LIMITS ON ANISOTROPY AND INHOMOGENEITY
FROM THE COSMIC BACKGROUND RADIATION

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Abstract

We consider directly the equations by which matter imposes anisotropies on freely propagating background radiation, leading to a new way of using anisotropy measurements to limit the deviations of the Universe from a Friedmann-Robertson-Walker (FRW) geometry. This approach is complementary to the usual Sachs-Wolfe approach: the limits obtained are not as detailed, but they are more model-independent. We also give new results about combined matter-radiation perturbations in an almost-FRW universe, and a new exact solution of the linearised equations.

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1 Introduction

The Cosmic Background Radiation (CBR) is the keystone of modern cosmological analysis, in particular through use of the results of COBE [1,2] and other [3,4] measurements of anisotropies in the CBR to help us understand the nature of inhomogeneities in the universe (see [5] for a review of these observations). This paper presents a way of analysing this relationship that is an alternative to the usual analyses based largely on the Sachs-Wolfe effect (modified by astrophysical effects). Our analysis proceeds from slightly different assumptions than those usually made (though essentially compatible with them); it is to a considerable degree more model-independent than they are.

In [6] we established a theoretical framework for investigating the direct implications of CBR anisotropies (i.e. those that follow without assuming particular inflationary or other evolutionary models for the universe). In that paper, we set up fully covariant and gauge invariant evolution and constraint equations governing the perturbations of the photon distribution and metric. These equations were then used to show that if

\[ A_1 \] all fundamental observers measure the CBR to be almost isotropic in some domain,

then it follows that

\[ A_2 \] the spacetime geometry is almost Friedmann-Robertson-Walker (\('FRW\') - i.e. the shear, vorticity, spatial gradients and Weyl tensor are almost zero, and the metric may be put into perturbed-\('FRW\) form in that domain.

The latter is the assumption which underlies the usual Sachs-Wolfe analyses - for it is the starting point used to set up the Sachs-Wolfe equations.

In this paper, we use the formalism of [6], extending its results to find quantitative limits on the anisotropy and inhomogeneity of spacetime set by anisotropies in the CBR, without assuming a particular model for the origin of such perturbations. We end up with a series of estimates (Section 4) relating the inhomogeneity and anisotropy of the universe directly to the background radiation anisotropies. In addition, we find some new exact results on perturbations in almost-\('FRW\) universes with both matter and radiation (Sections 2 and 5). For a class of almost flat-\('FRW\) universes, we reduce the full set of linearised dynamical equations to two linear ODE's, and give a new exact solution at late times (Section 5).

The difference from the more usual analyses is that they consider observations of the CBR made at one space-time event \( P \) (‘here and now’), relating them to assumed perturbations of the universe at the time of decoupling, these in turn taken to arise from some particular evolutionary history or other. Here we make no such evolutionary assumptions; however (cf. \( A_1 \)) we assume the nature of CBR anisotropies is known not only at our own space-time position \( P \) but also throughout an open domain containing that event (eventually chosen to represent the period between last scattering and the present, in a region containing our world-line and past light cone).

At first glance this seems to make our analysis far more dependent on unobservable data than the standard approach. However this is an illusion, for that approach builds in equivalent assumptions at the outset, but in a rather hidden way, because it assumes \( A_2 \), which cannot be proven on the basis of observational evidence alone [7,8]. It can only be deduced from such data on the basis of some kind of Copernican assumption such as \( A_1 \), which is not directly testable [6-8]. Our approach helps make clear precisely what these hidden assumptions are, and thereby enables one to test how weak they can be and still allow deduction of the desired results. In fact, as pointed out by George Smoot (private communication), the Copernican assumption is in principle partially testable. This follows since the Sunyaev-Zeldovich effect allows us to confirm to some degree that distant galaxies do see nearly isotropic CBR, for otherwise the scattered radiation would have a significantly distorted black-body spectrum.
In the covariant approach we work in the real non-FRW spacetime, rather than starting from an assumed FRW model and perturbing away from it. The former approach considers the full set of dynamical equations that govern the real variables. The latter approach risks missing certain effects [9,10] or masking underlying assumptions. We provide an example of the second kind by showing that the Boltzmann equation imposes constraints on the photon distribution if the monopole moment is assumed to be Planckian to first order. The covariant formalism for analysing fluid inhomogeneities [9,11,12] is a development of Hawking's approach [13], and gives a gauge invariant alternative to Bardeen's formalism [14]. The covariant approach to the photon distribution function in this paper and [6] is based on [15,16], and is an alternative to the application of a Bardeen-type formalism, as presented in [17,18] (in different contexts from that of this paper).

Despite the success of the standard inflationary models with dark matter and critical density [19,20], current CBR observations are consistent with alternative models, and do not by themselves give independent tests of inflation [21,22]. Furthermore, future observations could produce problems for the standard models. The covariant approach provides a clear and direct relation between observational and theoretical quantities, unobscured by particular gauge choices or by the complexities of harmonically determined variables. Furthermore, we do not impose any specific model to generate fluctuations in the CBR. Thus we investigate, as far as possible, what is determined directly by observations of the CBR made by the family of fundamental observers. Where we are forced to make additional assumptions, they are made about observational quantities - and are thus in principle falsifiable by observation, provided we make some kind of Copernican assumption such as $A_1$, stating that all fundamental observers see the same kind of thing (the nature of the required assumptions is clarified below). In this sense, we provide a framework for comparing and testing various models, in which there is as clear as possible a distinction between observed and assumed properties.

Notation:
The metric $g_{ab}$ has signature $(-,+,+,+)$. Einstein's gravitational constant, the speed of light in vacuum, and Planck's constant are one. Round brackets on indices denote symmetrisation, square brackets anti-symmetrisation. $\nabla_a$ is the covariant derivative defined by $g_{ab}$. Given a four-velocity $u^a$, the associated projection tensor is $h_{ab} = g_{ab} + u_a u_b$, and the comoving time derivative and spatial gradient are

\[
\hat{Q}_{a\ldots b} = u^c \nabla_c Q_{a\ldots b},
\]

\[
\hat{\nabla}_c Q_{a\ldots b} = h^\alpha_a u_\alpha h^\beta_b u_\beta \ldots h^\gamma_f u_\gamma \nabla_g Q_{c\ldots f}
\]

for any tensor $Q_{a\ldots b}$ (in [6] we used $^{3}\nabla_a$ for $\hat{\nabla}_a$). If the tensor is spatial, we define

\[
|Q_{a\ldots b}| = (Q_{a\ldots b} Q^{a\ldots b})^{1/2}.
\]

Given a smallness parameter $\epsilon$, $O[N]$ denotes $O(\epsilon^N)$ and $A \simeq B$ means $A - B = O[2]$ (i.e. these variables are equivalent to $O[1]$). When $A \simeq 0$ we shall regard $A$ as vanishing (for it is zero to the accuracy of the first-order calculations that are the concern of this paper).

2 Covariant and Gauge-Invariant Analysis

The fundamental observers are assumed comoving with the cosmological matter, which is modelled by dust with mass-energy density $\rho$, and which is non-interacting with the radiation (as we are considering the epoch after last scattering). The physically preferred four-velocity $u^a$ of this matter is a suitable average over peculiar velocities (which are small). The matter flow is characterised by $u^a$ and its rate of expansion $\Theta (= 3H = 3S/S > 0$, where $H$ is the Hubble parameter and $S$ the scale factor), shear $\sigma_{ab}$, and vorticity $\omega_{ab}$, all non-zero in general; however the flow lines are geodesic: $\dot{u}^a = 0$ (consequent on the vanishing of the matter pressure).

