Interactions of Strings based on the Lorentz Force in Loop Space

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PACS numbers: 11.17.+y, 11.15.-q

Abstract

Interactions of strings with local tensor fields are discussed in terms of the Lorentz force in loop space, the space of all loops in space-time. We consider the Nambu-Gotô string interacting with a U(1) gauge field on the loop space and derive a necessary condition to close the algebra which consists of constraints in the system. Examining this condition, we find that (a) strings interact with second-rank antisymmetric tensor fields; (b) only tensionless strings interact with massive vector fields in the Stueckelberg formalism; (c) strings do not interact with massive third-rank tensor fields in a generalized Stueckelberg formalism.

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The Kalb-Ramond interaction, that is, the interaction of strings with second-rank antisymmetric tensor fields has often been discussed in various contexts [1-5]. Kalb and Ramond originally introduced this interaction in the study of generalization of the action-at-a-distance theory between point particles [1]. Lund and Regge arrived at the Kalb-Ramond interaction in search of the Lorentz-covariant description of irrotational, incompressible fluid [2]. Radiation of Goldstone bosons from cosmic strings [3] and Fermi-Bose transmutation in (3+1)-dimensions [4] were also studied using the Kalb-Ramond interaction.

It has been pointed out that a second-rank antisymmetric tensor field is a constrained U(1) gauge field on loop space (the space of all loops in space-time) or a component of a U(1) gauge field on the loop space [6-9]. A gauge theory of the second-rank antisymmetric tensor field is derived from a U(1) gauge theory in loop space. As we will see later, the Kalb-Ramond interaction is expressed as the Lorentz force in loop space [7,8].

Since the U(1) gauge field on loop space is a functional field on space-time, it contains an infinite number of local tensor fields besides the second-rank antisymmetric tensor field. It has actually been shown that the U(1) gauge theory in loop space yields the Stueckelberg formalism for vector and third-rank tensor fields [9,10]. In the present paper we shall discuss whether interactions of a string with the vector and tensor fields are formulated in terms of the Lorentz force in loop space.

We define a loop space $\Omega M^D$ as the set of all loops in $D$-dimensional Minkowski space $M^D$. An arbitrary loop $x^\mu = x^\mu(\sigma)$ ($0 \leq \sigma \leq 2\pi$) in $M^D$ is represented as a point in $\Omega M^D$ denoted by coordinates $(x^{\mu \sigma})$ with $x^{\mu \sigma} = x^{\mu}(\sigma)$ \(^{1)}\). Let us recall the conditions for a U(1) gauge field $A_{\mu \sigma}[x]$ on $\Omega M^D$ [9]. Since the gauge transformation $\delta A_{\mu \sigma} = \partial_{\mu \sigma} \Lambda$ ($\partial_{\mu \sigma} \equiv \partial/\partial x^{\mu \sigma}$, and $\Lambda$ is an infinitesimal scalar function on $\Omega M^D$) has no relation with reparametrizations $\sigma \rightarrow \bar{\sigma}(\sigma)$, $A_{\mu \sigma}$ has to satisfy

$$x^{\mu}(\sigma)A_{\mu \sigma} = 0,$$  

where the prime indicates differentiation with respect to $\sigma$. Under the reparametrizations, $A_{\mu \sigma}$ transforms as $\bar{A}_{\mu \sigma}$ with $\bar{x}^{\mu \sigma} = x^{\mu \sigma}$, from which we find that the reparametrization-invariant condition for $A_{\mu \sigma}$ is

$$x^{\mu}(\sigma)\partial_{\mu \sigma}A_{\nu \rho} + \delta'(\sigma - \rho)A_{\nu \sigma} = 0.$$  

Combining the differential of (1) with respect to $x^{\nu \rho}$ and (2), we obtain

$$x^{\mu}(\rho)F_{\mu \nu \rho \sigma} = 0,$$  

where $F_{\mu \nu \rho \sigma} \equiv \partial_{\mu \nu}A_{\rho \sigma} - \partial_{\rho \sigma}A_{\mu \nu}$.

