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**Equations of Motion
for Anisotropic Nonlinear Elastic Continuum
in Gravitational Field**

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Abstract

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Equations of motion for anisotropic nonlinear elastic continuum in the gravitational field are written in the form convenient for numerical calculations. The energy-stress tensor is expressed through scalar and tensor products of three vectors imbedded in the continuum. Examples of expansion of the energy-stress tensor into scalar and tensor invariants corresponding to some crystal classes are given.

Аннотация

Соколов С.Н. Уравнения движения анизотропной нелинейной упругой среды в гравитационном поле: Препринт ИФВЭ 94-95. – Протвино, 1994. – 17 с., библиогр.: 47.

Уравнения движения для анизотропной нелинейной упругой среды в гравитационном поле написаны в форме, удобной для численных вычислений. Тензор энергии-натяжений выражен через скалярные и тензорные произведения трех векторов, вмороженных в среду. Даются примеры разложения тензора энергии-натяжений по скалярным и тензорным инвариантам, соответствующим некоторым кристаллическим классам.

1. INTRODUCTION

Paper describes a relatively simple general relativistic version of the nonrelativistic theory of large elastic deformations [1,3].

The general relativistic elasticity theory was proposed and refined in a large number of publications (the list of references [5-26] is just a sample) and for the purely elastic case is now formally completed (cf. books [22-26]). It was applied to some problems of astrophysics and of gravitational detector physics [27-36]. Nevertheless, the equations of motion for the elastic anisotropic continuum in GR, compared with the widely used and taught [4] equations of motion for gases and fluids, are still unpopular. In particular, in the numerous discussions of mechanical gravitational antennae, where the elasticity theory in the changeable gravitational field should be naturally applied, this theory is usually disregarded, and some common sense reasonings are used instead. One of the causes of such situation is the formal mathematical style of papers on elasticity in GR, especially of the latest publications. Besides, the articles are burdened by the discussions of notions of displacements, rotations, reference states, and so on. The continuing discussions of these notions (e.g. the discussion of rigid motion and of the local synchronization [38-45]) create the false impression that the elasticity in GR, referring to these notions, is still problematic.

Another possible cause of the unpopularity of equations of the elasticity theory in GR is that they are inconvenient, especially for the numerical calculations like [46]. The relation between the deformation tensor and the stress-energy tensor involving the 4-rank elasticity tensor (tensor of elastic coefficients) $\Lambda^{\alpha\beta\gamma\delta}$, which is not a very good idea even in the nonrelativistic case (due to the dependence of elements of Λ on the directions of the crystallographic axes and on deformations), becomes too complicated in the general case of nonlinear elasticity and changeable metric. Besides, often (e.g. in [16]) the elasticity tensor

is introduced at a symbolic level, and the reader is left to guess how this tensor is changing when the angles between crystallographic axes are changing.

The present paper describes the result of an attempt to modify the formulation of equations of motion of general (inhomogeneous anisotropic) elastic continuum to an equivalent, but more practical form and to give a relatively simple derivation of these equations. The idea is to apply the method preferred by the nonrelativistic elasticity theory in the nonlinear anisotropic case, where one uses, instead of deformation tensor and the 4-rank elasticity tensor, three vectors $\mathbf{r}_i$ defined as the derivatives of the laboratory coordinates with respect to matter coordinates. In case of crystalline matter, the matter coordinates are usually so chosen that these vectors are proportional to grid vectors or their linear combinations and the symmetry properties of the elastic matter are simply expressed by their transformations. The six scalar products n_{ij} of these vectors are then used as scalar arguments of the elastic potential and the stress tensor is expressed through the partial derivatives of this potential. If an anisotropic matter has some symmetry, it looks as an invariance of the elastic potential to certain transformations of n_{ij} and leads to the dependence of the potential on polynomial invariants built of n_{ij} .

In this method, neither the elasticity tensor, nor even the deformation tensor are truly needed. However, paying tribute to the traditions of linear elasticity, the scalar products are usually (e.g. in [1,25]) combined into the matrix $e_{ij} = (n_{ij} - \delta_{ij})/2$ that vanishes in the unstressed state and in the special case of Cartesian coordinates has elements equal to the elements of the deformation tensor. Sometimes, the elasticity tensor is computed from the elastic potential in the linear approximation for the sake of comparison with the linear theory [25]. In the nonlinear theory, the deformation tensor as a *tensor* is efficiently used [3] in the isotropic case only.

In the relativistic version of this method, given in this paper, we will use in the equations of motion the 4-dimensional analogues A_i of vectors $\mathbf{r}_i$ as basic quantities.