The frame defined by $u^a$ defines an invariant 3+1 splitting of tensors [23]. In particular, for a photon four-momentum

\[
p^a = E(u^a + \epsilon^a), \quad \epsilon_a u^a = 0, \quad \epsilon_a \epsilon^a = 1,
\]
where $E$ is the photon energy and $\epsilon_a$ the direction of photon momentum, relative to fundamental observers. Then the CBR distribution function may be expanded as [15]

$$f(x^c, E, \epsilon^d) = F(x^c, E) + F_a(x^c, E)\epsilon^a + F_{ab}(x^c, E)\epsilon^a\epsilon^b + \ldots$$

(2)

where the covariant harmonics (multipole moments) $F_{a_1\ldots a_L}(x^c, E)$ for $L \geq 1$ are symmetric trace-free tensors orthogonal to $\epsilon^a$, that provide a measure of the deviation of $\tilde{f}$ from exact isotropy (as measured by $u^a$). If the CBR is almost isotropic after last scattering for all fundamental observers, then [6]:

$$F, \dot{F} = O[0], \quad F_{a_1\ldots a_L}, \nabla_b F_{a_1\ldots a_L} = O[1], \quad L \geq 1. \quad (3)$$

Energy integrals of the first three harmonics define the radiation energy density, energy flux and anisotropic stress [15]:

$$\mu = 4\pi \int_0^\infty E^3 F dE = 3\rho = O[0], \quad (4)$$

$$q_a = \frac{4\pi}{3} \int_0^\infty E^3 F_a dE = O[1], \quad (5)$$

$$\tau_{ab} = \frac{8\pi}{15} \int_0^\infty E^3 F_{ab} dE = O[1] \quad (6)$$

(in [6] we used $\mu_R$ for $\mu$). We will also need the integral of the third harmonic:

$$\xi_{abc} = \frac{8\pi}{35} \int_0^\infty E^3 F_{abc} dE = O[1]. \quad (7)$$

Note that the total energy–momentum tensor is made up of matter and radiation contributions:

$$T_{ab} = (\rho + \mu)u_a u_b + \frac{1}{3} \mu h_{ab} + \tau_{ab} + 2u_a q_b.$$  

2.1 Covariant linearised harmonics of the Boltzmann equation

The Boltzmann equation in curved spacetime

$$p^a \left( \frac{\partial}{\partial x^a} - \Gamma^e_{ab} p^b \frac{\partial}{\partial p^e} \right) f(x^d, p^e) = C[f]$$

may be decomposed into covariant harmonic equations via (1), (2). The full (exact and non-linear) results are given in [16, p501]. For the collision–free and zero acceleration case, the linearised zero, first and second harmonic equations are:

$$E \tilde{F} - \frac{1}{3} E^2 \Theta \frac{\partial F}{\partial E} + \frac{1}{2} E \nabla_a F^a \approx 0, \quad (8)$$

$$E \tilde{F}_a - \frac{1}{3} E^2 \Theta \frac{\partial F_a}{\partial E} + E \nabla_a F + \frac{2}{3} E \nabla_b F^b_a \approx 0, \quad (9)$$

$$E \tilde{F}_{ab} - \frac{1}{3} E^2 \Theta \frac{\partial F_{ab}}{\partial E} - E^2 \tau_{ab} \frac{\partial F}{\partial E} + E \nabla_{(a} F_{b)} - \frac{1}{3} h_{ab} \nabla_c F^c + \frac{2}{3} E \nabla_a F_{ab} \approx 0. \quad (10)$$

These are the fundamental (covariant) equations governing the dynamics of radiation at a microscopic level.
2.2 Covariant linearised evolution and constraint equations

If (8–10) are multiplied by $E^2$ and integrated over all photon energies, then they produce the radiation conservation equations governing $\mu$ and $q_a$, as well as the crucial evolution equation for $\pi_{ab}$ (given for the first time in [6], in full non-linear form). These equations and the remaining (linearised) conservation, Einstein, Ricci and Bianchi equations are as follows, obtained from [6] and the general exact equations of [23] (with $\dot{u}_a = 0$ but allowing for an imperfect energy-momentum tensor).

a) Matter and radiation energy and momentum conservation:

$$\dot{\rho} + \Theta\rho = 0, \quad \dot{u}_a = 0,$$

$$\dot{\mu} + \frac{4}{3}\Theta\mu + \nabla_a q^a \simeq 0, \tag{11}$$

$$\dot{q}_a + \frac{4}{3}\Theta q_a + \frac{1}{2}\nabla_a \mu + \nabla_b \pi^b \simeq 0. \tag{12}$$

b) Evolution of radiation anisotropic stress tensor:

$$\dot{\pi}_{ab} + \frac{4}{3}\Theta \pi_{ab} + \frac{8}{3}\mu \sigma_{ab} + 2\nabla_{(a}\pi_{b)} - \frac{2}{3}h_{ab} \nabla q^c + \nabla_a \xi_{ab} \simeq 0. \tag{13}$$

c) Einstein, Ricci and Bianchi propagation equations:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \mu + \frac{1}{2}\rho \simeq 0, \tag{15}$$

$$\dot{\sigma}_{ab} + \frac{2}{3}\Theta \sigma_{ab} + E_{ab} - \frac{1}{2}\pi_{ab} \simeq 0, \tag{16}$$

$$\dot{\omega}_{ab} + \frac{1}{3}\Theta \omega_{ab} \simeq 0, \tag{17}$$

$$\dot{E}_{ab} + \Theta E_{ab} + \nabla^d H_{(a}^\varepsilon_{b)c d} + \left( \frac{1}{3}\rho + \frac{2}{3}\mu \right) \sigma_{ab} +$$

$$+ \frac{1}{2}\pi_{ab} + \frac{1}{6}\Theta \pi_{ab} + \frac{1}{2}\nabla_{(a}\pi_{b)} - \frac{2}{3}h_{ab} \nabla q^c \simeq 0, \tag{18}$$

$$\dot{H}_{ab} + \Theta H_{ab} - \nabla^d E_{(a}^\varepsilon_{b)c d} + \frac{1}{2}\nabla^d \pi_{(a}^\varepsilon_{b)c d} \simeq 0. \tag{19}$$

d) Einstein, Ricci and Bianchi constraint equations:

$$q_a - \frac{2}{3}\nabla_a \Theta + \nabla^d (\sigma_{ba} + \omega_{ba}) = 0, \tag{20}$$

$$\nabla_a \omega^a = 0, \tag{21}$$

$$H_{ab} + \nabla^d \left[ \sigma_{(a}^\varepsilon_{b)c d} + \omega_{(a}^\varepsilon_{b)c d} \right] \simeq 0, \tag{22}$$

$$\nabla_b E_{a}^b - \frac{1}{2}\nabla_a (\mu + \rho) + \frac{1}{2}\nabla_b \pi_{a}^b + \frac{1}{2}\Theta q_a \simeq 0, \tag{23}$$

$$\nabla_b H_{a}^b - (\rho + \frac{2}{3}\mu) \omega_a + \frac{1}{2}\pi_{abc} \nabla^d q^c \simeq 0, \tag{24}$$

where $E_{ab}$, $H_{ab}$ are the electric and magnetic parts of the Weyl tensor, and $\varepsilon_{abc} \equiv \eta_{abcd} u^d$, with $\eta_{abcd}$ the spacetime permutation tensor.

It follows [6] from these equations and (3–7) that:

$$\psi, \dot{\psi} = O[0], \quad \nabla_a \psi = O[1], \quad \sigma_{ab}, \omega_{ab}, E_{ab}, H_{ab} = O[1]. \tag{25}$$

where $\psi \equiv \mu$, $\Theta$, $\rho$. These qualitative results from [6] will be made more quantitative in section 4.
2.3 Integrability conditions and conserved quantities

In the case of zero acceleration, the linearised form of the identity [11] governing the commutation of time and spatial derivatives is:

\[ \nabla_a \Psi \simeq (\nabla_a \Psi)' + \frac{1}{2} \Theta \nabla_a \Psi \] (26)

where \( \Psi \) is any \( O[1] \) spatial tensor or any scalar with \( O[1] \) gradient. The commutation of spatial derivatives themselves is given by the projected Ricci identities [11], which imply the exact identity

\[ \nabla_{[a} \nabla_{b]} \psi = -\psi \omega_{ab} \] (27)

where \( \psi \) is any scalar, and for a nearly-FRW spacetime, the linearised conditions

\[ \nabla_{[a} \nabla_{b]} \psi_{cd} \simeq \frac{k}{S^{7}} h_{[a} \psi_{b]} \] (28)

where \( \psi_a \) is any \( O[1] \) spatial vector, and \( k = 0,1,-1 \) is the spatial curvature index of the limiting (background) spacetime; and

\[ \nabla_{[a} \nabla_{b]} \psi_{cd} \simeq \frac{k}{S^{7}} (h_{[a} \psi_{b]} d - \psi_{c[a} h_{b]d}) \] (29)

where \( \psi_{ab} \) is any \( O[1] \) spatial tensor. These identities could be overlooked in an approach that starts from an FRW background solution and perturbs away from it. The integrability conditions implicit in (27–29) lead to the new results:

(I) if the covariant vector perturbations are spatially homogeneous to first order, then the vorticity vanishes to first order (i.e. non-zero terms are at most second order), and either all vector perturbations vanish to first order, or the spacetime has a flat FRW background.

