ABSTRACT

We study the phase space of dual QCD-like systems in the presence of uniform backgrounds.

New York, NY, 10027, U.S.A.
Physics Department, Columbia University

Kyoung Ho Lee and Pihun Yi

Self-Dual Anyons in Uniform Background Fields
1 Introduction

In Abelian Maxwell-Higgs systems, there are well-known topologically stable vortex solutions in three dimensions. With a specific coupling constants, the energy functional can be saturated by the so-called self-dual configurations satisfying certain first-order differential equations [1]. When the kinetic term for the gauge field includes Chern-Simons term, the corresponding self-dual models are also found and studied [2, 3, 4].

An interesting generalization of such self-dual models arises when the system is coupled to an external background charge density or an external magnetic field. In particular, Maxwell-Higgs systems with the uniform external electric charge density was argued to describe the real superconductor more closely than ones without. The self-dual limit of these systems has been investigated extensively by one of the authors [5]. In this paper, we want to consider two more self-dual models, which incorporate a Chern-Simons term. A nonrelativistic version with an external magnetic field was studied before as an effective field theory for the fractional quantum Hall effects [6].

There are several novel features in (Maxwell) Chern-Simons-Higgs systems with the background charge density. Their structure is much richer than the systems without. First, homogeneous ground states are possible in two rather different manners: either a symmetric phase with a uniform magnetic field or an asymmetric phase with a uniform Higgs charge density. (But they obviously belong to two different superselection sectors, even if the spatial volume is finite.) Second, a magnetoroton mode is possible in the asymmetric phase for certain parameter ranges. This mode has the lowest energy at a nonzero wavelength. Third, such a roton mode, when its energy is imaginary, leads to the instability of the homogeneous asymmetric vacuum. When this happens, the resulting ground state may have a crystal structure in charge density. (If the symmetric phase is unstable on the other hand, the resulting ground state would be a vortex lattice.) Then the translation symmetry must be spontaneously broken and there will be a sound wave as Goldstone boson. Fourth, the CTP symmetry is explicitly broken and so solitons and antisolitons have in general different mass spectrums. The Lorentz symmetry is also broken and some solitons can have zero or negative rest mass even though it must have positive kinetic mass. Fifth, the angular momentum operator from the Noether theorem should be modified. Some of these features are studied previously for different models [6, 7, 8].

In this paper, we concentrate on the self-dual Chern-Simons Higgs models with a uniform background charge density. As we shall see shortly, the usual extended supersymmetry associated with many self-dual models is no longer manifest in the presence of the background charge density.

We start by introducing self-dual models in Sec.[2], and then go on to study the vacuum structure of the model in Sec.[3]. We study the homogeneous symmetric vacuum that exists for all self-dual couplings, and find it stable even when the mass of elementary excitations are imaginary. The vacuum structure of the asymmetric phase depends on the parameters of the model and turns out
to be quite rich. We repeat the stability analysis for each homogeneous asymmetric vacuum. Rather interesting species of self-dual solitons appear in some of these vacua, which will be investigated in Sec.[4]. Here, we will find that the symmetric phase is in fact infinitely degenerate for certain parameter range. The section [5] is devoted to the matter of conserved angular momentum. The conserved Noether angular momentum turns out to be inappropriate due to severe divergences, and needs to be modified. A satisfactory and physically well-motivated finite expression is found by considering the same system on a sphere. Some concluding remarks are discussed in Sec.[6].

2 Model

The first self-dual model, which is the one we will study in detail here, is the theory of a Higgs field \( \phi = f e^{i \theta}/\sqrt{2} \) coupled to a Chern-Simons gauge field \( A_\mu \). The gauge field is coupled to a uniform background electric charge density \( \rho_e \), so that the Lagrangian of the model can be written as,

\[
\mathcal{L} = \frac{\kappa}{2} \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho + |D_\mu \phi|^2 - U - \rho_e A_0, \tag{1}
\]

where the self-dual potential is given by a specific form:

\[
U = \frac{1}{4 \kappa^2} |\phi|^2 \left( 2 |\phi|^2 - v^2 \right) - \frac{\rho^2_e}{2 \kappa} \left( 2 |\phi|^2 - v^2 \right), \tag{2}
\]

with \( D_\mu \phi = (\partial_\mu + i A_\mu) \phi \). The parameters \( v^2 \) and \( \rho_e \) can be either positive or negative, but without loss of generality we may assume that \( \kappa > 0 \). By shifting \( A_i \rightarrow A_i - \rho_e \epsilon_{ij} x^j/2\kappa \), we can see that the background electric charge \( \rho_e \) is equivalent to a uniform background magnetic field \( F_{z\bar{z}} = \rho_e/\kappa \).