In fact, such vectors are widely used in the papers on the relativistic elasticity as intermediate quantities in the calculation of the deformation and other tensors, after what only the tensors are used in the equations and the vectors are put aside. This practice is computationally inefficient, since these vectors contain information about the current local position of moving crystallographic axes, and this information is lost in the deformation tensor. The lost information should be later somehow introduced into the equations. This may be done through the equations of motion for the elasticity tensor and through dependence of this tensor upon deformations, but this requires additional computing.

In terms of vectors A_i , the relativistic nonlinear elasticity properties are described by a scalar elastic potential depending on six scalar products $N_{ij} = A_i \cdot A_j$. If elastic properties of the matter have some symmetries, the corresponding transformations of the elastic potentials in the relativistic and nonrelativistic cases are the same, so the results of the nonrelativistic symmetry analysis for different crystal classes can be directly used in the relativistic domain. Though the anisotropic elastic matter is not realistic in astrophysics, it may be used in convenient models.

Expanding the energy-stress tensor around the unstressed state, one gets the expressions with the familiar constants of linear nonrelativistic theory.

In the paper, we refer to the simplest physical arguments, evade the introduction of notions that do not enter directly the equations of motion, and try to make the text as self-contained as possible assuming no knowledge of elasticity theory, but relying on the familiarity with GR.

Notations

Greek indices run 0,1,2,3, latin ones run 1,2,3, summations are implied between upper and lower repeated indices only, argument $[y]$ means "in the coordinate frame y ", comma means the partial derivative, semicolon means the covariant derivative, central dot denotes the scalar product using metric g , symbol $\otimes$ means the tensor product $(A \otimes B)^{\alpha\beta} = A^\alpha B^\beta$, and $\mathcal{L}_v$ means the Lie derivative $\mathcal{L}_v a = a_{,\alpha} v^\alpha - v_{,\alpha} a^\alpha$.

2. KINEMATICS

We mean here under kinematics an affine-geometrical part of equations that does not need the knowledge of either the metric, or of the elasticity properties and that use only the observable trajectories $\mathbf{x}(t, \mathbf{y})$, where parameters $\mathbf{y}$ label the points of the continuum (are the Lagrangian coordinates). In case of crystalline matter, parameters $\mathbf{y}$ are often chosen so that the derivative $\mathbf{Y}_i = \partial \mathbf{x} / \partial y_i$ shows the momentary direction of an crystallographic axis i at point $\mathbf{x}$. The physical state of deformed matter at some its point $\mathbf{y}$ can be computed (with the use of the metric and functions specifying elasticity), if the relative positions of a number of nearby points $\mathbf{s}_i = \mathbf{y} + \epsilon_i \mathbf{a}_i$, forming an infinitesimal simplex s , are known. Here, ϵ_i are infinitesimal parameters and vectors $\mathbf{a}_i$ (belonging to the tangent space at point $\mathbf{y}$) in the coordinate system $[y]$ do not depend on time: they are, like the system $[y]$, "frozen" in the continuum. In the observer's system $[x]$, $\mathbf{a}_i[x] = \partial \mathbf{s}_i / \partial \epsilon_i = a_i^j[y] \mathbf{Y}_j$. (Often the system $[y]$ is so chosen that $\mathbf{a}_i[x] = \mathbf{Y}_i$.) The aim of this section is to express the evolution of vectors

$\mathbf{a}_i[x]$ at fixed observation point $\mathbf{x}$ through the velocity $\mathbf{V}$ of matter and spatial derivatives of $\mathbf{a}_i[x]$ and clarify their relations with 4-dimensional quantities.

In the 3-dimensional approach, one writes $d\mathbf{Y} = \mathbf{Y}_{,t}dt + \mathbf{Y}_{,j}dx^j$, where the first term $\mathbf{Y}_{,t}dt = \mathbf{x}_{,t}dt = \mathbf{V}_{,j}dt = \mathbf{V}_{,j}Y^jdt$ is the change of $\mathbf{Y}$ due to the gradient of velocity; the observer looking always at the same point of matter would see only this term. The second term is due to the change of observed point of matter at the observation point. The fixation of observation point $\mathbf{x}$ gives $dx^j + V^jdt = 0$. Combining these equalities, we get the well-known equation with the Lie derivative

$$\dot{\mathbf{Y}}_i = \mathbf{V}_{,j}Y_i^j - \mathbf{Y}_{i,j}V^j = -\mathcal{L}_{\mathbf{V}}\mathbf{Y}_i.$$

Similar equation holds for linear combinations of $\mathbf{Y}_i$:

$$\dot{\mathbf{a}}_i = \mathbf{V}_{,j}a_i^j - \mathbf{a}_{i,j}V^j = -\mathcal{L}_{\mathbf{V}}\mathbf{a}_i. \quad (1)$$

Note that these equations are exact for any velocity $\mathbf{V}$ and can be used in the relativistic theory as well. One has only to relate the 3-dimensional quantities entering (1) with 4-vectors used in relativistic equations.