(II) if the covariant tensor perturbations are spatially homogeneous to first order, then the spacetime is either FRW to first order, or it has a flat FRW background.

The first result follows from (27), which shows that \( \omega_{ab} \simeq 0 \) since \( \nabla_a \psi \) is a vector perturbation for \( \psi = \mu, \rho \), and from (28), which implies \( kS^{-2} \psi_a \simeq 0 \) for \( \psi_a = q_a \) or any other vector perturbation. The second result follows from (28) and (29), which imply

\[ kS^{-2} \psi_a \simeq 0 \simeq kS^{-2} \psi_{ab} \]

for any vector perturbation \( \psi_a \) (since \( \nabla_a \psi_b \) is a tensor) or tensor perturbation \( \psi_{ab} \). Thus either \( k/S^2 \simeq 0 \) (the background FRW model is flat to the accuracy we are working), or

\[ \nabla_a \mu \simeq 0 \simeq \nabla_a \rho, \quad q_a \simeq 0, \quad \pi_{ab} \simeq 0, \quad \nabla_{[a} \xi_{b]} \simeq 0. \]

In the latter case, (27) implies \( \omega_{ab} \simeq 0 \), (14) implies \( \sigma_{ab} \simeq 0 \), and then (20) gives \( \nabla_a \Theta \simeq 0 \), while (16) and (22) give \( E_{ab} \simeq 0 \simeq H_{ab} \). Thus the spacetime is FRW to first order (i.e. it differs from FRW at most by second order terms).

We can derive further linearised integrability results that hold in general (i.e. without assuming homogeneity of vector or tensor perturbations), by covariant differentiation of the dynamical equations. For example, taking the gradient of (24), and using (21) and \( \nabla_{[a} \nabla_{b]} q_{cd} \simeq 0 \) (which follows from (28)), we get

\[ \nabla_a \nabla_b H^{ab} \simeq 0. \] (30)

Similarly, using the contraction of (29) for \( \psi_{cd} = \omega_{cd} \), the gradient of (20) gives

\[ \nabla_a \nabla_b \sigma^{ab} + \nabla_a q^a - \frac{2}{3} \nabla^a \Theta \simeq 0, \] (31)

while (23) yields

\[ \nabla_a \nabla_b (E^{ab} + \frac{1}{2} \pi^{ab}) + \frac{1}{2} \Theta \nabla_a q^a - \frac{1}{3} \nabla^a (\mu + \rho) \simeq 0. \] (32)
Finally, we note the existence of various quantities that are conserved to first order along the matter flow. For example, (17) immediately shows that

\[(S^2 \omega_{ab})' \simeq 0,\]

while (26), (19) and (16) imply that

\[
\dot{H}_{ab}^* + \Theta H_{ab}^* \simeq 0, \quad H_{ab}^* \equiv H_{ab} + \hat{\nabla}^d \sigma_{(c \epsilon_k)c}d.
\]

Then (33) and (11) give

\[
\left( \frac{H_{ab} + \hat{\nabla}^d \sigma_{(c \epsilon_k)c}d}{\rho} \right)' \simeq 0.
\]

It therefore follows that if either the vorticity or \(H_{ab}^*/\rho\) are negligible (i.e. \(O[2]\)) at last scattering, they remain so at all subsequent times.

### 3 Temperature Anisotropy of the CBR

It is important to realise that the covariant dipole moment \(F_a\) of the CBR distribution (see (2)), although dependent upon the choice of \(u^a\), cannot be set to zero by this choice, since it is frequency-dependent and its vanishing implies special conditions on the anisotropy of \(f\). Since the \(u^a\)-frame is physically defined by the matter, and is already assumed to have been corrected for local peculiar velocities, \(F_a\) represents a possible residual intrinsic frequency dependent dipole moment of the CBR distribution relative to the matter, with invariant significance. This distributional dipole however contains much more information than the dipole of CBR temperature anisotropy, which is in fact proportional to the energy flux \(q_a\) given by (5) (see (41) below).

Even for the temperature dipole however, one cannot separate the intrinsic dipole from that induced by peculiar velocity of the observer [19,24]. It is standard to assume that the intrinsic temperature dipole is negligible after correction for peculiar velocity. This is equivalent to the non-trivial assumption that the average four-velocity of the matter coincides with the energy-frame [12] four-velocity of radiation. Although it can be justified for adiabatic perturbations with the standard model [24], we will not make this special assumption, so that we allow for an intrinsic dipole in the temperature (i.e. non-zero \(q_a\)) after correction for local peculiar velocities. This approach accommodates future improvement of observational results for the peculiar velocity which are independent of the CBR observations, and which may reveal a non-negligible residual temperature dipole. From this point of view, the current limits on the intrinsic dipole should be related to the current uncertainties in the local peculiar velocity.

Another aspect of the dipole moment \(F_a\) which appears not to have been previously recognised is its link to deviations from a thermal Planck spectrum in the monopole moment \(F\). This aspect is hidden if one starts from a background FRW solution and then perturbs - rather than considering the real non-FRW solution. The latter approach shows via the Boltzmann equation that non-trivial constraints are imposed on the dipole if \(F\) is Planckian to first order (as strongly indicated by observations [5]). For suppose that

\[
F(x^a, E) \simeq 2 \left[ \exp \left( \frac{E}{kT(x^a)} \right) - 1 \right]^{-1}
\]

where \(k\) is Boltzmann’s constant. Then (34) and the Boltzmann monopole harmonic equation (8) imply

\[
\hat{\nabla}_a F^a \simeq \left( \frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right) \frac{3E/kT}{1 - \cosh(E/kT)}.
\]

Thus the dipole moment is subject to the restriction (35) if the monopole moment is Planckian to \(O[1]\).
One consequence of (35) is
\[ \frac{\dot{T}}{T} = -\frac{\dot{S}}{S} + O[1]. \]  

(36)

In contrast to many other treatments (where \( TS = \text{constant} \)), \( T \) is not a fictitious background temperature, but is the gauge-invariant average temperature in the actual spacetime. Now observations of the CBR measure temperatures in different directions on the sky. The full-sky average temperature \( T(x^a) \) at event \( x^a \) is determined by the monopole harmonic of the photon distribution:
\[ \mu(x^a) = a[T(x^a)]^4 \]  
\[ = 4\pi \int_0^\infty E^3 F(x^a, E) dE \]  

(37)
on using (4), where \( a \) is the Stefan–Boltzmann constant. A directional temperature is determined by all the harmonics (2) via the directional energy density per unit solid angle that is defined by the integrated (bolometric) brightness [25] (see also [26–28])
\[ I(x^a, \epsilon^b) = \int_0^\infty E^3 f(x^a, E, \epsilon^b) dE = \frac{a}{4\pi} [T(x^a) + \delta T(x^a, \epsilon^b)]^4 \]  

(38)

which defines the gauge-invariant fluctuation \( \delta T(x^a, \epsilon^b) \).