\(^{1)}\) In the present paper, the indices $\kappa, \lambda, \mu, \nu$ and $\xi$ take the values 0, 1, 2, ..., $D-1$, while the indices $\rho$ and $\sigma$ take continuous values from 0 to $2\pi$. 

2
Consider now a charged point particle in $\Omega M^D$. A trajectory of the particle is specified by the world line $x^\mu(\tau) (\equiv x^\mu(\tau, \sigma))$ parametrized by $\tau$. The Lorentz force due to $A_\mu$ is described by

$$S_L = - \int_{\tau_i}^{\tau_f} d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \dot{x}^\mu(\tau) A_\mu [x(\tau)] ,$$  \hspace{1cm} (4)

where the overdot indicates differentiation with respect to $\tau$. The interaction term $S_L$ is gauge invariant if $\Lambda[x(\tau_i)] = \Lambda[x(\tau_f)]$. In addition $S_L$ is invariant under reparametrizations $\tau \to \bar{\tau}(\tau)$, $\sigma \to \bar{\sigma}(\sigma)$, which we hereafter call restricted reparametrizations. The simplest solution of (1) and (2) is

$$A_\mu^{(1)}[x] = q_1 x^\nu(\sigma) B_{\mu \nu}(x(\sigma)) ,$$  \hspace{1cm} (5)

where $q_1$ is a constant with dimensions of $[\text{length}]^{-1}$ and $B_{\mu \nu}$ is a second-rank antisymmetric tensor field on $M^D$. For this solution, (4) becomes the Kalb-Ramond interaction term

$$S_{\text{KR}} = -(q_1/4\pi) \int d\tau \int d\Sigma B_{\mu \nu}(x(\tau, \sigma)) \text{ with } \dot{x}^\mu x^\nu - \dot{x}^\nu x^\mu .$$

Evidently $S_{\text{KR}}$ is invariant under two-dimensional reparametrizations $\tau \to \bar{\tau}(\tau, \sigma)$, $\sigma \to \bar{\sigma}(\tau, \sigma)$ although $S_L$ is not.

The point particle in $\Omega M^D$ may be regarded as a closed string in $M^D$; the world line $x^\mu(\tau)$ then represents a world sheet of the string. To define motion of the point particle in $\Omega M^D$, we now take the Nambu-Gotô action

$$S_{\text{NG}} = -\frac{T}{\sqrt{2}} \int_{\tau_i}^{\tau_f} d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \sqrt{-\Sigma_{\mu \nu} \Sigma^{\mu \nu}} ,$$  \hspace{1cm} (6)

where $T$ is a tension parameter. The action $S_{\text{NG}}$ is invariant under the two-dimensional reparametrizations.

In connection with this invariance, the total action $S_{\text{NG}} + S_L$ yields two primary constraints:

$$\mathcal{K}(\sigma) \equiv \frac{1}{2} (\mathcal{P}(\sigma)^2 + T x'(\sigma)^2) = 0 ,$$  \hspace{1cm} (7)

$$\mathcal{M}(\sigma) \equiv x^\mu(\sigma) p_\mu(\sigma) = 0 ,$$  \hspace{1cm} (8)

where $\mathcal{P}(\sigma)^2 \equiv \mathcal{P}_\mu(\sigma) \mathcal{P}^{\mu}(\sigma)$, $x'(\sigma)^2 \equiv x'_\mu(\sigma) x^{\mu}(\sigma)$, and $\mathcal{P}_\mu(\sigma)$ is defined by $\mathcal{P}_\mu(\sigma) \equiv p_\mu(\sigma) + A_{\mu \sigma}$ with the canonical momentum $p_\mu(\sigma) \equiv \partial L / \partial \dot{x}^\mu(\sigma)$ ($\int d\tau L = S_{\text{NG}} + S_L$). Because of (1), the constraint (8) reduces to the reparametrization-invariant condition for strings: $x^\mu(\sigma) p_\mu(\sigma) = 0$. Assuming the Poisson bracket $\{ x^\mu(\rho), p_\nu(\sigma) \}_\text{PB} = \delta_\mu^\nu \delta(\rho - \sigma)$, we obtain