In the 4-dimensional approach, the "frozen" system $[y]$ gets its own time coordinate τ and the infinitesimal simplex is supplied by the fourth point $s_0 = y + \epsilon_0 v$, $v[y] = (1, 0, 0, 0)$. Vectors v, a_i in the observer's frame $[x]$ with coordinates t, x^i can be expressed as derivatives

$$v = x(s_0)_{,\epsilon_0} = x_{,\tau}, \quad a_i = x(s_i)_{,\epsilon_i}.$$

Since ϵ_i and τ are independent parameters, in the $[x]$ frame we have identities

$$x_{,\tau\epsilon} = v_{,x^\alpha}x_{,\epsilon}^\alpha = v_{,x^\alpha}a^\alpha =$$

$$x_{,\epsilon\tau} = a_{,x^\alpha}x_{,\tau}^\alpha = a_{,x^\alpha}v^\alpha.$$

Subtracting the third expression from last one, we obtain the equation

$$a_{,\beta}^\alpha v^\beta - v_{,\beta}^\alpha a^\beta = 0, \quad (2)$$

compactly written as $\mathcal{L}_v a = 0$. Solving it with respect to the time derivatives of a_i , we get in the observer's frame the evolution equation

$$a_{i,0} = (v_{,\beta}a_i^\beta - a_{i,j}v^j)/v^0. \quad (3)$$

If one sets in (3) $\tau = t$, that is if one synchronizes the points of the simplex in the observer's system, then v turns into velocity of matter in the observer's frame

$$v[x] = V = x_{,t} = (1, \dot{\mathbf{x}}),$$

and eq.(3) reduces to eq.(1) written in the 4-dimensional form. Vectors a_i are defined there as the derivatives $a_i = x(s_i)_{,\epsilon_i}$ at fixed t and have zero time components. We will denote them $\hat{a}_i$. They are observer-dependent and are connected with (observer-independent) 4-vectors a_i , equal to derivatives $x(s_i)_{,\epsilon_i}$ at fixed τ , by the relation

$$\hat{a}_i = a_i - a_i^0 V.$$

Clearly, for different observers values $\hat{a}_i$ differ by a vector proportional to V . It means that quantities $\hat{a}_i$ are not 4-vectors, but are observer-dependent elements of 4-vector rays $\bar{a}_i = a_i + kV$ with indefinite parameter k (in other terminology, rays $\bar{a}_i$ are equivalence classes modulo V).

If metric is given, parameter τ can be defined as the proper time, and v in (3) turns into the normalized 4-velocity U , $U^2 = -1$ (in this paper time-like vectors have negative squares). So, if the initial distribution of 4-vectors a_i at $t = 0$ is given, their evolution can be calculated by means of (3). As is well-known, the Lie derivative does not change, if the partial derivatives are replaced by the covariant derivatives with respect to any metric. So, (2) can be equivalently written in the explicitly covariant form

$$a_{i;\beta}^\alpha v^\beta - v_{i;\beta}^\alpha a_i^\beta = 0. \quad (2')$$

In the dynamical equations below, only the projections A_i of vectors a_i , or of rays $\bar{a}_i$, or of their elements $\hat{a}_i$, on the subspace orthogonal to U are used. These projections coincide

$$A_i = a_i + U U \cdot a_i = \bar{a}_i + U U \cdot \bar{a}_i = \hat{a}_i + U U \cdot \hat{a}_i.$$

Vectors a_i look more elegant than rays $\bar{a}_i$ and are preferred in the papers on relativistic elasticity. However, the definition of a_i has some subtle points (e.g. the surfaces $\tau = \text{const}$ may happen to be not space-like) and the calculation of a_i involves point-dependent and metric-dependent transformation of coordinates, while quantities $\bar{a}_i$ are simply linear combinations of $Y_i = (0, \mathbf{Y}_i)$ plus the term kV (or kU) with indefinite k . So, it is easier and safer to use $\hat{a}_i$ and eq.(1) (or eq.(3) with $\tau = t$) remembering that $\hat{a}_i$ are just ray elements.

3. DYNAMICS

Stresses depend on distances between particles, so metric tensor g enters the dynamical equations.

The energy-stress tensor of matter T in the most general case is an arbitrary symmetric tensor-valued function¹ of the metric tensor and of all other available vectors, tensors, and scalars: vectors v, a_i , electromagnetic fields, temperature, ages of deformations, *etc.* In the purely elastic case, only the metric g and vectors $U = v/\sqrt{-v^2}$, a_i (or v and rays $\bar{a}_i$) are available as arguments. The most general tensor-valued function of g, U, a_i has form

$$F = c\bar{g} + c_{00}U \otimes U + \sum_i c_{0i}U \otimes a_i + \sum_i c_{i0}a_i \otimes U + \sum_{ij} c_{ij}a_i \otimes a_j + ()^*,$$

where $\bar{g}$ means the metric tensor with upper indices, coefficients c are some scalar (invariant) functions of all available scalars $U \cdot a_i, a_i \cdot a_j$, and $()^*$ denotes terms with dual tensors. The stress-energy tensor T is a particular case of F satisfying certain physical requirements.