The covariant multipole moments \( \tau_{a1...a_L} (x^a) \) \( (L \geq 1) \) of temperature anisotropy are trace-free, symmetric tensors orthogonal to \( u^a \), defined by
\[ \tau \equiv \frac{\delta T}{T} = \tau_{a} e^{a} + \tau_{ab} e^{a} e^{b} + \tau_{abc} e^{a} e^{b} e^{c} + \ldots \]  

(39)

By (38), (37), (2) and (4) they are given in general, to a good approximation, as normalised integrals of the covariant distribution harmonics:
\[ \tau_{a1...a_L} \simeq \left( \frac{1}{4\mu} \right) 4\pi \int_0^\infty E^3 F_{a1...a_L} dE. \]  

(40)

These moments give a covariant and gauge–invariant description of the CBR temperature variation, with spectral information integrated out. By (5–7), (40) gives the dipole, quadrupole and octopole as:
\[ \tau_{a} \simeq \frac{3\mu_{a}}{4\mu}, \quad \tau_{ab} \simeq \frac{15\mu_{ab}}{8\mu}, \quad \tau_{abc} \simeq \frac{35\xi_{abc}}{8\mu}. \]  

(41)

The \( \tau_{a1...a_L} \) are a frame-independent alternative to the usual multipole coefficients \( A_{LM} \) in an expansion in spherical harmonics \( Y_{LM} \). If we choose a standard triad in the rest space of \( u^a \) such that \( e^a = (0, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), then the two approaches are linked (cf. [15]) by
\[ \tau(x, \epsilon) = \sum_{L=1}^{\infty} \sum_{M=-L}^{L} A_{LM}(x) Y_{LM}(\theta, \phi) = \sum_{L=1}^{\infty} \sum_{a1...a_L} \tilde{A}_{a1...a_L}(x) \epsilon^{a1} \ldots \epsilon^{a_L}. \]  

The correlation function
\[ C(\alpha) = \langle \tau(x, \epsilon) \rangle \langle \tau(x, \epsilon') \rangle, \quad \epsilon^{\alpha} \epsilon^{\alpha'} = \cos \alpha, \]  

where \( \langle \ldots \rangle \) denotes a statistical average, is the key quantity in actual observations. In this paper we will not consider the detailed statistical analysis of the correlation function, which is given for example in [19–21, 24, 26, 29–31], where an inflationary model for perturbations is assumed. Our concern here is with the underlying principles of how to relate observational limits to properties of the spacetime geometry in a model-independent way.

Current CBR observations place limits on \( \tau_{a1...a_L}(t_0, y)|_C \) where \( t_0 \) is the proper time along our worldline \( C \) since last scattering and \( y|_C \) are comoving coordinates of \( C \). By A1, these limits may be extended to hold on each worldline \( y \) at a proper time \( t_0 \) along the worldline after last scattering:
\[ | \tau_{a1...a_L}(t_0, y) | < \xi_{L}(t_0). \]  

As in [6], we assume that anisotropies are \( O[1] \) back to last scattering, and extend the limit to hold for all times \( 0 < t \leq t_0 \), thus obtaining the assumption
**B1:** there exist $O[1]$ constants $\epsilon_L$ such that $\epsilon_L(t) \leq \epsilon_L$. Hence for any event after last scattering

$$|\tau_{a_1...a_L}| < \epsilon_L.$$  \hspace{1cm} (42)

In principle (and possibly not too far off in practice), observations place limits on the comoving time derivatives of the multipoles. As before we assume

**B2:** there exist $O[1]$ constants $\epsilon_L^*, \epsilon_L^{**}, \epsilon_L^{***}$ such that

$$|\tilde{\tau}_{a_1...a_L}| < \epsilon_L^* \Theta, \quad |\tilde{\tau}_{a_1...a_L}| < \epsilon_L^{**} \Theta^2, \quad |\tilde{\tau}_{a_1...a_L}| < \epsilon_L^{***} \Theta^3,$$  \hspace{1cm} (43)

where we have normalised the derivatives relative to the expansion of the universe (recall that $\Theta > 0$ is $O[0]$), and we will not need the higher derivative limits. Since it is effectively impossible for us to move cosmological distances off $C$, there will not be direct observations of the spatial derivatives of the multipoles. However we will need limits on spatial and mixed derivatives up to third order, and so assume also, on general plausibility grounds (given the basic Copernican assumption), that additionally:

**B3:** there exist $O[1]$ constants $\epsilon_L^*, \epsilon_L^{**}, \epsilon_L^{***}$ such that

$$|\tilde{\nabla}_c \tau_{a_1...a_L}| < \epsilon_L^* \Theta, \quad |\tilde{\nabla}_c \tilde{\nabla}_c \tau_{a_1...a_L}| < \epsilon_L^{**} \Theta^2, \quad |\tilde{\nabla}_c \tilde{\nabla}_c \tau_{a_1...a_L}| < \epsilon_L^{***} \Theta^3,$$  \hspace{1cm} (44)

and constants $\epsilon_L^*, \epsilon_L^{**}, \epsilon_L^{***}$ such that

$$|\tilde{\nabla}_c \tau_{a_1...a_L}| < \epsilon_L^* \Theta^2, \quad |\tilde{\nabla}_c \tau_{a_1...a_L}| < \epsilon_L^{**} \Theta^3, \quad |\tilde{\nabla}_c \tau_{a_1...a_L}| < \epsilon_L^{***} \Theta^4.$$  \hspace{1cm} (45)

One should note that we expect all the quantities $\epsilon$ defined here to be very small; probably at most $10^{-3}$. The exact isotropic case considered by Ehlers, Geren, and Sachs [32] corresponds to $\epsilon_L = \epsilon_L^* = \epsilon_L^{**} = 0$, and is a special case of what follows. In fact, our special case is a small generalisation of the EGS result to include non-interacting dust matter in the source of the gravitational field.

### 4 Model-independent Limits on Spacetime Anisotropy and Inhomogeneity

The observationally based limits (42) on temperature anisotropy lead directly via (41) to the limits

$$|\tilde{q}_{ab}| < \frac{4}{3} \mu \epsilon_1, \quad |\pi_{ab}| < \frac{8}{3} \mu \epsilon_2, \quad \xi_{abc} < \frac{8}{3} \mu \epsilon_3,$$  \hspace{1cm} (46)

on the radiation anisotropy tensors. Limits on the derivatives of the radiation tensors arise from differentiating (41), using (42-45) and the evolution and constraint equations of Section 2.2. For example, the limits on derivatives of $\tilde{q}_{ab}$ that we will need are

$$|\tilde{q}_{ab}| < \frac{4}{3} \mu \Theta (\frac{4}{3} \epsilon_2 + \epsilon_1^*), \quad |\tilde{\nabla}_a \tilde{q}_{ab}| < \frac{4}{3} \mu \Theta \epsilon_1^*,$$  \hspace{1cm} (47)

$$|\tilde{q}_{ab}| < \frac{4}{3} \mu \Theta^2 [(20 + 4 \Omega_R + 2 \Omega_M) \epsilon_1 + 24 \epsilon_1^* + 9 \epsilon_1^{**} + 9 \epsilon_1^{***}], \quad |\tilde{\nabla}_a \tilde{\nabla}_b \tilde{q}_{ab}| < \frac{4}{3} \mu \Theta^3 \epsilon_1^*,$$  \hspace{1cm} (48)

$$|\tilde{\nabla}_a \tilde{\nabla}_b \tilde{q}_{ab}| < \frac{4}{3} \mu \Theta^3 (3 \epsilon_1^* + 4 \epsilon_1^*), \quad |\tilde{\nabla}_a \tilde{\nabla}_b \tilde{q}_{ab}| < \frac{4}{3} \mu \Theta^3 (3 \epsilon_1^{**} + 4 \epsilon_1^{***}),$$  \hspace{1cm} (49)

with similar expressions for $\pi_{ab}, \xi_{abc}$. The density parameters are

$$\Omega_R = \frac{\mu}{3 H^2}, \quad \Omega_M = \frac{\rho}{3 H^2}.$$  

Now (46) and (47-49) (with their counterparts for $\pi_{ab}$ and $\xi_{abc}$) are used in the evolution and constraint equations and their derivatives in order to find limits on the geometric and dynamic quantities that characterise the deviation of the universe from FRW form. We emphasise that the following limits are gauge-invariant, covariant and imposed directly by observational quantities, i.e. the temperature dipole, quadrupole, octopole and their derivatives.