$$\{ \mathcal{K}(\rho), \mathcal{K}(\sigma) \}_\text{PB} = T^2 (\mathcal{M}(\rho) + \mathcal{M}(\sigma)) \delta'(\rho - \sigma) - \mathcal{P}(\rho) \mathcal{P}(\sigma) \mathcal{F}_{\mu \nu, \rho \sigma} ,$$  \hspace{1cm} (9)

$$\{ \mathcal{K}(\rho), \mathcal{M}(\sigma) \}_\text{PB} = (\mathcal{K}(\rho) + \mathcal{K}(\sigma)) \delta'(\rho - \sigma) - \mathcal{P}(\rho) x^\nu(\sigma) \mathcal{F}_{\mu \nu, \rho \sigma} ,$$  \hspace{1cm} (10)

$$\{ \mathcal{M}(\rho), \mathcal{M}(\sigma) \}_\text{PB} = (\mathcal{M}(\rho) + \mathcal{M}(\sigma)) \delta'(\rho - \sigma) - \dot{x}^\mu(\rho) x^\nu(\sigma) \mathcal{F}_{\mu \nu, \rho \sigma} .$$  \hspace{1cm} (11)
The second terms on the right-hand sides of (10) and (11) vanish on account of (3). To close the algebra consisting of the constraints, \( \mathcal{F}_{\mu\nu,\sigma} \) has to satisfy

\[
\mathcal{P}^\mu(\rho)\mathcal{P}^\nu(\sigma)\mathcal{F}_{\mu\nu,\sigma} = 0 .
\]  

(12)

A question is whether or not the two-dimensional reparametrization invariance of (6) is essential to obtain (7) and (8). To answer this question, we consider the simpler action\(^{2})\)

\[
S_T = -T \int_{t_1}^{t_2} dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \sqrt{-\dot{z}^2} z^\mu z^\nu .
\]  

(13)

Since \( S_T \) is invariant only under the restricted reparametrizations, the total action \( S_T + S_L \) yields the primary constraint (7) alone. Then we have a secondary constraint that is given by setting the right-hand side of (9) to be zero. Now consider the solutions of both (1) and (2) associated with the gauge parameters \( \lambda^A(\sigma) \) that are described in terms of \( x^\mu(\sigma) \) and infinitesimal functions on \( M^D \). These solutions consist of local fields on \( M^D \). (For example, see (5), (15) and (19).) The field strength \( \mathcal{F}_{\mu\nu,\sigma} \) for each of the solutions has the following form: \( \mathcal{F}_{\mu\nu,\sigma} = \delta(\rho - \sigma)I_\mu(\sigma) + \delta(\rho - \sigma)J_\mu(\sigma) \), where \( I_\mu(\sigma) \) and \( J_\mu(\sigma) \) are smooth functions of \( \sigma \) satisfying \( I_\mu = -I_{\nu \mu} \) and \( J_\mu = J_{\nu \mu} \). (See (14), (18) and (20).) Then, using the simple formula \( f(\rho)\delta(\rho - \sigma) = f(\sigma)\delta(\rho - \sigma)\) \( (f(\sigma) \) is an arbitrary smooth function of \( \sigma \), we can show that the secondary constraint reduces to \( T^2 A_M(\sigma) - \mathcal{P}^\mu(\sigma)\mathcal{P}^\nu(\sigma)J_{\mu \nu}(\sigma) = 0 \). In order that this equation holds together with (10) and (11), we conclude (8) and \( \mathcal{P}^\mu(\sigma)\mathcal{P}^\nu(\sigma)J_{\mu \nu}(\sigma) = 0 \). We can thus derive (7) and (8) from \( S_T + S_L \); the two-dimensional reparametrization invariance is not essential to define motion of the point particle interacting with \( A_{\mu \sigma} \). Reparametrization invariance required for actions of point particles in \( \Omega M^D \) is only invariance under the restricted reparametrizations. Accordingly, it seems that \( S_T \) is more suitable for our model than \( S_{NG} \).