The C, P, and T are all broken: the Chern-Simons term breaks the P and T, while the background charge term breaks the C and CTP transformation.

The second model is a Maxwell Chern-Simons Higgs theory with a neutral scalar field \( N \), whose Lagrangian is

\[
\mathcal{L} = -\frac{1}{4e^2} F_{\mu \nu} F^\mu_\nu + \frac{\kappa}{2} \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho + \frac{1}{2\epsilon^2} (\partial_\mu N)^2 + |D_\mu \phi|^2 - \bar{U} - \rho_e A_0, \tag{3}
\]

where the potential is

\[
\bar{U} = N^2 |\phi|^2 + \frac{\epsilon^2}{8} \left( 2 |\phi|^2 - v^2 - 2\kappa N \right)^2 - \rho_e N. \tag{4}
\]

Actually, the first model can be obtained from the second by taking the limit \( \epsilon^2 \rightarrow 0 \). The Maxwell term and the kinetic term for \( N \) become negligible, while the field equation for \( N \) may be used to recover \( \mathcal{L} \) from \( \bar{\mathcal{L}} \). Similar models without the background charge density have been studied before [4], and the pure Maxwell-Higgs system with \( \kappa = 0 \) was studied extensively by one of the authors [5]. Except for the case \( \kappa = 0 \) where the parity is a good symmetry, there is no qualitative difference between the two models and so we will focus on the simpler pure Chern-Simons-Higgs model \( \mathcal{L} \).
The system is invariant under the local gauge transformations. Gauss's constraint obtained by varying $A_0$ is
\[ \kappa F_{12} + f^2(\dot{\theta} + A_0) - \rho_e = 0, \quad (5) \]
where the dot denotes the time derivative. The electric charge current $-f^2(\partial_\mu \theta + A_\mu)$ is conserved and the corresponding conserved charge is $Q = -\int d^2 x f^2(\dot{\theta} + A_0)$.

Note that both the total electric charge and the total magnetic flux must be conserved independently, satisfying Gauss's constraint. (One can imagine the theory is on a spatially compact manifold so that “total” magnetic flux and “total” electric charge make sense.) Therefore, we find infinitely many superselection sectors, each of which is labeled by the total electric charge or magnetic flux of the system. Later on, we shall consider cases where the background charge density $\rho_e$ is cancelled at spatial infinity by either magnetic flux $F_{12}$ or the Higgs charge density $-f^2(\dot{\theta} + A_0)$. Collectively, the first class of sectors are to be called the symmetric phase and the latter the asymmetric phase, respectively, for the obvious reason.

Since the background charge density is uniform, the space-time translation symmetry is preserved. The corresponding energy-momentum tensor $T_{\mu\nu}$ obtained through the Noether theorem is
\[ T_{\mu\nu} = \frac{\kappa}{2} \epsilon_{\mu\nu\sigma\rho} A^\sigma \partial_\nu A^\rho + \partial_\mu f \partial_\nu f + f^2(\partial_\mu \theta + A_\mu) \partial_\nu \theta - \eta_{\mu\nu} \mathcal{L}. \quad (6) \]
The conserved energy of the system is then $E = \int d^2 x T_{00}$. After a partial integration, Gauss's constraint (5) can be used to show that
\[ E = \int d^2 x \left\{ \frac{1}{2} f^2 + \frac{1}{2} (\partial_i f)^2 + \frac{1}{2} f^2(\dot{\theta} + A_0)^2 + \frac{1}{2} f^2(\partial_i \theta + A_i)^2 + U \right\}. \quad (7) \]
By using Gauss's constraint (5) again and integrating by parts, we rewrite the energy as
\[ E = \int d^2 x \left\{ \frac{1}{2} f^2 + \frac{1}{2} (\partial_i f + \epsilon_{ij} f(\partial_j \theta + A_j))^2 + \frac{1}{2} f^2(\dot{\theta} + A_0 - \frac{1}{2\kappa}(f^2 - v^2))^2 \right\} + \frac{v^2}{2} \int d^2 x F_{12} - \int d^2 x \partial_i \left\{ \frac{1}{2} \epsilon_{ij} f^2(\partial_j \theta + A_j) \right\}. \quad (8) \]
The last term vanishes identically, as long as $F_{12} = 0$ or $f^2 = 0$ asymptotically. Recall that these conditions are characteristic of the asymmetric phase and the symmetric phase alluded to earlier.