Tensor T should be symmetric. Hence, $()^* = 0$ and matrix $c_{\alpha\beta}$ is symmetric.

The state of the matter (the pressure, *etc.*) near the point x cannot depend on the choice of the synchronization of points of the infinitesimal simplex. Hence, tensor T should be invariant to the replacement of vectors a_i by $a_i + k_i U$, where k_i are arbitrary scalars. Other words, T actually is a function of rays $\bar{a}_i$. Equivalently, it is a function of vectors A_i orthogonal to U : $U \cdot A_i = 0$.

Tensor T , satisfying these requirements, can be written as

$$T = c\bar{g} + c_{00}U \otimes U + \sum_i c_{0i}(U \otimes A_i + A_i \otimes U) + \sum_{ij} c_{ij}A_i \otimes A_j.$$

The components T^{0i} in the rest frame $\bar{g}^{0i} = 0$ should be zero. Hence, $c_{0i} = c_{i0} = 0$ and T reduces to

$$T = c\bar{g} + c_{00}U \otimes U + \sum_{ij} c_{ij}A_i \otimes A_j,$$

where coefficients are functions of scalars $N_{ij} = A_i \cdot A_j$ and matrix c_{ij} is symmetric.

The expression for T can be further reduced, if one notes that

$$\bar{g} = -U \otimes U + \bar{h}, \quad \bar{h} = \sum_{ij} M_{ij}A_i \otimes A_j, \quad (4)$$

where matrix $M = N^{-1}$ and where $\bar{h}$ plays the role of $\bar{g}$ in the subspace orthogonal to U ($\bar{h}$ is often referred in the literature as the projection operator

¹In the usual approach, the physical uncertainties concerning the reference state in the definition of the deformation tensor lead some authors to a viewpoint [17,18] that one should relate time derivatives of the deformation and the stress tensors, and not the tensors themselves. We use no tensors related with reference states, so this complication (leading to the third order equations of motion) does not occur here.

on the subspace orthogonal to U). Hence, replacing c_{ij} by $c_{ij} - M_{ij}c$ and c_{00} by $\rho = c_{00} - c$, we may absorb the term $c\bar{g}$ into other terms and reduce tensor T to the expression

$$T = \rho U \otimes U + \sum_{ij} c_{ij} A_i \otimes A_j. \quad (5)$$

The stress part

$$P = T - \rho U \otimes U = \sum_{ij} c_{ij} A_i \otimes A_j$$

of the energy-stress tensor involves 6 independent scalar functions c that contain all the information about the elasticity properties of the matter. Three vectors A_i contain all the information about the deformations of the matter and of the positions of the crystallographic axes at point $\mathbf{x}$ at time t . Function ρ is, obviously, the density of the matter including the density of the stress energy. It depends on the choice of c .

Functions c are fully independent for the generalized elastic matter, but for the (strictly) elastic matter they should satisfy elasticity relations

$$\partial P^{\alpha\beta} / \partial g_{\mu\nu} = \partial P^{\mu\nu} / \partial g_{\alpha\beta}.$$

Among these relations, only 15 relations

$$\partial P^a / \partial g_b = \partial P^b / \partial g_a, \quad (6)$$

where $a \neq b$ and a, b are collective indices ($a = \overline{1,6} \rightarrow ij = 11, 22, 33, 23, 31, 12$), are independent. They correspond to 15 symmetry relations $\Lambda^{ijkl} = \Lambda^{klij}$ of the traditional theory and can be used as *definition* of an elastic matter. If functions c are polynomials of N_{ij} , relations (6) reduce the number of independent coefficients in these polynomials.

Relations (6) formalize the reversibility of elastic deformations. The reversibility implies the existence of a scalar potential W , entering the Lagrangian, so that the elements of tensor P are the partial derivatives² $\partial W / \partial g_{ij}$. Since $\partial^2 W / \partial g_{ij} \partial g_{kl} = \partial^2 W / \partial g_{kl} \partial g_{ij}$, the elements of T should satisfy the relations (6). When calculating derivatives of W , it is convenient to denote

$$A_{ij} = (A_i \otimes A_j + A_j \otimes A_i) / 2$$

and use the relation $\frac{1}{2}(\partial N_{ij} / \partial g_{\alpha\beta} + \partial N_{ij} / \partial g_{\beta\alpha}) = A_{ij}^{\alpha\beta}$, or, shortly, $A_{ij} = \partial N_{ij} / \partial g$.