Let us now examine the condition (12) in order to know what local fields interact with strings. The field strength of (5) is

\[
\mathcal{F}^{(1)}_{\mu\nu,\sigma}[x] = q_4 \delta(\rho - \sigma) z^\lambda(\sigma) F_{\lambda \mu \nu}(z(\sigma))
\]  

(14)

with \( F_{\lambda \mu \nu} \equiv \partial_\lambda B_{\mu \nu} + \partial_\mu B_{\nu \lambda} + \partial_\nu B_{\lambda \mu} \). Using the formula \( f(\rho)\delta(\rho - \sigma) = f(\sigma)\delta(\rho - \sigma) \), we see that (14) satisfies (12) by virtue of the antisymmetric property \( F_{\lambda \mu \nu} = -F_{\lambda \nu \mu} \). This result justifies the Kalb-Ramond interaction.

Next we consider a solution of (1) and (2) consisting of a vector field \( A_\mu \) and a scalar field \( \phi \) on \( M^D \) [9]:

\[
A^{(2)}_{\mu \nu}[x] = q_2 \sqrt{-z^\prime(\sigma)^2} \Pi_{\mu \nu}(\sigma) A^\sigma(z(\sigma)) + e_2 Q_{\mu}(\sigma) \phi(z(\sigma))
\]  

(15)

\(^{2})\) Takabayasi has discussed the action \( S_T \) in connection with the multilocal model [11].
with
\[
\Pi_{\mu\nu}(\sigma) \equiv \frac{1}{-x'(\sigma)^2} \left( x'_\mu(\sigma)x'_\nu(\sigma) - \eta_{\mu\nu}x'(\sigma)^2 \right), \quad (16)
\]
\[
Q_{\mu}(\sigma) \equiv \frac{x'_\mu(\sigma)}{\sqrt{-x'(\sigma)^2}}, \quad (17)
\]

where \( q_2 \) is a constant with dimensions of [length]\(^{-1} \) and \( e_2 \) a dimensionless constant. The field strength of (15) is
\[
\mathcal{F}_{\mu\nu(\sigma)}^{(2)}[x] = q_2 \delta(\rho - \sigma) \left\{ \sqrt{-x'(\sigma)^2} F_{\mu\nu}(x(\sigma)) - x'^{\lambda}(\sigma)Q_{\mu}(\sigma)F_{\nu\lambda}(x(\sigma)) \right. \\
+ Q_{[\mu}(\sigma)\tilde{A}_{\nu]}(x(\sigma)) \left. \right\} - q_2 \delta'(\rho - \sigma) \{ \Pi_{\mu\nu}(\rho)Q_{\lambda}(\rho)\tilde{A}^\lambda(x(\rho)) + \Pi_{\mu\nu}(\sigma)Q_{\lambda}(\sigma)\tilde{A}^\lambda(x(\sigma)) \} , \quad (18)
\]

where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \tilde{A}_\mu \equiv A_\mu - (e_2/q_2)\partial_\mu \phi \). It was shown in Ref.[9] that \( \tilde{A}_\mu \) is a massive vector field in the Stueckelberg formalism. Substituting (18) into (12), we see that the terms proportional to \( \delta(\rho - \sigma) \) vanish as in the case of \( B_{\mu\nu} \). By virtue of the symmetric property \( \Pi_{\mu\nu} = \Pi_{\nu\mu} \), the condition \( \mathcal{P}^{\nu}(\rho)\mathcal{P}^{\mu}(\sigma)\mathcal{F}_{\mu\nu(\sigma)}^{(2)} = 0 \) turns out to be \( \delta'(\rho - \sigma) \{ \mathcal{P}(\rho)^2 Q_{\mu}(\rho)\tilde{A}^{\mu}(x(\rho)) + \mathcal{P}(\sigma)^2 Q_{\lambda}(\sigma)\tilde{A}^{\lambda}(x(\sigma)) \} = 0 \) after the use of (8). Since \( A_\mu \) and \( \phi \) are arbitrary local fields, we conclude \( \mathcal{P}(\sigma)^2 = 0 \). Comparing this with (7), we have \( T = 0 \). This result shows that only tensionless (or null) strings [12] can interact with \( \tilde{A}_\mu \).