In the asymmetric phase, we naturally consider the excited states of finite total magnetic flux $\Psi = \int d^2 x F_{12}$, and for those configurations we find the following self-dual bound,
\[ E \geq \frac{v^2}{2} \Psi. \quad (9) \]
As we will see later, within certain parameter range, this bound can be saturated even when $v^2 \Psi$ is actually negative.

In the symmetric phase, however, there exists a finite average magnetic flux density, $F_{12} = \rho_e / \kappa$. Then, the symmetric phase comes with a nonzero vacuum energy, $E_0 = (v^2/2) \int F_{12}^2$, as is evident
in Eq. (8). On top of this, of course, there will be localized net fluxes of solitonic configurations. We must then consider the excited states of finite net excess magnetic flux \( \Psi - \Psi_0 = \int d^2 x \left( F_{12} - F_{12}^e \right) \), the excess energy of which is again bounded below

\[
E - E_0 \geq \frac{v^2}{2} (\Psi - \Psi_0) = -\frac{v^2}{2} \int d^2 x f^2 (\theta + A_0).
\]

In order to saturate these self-dual bounds, the modulus field \( f \) must be static and, together with the gauge field \( A \), solve the following set of the first order self-dual equations:

\[
\partial_i f + \epsilon_{ij} f (\partial_j \theta + A_j) = 0
\]

\[
\kappa F_{12} + \frac{1}{2\kappa} f^2 (f^2 - v^2) - \rho_e = 0
\]

Here we have used Gauss’ law to remove \( \theta + A_0 \) in favor of \( \kappa F_{12} \).

Are there any vortex-like configurations solving these self-dual equations? First of all, the modulus field \( f \) must vanish at the center of a vortex in order for the Higgs field to be well-defined, and this combined with Eq. (11) implies that a self-dual vortex-like configuration always consists of anti-vortices only, \( \theta = -\sum_a \text{Arg} (\bar{x} - \bar{q}_a) + \eta \) where \( \bar{q}_a \) are the positions of the anti-vortices and \( \eta \) is a single valued function.

Then, we may combine the coupled first-order equations above to produce a single second order equation for \( f \) with sources at the sites of anti-vortices.

\[
\nabla^2 \ln f^2 - \frac{1}{\kappa^2} \left[ f^2 (f^2 - v^2) - 2\kappa \rho_e \right] = 4\pi \sum_a \delta (\bar{x} - \bar{q}_a).
\]

In the symmetric phase, for instance, we see that, far from the solitons, the modulus field behaves exponentially \( f \propto e^{-\rho_e \kappa^2 / 4\kappa} \) so that there is no self-dual configuration in the symmetric phase whenever \( \rho_e < 0 \).

(The solutions of this last equation in case the external charge density vanishes were studied in detail before [2]. In the symmetric phase, there are q-balls and q-balls with vortices, while, in the asymmetric phase, there are topological vortices. One of interesting properties of these solitons is that they carry the fractional spin and satisfy the fractional spin statistics therein.)

3 Vacuum Structure

Next, let us consider the vacuum structure. Of particular interest are those superselection sectors with homogeneous vacuum states in it. One may naively expect to find a homogeneous ground state in any given sector, but it is not difficult to see that there exist only two cases where this is possible: either \( \int f^2 (\theta + A_0) = 0 \) or \( \int F_{12} = 0 \).

