Continuous media interpretation of
supersymmetric Wess-Zumino type models

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Abstract

Supersymmetric Wess-Zumino type models are considered as classical material media that can be interpreted as fluids of ordered strings with heat flow along the strings or a mixture of fluids of ordered strings with either a cloud of particles or a flux of directed radiation.
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Introduction. The continuous media interpretation of a given field theory has been of great utility in finding specific models of either perfect or anisotropic fluids with simple physical and mathematical properties. For instance, it is well known that scalar fields may be used to model different material media such as irrotational perfect fluids [1] and anisotropic fluids [2]. Non-linear sigma models may model topological defects as cosmic strings [3], cosmic walls and bubbles [4].

Following the same idea, Dirac and Weyl spinor fields also have been used to construct models of continuous matter. In particular, massive neutrinos can behave as perfect fluids [5]. In general, these fields can be used to represent non-perfect fluids [6, 7, 8]. Furthermore, anisotropic fluids with interesting physical properties and fluids of cosmic strings with heat flow along the strings can also be constructed from non-compact sigma models [9].

In such a formalism the fields are considered as potentials that describe the fluids, i.e., the usual quantum field theory interpretation is abandon. Then, following this principle, it is a natural question to ask about the “fluid interpretation” of supersymmetric fields. In this paper we study the Wess-Zumino model and following the principle of considering each field as a matter potential we construct some physically meaningful material media. We would like to mention that a model of supersymmetric fluid that does not follow the underlying principle mentioned above has already been discussed [10]. This model, as the author mention, has not a clear physical interpretation.

We show that it is possible to have an interpretation of the supersym-
metric Wess-Zumino model as continuous media if the fields are taken classically. In such a case, the spinor field does not contribute to the metric (symmetrized) energy-momentum tensor (EMT) being a *gravitational ghost field* [7]. The resulting EMT is due only to the scalar and pseudo-scalar fields of the model. Then, in order to have a contribution to the EMT from the spinor field, it is necessary to introduce in the standard Wess-Zumino Lagrangean terms that are usually neglected, as total divergences or non-renormalizable (even though regularizable) terms [11]. It is shown that when such terms are taken into account the spinor field can have a significant contribution in the resulting EMT.

Considering a particular interaction term we find that it is possible to construct models of anisotropic fluids wherein the pressure along the anisotropy direction (perpendicular) is smaller than the isotropic (parallel) pressure. This fact is of particular interest because it represents a physical condition necessary to model objects in astrophysics [12] and cosmology [13]. A second model that can be constructed from the present approach is a cosmic string fluid with heat flow along the strings. Strings with heat flow along its length have been considered in the context of geometric extensions of Nambu strings [14]. It is also found models that are a mixture of cosmic string fluid with either a cloud of particles or a flux of directed radiation.

*The energy-momentum tensor of Wess-Zumino model.* The supersymmetric Wess-Zumino model is given by the following Lagrangean [15]

\[
L = \frac{1}{2} [A^\mu A_{,\mu} + B^{\mu B_{,\mu}} + \bar{\Psi}(i\partial - m)\Psi - g\Psi(A + i\gamma^5 B)\Psi - V(A, B)],
\]  

(1)
with

\[ V(A, B) \equiv m^2(A^2 + B^2) + 2mgA(A^2 + B^2) + g^2(A^2 + B^2)^2, \quad (2) \]

where \( A \) and \( B \) are both Hermitians spin-0 fields (scalar and pseudo-scalar, respectively) and \( \Psi \) is a spin-1/2 Majorana field. The base space of the model is the superspace [16], being the sector related to the space-time coordinates the usual Minkowski space. The results that we shall be concern in this paper do not depend on this particular choice and may be easily generalized in the usual way to a local Lorentzian subspace what makes possible the coupling with Einstein equations.

The Lagrangean (1) in fact corresponds to the on shell representation of the model where the fields satisfy the equations

\[
(+m^2) A = -g \left[ \bar{\Psi} \Psi + mA(3A^2 + B^2) + 2g A(A^2 + B^2) \right], 
\]

\[
(+m^2) B = -g \left[ i\bar{\Psi} \gamma^5 \Psi + 2mA B + 2g B(A^2 + B^2) \right], 
\]

\[
(i\partial - m) \Psi = 2g(A + i\gamma^5 B)\Psi. 
\]

We may consider in the Lagrangean (1) additional total divergence terms (topological terms) as

\[
L_D = \Omega_{\mu, \nu}, \quad (6) 
\]

that without changing the dynamics of the model can yield non-trivial changes in the EMT. For instance, (6) contributes with a new term of the form [17]

\[
T_{\mu \nu} = \Omega_{\mu, \nu} + \Omega_{\nu, \mu}, \quad (7) 
\]
which may modify the resulting continuous media changing for instance the state equations.

