Geometry of Quantum Mechanics

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Abstract. Quantum mechanics is formulated on the true space of physical states—the projective Hilbert space. It is found that the postulates of quantum mechanics assume an intrinsically geometric form. In particular, observables are represented by real-valued functions, the Lie bracket on the space of observables is given by a Poisson structure, and the Schrödinger evolution is equivalent to the flow along a Hamiltonian vector field.

I. INTRODUCTION

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In classical mechanics, states are represented by points of a symplectic manifold and the space of observables consists of the (smooth) real-valued functions on this manifold. To each classical observable is associated a so-called Hamiltonian vector field and the dynamics is simply given by the flow along the Hamiltonian vector field corresponding to the Hamilton function of the system. Classical mechanics is thus intrinsically *geometric*. The arena for quantum mechanics, on the other hand, is a Hilbert space. States of a quantum system are represented by rays in this Hilbert space, while the observables are self-adjoint linear operators. The evolution is determined by the Hamiltonian operator via the Schrödinger equation; the evolution is *linear*. The formalism of quantum mechanics is, it seems, intrinsically *algebraic*. Algebraic notions seem to be essential also in the formulation of measurement theory.

In spite of these striking differences, there are certain heuristic similarities. In both descriptions, the space of observables possesses two algebraic structures—a Lie bracket and an abelian operation. In classical mechanics, the Lie algebra is provided by the Poisson bracket and the abelian product by the pointwise multiplication of functions on the phase space. In quantum mechanics, the Lie bracket is given by ($-i$ times) the commutator while the anti-commutator—the *Jordan product*—provides the abelian product. In each

case, the Lie bracket acts as a derivation on the commutative algebra. The difference is that the classical algebra is *associative*, while the quantum (Jordan) algebra is not. In each case, the evolution is obtained by exponentiating the infinitesimal transformation generated by a preferred observable.

We will see in this paper that these similarities are in fact much more natural than they might appear at first. To see this, recall first that the space of quantum states is not the Hilbert space itself, but the space of rays, i.e. the *projective* Hilbert space. It is natural to ask whether quantum mechanics admits a formulation which is intrinsic to this true space of states. We will see that when expressed in terms of structures inherent to the projective Hilbert space, quantum mechanics assumes an intrinsically *geometric* form. Moreover, we will find that the quantum description is, in a certain sense, a special case of classical mechanics!

This work was intended to be an extension of a paper by Kibble [1]. However, after completing it, we learned of similar results by a number of other authors (see, e.g., [2-4]). Unfortunately, due to space limitation, we will not be able to provide a chronological account of these developments. Our goal here is only to present a pedagogical review in which the structural issues are brought to forefront. Thus, we will also omit proofs and, furthermore, restrict attention to Hilbert spaces of finite dimension (e.g. spin systems). However, most of our results admit extensions to the infinite-dimensional case. A more complete treatment will appear elsewhere.

II. GEOMETRIC STRUCTURES

Denote by $\mathcal{P}$ the space of rays in the Hilbert space $\mathcal{H}$. Since any finite-dimensional Hilbert space can be identified with C^n , $\mathcal{P}$ is simply the complex projective space—the archetypical Kähler manifold. Thus, $\mathcal{P}$ is a genuine manifold with two geometric structures—a Riemannian metric and a symplectic form. In section II.A, we construct the projective space and explain the relationship between these structures and the Hermitian inner-product on $\mathcal{H}$. In II.B, we discuss the notion of the observables of the theory and consider the role of the symplectic structure. Section II.C is concerned with the aspects of quantum mechanics which have no classical analogs. These are related to the presence of the metric.

A. The projective Hilbert space

Let us begin by viewing $\mathcal{H}$ as a *real* vector space equipped with a complex structure, J — a linear operator on $\mathcal{H}$ satisfying $J^2 = -1$. Let us decompose the Hermitian inner-product into real and imaginary parts:

$$\langle \Psi | \Phi \rangle = \frac{1}{2}G(\Psi, \Phi) - \frac{i}{2}\Omega(\Psi, \Phi). \quad (1)$$

It is straightforward to verify that G is a positive definite (real) inner-product on $\mathcal{H}$, and that Ω is a symplectic form.