Finally, we consider a solution of (1) and (2) that consists of a tensor field \( A_{\lambda\mu\nu} \) on \( M^D \) with the symmetric property \( A_{\lambda\mu\nu} = A_{\nu\lambda\mu} \) and a symmetric tensor field \( \phi_{\mu\nu} \) on \( M^D \) [10]:
\[
A^{(3)}_{\mu\nu}(x) = q_3 \sqrt{-x'(\sigma)^2} \left\{ \delta_{\mu\nu}Q^{\lambda}(\sigma)Q^{\lambda}(\sigma) - \Pi_{\mu}^{\lambda}(\sigma)Q^{\nu}(\sigma)Q^{\nu}(\sigma) \right\} A_{\nu\kappa\lambda}(x(\sigma)) \\
- e_3 \left( Q'(\sigma)Q^{\sigma}(\sigma)Q^{\lambda}(\sigma) + 2\Pi_{\mu}^{\sigma}(\sigma)Q^{\nu}(\sigma) \right) \phi_{\lambda\nu}(x(\sigma)) , \quad (19)
\]

where \( q_3 \) is a constant with dimensions of [length]\(^{-1} \) and \( e_3 \) a dimensionless constant. The field strength of (19) is
\[
\mathcal{F}_{\mu\nu(\sigma)}^{(3)}[x] = q_3 \delta(\rho - \sigma) \left\{ \sqrt{-x'(\sigma)^2} [Q^{\sigma}(\sigma)Q^{\lambda}(\sigma)F_{\mu\nu\kappa\lambda}(x(\sigma)) \right. \\
+ Q^{\xi}(\sigma) \left\{ 2\delta_{\mu\nu}Q^{\lambda}(\sigma)F_{\nu\kappa\lambda}(x(\sigma)) + Q^{\sigma}(\sigma)Q^{\lambda}(\sigma)Q_{[\mu}(\sigma)F_{\nu]\kappa\lambda}(x(\sigma)) \right\} \right. \\
+ \left. \left\{ \eta_{\mu\nu}\epsilon^{\kappa\lambda}(\sigma)Q^{\lambda}(\sigma)Q^{\sigma}(\sigma) + 2\eta_{\mu\nu}\Pi_{\sigma}^{\kappa}(\sigma)Q^{\lambda}(\sigma) \right\} \tilde{A}^{\kappa\lambda}(x(\sigma)) \right\} \\
+ q_3 \delta'(\rho - \sigma) \{ \Pi_{\mu\nu}(\rho)Q^{\kappa}(\rho)Q^{\lambda}(\rho) + 2\Pi_{\mu\nu}(\rho)Q^{\kappa}(\rho) \} Q_{\xi}(\rho)\tilde{A}^{\kappa\lambda}(x(\rho)) \\
+ \left\{ \Pi_{\mu\nu}(\rho)Q^{\kappa}(\rho)Q^{\lambda}(\rho) + 2\Pi_{\mu\nu}(\rho)Q^{\kappa}(\rho) \right\} \xi(\rho)\tilde{A}^{\kappa\lambda}(x(\rho)), \quad (20)
\]