Another way to modify the resulting EMT without breaking the supersymmetric character of the model is by considering in the Lagrangean non-renormalizable terms. Following Weinberg [11] we may also consider “regularizable” terms which do not break the symmetry of the model and there is no physical reason to neglect. Such terms can be written using the “invariants” constructed with the conserved currents of the original model.

For our purposes it is also interesting to consider other kinds of non-renormalizable term like

$$L_I = \alpha A_\mu \bar{\Psi} \gamma^\mu \Psi.$$  \hspace{1cm} (8)

This simple term is particularly interesting because when \( g = 0 \) and \( m = 0 \) [cf. (1)] is an invariant constructed with two conserved currents: \( I^\mu = A_\mu \) and \( J^\mu = \bar{\Psi} \gamma^\mu \Psi \).

Let us write the (symmetrized) EMT obtained from (1) with the additional term \( L^1 \)

$$T_{\mu\nu} = A_\mu A_\nu + B_\mu B_\nu - \frac{1}{2} g_{\mu\nu} [A^\rho A_\rho + B^\rho B_\rho - V(A, B)]$$

$$+ \frac{i}{8} (\bar{\Psi} \gamma_\mu \Psi_\nu - \bar{\Psi}_\nu \gamma_\mu \Psi + \bar{\Psi}_\nu \gamma_\nu \Psi_\mu - \bar{\Psi}_\mu \gamma_\nu \Psi + \bar{\Psi}_\mu \gamma_\nu \Psi_\nu + T_{\mu\nu}^1),$$  \hspace{1cm} (9)

where \( T_{\mu\nu}^1 \) stands for the part of the EMT associated with (6) and/or (8).

To have a consistent supersymmetric theory the components of the spinor field \( \Psi \) must be anticommuting Grassmann numbers. This implies that the EMT (9) cannot be considered as a source of the classical gravitational field,
because its components belong to the Grassmann algebra while the components of usual Einstein equations are real. In order to overcome this difficulty, it is usually taken the expected values of the energy-momentum tensor operator $T_{\mu\nu}$ as the source of the Einstein equations. In this paper we are interested in the interpretation of each field as a “matter potential”, in the sense of the Debye potentials for fluids. In other words, we shall rewrite (9) in an appropriated form and compare it to the energy-momentum tensor of a “fluid”. A simple way of solving this problem for the present case is to consider all the fields of the model classically, i.e., the fields $A$ and $B$ as scalar fields, while the spinor field $\Psi$ as a classical (commuting) Majorana field with $c$-number components. We shall adopt this strategy and call the resulting model as the “classical interpretation of the Wess-Zumino model”.

An important property of the classical Majorana spinor field is that the corresponding symmetrized EMT vanish identically, i.e., it is a gravitational ghost field [6, 7]. In the Weyl representation the commuting bi-spinors satisfy [18], $\bar{\psi} \sigma_{\mu} \psi_{,\nu} = \bar{\psi}_{,\nu} \sigma_{\mu} \psi$. Then identical terms in (9) cancel and the resulting EMT for the classical interpretation of the on shell Wess-Zumino model reduces to

$$T_{\mu\nu} = A_{,\mu} A_{,\nu} + B_{,\mu} B_{,\nu} - \frac{1}{2} g_{\mu\nu} \left[ A^{,\rho} A_{,\rho} + B^{,\rho} B_{,\rho} - V(A, B) \right] + T^1_{\mu\nu}, \quad (10)$$

The contribution of the Majorana spinor field appears in $T^1_{\mu\nu}$, i.e., in the case that $T^1_{\mu\nu} = 0$, the spinorial part of the usual supersymmetric Wess-Zumino model does not contributes to the resulting continuous media that follows from (10). However, contribution of this field can appear when we
consider interactions or topological terms as the above mentioned. To be more specific we choose the particular Wess-Zumino model (1) with the interacting term (8)]. In such a case Eq. (10) reads

\[ T_{\mu\nu} = A_{\mu}A_{\nu} + B_{\mu}B_{\nu} - \frac{1}{2} g_{\mu\nu} [A_{\rho\sigma}A^{\rho\sigma} + B_{\rho\sigma}B^{\rho\sigma} - V(A, B)] + \frac{1}{2} \alpha [A_{\mu}J_{\nu} + A_{\nu}J_{\mu}], \]

(11)

where \( \alpha \) and \( J_{\mu} \) are the same already defined quantities.