²Generally, $T^{\alpha\beta} = 2\delta L / \delta g_{\alpha\beta}$, where L is the Lagrangian and δ is the Euler variation. Assuming that the state of the elastic body does not depend on the derivatives of the metric, we reduce variations of P to the partial derivatives.

Both T_{ij} and A_i are directly measurable quantities. Therefore, if sufficient data on $T(A)$ are available, functions c can be found by solving (5) with respect to c . Nothing prevents to do it in the rest frame, so dependence of functions c on the scalar products can be taken from the results of the usual static experiments quoted in the nonrelativistic nonlinear elasticity theory. In astrophysical cases, functions c can be chosen on the basis of some physical arguments (like used in [37]).

The eqs.(3), (5) together with the Bianchi identity

$$T^{\alpha\beta}_{;\beta} = 0 \quad (7)$$

make a complete set of exact equations of motion of arbitrary inhomogeneous anisotropic nonlinear-elastic medium in GR, if metric g is given. Four eqs.(7) correspond to four unknown functions ρ, U^i , 12 eqs.(3) define the evolution of 12 components of vectors a_i , A_i are projections of a_i , eq.(5) gives the relation between A and T .

If the gravitational field of the elastic matter itself is important, the Einstein equations

$$E(g) = -8\pi T, \quad (8)$$

where E means the Einstein tensor, should be added to the system (3),(5),(7). In case, when dependence of ρ on vectors A is known (calculated in advance), one of the equations (7) (with $\alpha = 0$) may be omitted or used as an accuracy check. Sometimes, the dependence of ρ, c on the scalar products may be partially replaced by the dependence on ρ , if this simplifies the functions.

4. SYMMETRIES

If the elastic properties of matter have some symmetries, they place some limitations on functions c .

Let G be a group represented by 3×3 real matrices K with elements K_j^i .

Definition. *Elastic continuum at point x is symmetric to G , if at that point*

$$T(A) = T(KA),$$

where $(KA)_j = \sum_i K_j^i A_i$.

Evidently, the existence or the absence of a symmetry is not affected by the replacement of A_i by their linear combinations, only the representation K changes. A good choice of A may simplify K .

Note that this symmetry is defined as a symmetry of the elasticity properties and not as a geometrical one. Unlike the traditional definitions involving

reflections and rigid rotations of an unstressed crystal by angles $2m\pi/n$ around some axes, this definition refers to vectors A in the current stressed state only and does not suppose either any specific form of the metric g , or the existence of the unstressed (reference) state. In the stressed state, the angles between A_i do not generally coincide with $2m\pi/n$ for some integer m, n and, if $K^n A = A$, the transformation $A \rightarrow KA$ is not generally the rotation by the angle $2m\pi/n$ (what is usually the case in the unstressed state) but is an affine transformation changing the scalar products of vectors "frozen" in the matter. Of course, in case of small stresses one may neglect the difference and imagine K as rotations and reflections.

The assumption that functions c depend solely on N_{ij} implies that stresses do not change, if all 3 A_i simultaneously change sign, what means that we consider only the matter the elasticity of which has the right-left symmetry.

Ideal uniformly stressed crystals do not change their properties, if adjacent layers slid exactly by one step of the grid. If vectors A are proportional to the grid vectors, this symmetry in macroscopic scale means the invariance of T to certain large shears corresponding to the transformations $A_i \rightarrow A_i \pm A_j$. This symmetry does not survive in general conditions and we shall not discuss it further.

Existence of symmetries allows one to define scalar and tensor invariants S, T of the group G ($S : S(A) = S(KA)$, $T : T(A) = T(KA)$) and to expand T into these invariants

$$T = \rho U \otimes U + \sum_u C_u T^u, \quad (9)$$

where ρ, C_u are invariant functions of scalar invariants S and tensor invariants T (for elastic matter (6)) are the derivatives of S with respect to g .

To make the expansion (9) unique, we (using the idea of polynomial basis [1]) will define S as polynomials of N_{ij} making a basis in the set of all invariant polynomials of N_{ij} . It means, that any invariant polynomial can be expressed as a polynomial of invariants S . As usual, if functions C_u are not polynomial, they may be considered as the limits of sequences of polynomial approximations.

The number n of elements in basis S and, hence, the number of functions C_u is a unique function of G , while the freedom in the choice of S can be used to single out invariants that conserve their independence in the linear approximation.

If the unstressed state exists,³ an expansion near this state becomes useful.

³This assumption is physically restrictive. For example, the neutron elastic matter does not exist without an external pressure. In such cases, an expansion near some stressed state may be useful (cf. [18]).