The homogeneous ground state one finds in the first sector is just the uniform magnetic field configuration, \( F_{12} = \rho_e / \kappa \), without any Higgs field expectation value, \( \langle \phi \rangle = 0 \). Since this configuration saturates the self-dual equations trivially, this symmetric homogeneous vacuum must be an absolute minimum within this superselection sector.
From the form of the self-dual potential $U(f)$, however, one can see that for sufficiently large and positive $\rho_\epsilon$, the potential is concave at origin $|\phi| \equiv f/\sqrt{2} = 0$, and the homogeneous symmetric vacuum appears to be at a saddle point, unstable even perturbatively. What are we missing? The uniform magnetic field $\rho_\epsilon/\kappa$ affects the dynamics of the small perturbation profoundly, among other things. To the first order in perturbation, the small deviation $\delta \phi = \phi$ obeys the following equation with a nontrivial kinetic part,

$$- \partial_0^2 \delta \phi = - (\partial_i - i A_{\ast i})^2 \delta \phi + U''(0) \delta \phi,$$

where the gauge field configuration $A_{\ast i}$ satisfies $\partial_1 A_{\ast 2} - \partial_2 A_{\ast 1} = \rho_\epsilon/\kappa$. To this leading order, the gauge field fluctuation $A - A_{\ast}$ decouples from the Higgs mode, and actually vanishes identically.

Note that a perturbative instability is possible if and only if the operator on the right hand side of Eq. (14) possesses a negative eigenvalue. But, this eigenvalue problem is just the well-known Landau level problem with an extra “potential” term $U''(0)$. The eigenvalue spectrum of the kinetic part is $(2n + 1) |\rho_\epsilon|/\kappa$ for all integers $n \geq 0$, which implies that the operator is bounded below by $|\rho_\epsilon|/\kappa + U''(0) = |\rho_\epsilon|/\kappa + \left[v^4/4\kappa^2 - \rho_\epsilon/\kappa\right] \geq 0$. Hence, the perturbative calculation predicts that the homogeneous symmetric vacua is indeed linearly stable. By the way, the infinite number of zero modes that appear when $v^2 = 0$ (with positive $\rho_\epsilon$) is intimately related to the vanishing self-dual energy bound (9).

In the second sector, where Gauss's constraint is satisfied by introducing an opposite scalar charge density, it is again possible to obtain a homogeneous configuration that solves the field equations. With $F_{\ast 2} = 0$ everywhere, the constraint is then satisfied as $a(f) \equiv \rho_\epsilon/f^2 = \hat{\Theta} + A_0$. The effective energy density of such configurations is then given by the sum of the scalar potential and an “electrostatic” contribution coming from the scalar kinetic term.

$$\mathcal{E} = \rho_\epsilon^2/2f^2 + U(f) = \frac{1}{8\kappa^2 f^2} (f^2 - v^2 - 2\kappa \rho_\epsilon)^2. \quad (15)$$

Searching for homogeneous vacua in this second superelection sector is now simply a matter of minimizing $\mathcal{E}$ with respect to $f$.

Once we find a local minimum at $f = a$, the next logical step is to test its stability. One convenient way is to check whether the configuration saturates the self-dual bound. Since there is no net magnetic flux, the self-dual bound of the energy functional vanishes, and a minimum of $\mathcal{E}$ saturates the bound if and only if the value of $\mathcal{E}$ there is identically zero. Later on, we will find that this does not necessarily hold for all available minima.

On the other hand, we also have the option of carrying out a perturbation to test the linear stability. For this, one again expands the field equations around the homogeneous vacuum. But unlike the previous case of symmetric vacuum, the gauge field fluctuation does not decouple, and we find a system of coupled linear partial differential equations with uniform coefficients:

$$[-\partial_0 \delta \partial_0 + \partial_i \partial_i + a^2(u) - U''(u)] \delta f + 2a(u) \delta A_0 = 0,$$
\[ \kappa \epsilon_{ij} \partial_i \delta A_j + u^2 \delta A_0 + 2 u a(u) \delta f = 0, \]  
\[ \kappa \epsilon_{ij} (\partial_0 \delta A_j - \partial_j \delta A_0) + u^2 \delta A_i = 0, \]  
(16)