9
From (13) and (14) we get immediately
\[
\frac{|
abla_a \mu|}{\mu} = 4 \frac{|
abla_a T|}{T} < H(8\epsilon_1 + 12\epsilon'_1 + \frac{72}{5}\epsilon_2') ,
\] (50)
\[
\frac{|
abla_{ab}|}{\Theta} < \frac{8}{3}\epsilon_2 + \epsilon'_2 + 5\epsilon_1' + \frac{3}{7}\epsilon_3' .
\] (51)

The remaining limits require more complicated manipulation of the evolution and constraint equations.

From (27), using \(\nabla_{ab}(13)\) to get \(|\nabla_{ab} \nabla_{ab}|\), we find
\[
\frac{|\omega_{ab}|}{\Theta} < 9\epsilon'_1 + 3\epsilon''_1 + \frac{6}{5}\epsilon''_2 .
\] (52)

From (23), using \(\nabla_{c}(16), \nabla_{c}(14)\) and \(|\nabla_{c}(14)|^2\) to get limits on \(|\nabla_{c} E_{ab}|, |\nabla_{c} \sigma_{ab}|\) and \(|(\nabla_{c} \omega_{ab})|^2\), we find
\[
\frac{|\nabla_{c} \rho|}{\Theta} < \frac{27}{\Theta} H \epsilon'_2 + \left(\frac{\Omega_R}{\Omega_M}\right) H[12\epsilon_1 + 12\epsilon'_1 + 61\epsilon'_2] + \frac{3}{\Omega_M} H[165\epsilon''_1 + 45\epsilon''_2 + 110\epsilon'_2 + 69\epsilon''_2 + 9\epsilon''_3 + 18\epsilon''_4] .
\] (53)

From \(\nabla_{c}(12)\), using (13) and (26) to get \(|\nabla_{c} \mu|^2\), we find
\[
\frac{|\nabla_{c} \Theta|}{\Theta} < H[50\epsilon'_1 + 51\epsilon''_1 + 9\epsilon''_2 + 3\epsilon''_3 + \frac{24}{5}\epsilon'_2 + 18\epsilon'_2 + \frac{18}{7}\epsilon''_2 + 4(\Omega_R + \Omega_M)H\epsilon_1] .
\] (54)

From (16), using (14) to get \(|\sigma_{ab}|\) and (12), (15), (26), we find
\[
\frac{|E_{ab}|}{\Theta} < H[50\epsilon'_1 + 15\epsilon''_1 + \frac{88}{3}\epsilon_2 + 14\epsilon'_2 + 3\epsilon''_2 + \frac{66}{5}\epsilon'_3 + \frac{2}{7}\epsilon''_3 + \frac{4}{37}(11\Omega_R + 15\Omega_M)H\epsilon_2] .
\] (55)

From (22), using \(\nabla_{c}(27)\) (with \(\psi = \mu\)) and \(\nabla_{c}(13)\) to get \(|\nabla_{c} \omega_{ab}|\), and \(\nabla_{c}(14)\) to get \(|\nabla_{c} \sigma_{ab}|\), we find
\[
\frac{|H_{ab}|}{\Theta} < H[45\epsilon''_1 + 9\epsilon''_2 + 9\epsilon'_2 + 3\epsilon''_2 + \frac{18}{7}\epsilon''_2 + \frac{2}{7}\epsilon''_3] .
\] (56)

These equations show explicitly the role of the dipole, quadrupole and octopole. For example, (50) shows that the gradient of radiation energy density or average temperature, which reflects inhomogeneous deviations from FRW spacetime, is bounded by the limits on both the dipole and quadrupole of temperature anisotropy, while by (51), the shear, which reflects anisotropic deviations from FRW spacetime, is bounded by the limits on the dipole, quadrupole and octopole.

Note that if we follow the usual assumption that the CBR temperature dipole is negligible (after correction for local peculiar velocities), then by (41), (42−45) it follows that all the \(\epsilon_1\)'s vanish:
\[
\tau_a \simeq 0 \iff q_a \simeq 0 \iff \epsilon_1 = \epsilon'_1 = \cdots = \epsilon''_1 = 0 .
\]

In this case, the bounds in (50–56) are reduced, so that a negligible dipole reduces the limits on anisotropy and inhomogeneity.

Using (12) and (37), (50) may be re-written
\[
\frac{|\nabla_a T|}{T} < \frac{1}{T} \epsilon_1 + 3\epsilon'_1 + \frac{18}{7}\epsilon'_2 ,
\]
from which it follows that
\[
\frac{|\nabla_a T|}{T} < \frac{1}{T} \epsilon_1 \iff d_R \gg t_R ,
\]
are characteristic length and time scales defined by the CBR. In practice $\epsilon'_L, \epsilon''_L$ and especially $\epsilon'_L, \epsilon''_L, \ldots$ are not known from observations. In order to produce more useful versions of the results, we need a reasonable estimate of these quantities. First, we make the reasonable assumption

**C1:** the spatial gradients of the temperature multipoles are not greater than their time derivatives:

$$\epsilon'_L \leq \epsilon'_M; \epsilon''_L, \epsilon''_M \leq \epsilon''_M; \epsilon''_L, \epsilon''_M, \epsilon''_L \leq \epsilon''_M.$$

Next, we can estimate $\epsilon'_L, \epsilon''_M, \epsilon''_M$ by invoking the characteristic time $t_R$ defined by (57), leading to assumption

**C2:** the bounds on the time derivatives of the temperature harmonics are estimated by $\Theta \epsilon'_L \approx \epsilon_L/t_R$, $\Theta^2 \epsilon''_L \approx \epsilon_L/t_R^2$, $\Theta^3 \epsilon''_M \approx \epsilon_L/t_R^3$, so that, using (36)

$$\epsilon'_L \approx \frac{1}{2} \epsilon_L, \epsilon''_M \approx \frac{1}{3} \epsilon_L, \epsilon''_M \approx \frac{1}{7} \epsilon_L.$$

We can now use **C1–2** to re-cast (58–66) in terms of the observationally realistic $\epsilon_L$:

$$\frac{\left|\overline{\nabla} a \right|}{\mu} = 4 \frac{\left|\overline{\nabla} a T\right|}{T} < H \left(12 \epsilon_1 + \frac{24}{5} \epsilon_2\right),$$

$$\frac{|\sigma_{ab}|}{\Theta} \approx \left\{ \begin{array}{ll}
\frac{5}{3} \epsilon_1 + 3 \epsilon_2 + \frac{2}{7} \epsilon_3,
\end{array} \right.$$ (59)

$$\frac{|\omega_{ab}|}{\Theta} \approx \left\{ \begin{array}{ll}
\frac{10}{3} \epsilon_1 + \frac{2}{15} \epsilon_2,
\end{array} \right.$$ (60)

$$\frac{\left|\overline{\nabla} a \right|}{\Theta} < \frac{9}{7} H \epsilon_2 + \left(\frac{H}{\Omega_M}\right) \left[60 \epsilon_1 + 134 \epsilon_2 + 6 \epsilon_3\right] + \left(\frac{\Omega_R}{\Omega_M}\right) H[16 \epsilon_1 + \frac{31}{18} \epsilon_2],$$ (61)

$$\frac{\left|\overline{\nabla} a \Theta\right|}{\Theta} < H(\frac{285}{3} \epsilon_1 + 8 \epsilon_2) + 4(2 \Omega_R + \Omega_M)H \epsilon_1,$$ (62)

$$\frac{|E_{ab}|}{\Theta} < H(\frac{55}{3} \epsilon_1 + \frac{183}{3} \epsilon_2 + \frac{23}{7} \epsilon_3) + \frac{4}{45}(11 \Omega_R + 15 \Omega_M)H \epsilon_2,$$ (63)

$$\frac{|H_{ab}|}{\Theta} < H(\frac{16}{3} \epsilon_1 + \frac{32}{15} \epsilon_2 + \frac{1}{31} \epsilon_3).$$ (64)

If the dipole is neglected, then we can set $\epsilon_1 = 0$ in (58–64). Note also that for the $\Omega_R, \Omega_M$ and $H$ on the right sides of (58–66) and (58–64) we may use the $O[1]$ values, i.e. the values they take in the limiting (background) FRW spacetime, which has non-interacting dust and isotropic radiation - exact solutions are given in [33].