Let scalar products N_{ij} and invariants S have in the unstressed state the values $\hat{N}_{ij}, \hat{S}$ (these values are constants, symmetry transformations do not affect them). Consider the differences $\Delta^u = S^u - \hat{S}^u$ as polynomials of $\Delta_{ij} = N_{ij} - \hat{N}_{ij}$. Suppose that basis S contains the subset S_1 of n_1 first order invariants having either nonzero unstressed values or single zero at the unstressed state, the subset S_2 of n_2 second order invariants having double zero at this state, and the subset S_h of n_h high order invariants having at least triple zero at this state. It is convenient, instead of using invariants that may become dependent in approximation linear in Δ^u , to choose the basis so that the numbers n_1, n_2 of low order invariants are minimal. We will call such basis laconic. In the laconic basis, the first order invariants will remain independent in the linear approximation, and the elastic potential can be written as⁴

$$W = \sum_{u \leq v} c_{uv} \Delta^u \Delta^v + \sum_m c_m S^m + W_h, \quad (10)$$

where c_{uv}, c_m are constants, Δ 's contain first order invariants, S^m are second order invariants, and function W_h has high order zero at the unstressed state. Expansion (10) contains $n_1(n_1 + 1)/2 + n_2$ constants. The values of constants are limited by the stability condition $W \leq 0$.

In simple models and in case of small deformations, function W_h can be set to zero. This does not spoil the exact invariance to the symmetry transformations. If $W_h = 0$, stress tensor P is

$$P = \sum_{u \leq v} c_{uv} (\mathcal{T}^u \Delta^v + \mathcal{T}^v \Delta^u) + \sum_m c_m \mathcal{T}^m. \quad (11)$$

One can further simplify P retaining there only the terms linear in $N_{ij} - \hat{N}_{ij}$. Note that such linear approximation does not reduce the number of constants ($n_1(n_1 + 1)/2 + n_2$), but makes the invariance approximate.

The scalar invariants S and scalar invariants of the nonrelativistic theory [1,25] result from the same algebraic considerations and coincide in their formal structure. Therefore, invariants S can be obtained from the nonrelativistic ones by the formal replacement of quantities e_{ij} by scalar products N_{ij} . To find the linear approximations of invariants and to combine them to laconic basis is then easy. Let us consider briefly some examples.

- In case of no symmetry, group G is trivial, every A_{ij} is a tensor invariant and every N_{ij} is a scalar invariant, so we have 6 functions C_u of 6 invariants in

⁴Note that W contains the differences of invariants $\Delta = S - \hat{S}$ and is not a polynomial of the differences $N_{ij} - \hat{N}_{ij}$ similar to polynomials of e_{ij} used in [1]. Treating $\hat{N}_{ij}$ as constants we evade the introduction of ambiguous reference vectors $\hat{A}_i$.

the expansion (9) and $6 \times 7/2 = 21$ constant c_{uv} in the expansions (10),(11). In P, T , these constants are the coefficients before the tensors

$$A_{kl}\Delta_{ij} + A_{ij}\Delta_{kl}.$$

• A monoclinic crystal has one preferred direction. Let A_3 (i.e. $\mathbf{a}_3$) be in this direction. The vectors A_1, A_2 can be so chosen that the symmetry group G is represented by their simultaneous transformations $A_1 \rightarrow -A_1, A_2 \rightarrow -A_2$ and in the unstressed state $\hat{N}_{13} = \hat{N}_{23} = 0$. Then 7 independent scalar invariants S may be chosen as

$$S = \{N_{11}, \quad N_{22}, \quad N_{33}, \quad N_{12}, \\ N_{13}^2, \quad N_{13}N_{23}, \quad N_{23}^2\}.$$

The first 4 of them are of the first order, the rest, of the second order. The choice is laconic.

Therefore, expansion (9) contains 7 functions C_u , and expansions (10), (11) contain $4 \times 5/2 + 3 = 13$ constants c_{uv}, c_m .

• A cubic crystal has 3 equivalent grid vectors. Let A_i coincide up to a common scale factor with grid vectors. Then the group G may be represented by the interchanges of $A_i \leftrightarrow A_j$ and sign changes $A_i \rightarrow -A_i, A_j \rightarrow -A_j$, numbers $\hat{N}_{ij}$ vanish for $i \neq j$, and all $\hat{N}_{ii}$ coincide. Let $\hat{N}_{ii} = 1$. The basis S includes 9 independent scalar invariants that by the analogy with the nonrelativistic case may be built as

$$s_{01} = \sum N_i, \quad s_{02} = \sum N_i N_j, \quad p_{03} = \prod N_i, \\ s_{10} = \sum M_i, \quad s_{20} = \sum M_i M_j, \quad p_{30} = \prod N_{ij}, \\ s_{11} = \sum M_i N_i, \quad s_{12} = \sum M_i N_j N_k, \quad s_{21} = \sum M_i M_j N_k,$$

where $N_i = N_{ii}, M_i = N_{jk}^2$ and $i \neq j \neq k$.