where \( F_{\mu\nu\kappa\lambda} \equiv \partial_\mu A_{\nu\kappa\lambda} - \partial_\nu A_{\mu\kappa\lambda} \), \( \tilde{A}_{\lambda\mu\nu} \equiv A_{\lambda\mu\nu} - (e_3/q_3)\partial_\lambda \phi_{\mu\nu} \), and \( \eta_{\mu\nu} \), diag(\( \eta_{\mu\nu} = (1, -1, -1, \ldots, -1) \)), is the metric tensor on \( M^D \). As shown in ref.[10], \( \tilde{A}_{\lambda\mu\nu} \) is a massive tensor field in the Stueckelberg formalism extended to third-rank tensor fields. Even if \( T = 0 \), (20) does not satisfy (12). This is because the term \( \mathcal{P}^{\nu}(\sigma)\mathcal{P}^{\mu}(\sigma)A_{\lambda\mu\nu}(x(\sigma)) \) occurring in

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3) \( X_{[\mu}Y_{\nu]} \equiv X_\mu Y_\nu - X_\nu Y_\mu \).
\[ \mathcal{P}^\mu(\sigma)\mathcal{P}^\nu(\sigma)F^{(3)}_{\mu\nu,\rho\sigma} \] does not vanish. As a result, it is concluded that strings do not interact with \( \tilde{A}_{\lambda\mu} \) (at least in the form of the Lorentz force in loop space).

Besides (14), (18) and (20), we can obtain, from a solution \( \mathcal{A}^{(p)}_{\mu\nu} \) \((p = 4, 5, 6, \ldots)\) of (1) and (2), the field strength \( F^{(p)}_{\mu\nu,\rho\sigma} \equiv \partial_{\mu\nu}\mathcal{A}^{(p)}_{\rho\sigma} - \partial_{\rho\sigma}\mathcal{A}^{(p)}_{\mu\nu} \) written in terms of a \( p \)-th-rank tensor field on \( M^D \). Since \( F^{(p)}_{\mu\nu,\rho\sigma} \) is more complicated than (20), it will not satisfy (12). We thus infer that strings do not interact with the \( p \)-th-rank tensor field, at least in the form of the Lorentz force in loop space.

In conclusion, we have studied interactions of a string with the local fields \( B_{\mu\nu}, \tilde{A}_{\lambda\mu} \) and \( \tilde{A}_{\lambda\mu} \) from the point of view of the Lorentz force in loop space. Examining the condition (12), we found that (a) as is expected, strings interact with \( B_{\mu\nu} \); (b) only tensionless strings interact with \( \tilde{A}_{\mu} \); (c) strings do not interact with \( \tilde{A}_{\lambda\mu} \).

Is it possible to formulate an interaction between \( \tilde{A}_{\mu} \) and a string with tension? We might be able to introduce tension into the tensionless string interacting with \( \tilde{A}_{\mu} \) by utilizing dimensional reduction in the Kaluza-Klein theory [8,13]. Alternatively, following the discussion in Refs[14], we could construct interaction terms of a string and \( \tilde{A}_{\mu} \) with the help of "internal" degree of freedom.

Lund and Regge showed that the action \( S_{KR} + \frac{1}{12} \int d^4xF_{\lambda\mu\nu}F_{\lambda\mu\nu} \) defines the Lorentz-invariant theory of vortex motion in an irrotational, incompressible fluid and that the Nambu-Goto action is required to regularize the self-interaction of a vortex [2]. Since tensionless strings interact with \( \tilde{A}_{\mu} \), we can consider the action \( S_{L}^{(2)} + \int d^4x[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2\tilde{A}_{\mu}\tilde{A}^{\mu}] \), where \( S_{L}^{(2)} \) is defined by (4) with (15) and \( m \) is a mass parameter. It is an interesting subject to study what kind of vortex motion is described by this action and how to regularize the self-interaction of a vortex.

**Acknowledgments**

I would like to thank Y. Hosotani, H. Suura and A. Vainshtein for their kind hospitality at University of Minnesota, where this work was done. I am grateful to O. Hara, S. Ishida, S. Y. Tsai, S. Naka and other members of Theoretical Physics Group at Nihon University for continuous encouragements. Thanks are also due to T. Fujita for careful reading of the manuscript.
References