The bounds (58–64) are the main results of the quest for a direct link from feasible observational limits on the CBR to limits on the deviations of the universe from FRW since last scattering (within our past light cone). They are based on the reasonable assumptions **B1–3, C1–2**. These results give for the first time a direct relation between CBR observational limits and limits to anisotropy and inhomogeneity, without assuming a specific evolutionary model, or the curvature index $k$ of the background.

From these limits we can obtain conservative estimates of present–time bounds on the anisotropy and inhomogeneity of the universe. Let

$$\epsilon \equiv \max(\epsilon_1, \epsilon_2, \epsilon_3)$$
denote the upper limit of currently observed anisotropy in the CBR temperature variation, and take \((\Omega_M)_0 \ll 1\). Then (58-64) imply
\[
\left( \frac{\nabla_a H}{\mu} \right)_0 < 17 H_0 \epsilon , \quad \left( \frac{|\sigma_{ab}|}{\Theta} \right)_0 < 6 \epsilon , \quad \left( \frac{|\omega_{ab}|}{\Theta} \right)_0 < 4 \epsilon . \tag{65}
\]
\[
\left( \frac{|E_{ab}|}{\Theta} \right)_0 < \left[ \frac{4}{3} (\Omega_M)_0 + 56 \right] H_0 \epsilon , \quad \left( \frac{|H_{ab}|}{\Theta} \right)_0 < 10 H_0 \epsilon , \tag{66}
\]
\[
\left( \frac{\nabla_a \Theta}{\Theta} \right)_0 < \left[ 4 (\Omega_M)_0 + 77 \right] H_0 \epsilon , \quad \left( \frac{|\nabla_a \rho|}{\rho} \right)_0 < C(\Omega) H_0 \epsilon , \tag{67}
\]
where \(C(\Omega) \equiv 5 + 200/(\Omega_M)_0\). The latter is a relatively weak limit, consistent with inhomogeneities in the large-scale matter distribution. If we are willing to assume that \((\Omega_M)_0 \approx 1\) today, we get a reasonably tight limit from (67). However, the observational evidence points towards a range of values between 0.1 and 0.3 as more plausible [34]. Including the lowest limits implied by nucleosynthesis, we can represent the range of possibilities by a table of values:

<table>
<thead>
<tr>
<th>((\Omega_M)_0)</th>
<th>0.02</th>
<th>0.1</th>
<th>0.3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C(\Omega))</td>
<td>10005</td>
<td>2005</td>
<td>672</td>
<td>205</td>
</tr>
</tbody>
</table>

As we go back in time towards last scattering, whatever value it has today, \(\Omega_M\) will rapidly approach 1.

We note that if the dipole is neglected, then we can set \(\epsilon_1 = 0\) in (58-64) to obtain the revised "dipole-free" estimates:
\[
\left( \frac{\nabla_a H}{\mu} \right)_0 < 5 H_0 \epsilon , \quad \left( \frac{|\sigma_{ab}|}{\Theta} \right)_0 < 4 \epsilon , \quad \left( \frac{|\omega_{ab}|}{\Theta} \right)_0 < \epsilon , \tag{65a}
\]
\[
\left( \frac{|E_{ab}|}{\Theta} \right)_0 < \left[ \frac{4}{3} (\Omega_M)_0 + 38 \right] H_0 \epsilon , \quad \left( \frac{|H_{ab}|}{\Theta} \right)_0 < 4 H_0 \epsilon , \tag{66a}
\]
\[
\left( \frac{\nabla_a \Theta}{\Theta} \right)_0 < 4 (\Omega_M)_0 + 2 \right] H_0 \epsilon , \quad \left( \frac{|\nabla_a \rho|}{\rho} \right)_0 < \left[ \frac{140}{(\Omega_M)_0} + 5 \right] H_0 \epsilon , \tag{67a}
\]
which are considerably sharper. We recover the EGS result [32] on setting \(\epsilon = 0\) (exact isotropy implies an exact FRW solution).

5 Almost Flat–FRW Solutions

As already pointed out, we are unlikely to have direct observational information about the spatial gradients of the temperature multipoles. In section 4 we used the estimate C1 that the spatial gradients are not greater than the time derivatives. Here we consider a more stringent, but apparently not unreasonable, assumption on the spatial gradients. In effect we extend to the spatial gradient bounds the assumption C2 already made on the time derivative bounds - i.e. we replace assumptions C1–2 by

\[ \mathbf{D}: \text{the bounds on both time and spatial derivatives of the temperature multipoles are determined via the characteristic scales (57) of the CBR:} \]
\[ \Theta \sigma_L \simeq \frac{\epsilon_L}{t_R} \Rightarrow \sigma_L \simeq \frac{\epsilon_L}{3} \epsilon_L , \quad \Theta \sigma_L \simeq \frac{\epsilon_L}{d_R} \Rightarrow \sigma_L < (\epsilon_1 + \frac{2}{3} \epsilon_2) \epsilon_L , \] (68)

where we have used (36) and (58). It immediately follows from (68) that \( \sigma_L = O[2] \), and hence by (44) and (41)

\[ \hat{\nabla} \sigma \approx 0 , \quad \hat{\nabla}_a \pi_{bc} \approx 0 , \quad \hat{\nabla}_a \hat{\epsilon}_{bcd} \approx 0 , \quad \ldots \] (69)

Thus from an apparently reasonable assumption on the spatial gradients, we are led to the vanishing at first order of vector and tensor inhomogeneities in the radiation. The point is that the radiation time scale \( t_R \) is a zero order quantity (it exists for exactly isotropic CBR), while the radiation length scale \( d_R \) is first order - it is only finite when there are inhomogeneities in the CBR. If inhomogeneities in the temperature fluctuation \( \tau \) are determined by the characteristic inhomogeneity scale, then, as shown by (68), they are negligible in comparison with anisotropies. This in turn leads to a very restrictive condition:

**if the radiation energy flux and anisotropic stress are homogeneous to first order,**

i.e. if (69) holds, which will follow if \( \mathbf{D} \) is true, then

**either the spacetime is FRW to first order, or the spatial curvature vanishes to first order and so the background has a flat FRW geometry.**

This follows as a special case of the results (I), (II) derived in section 2.3. Note that it may also be derived (but with greater effort) by considering the integrability conditions of the constraint equations (20–24) in the case that (69) holds.

This result may be viewed as providing an alternative motivation for the almost flat–FRW model of the universe. Furthermore, we are able to reduce the system of dynamical equations in this model to a pair of linear evolution equations, for the shear and energy flux. Using the \( O[1] \) identity (26) with (69), the spatial gradients of (13) and (14) imply

\[ \hat{\nabla}_a \hat{\nabla}_b \mu \simeq 0 \Rightarrow \omega_{ab} \simeq 0 , \quad \hat{\nabla}_a \sigma_{bc} \simeq 0 , \] (70)

where we have used (27). Then (70) reduces (22) and the spatial gradient of (16) to

\[ H_{ab} \approx 0 , \quad \hat{\nabla}_a E_{bc} \simeq 0 . \] (71)