**Matter interpretation of the model.** A particularly simple, albeit interesting, example is obtained assuming \( A_{\mu}A^{\mu} > 0 \) and \( B_{\mu}B^{\mu} > 0 \) in (11). In this case we can define two timelike unit vectors by

\[ u_{\mu} = A_{\mu} / \sqrt{A_{\nu}A^{\nu}}, \quad v_{\mu} = B_{\mu} / \sqrt{B_{\nu}B^{\nu}}. \]

(12)

Moreover, the lightlike or null vector \( J_{\mu} (\equiv \bar{\Psi} \gamma_{\mu} \Psi) \) may be decomposed as

\[ J_{\mu} = N(u_{\mu} + z_{\mu}), \quad N = J_{\mu}u^{\mu} = J_{\mu}A^{\mu} / \sqrt{A_{\nu}A^{\nu}}; \]

(13)

where the spacelike vector \( z_{\mu} \) obeys,

\[ u^{\mu}z_{\mu} = 0, \quad z^{\mu}z_{\mu} = -1. \]

(14)

Therefore the EMT (11) can be written into the form

\[ T_{\mu\nu} = (r + 2q)u_{\mu}u_{\nu} + sv_{\mu}v_{\nu} + q[u_{\mu}z_{\nu} + u_{\nu}z_{\mu}] - \frac{1}{2} g_{\mu\nu} [r + s - V(A, B)], \]

(15)

where we have introduced the definitions

\[ r \equiv A_{\mu}A^{\mu}, \quad s \equiv B_{\mu}B^{\mu}, \quad q \equiv \frac{1}{2} \alpha A_{\mu}J^{\mu}. \]

(16)
Now, the EMT (15) may be easily cast into its canonical forms [19] and interpreted as describing different continuous media depending mainly upon the constant $\alpha$. The algebraic structure of such a tensor can be analyzed via its associated secular or eigenvalue equation which in the present case is equivalent the quartic equation

$$
\Lambda \left[ \Lambda^3 - \Lambda^2 (r + s + 2q) + \Lambda [q^2 + (r + 2q)s(1 - (u_{\mu}u^\mu)^2) - 2qs(v_\mu z^\mu)(v_\nu u^\nu)]
+ q^2s((v_\mu u^\mu)^2 - (v_\mu z^\mu)^2 - 1) \right] = 0,
$$

(17)

where the eigenvalues $\lambda$ of (15) are related to $\Lambda$ by $\lambda = \Lambda - (r + s - V)/2$. From Eq. (17) we have the root $\Lambda = 0$, i.e., $2\lambda_1 = -r - s + V$. The explicit expressions for the other three eigenvectors are not simple and they shall be presented elsewhere. In this paper we shall rather study some significant particular cases.

(i) From (17) we see that when $q = 0$ and $A_{\mu} \neq 0$ the resultant EMT represents the sum of two irrotational perfect fluids. This particular case was already studied by one of us in a different context [2].

(ii) Another simple situation is when both vectors $v_\mu$ and $u_\mu$ are parallel each other, i.e., when $u^\mu v_\mu = 1$ and consequently $z^\mu v_\mu = 0$. In this case the resultant EMT is equivalent to the one of a nonviscous fluid with heat flow which can be of interest in some applications, for instance, to construct models of anisotropic universes and anisotropic fluid spheres [12], and also to describe the non-adiabatic collapse of relativistic stars [20]. Under the
mentioned conditions the eigenvalue equation reduces to

$$A^2 \left[A^2 - A(r + s + 2q + q^2)\right] = 0. \quad (18)$$

This last equation yields four eigenvalues, two of them equal, $\lambda_1 = \lambda_2 = -(r + s - V)/2$, and $[\lambda_\pm = \Lambda_\pm - (r + s - V)/2]$ given by

$$\lambda_\pm = (2q + V \pm \sqrt{(r + s + 2q)^2 - 4q^2})/2, \quad (19)$$

Then, it follows that $\Lambda_\pm$ may be either real if $(r + s + 2q)^2 \geq 4q^2$ or complex if $(r + s + 2q)^2 < 4q^2$. In other words we have three different cases.

\textbf{a)} First we consider the case with $\Lambda_+ \geq \Lambda_-$. If we further assume that $V = V(A, B) \geq 0$, the relation $\lambda_+ \geq \lambda_-$ holds and $\lambda_+$ is associated to the timelike eigenvector of (11), and $\lambda_-$ to a spacelike eigenvector. In this case the EMT (15) can be cast in the canonical form

$$T_{\mu\nu} = (\rho + \pi)U_\mu U_\nu + (\sigma - \pi)X_\mu X_\nu - \pi g_{\mu\nu}, \quad (20)$$