A laconic basis may be chosen as

$$S_1 : \quad s_1 = s_{01}; \\ S_2 : \quad s_2 = s_{10}, \quad s_3 = s_{02} - 2s_{01} + 3; \\ S_3 : \quad s_4 = p_{30}, \quad s_5 = s_{11} - s_{10}, \quad s_6 = p_{03} - s_{02} + s_{01} - 3; \\ S_4 : \quad s_7 = s_{20}, \quad s_8 = s_{21}, \quad s_9 = s_{11} + s_{12} - s_{10}(s_{10} - 1).$$

Expansions (9) and (10),(11) involve, respectively, 9 functions and 3 constants. Explicitly, the elastic potential is

$$W = c_1 \left(\sum \Delta_{ii} \right)^2 + c_2 \sum_{i \neq j} N_{ij}^2 + c_3 \sum \Delta_{ii}^2 + W_h.$$

• Let continuum have one preferred direction and be isotropic in two other directions ("transversely isotropic"). It is convenient to choose A so that the group G is represented by $O(2)$ -rotations

$$A_1 \rightarrow A_1 \cos \phi + A_2 \sin \phi, \quad A_2 \rightarrow A_2 \cos \phi - A_1 \sin \phi,$$

and in the unstressed state $\hat{N}_{ii} = 1$ and $\hat{N}_{ij} = 0$ for $i \neq j$. Then 6 scalar invariants⁵ can be chosen as

$$S_1 = \{N_{33}, \quad N_{11} + N_{22}\},$$

$$\mathcal{J}_2 = \{N_{13}^2 + N_{23}^2, \quad n_1^2 + n_2^2\},$$

$$S_h = \{n_1 m_1 + n_2 m_2, \quad n_1 m_2 - n_2 m_1\},$$

where

$$n_1 = (N_{11} - N_{22})/2, \quad n_2 = N_{12}, \quad m_1 = (N_{13}^2 - N_{23}^2)/2, \quad m_2 = N_{13}N_{23}.$$

So, we need 6 functions of 6 scalar invariants in expansion (9) and 5 constants in (10),(11):

$$W = c_1 \Delta_{33}^2 + c_2 (\Delta_{11} + \Delta_{22})^2 + c_3 \Delta_{33} (\Delta_{11} + \Delta_{22}) \\ + c_4 (N_{13}^2 + N_{23}^2) + c_5 (n_1^2 + n_2^2) + W_h.$$

• If the continuum is isotropic, its invariance group is $O(3)$. Let A_i be chosen orthonormalized in the unstressed state, so that group G is represented by orthogonal matrices K . Then the simplest choice of its 3 scalar and 3 tensor invariants is

$$s_1 = \sum N_i, \quad s_2 = \sum_{ij} N_{ij}^2, \quad s_3 = \sum_{ijk} N_{ij} N_{ik} N_{kj},$$

$$t_1 = \sum A_{ii}, \quad t_2 = t_1 \cdot t_1 = \sum_{ij} A_{ij} N_{ij}, \quad t_3 = t_1 \cdot t_1 \cdot t_1 = \sum_{ijk} A_{ij} N_{ik} N_{kj},$$

where sums include terms with $i = j$, $i = j = k$. Quantities s are related with invariants of matrix N as follows

$$\text{tr} N = s_1, \quad \sum_{i \neq j} (N_i N_j - N_{ij}^2) = s_1^2 - s_2, \quad \det N = s_3/3 - s_1 s_2/2 + s_1^3/6.$$

⁵The basis given in [1,25] is incomplete: the invariant similar to $n_1 m_2 - n_2 m_1$ is independent of the five invariants listed there.

The simplest choice is not laconic. A laconic version of the scalar basis is

$$s_1 = \sum N_{ii}, \quad s'_2 = s_2 - s_1^2/3 = \sum_{i \neq j} (N_{ij}^2 + (N_{ii} - N_{jj})^2/6),$$

$$s'_3 = 27(s_3 - s_1 s_2) + 6s_1^3 = \sum n_i^3 - 3 \sum n_i m_i^2 + 6m_1 m_2 m_3,$$

where $n_i = 2N_{ii} - N_{jj} - N_{kk}$, $m_i = 3N_{jk}$. Evidently, s'_2, s'_3 have the double and triple zero at the unstressed state. The corresponding tensor invariants are

$$\mathcal{T}^1 = \sum A_{ii},$$

$$\mathcal{T}^2 = \sum_{i \neq j} (A_{ij} N_{ij} + (A_{ii} - A_{jj})(N_{ii} - N_{jj})/6),$$

$$\mathcal{T}^3 = \sum B_i n_i^2 - \sum_i (B_i m_i^2 + 6A_{jk} n_i m_i) + 6 \sum_{j < k} A_{jk} m_j m_k,$$

where $B_i = 2A_{ii} - A_{jj} - A_{kk}$. Invariants $\mathcal{T}^2, \mathcal{T}^3$ have single and double zero at the unstressed state.

