$H(t) = A^2 \lambda \coth(2\lambda t),$$

where
\[ a(t) = a_0 [\sinh(2\lambda t)]^{1/2} \]  
\[ U(\phi) = A^2 \lambda^2 \left[ (3A^2 - 2) \cosh^2 \left( \frac{\phi}{A} \right) + 2 \right] \]
as solution. Thus we have recovered one of the recent solutions of Barrow \[9\], for further details see \[10\].

Now we have our two possible descriptions of the investigations of the critical points. The potential \( U(\phi) \) has to be shifted to

\[ U(\phi) = A^2 \lambda^2 (3A^2 - 2) \cosh^2 \left( \frac{\phi}{A} \right), \]

so that the only critical point is at the origin \( \phi = 0 \), which is a minimum of \( U \) and hence a saddle point of \( W \). Alternatively, we can use \( g(H) \), which also has to be shifted into

\[ g(H) = -2H^2/A^2. \]

The non-Morse potential \( \overline{W} = 2H^3/3A^2 \) possesses, too, a saddle point at \( H = 0 \). In the two equivalent descriptions we were able to find that the saddle point corresponds to the reheating phase.

Furthermore, it is now possible to relate the “slow-roll” condition, for the velocity of the inflationary phase, to the critical points resulting from catastrophe theory. For inflation (with \( \ddot{a} > 0 \)) the two “slow-roll parameters” are given \[10\], in first order approximation, by \( \epsilon = -g/H^2 \) and \( \eta = -dg/d(H^2) \), where \( g \) is the “graceful exit function” given in \(19\). In this reduced dynamics, they are effectively determined by the first and second derivatives of the reduced non-Morse function \( W(H) \), i.e., more precisely by \( \epsilon = -(1/H^2)(dW/dH) \) and \( \eta = (1/2H)d^2W/(dH)^2 \). They will also determine the density fluctuations \[21\].

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