In the text-book treatments, one restricts attention to the sphere of vectors with unit norm, which we will denote by S . However, since any two normalized states which differ by a phase represent the same physical state, the true space of states of the quantum system is given by the quotient of S by the $U(1)$ -action, i.e. the projective Hilbert space $\mathcal{P}$ (see Fig. 1). Denote by $\tilde{G}$ and $\tilde{\Omega}$ the pull-backs to S of the metric and symplectic structure, respectively. Notice that the phase rotations are simply given by the flow along the vector field $\xi|_S := \frac{d}{dt}e^{-Jt}\Psi|_{t=0}$. Since ξ is the generator of a one-parameter subgroup of unitary transformations on $\mathcal{H}$, its action preserves both $\tilde{G}$ and $\tilde{\Omega}$. Now, since ξ is a Killing vector field, and $\mathcal{P}$ is the manifold which consists of the integral curves of ξ , there is a unique projection, g , of the metric $\tilde{G}$ to $\mathcal{P}$. Similarly, since ξ preserves the 2-form $\tilde{\Omega}$ and lies in the direction of degeneracy of $\tilde{\Omega}$, there is a unique symplectic 2-form, ω on $\mathcal{P}$, such that $\pi_*\omega = \tilde{\Omega}$, where $\pi : S \rightarrow \mathcal{P}$ is the projection defining $\mathcal{P}$.

Therefore, the structure on $\mathcal{P}$ which is induced by the Hermitian inner-product is given by a Riemannian metric g , and a symplectic 2-form ω . It is easy to show that the pair (g, ω) is compatible in the sense that it defines a Kähler structure on $\mathcal{P}$.

B. Symplectic geometry

Since the space $\mathcal{P}$ of quantum states is a Kähler manifold it is, in particular, a *symplectic* space and can therefore be regarded as a “phase space.” It is then natural to ask if the mathematical notions from classical mechanics play a direct role in quantum mechanics.

Let us first focus on observables. The analogy with classical mechanics suggests that to each self-adjoint operator F on $\mathcal{H}$, we attempt to associate a real-valued function on $\mathcal{P}$. Fortunately, there is an obvious candidate. The expectation value of F is a real-valued function on $\mathcal{H}$ whose restriction to S is constant along the orbits of the phase rotations (Fig. 1). Thus, there exists a function $f : \mathcal{P} \rightarrow \mathbb{R}$ such that $\pi_*f = \langle F \rangle|_S$; we will write simply $f = \langle F \rangle$. One

Figure 1: The projection map π from the unit sphere S in the Hilbert space $\mathcal{H}$ to the space of states $\mathcal{P}$.

can show that the Hamiltonian vector field X_f associated to this function is a *Killing vector field*: the motion generated by such a function preserves not only the symplectic structure –as does *any* Hamiltonian vector field– but also the metric. The converse also holds. Thus, there is a one-to-one correspondence between the self-adjoint operators on $\mathcal{H}$ and those real-valued functions on $\mathcal{P}$ whose Hamiltonian vector fields are the Killing vectors of g . Let us refer to these functions as “observables”, and denote the space of such observable functions by $\mathcal{O}$.

The association of self-adjoint operators on $\mathcal{H}$ to the observable functions on $\mathcal{P}$ has a striking consequence. Notice that the operator F determines a vector field, $X_F|_\Psi := \frac{d}{dt}e^{-JFt}\Psi|_{t=0} = -JF\Psi$, on S . (Recall that J is the complex structure representing $\sqrt{-1}$.) Since X_F generates a one-parameter group of unitary transformations on $\mathcal{H}$, it preserves the Hermitian inner-product, and hence both $\hat{G}$ and $\hat{\Omega}$. Now, the key point is that X_F on S projects down unambiguously under π and that the projection is precisely the Hamiltonian vector field X_f on the symplectic manifold $(\mathcal{P}, \omega)$. Let us set $F = H$, the Hamiltonian operator of the quantum system. Then, X_H is the dynamical flow on S , given by the Schrödinger equation, and the induced evolution on $\mathcal{P}$ coincides with the (“classical”) Hamiltonian flow X_h , where $h = \langle H \rangle$, the expectation value function. Thus, on $\mathcal{P}$, the space of “true” quantum states, the Schrödinger evolution is replaced by the Hamiltonian dynamics, exactly as in classical mechanics!