Therefore, in the isotropic case 3 functions of 3 invariants are needed for the description (9) of nonlinear elasticity, or 2 constants in the linear case

$$T = \rho U \otimes U + c_1 \mathcal{T}^1 \sum \Delta_{ii} + c_2 \mathcal{T}^2. \quad (12)$$

Constants c are related to the Lamé coefficients by

$$c_1 = -\frac{\lambda}{2} - \frac{\mu}{3}, \quad c_2 = -\mu.$$

Note that the usual stability conditions $3\lambda + 2\mu > 0$, $\mu > 0$ turn here into simpler conditions $c_1 < 0$, $c_2 < 0$.

5. COMMENT ON THE DEFORMATION TENSOR

We see that in terms natural for GR the equations of motion for an elastic continuum can be closed and the elasticity properties (including the symmetries of the elastic matter) can be specified without the appeal to the deformation tensor, or to the elasticity 4-rank tensor. Equally, there is no need to consider the elements of tensors in the Cartesian coordinate system, to discuss how the element of the continuum rotates in the course of its motion, or to introduce two-point tensors [3]. The only place above, where the deformation tensor was about to appear, was the isotropic case. There one may recognize in the invariants t_1, s_1 the dilatation tensor and its trace, and to identify $\sum \Delta_{ii}/2$ with

the trace of deformation tensor. To make this relation more explicit, we will define the dilatation and deformation tensors for arbitrary choice of A_i and cast (12) into more usual form.

Let the unstressed state (where $T = \rho U \otimes U$) exists, let in this state scalar products N_{ij} have values $\hat{N}_{ij}$, and matrix $\hat{M} = \hat{N}^{-1}$. Then the dilatation tensor can be defined as the simplest (polynomial) tensor invariant of $O(3)$ group

$$D = \sum_{ij} \hat{M}_{ij} A_i \otimes A_j$$

and the deformation tensor, as the difference

$$\mathcal{E} = (D - \bar{h})/2.$$

Since

$$\mathcal{E} = \frac{1}{2} \sum_{ij} (\hat{M}_{ij} - M_{ij}) A_i \otimes A_j,$$

$\mathcal{E}$ vanishes in the unstressed state. Tensors $D, \mathcal{E}$ are 4×4 matrices, but, if $\mathbf{U} = 0$, only their diagonal 3×3 block is nonzero and this block coincides with the similar nonrelativistic tensors. This definition is equivalent to the widely accepted one (e.g. in [16], [28]), if the reference state is unstressed.

In the expansion of T , we may replace the set of 3 polynomial tensor invariants $\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3$ of $O(3)$ group by a set of 3 tensor invariants $\bar{h}, \mathcal{E}, \mathcal{E}^{(2)} = \mathcal{E} \cdot \mathcal{E}$, involving non-polynomial quantity $\bar{h}$ (see [3] for the properties of the nonrelativistic version of this set). Then, in the linear approximation, instead of (12) we get

$$T = \rho U \otimes U - \lambda \bar{h} \text{tr} \underline{\mathcal{E}} - 2\mu \mathcal{E},$$

where $\underline{\mathcal{E}} = \mathcal{E}_\beta^\alpha$ and we used the identity $\sum_i N_{ii} - 3 = 2\mathcal{E}^{\alpha\beta} g_{\alpha\beta} = 2\mathcal{E}_\alpha^\alpha$.

In the anisotropic cases, however, deformation tensor does not enter the set of tensor invariants: different symmetry groups are associated with different tensor invariants, tensors $D, \mathcal{E}$ are specific for the group $O(3)$. So, trying to express T through $\mathcal{E}$, one has to introduce the elasticity tensor Λ (or rotation matrices) relating invariants of different symmetry groups (and specifying the current position of crystallographic axes). Then the classical relation $\sigma = \Lambda \mathcal{E}$ expressing stresses σ through deformations $\mathcal{E}$ becomes actually a definition of tensor Λ as solution of the equation

$$\sum_u C_u \mathcal{T}^u = -\Lambda \mathcal{E}$$

relating tensor invariants $\mathcal{T}$ (and scalar invariants entering C_u) of the group G with a tensor invariant $\mathcal{E}$ of group $O(3)$.

6. CONCLUSION

The formulation of equations of motion for elastic matter in terms of three vectors "frozen" in the matter makes the theory less formal and closer to the nonrelativistic nonlinear elasticity theory of anisotropic materials formulated in [1].

Different groups of elasticity symmetry give rise to different scalar and tensor invariants useful for the expansions of the stress tensor. The choice of laconic sets of polynomial invariants helps the separation of the linear part of the stress tensor in an invariant manner.

A number of structures (like the elasticity tensor and two-point tensors) and procedures (like polar decomposition, finding proper values, and rigid motion), used in the literature, are not needed in the given approach.

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