where

$$\begin{align*}
\rho &= \left[2q + V + \sqrt{(r + s + 2q)^2 - 4q^2}\right]/2, \\
\sigma &= \left[-2q - V + \sqrt{(r + s + 2q)^2 - 4q^2}\right]/2, \\
\pi &= (r + s - V)/2, \\
U_\mu &= C_+ u_\mu + D_+ v_\mu, \\
X_\mu &= C_- u_\mu + D_- v_\mu. \quad (21)
\end{align*}$$
With
\[ C_\pm \equiv \Lambda_\pm \sqrt{|\Lambda_\pm^2 - q^2|}, \quad D_\pm \equiv -q \sqrt{|\Lambda_\pm^2 - q^2|}, \]
and \( \Lambda_\pm = \lambda_\pm + \pi \). One can verify that the eigenvectors \( U_\mu \) and \( X_\mu \) are orthonormal, i.e.
\[ U_\mu U^\nu = -X_\mu X^\nu = 1; \quad X_\mu U^\nu = 0. \]

The EMT \((20)\) is the energy-momentum tensor of an anisotropic fluid with pressure \( \sigma \) along \( X_\mu \) (the direction of the anisotropy) and pressure \( \pi \) on the perpendicular plane to the anisotropy.

The variable \( \sigma \) can also take negative values. In such a situation (tension along the anisotropy direction) Eq. \((20)\) may be put in the form
\[ T_{\mu \nu} = (\lambda + \pi)(U_\mu U_\nu - X_\mu X_\nu) - \pi g_{\mu \nu} + \rho U_\mu U_\nu, \] where \( \lambda = -\sigma \) and \( \rho = \rho + \sigma = \sqrt{(r + s + 2q)^2 - 4q^2} \). The first part of the right hand side of Eq. \((24)\) is a fluid of ordered strings \([21]\) while the second is a cloud of particles. Fluids of the kind represented by \((24)\) are a simple generalizations of the clouds of “realistics” strings considered in \([22]\).

b) In the case \((r + s + 2q)^2 \leq 4q^2\), the tensor \((15)\) has the canonical form
\[ T_{\mu \nu} = (\rho + \pi)(U_\mu U_\nu - X_\mu X_\nu) - \pi g_{\mu \nu} + \beta(U_\mu X_\nu + U_\nu X_\mu), \] where
\[ \rho = (2q + V)/2, \]
\[
\begin{align*}
\pi &= (r + s - V)/2, \\
\beta &= \frac{1}{2} \sqrt{4q^2 - (r + s + 2q)^2},
\end{align*}
\] (26)

The EMT (25) represents a fluid made of parallel strings with a heat flow \( q_\mu = \beta X_\mu \) along the strings. \( U_\mu \) is a timelike vector while \( X_\mu \) is spacelike and satisfies the orthonormality conditions (23). Their relations with the initial fluid quantities are quite involved and they will be presented elsewhere. It is worth to mention that strings with heat flow along its length arise naturally when one considers cosmic strings described by the Nambu action corrected by terms built with the curvature of the string world sheet, these corrections represent a geometric description of the string width [14].

c) In the case \( (r + s + 2q)^2 = 4q^2 \) the canonical form of (15) is

\[
T_{\mu\nu} = (\rho + \pi)(U_\mu U_\nu - X_\mu X_\nu) - \pi g_{\mu\nu} + \beta J_\mu J_\nu,
\] (27)

The EMT (27) is the sum of a fluid of ordered strings and a null fluid, i.e., a fluid of radiation directed along the strings. This situation may be interesting to generalize the model of interacting strings and electromagnetic radiation considered in [23].

Similar results to ones presented in the cases \( a, b \) and \( c \) above have been also found considering the matter interpretation of non-compact sigma models [9]. Besides the cases above considered there are other eight different situations, namely, the quantities \( A_{\mu\nu} A^\nu \) and \( B_{\mu\nu} B^\nu \) may be positive, negative or zero. From this we must consider all possible (different) combinations
of these values, which yields nine cases that should be studied separately. The algebraic structure of the EMT (11) is different for each case, and so are the resultant (if any) fluid interpretations. However, we restricted ourselves to the case presented above because it seems to be the ones with more physical content.

Conclusions. We have shown that the supersymmetric Wess-Zumino model may be used as motivation to find models of continuous matter.

New simple continuous matter models can be obtained when total divergences or other non-renormalizable interaction terms are considered. In particular, the presence of the interaction term (8) made possible to find different models: i) The anisotropic fluid (20) with anisotropic pressure $\sigma$ (along the direction of anisotropy $X_\mu$) smaller than the isotropic pressure $\pi$. $\sigma$ may even be negative, representing a tension along the anisotropy. ii) A fluid of ordered strings with heat flow along the strings, cf. Eq. (25). iii) A mixture of a fluid of ordered strings with a fluid of directed radiation along the strings. This model follows from Eq. (15) in the degenerated case where $(r + s + 2q)^2 = 4q^2$. 


References