Finally, we can examine the meaning of the Poisson bracket. Let ω^{ab} be the inverse of ω_{ab} (i.e., the Poisson structure): $\omega^{ac}\omega_{bc} = \delta^a_b$. The Poisson bracket is then defined as

$$\{f, g\} := \omega(X_g, X_f) = \omega^{ab}(\partial_a f)(\partial_b g). \quad (2)$$

Of course, the Poisson bracket is well-defined for any functions on $\mathcal{P}$. However, when restricted to observable, one again finds a striking result:

$$\langle -i[F, G] \rangle = \{f, g\}. \quad (3)$$

Note the close similarity between (3) and Dirac's heuristic correspondence between the Poisson bracket of *classical* observables and the commutator of the “corresponding” quantum operators. If one uses the “quantum phase space” $\mathcal{P}$, the correspondence is in fact exact and holds for *all* self-adjoint operators.

Let us summarize. The role of the symplectic structure on the “quantum phase space” is completely analogous to that of classical mechanics. The Schrödinger evolution is replaced by the flow along the vector field associated with the Hamiltonian *function* on $\mathcal{P}$. The canonically defined Poisson bracket on the algebra of observables corresponds *precisely* to the commutator of self-adjoint linear operators on $\mathcal{H}$. Recall, however, that only those functions whose Hamiltonian vector fields preserve the metric are observable. For systems with finite-dimensional Hilbert spaces, the algebra of all smooth functions on $\mathcal{P}$ is infinite dimensional, while the algebra of observables is finite dimensional. Thus, observables constitute a very small subset of the set of functions on $\mathcal{P}$.

C. Riemannian geometry

While Riemannian geometry is not an element of the classical description, it *is* an essential ingredient of quantum mechanics. Recall that the space of self-adjoint operators on $\mathcal{H}$ possesses the structure of a Jordan algebra, which is given by the *anti*-commutator. The role of the metric is most easily illuminated by representing the Jordan product in terms of structure inherent to the projective space. By analogy with the definition (2) of the Poisson bracket, we define the following “Riemannian product” on $\mathcal{O}$:

$$(f, g) := g(X_f, X_g) = g^{ab}(\partial_a f)(\partial_b g). \quad (4)$$

After a short calculation, one finds that the anti-commutator is related to this operation:

$$\langle [F, G]_+ \rangle = \{f, g\}_+ := (f, g) + 2fg. \quad (5)$$

It is the *symmetric bracket* $\{f, g\}_+$ which corresponds to the anti-commutator of the associated self-adjoint operators. In particular, the algebra of observables is not closed under the operation of pointwise multiplication of functions nor under the Riemannian product of Eq. (4).

The Riemannian bracket does, nonetheless, admit a physical interpretation. First consider the Riemannian product of an observable f with itself. Eq. (5) implies:

$$(f, f) = \{f, f\}_+ - 2f^2 = 2(\langle F^2 \rangle - \langle F \rangle^2) = 2(\Delta f)^2. \quad (6)$$

Therefore, the length of X_f is the *uncertainty* of the observable f ! (The factor of $\sqrt{2}$ arose due to our convention in (1).)

More generally, the uncertainties of the observables f, g satisfy the inequality

$$(\Delta f)^2(\Delta g)^2 \geq \left[\frac{1}{2}\{f, g\} \right]^2 + \left[\frac{1}{2}(f, g) \right]^2. \quad (7)$$

Notice that this relation is *stronger* than the usual Heisenberg uncertainty relation, which is obtained by dropping the term involving the Riemannian product. Clearly, the Heisenberg uncertainty relation is then saturated only if the Riemannian product vanishes. In this sense, $(f, g)(\psi)$ ‘measures’ the coherence of the state ψ with respect to the observables f and g .

Some of the most interesting issues in quantum mechanics are related to the measurement problem. Note that the eigenvectors of the self-adjoint operator associated to f correspond to points $\psi_i \in \mathcal{P}$ at which f attains local extrema; the eigenstates are the zeros of the Hamiltonian vector field X_f . Further, the associated eigenvalues are simply the values $f_i = f(\psi_i)$ at these zeros. Thus, measurement of f must yield one of the values f_i .