By (69–71), the system of dynamical equations (11–24) closes at \( O[1] \) and reduces to a sub–system for \( \sigma_{ab} , \pi_{ab} , E_{ab} \)

\[ \dot{\sigma}_{ab} + \frac{2}{3} \Theta \sigma_{ab} + E_{ab} - \frac{1}{2} \pi_{ab} \simeq 0 , \] (72)

\[ \dot{\pi}_{ab} + \frac{2}{3} \Theta \pi_{ab} + \frac{8}{27} \mu \pi_{ab} \simeq 0 , \] (73)

\[ \dot{E}_{ab} + \Theta E_{ab} + (\frac{1}{2} \rho + \frac{2}{3} \mu) \sigma_{ab} - \frac{1}{2} \Theta \pi_{ab} \simeq 0 , \] (74)

and a sub–system for \( q_a , \hat{\nabla}_a \psi \) (\( \psi \equiv \mu , \rho , \Theta \)):

\[ \dot{q}_a + \frac{4}{3} \Theta q_a + \frac{1}{3} \hat{\nabla}_a \mu \simeq 0 , \] (75)

\[ 2 \hat{\nabla}_a \Theta - 3 q_a \simeq 0 , \] (76)

\[ \hat{\nabla}_a (\rho + \mu) - q_a \simeq 0 . \] (77)

These sub–systems represent a decoupling of the anisotropy and inhomogeneity, since (72–74) contain no spatial gradients (because \( H_{ab} , \omega_{ab} \) and the acceleration vanish to the order of the calculation, this solution belongs to the ‘silent universe’ class recently examined by Mataresse et al [35,36]). In the special case where we assume that the temperature dipole is negligible, (41) and (75–77) show

\[ q_a \simeq 0 \Rightarrow \hat{\nabla}_a \mu \simeq \hat{\nabla}_a \rho \simeq \hat{\nabla}_a \Theta \simeq 0 \]

which, together with (69–71), shows that the spacetime is homogeneous to first order. Thus in the dipole–free case when \( \mathbf{D} \) is assumed, the spacetime is Bianchi I to the accuracy of the calculation. Conversely,
if there is inhomogeneity in the radiation and matter, and if the CBR characteristic
length scale determines the CBR temperature inhomogeneity (i.e. if D holds), then the
dipole cannot be negligible after correction for peculiar velocities.

From now on we will assume that the dipole is not negligible, i.e. \( q_a \neq O[2] \). In (72-77), the
coefficients \( \mu, \rho \) and \( \Theta \) may be given their FRW zero-order forms, which are in fact the solutions of
(11,12) (using (69)) and (15). In particular, these equations imply the \( O[1] \) Friedmann equation [6]
\[
\Theta'^2 - 3(\rho + \mu) \simeq 0 .
\] (78)

Before we consider the decoupling and solving of (72-74) and (75-77), we give the limits that they
imply, using (46,47) and (68,69):
\[
\left| \nabla_a \mu \right| \mu < 12H\epsilon_1 , \quad \left| \nabla_a \rho \right| \rho < 16 \left( \frac{\Omega_R}{\Omega_M} \right) H\epsilon_1 , \quad \left| \nabla_a \Theta \right| \Theta < 2\Omega_R H\epsilon_1 , \quad \left| F_{ab} \right| < 3\epsilon_2 .
\] (79)

Note that a bound on \( E_{ab} \) does not follow directly. These results sharpen the bounds given by
(58,59,61,62) (recalling that C1-2 have been replaced by D). In fact, the bound on the matter
inhomogeneity is drastically sharpened at late times, when \( \Omega_R \ll \Omega_M \). The source of this is the
disappearance of the gradient of the electric Weyl tensor which produced the \( H/\Omega_M \) term of (61).
This gradient, along with all tensor gradients (i.e. gradients of vectors and tensors), drops out by
virtue of D. In other words:

if only scalar inhomogeneities occur (i.e. all vectors and tensors are gradients of scalars), then the matter inhomogeneities are rapidly suppressed at late times.

A potential problem arising from (79) is that the limits suggest the matter inhomogeneity is much
less than the radiation inhomogeneity at late times. However, the upper limits do not force this to
occur, they simply allow the possibility. Below we will give an example of a late-time solution where
the matter inhomogeneity is greater than the radiation inhomogeneity.

By taking \( u^a \nabla_a \) derivatives and using (15) and (78), the sub-systems (69-71) and (72-74) may be
decoupled:
\[
\begin{align*}
\ddot{\sigma}_{ab} + \Theta^{-1} & \left( \frac{16}{3}, \Theta^2 + \mu + \frac{1}{2}\rho \right) \dot{\sigma}_{ab} + \left( \frac{2}{3}, \Theta + \frac{43}{4}\rho \right) \sigma_{ab} + \\
+ \Theta^{-1} & \left( \frac{44}{12}, \Theta^2 + \frac{7}{8}\rho \Theta^2 - \frac{44}{12}, \Theta^2 - \frac{5}{3}\rho^2 - \frac{26}{3}, \Theta \rho \right) \sigma_{ab} \simeq 0 , \\
\dot{\dot{q}}_a + 3\Theta \dot{q}_a + & \left( \frac{11}{9}, \Theta^3 - 2\mu - \frac{2}{5}, \rho \right) q_a \simeq 0 .
\end{align*}
\] (80)

In principle we can obtain the \( O[1] \) solution for the almost flat-FRW model after last scattering as
follows. The solutions of (11), (12) and (78) imply
\[
\mu \simeq rS^{-4} , \quad \rho \simeq mS^{-3} , \quad \Theta \simeq S^{-2} [3(r + mS)]^{1/2} , \quad \dot{r} \simeq 0 = \ddot{m}
\] (82)

which allow us to reduce (80,81) to linear ODE’s in \( S \). We write
\[
\begin{align*}
\sigma_{ab} & \simeq A_a^{(I)} \Sigma_{(I)}(S) , \quad I = 1,2,3 \\
q_a & \simeq B_a^{(\Lambda)} Q_{(\Lambda)}(S) , \quad \Lambda = 1,2
\end{align*}
\] (83)

with \( A_a^{(I)} \simeq 0 \simeq B_a^{(\Lambda)} \). Then \( \Sigma_{(I)} \) are linearly independent solutions of
\[
\begin{align*}
\Sigma'' + & \left[ \frac{2(8r + 9mS)}{S(r + mS)} \right] \Sigma'' + \left[ \frac{r(94r + 89mS)}{S^2(r + mS)^2} \right] \Sigma' - \\
& - \left[ \frac{480r^3 + 1006rmS + 525m^2S^2}{20S^3(r + mS)^2} \right] \Sigma \simeq 0
\end{align*}
\] (85)
and $Q_{(\lambda)}$ are linearly independent solutions of
\[
Q'' + \left[\frac{3(4r + 5mS)}{2S(r + mS)}\right] Q' + \left[\frac{2(5r + 7mS)}{S^2(r + mS)}\right] Q \simeq 0.
\] (86)

Given the solution $\Sigma(S)$ of (85), we find from (72,73) that
\[
\tau_{ab} \simeq \frac{C_{ab}}{S^4} - \left(\frac{8r}{5S^4}\right) A_{ab}^{(I)} \int \left(\frac{S \Sigma_{(r)}}{[3(r + mS)]^{1/2}}\right) dS,
\] (87)
\[
E_{ab} \simeq \frac{C_{ab}}{2S^4} - A_{ab}^{(I)} \left[\left(\frac{3(r + mS)^{1/2}}{3S}\right) \Sigma_{(r)} + \left(\frac{4r}{5S^4}\right) \int \left(\frac{S \Sigma_{(r)}}{[3(r + mS)]^{1/2}}\right) dS\right],
\] (88)
where $\dot{C}_{ab} \simeq 0$.

Similarly, given the solution $Q(S)$ of (86), we find from (75–77) that
\[
\hat{\nabla}_{ab} \mu \simeq -[3(r + mS)]^{1/2} B_{ab}^{(A)} [Q'_{(A)} + \frac{4}{S} Q_{(A)}],
\] (89)
\[
\hat{\nabla}_{ab} \Theta \simeq \frac{3}{2} B_{ab}^{(A)} Q_{(A)},
\] (90)
\[
\hat{\nabla}_{ab} \rho \simeq -[3(r + mS)]^{1/2} B_{ab}^{(A)} [Q'_{(A)} + \frac{5}{S} Q_{(A)}].
\] (91)

Note that because two time derivatives were needed in the decoupling that led to (80), there will be a consistency condition imposed on the integration constants $A_{ab}^{(I)}$, $C_{ab}$ via (74).