But what are the corresponding probabilities? A detailed calculation provides a beautiful correspondence between the geodesic separation of points on $\mathcal{P}$ and the (squared) modulus of the Hermitian inner-product; if $\pi(\Psi) = \psi$ and $\pi(\Phi) = \phi$, then

$$|\langle \Psi | \Phi \rangle|^2 = \cos^2 \left(\frac{\sigma(\psi, \phi)}{\sqrt{2}} \right), \quad (8)$$

where $\sigma(\psi, \phi)$ is the geodesic distance separating ψ and ϕ . In particular, if immediately prior to measurement of f , the system is in the state labelled by the point $\psi \in \mathcal{P}$, then the probability $P_i(\psi)$ of obtaining the eigenvalue f_i is simply given by

$$P_i(\psi) = \cos^2 \left(\frac{\sigma(\psi_i, \psi)}{\sqrt{2}} \right). \quad (9)$$

If f_i is a degenerate eigenvalue, there is a submanifold $E_i \subset \mathcal{P}$ of associated eigenstates. The probability that measurement of f will yield the value f_i then

becomes

$$P_i(\psi) = \cos^2 \left(\frac{\sigma(E_i, \psi)}{\sqrt{2}} \right), \quad (10)$$

where $\sigma(E_i, \psi)$ is the length of the shortest curve with initial point ψ and terminal point on E_i . Further, if measurement of f yields this degenerate eigenvalue, the state of the system is the terminal point of the curve defining $\sigma(E_i, \psi)$.

There is then a precise sense in which the measurement process is more likely to force a “collapse” to a nearby state than to a distant state.

III. DISCUSSION

Results of Section 2 demonstrate that quantum mechanics can also be treated in a geometric manner. In fact, one may forget about the linear space altogether and formulate the postulates of quantum mechanics in terms of Kähler geometry: The space of states of a quantum mechanical system is a projective Hilbert space $(\mathcal{P}, \langle \cdot, \cdot \rangle, \omega)$ —a Kähler manifold. The observables are those functions on $\mathcal{P}$ whose Hamiltonian vector fields are Killing vectors of g . This space of observables is closed under the Poisson bracket as well as the symmetric bracket defined by Eq. (5). Time evolution is given by the flow along the Hamiltonian vector field associated to a preferred observable, just as in the classical description. However if one performs a measurement of the observable f , one will obtain one of the extremal values of f . Immediately after obtaining such a value, the state will lie on the corresponding eigen-submanifold, with probabilities given by certain geodesic distances (10).

Notice that the symplectic structure plays the same role as in classical mechanics. Hence, the geometric formalism provides a natural language to analyze the problem of taking the classical limit. Even more striking is the fact that, from the mathematical point of view, the intrinsically quantum mechanical notions (uncertainty and “collapse of the wave-function”) are governed by the metric, which does not appear in the classical description. The unitary evolution is described by the symplectic structure, while the “reduction” part of the evolution is described by the metric. This separation of the dynamics is of considerable interest to the analysis of the measurement problem.

Potentially, the most powerful application of the geometric formulation could come from the fact that it suggests natural generalizations of quantum mechanics. To streamline one’s efforts, however, it is important to first obtain a characterization of *standard* quantum mechanics. Fortunately such a characterization now exists, thanks to the following theorem: *If $\mathcal{M}$, is a compact, simply-connected, complete, Kähler manifold which admits the maximum num-*

ber of observables, then $\mathcal{M}$ is a projective Hilbert space. This result suggests extensions in several directions. For example, one may use a Kähler manifold which is *not* a projective Hilbert space for one's space of states, admit a wider class of observables or consider evolution along the flow generated by more general functions. (The last strategy reproduces the known non-linear generalizations of the Schrödinger equation.) The theorem provides criteria to measure “how much of a generalization one has made”.

It has taught us that, to be viable, such generalizations have to be quite subtle. In particular, they must be finely tuned to ensure compatibility with the experimentally verified probabilistic predictions of ordinary quantum mechanics. Finally, there is an important caveat: although most of the postulates of quantum mechanics assume an elegant form when expressed in geometrical terms, statements involving spin statistics are *not* naturally phrased in such a way essentially because the projective space arising from the direct product of two Hilbert spaces is not conveniently described in terms of the two projective spaces alone. Any viable generalization will have to overcome this difficulty.

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