Thus (78), (83,87,88) and (84,89–91) represent an exact solution of the linearised equations governing the metric, CBR and matter after last scattering in a universe subject to the observational assumption $D$, and with anisotropic radiation. The explicit analytic form of the solutions depends on finding explicitly the solutions to the ODE’s (85,86). We can provide explicit solutions for late times (i.e. long after last scattering) when
\[
\mu \ll \rho \simeq \frac{1}{3} \Theta^2 \ll 1
\] (92)
using (78). By (92), (80) simplifies to
\[
\ddot{\sigma}_{ab} + 7\frac{\Theta}{S} \dot{\sigma}_{ab} + \frac{43}{18} \Theta^3 \sigma_{ab} + \frac{1}{9S} \Theta^3 \sigma_{ab} \simeq 0.
\] (93)

Writing $\sigma_{ab} \simeq U_{ab} \Theta^n$, where $\dot{U}_{ab} \simeq 0$, we find from (93) that $n = \frac{1}{3}$, $\frac{5}{3}$, $2$. However $n = \frac{1}{3}$ violates (51) at late times. Thus an acceptable solution to (93) is
\[
\sigma_{ab} \simeq U_{ab} \Theta^{5/3} + V_{ab} \Theta^2, \quad U_{ab} \simeq 0 \simeq V_{ab}.
\] (94)

Using (92) and (94) in (72–74) we find that the consistency condition on the constants of integration gives $V_{ab} \simeq 0$ and the late-time solution is
\[
\sigma_{ab} \simeq U_{ab} \Theta^{5/3}, \quad \tau_{ab} \simeq \frac{1}{6} U_{ab} \Theta^{8/3} \simeq \frac{1}{6} \Theta \sigma_{ab}, \quad E_{ab} \simeq \frac{1}{4} U_{ab} \Theta^{8/3} \simeq \frac{3}{2} \pi_{ab}.
\] (95)

Now we use (92) to simplify (81) to
\[
\dot{q}_a + 3\Theta q_a + \frac{14}{3} \Theta^2 q_a \simeq 0.
\] (96)
With $q_a \simeq J_a \Theta^n$, $\dot{J}_a \simeq 0$, we get $n = \frac{5}{3}$, $\frac{7}{3}$, neither of which violates the limit (46). Thus a solution of (96) is
\[
q_a \simeq J_a \Theta^{7/3} + K_a \Theta^{8/3}, \quad \dot{J}_a \simeq 0 \simeq \dot{K}_a.
\] (97)
and this leads via (75–77) to
\[ \hat{\nabla}_a \mu \simeq -\frac{1}{2} J_a \Theta^{1/3}, \quad \hat{\nabla}_a \Theta \simeq \frac{3}{2} q_a, \quad \hat{\nabla}_a \rho \simeq \frac{3}{2} J_a \Theta^{1/3} + K_a \Theta^{1/3}. \] (98)

Thus we have presented an exact and explicit late-time solution (94, 95, 97, 98), which in particular provides a quantitative measure for the rate of approach towards the limiting background solution. This solution also allows us to tighten the limit in (79) on shear at late times, and to give a late-time limit on the Weyl tensor: by (46) and (95) we have
\[ \frac{|\sigma_{ab}|}{\Theta} < \left( \frac{16 \Omega_R}{15 \Omega_M} \right) \epsilon_2, \quad \frac{|E_{ab}|}{\Theta} < \frac{4 \Omega_R H \epsilon_2}{\Omega_M}. \] (99)

Furthermore, the solution contains spacetimes in which the late-time radiation inhomogeneity is negligible in comparison with the matter inhomogeneity. This arises when \( J_a \) and \( K_a \) in (98) are chosen to ensure that
\[ \frac{|\nabla_a \mu|}{\mu} \ll \frac{|\nabla_a \rho|}{\rho} \ll 16 \left( \frac{\Omega_R}{\Omega_M} \right) H \epsilon_1, \] (100)
where the final inequality follows from (79), consistently with (97, 98). The particular choice of \( J_a \simeq 0 \), i.e. radiation inhomogeneity vanishing to first order at late times, will achieve (100), with
\[ \nabla_a \mu \simeq 0, \quad \frac{|\nabla_a \rho|}{\rho} \ll 4 \left( \frac{\Omega_R}{\Omega_M} \right) H \epsilon_1, \] (101)
by (97, 98) and (46).

6 Conclusion

We have seen how a series of assumptions of increasing sharpness (incorporating the inevitable Copernican supposition) leads to increasingly powerful deductions from the CBR anisotropy. As emphasized before, these limits are independent of detailed assumptions about the dynamical history of matter in the universe, and provide an alternative mode of analysis to the usual approaches. This analysis has the advantage of being both covariant and gauge-invariant [11, 12]. It gives somewhat less information than the usual approaches based on the Sachs-Wolfe effect and its generalisations, precisely because it is more model independent; however this also means that its conclusions are more robust than those more standard analyses, as they do not depend so much on the assumptions of particular evolutionary models. In particular they are not dependent on whether or not inflation took place, and whether or not the density parameter \( \Omega \) is near the critical value.

The analysis proceeds through a series of increasingly restrictive Copernican assumptions about the nature of CBR anisotropies in an open neighbourhood about our world line, which is envisaged as including the observable region of the universe. Such assumptions are inevitable if we wish to justify the assumption of an almost-FRW model [7, 8]. The qualitative assumption A1 (Section 1) is sharpened to the quantitative assumptions B1–3 (Section 2) leading to the limits (50–56), which can be sharpened a little if we assume that the matter and radiation frames coincide (i.e. if there is no CBR temperature dipole). Somewhat sharper assumptions C1–2 (Section 4) give better limits, leading to our estimates (58–64) and (65–67), which are further sharpened if the dipole can be neglected to give (68a, 68a, 67a). These equations make quantitative the results of [6] (the universe is almost FRW, see A2) and include as a special case the Ehlers-Geren-Sachs exact theorem [32]. They confirm previous estimates based on the CBR anisotropies that the shear and vorticity are at most about \( 10^{-3}\) of the expansion (on choosing \( \epsilon = 10^{-4}\), to concur with recent CBR anisotropy measurements).

We regard these assumptions and limits as highly plausible, and believe they are useful not only in terms of the limits obtained but also in making quite explicit the kind of Copernican assumptions one has to make in order to extract information from the CBR anisotropies (such assumptions necessary
underlie the standard Sachs-Wolfe type analyses, because these analyses assume A2 as their starting point, but they do so in a somewhat hidden way). More debatable are the stronger assumptions D of section 5, which seem on the face of it quite plausible but then lead to very restrictive conclusions: either the universe is FRW to first order (that is, its difference from a FRW geometry is at most second order) or it has a flat background FRW geometry (to the accuracy of our first-order calculation). When this is true we can get explicit solutions of the equations representing the combined matter and radiation system - but they only allow spatial inhomogeneity when the dipole term cannot be neglected.

Thus some may wish to adopt these stronger assumptions, while others may feel the conclusions are too strong and therefore the assumptions should be questioned. We are open minded in this matter; the main point is that the analysis presented here makes quite clear the range of possible assumptions, and their consequences when we take the Einstein-Liouville equations (and consequent propagation and conservation equations) into account.

Finally we believe this paper shows well the utility of the covariant harmonic approach to both perturbations and to kinetic theory. In particular in examining kinetic effects, it makes quite clear how only the first three harmonic terms in the distribution function explicitly enter the field equations; the anisotropies represented by higher harmonics can affect the geometry only by cascading \cite{15}; that is, by inducing anisotropies in the lower order harmonics through divergence or gradient terms, as in equation (10), or through collisions \cite{16,37}.

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REFERENCES

2. EL Wright et al. (1994) *Astrophys J* **420**, 1
18. AK Rebhan and DJ Schwarz (1994) *Preprint GR-QC* 9403032
33. GFR Ellis (1987) in *VIth Brazilian School of Cosmology and Gravitation* ed M Novello (World Scientific)