where we used the gauge 0 = 0. A mode expansion with the frequency \( \omega \) and the spatial momentum \( p \) yields the following dispersion relation:

\[ w^2 = p^2 + \frac{1}{2} \left( \mathcal{E}''(u) + \frac{u^4}{\kappa^2} \right) \pm \frac{1}{2} \sqrt{\left( \mathcal{E}''(u) - \frac{u^4}{\kappa^2} \right)^2 + 16 a^2(u) p^2}. \]  
(17)

Since \( \mathcal{E}'' \) is necessarily nonnegative at local minima, large wavelength fluctuations are massive ones with the masses squared given by

\[ m_H^2(u) \equiv \mathcal{E}''(u), \quad m_A^2(u) \equiv \frac{u^4}{\kappa^2}. \]  
(18)

These are associated with small fluctuations in the Higgs mode and in the massive gauge field mode respectively. When \( \rho_e = 0 \), the spin of the Higgs particle of mass \( m_H \) is zero, while that of the vector particle of mass \( m_A \) is one. By the continuity, we expect this aspect persists even with nonzero charge background. At the zero momentum, Eq. (16) implies that the Higgs mode and the vector field mode are decoupled, confirming that the spin of the Higgs particles and the vector bosons are zero and one, respectively. We also note that two branches of the dispersion relation do not cross each other except at \( p^2 = 0 \) when \( m_A = m_H \).

A linear instability occurs if and only if \( w^2 \) becomes negative at some nonnegative value of \( p^2 \). We find the homogeneous vacuum of \( f = u \) is linearly unstable if and only if the following inequality is satisfied.

\[ 4a^2 > (m_H + m_A)^2. \]  
(19)

This can be easily seen from the explicit forms of \( w^2 \) and \( p^2 \) when \( w^2 \) takes the minimum value, given the dispersion relation above:

\[ p^2_* = - \frac{1}{16a^2} \left[ (m_H^2 - m_A^2)^2 - 16a^4 \right], \]
\[ w^2_* = - \frac{1}{16a^2} \left[ (m_H + m_A)^2 - 4a^2 \right] \left[ (m_H - m_A)^2 - 4a^2 \right]. \]  
(20)

The \( w^2_* \) can take a negative value if either \( 4a^2 > (m_H + m_A)^2 \) or \( (m_H - m_A)^2 > 4a^2 \), but the latter implies \( p^2_* < 0 \), and thus is unphysical.

In some cases, there exists a so-called magnetoroton mode that are excitations with the least energy but at nonzero momentum. In the present context, magnetorotons should appear whenever \( p^2_* > 0 \). Depending on whether \( m_H > m_A \) or not, the roton mode occurs on the branch of the Higgs mode or the vector boson mode. One thus naively expect that the spin of the roton mode is zero or one, depending on whether \( m_H > m_A \) or not. However, there is no rest frame for rotons and it is not clear at this moment whether it is possible to separate the orbital and spin angular momentum.
for rotons. As we shall see shortly, magnetorotons in a stable vacuum exist for a limited range of \( \kappa \rho_c \) and that only for positive \( v^2 \).

Now let us list all possible minima of \( \mathcal{E} \) over various ranges of the parameters \( v^2 \) and \( \kappa \rho_c \). Extremizing \( \mathcal{E} \), we find the following four solutions:

\[
\frac{\bar{u}^2_\pm}{2} \equiv \frac{v^2 + \sqrt{v^4 + 8\kappa \rho_c}}{2},
\]

\[
\frac{\bar{u}^2_\pm}{6} \equiv \frac{v^2 + \sqrt{v^4 - 24\kappa \rho_c}}{6}. \tag{21}
\]

Not all of these represent a physical vacuum. Depending on the parameters of the theory, some of them become negative or even complex. Also, a \( u^2 \geq 0 \) may correspond to a local maximum instead of a minimum. Through a careful but elementary study of \( \mathcal{E} \), one finds the following homogeneous vacuum structure:

(1) \( v^2 < 0, \ 0 < \kappa \rho_c ; \ u_+ \) is an absolute minimum and saturates the self-dual bound.
(2) \( v^2 < 0, \ \kappa \rho_c < 0 ; \ \bar{u}_+ \) is an absolute minimum but does not saturate the self-dual bound.
(3) \( v^2 > 0, \ \kappa \rho_c > v^4/24 ; \ u_+ \) is an absolute minimum and saturates the self-dual bound.
(4) \( v^2 > 0, \ v^4/24 > \kappa \rho_c > 0 ; \ u_+ \) is an absolute minimum and saturates the self-dual bound; \( \bar{u}_- \) is a local minimum that does not saturate the self-dual bound.
(5) \( v^2 > 0, \ 0 > \kappa \rho_c > -v^4/8 ; \ u_\pm \) are two degenerate absolute minima and saturates the self-dual bound.
(6) \( v^2 > 0, \ -v^4/8 > \kappa \rho_c ; \ \bar{u}_+ \) is an absolute minimum but does not saturate the self-dual bound.

The cases where the equality rather than the inequality holds can be understood as the degenerate limits of the cases in the above list.

Despite this long list of different cases, the linear stabilities are more or less determined by the vacuum expectation value \( u \) alone. For example, the homogeneous vacua \( u = u_\pm \), whenever they are real, do saturate the self-dual bound \( \mathcal{E} = 0 \), implying that they are always stable under linear perturbations. Indeed, an explicit calculation shows that

\[
4a^2(u_+) = (m_H(u_+) - m_A(u_+))^2,
\]

\[
4a^2(u_-) = (m_H(u_-) + m_A(u_+))^2. \tag{22}
\]

Clearly both \( u_\pm \) vacua are linearly stable, in view of the instability criterion (19). The above equation implies \( p_0^2(u_+) < 0 \) so that there is no roton in the \( u_+ \) vacuum. Interestingly enough, \( u_- \) vacuum is actually right at the borderline of instability, and generically possesses a massless roton mode at \( p_0^2(u_-) = m_H(u_-) m_A(u_-) \).

The linear fluctuations around \( \bar{u}_\pm \) vacua are a bit more complicated. First of all, neither \( \bar{u}_+ \) nor \( \bar{u}_- \) saturates the self-dual bound since \( \mathcal{E}(\bar{u}_\pm) > 0 \) unless \( \kappa \rho_c = 0 \) or \( \kappa \rho_c = -v^4/8 \): Even the
linear instability becomes a nontrivial issue. Using Eq. (21), we find the following behaviour of $\omega^2_\pm$ and $p^2_\pm$ in terms of the vacuum expectation values $\tilde{u}_\pm$:

\[
\begin{align*}
 p^2_\pm &= \frac{\tilde{u}_\pm^2 (v^2 - 5\tilde{u}_\pm^2) (v^2 - 2\tilde{u}_\pm^2) (v^2 - 3\tilde{u}_\pm^2)}{\kappa^2 (v^2 - 3\tilde{u}_\pm^2)^2}, \\
 \omega^2_\pm &= \frac{4\tilde{u}_\pm^2 (2\tilde{u}_\pm^2 - v^2)}{\kappa^2 (v^2 - 3\tilde{u}_\pm^2)^2},
\end{align*}
\]

where we have traded off $\rho_c$ in favor of the vacuum expectation value $\tilde{u}_\pm$.

In $\tilde{u}_+$ vacua for both cases (2) and (6), it is easy to see that $\omega^2_\pm$ is always nonnegative, which again translates into the linear stability. Are there roton modes? The answer is yes, but only for case (6). $p^2_\pm$ is positive in this case, provided that $\tilde{u}_+^2 \leq v^2$. In terms of $\kappa \rho_c$, the corresponding range is given by $-v^4/8 > \kappa \rho_c \geq -v^4$.

Combined with the case (5) where a roton mode is found for $u_-$ vacuum, this means that there exist a roton mode around a homogeneous asymmetric vacuum only for theories with $v^2 > 0$ and $0 > \kappa \rho_c \geq -v^4$. This roton is massless in $u_-$ vacuum while massive in $\tilde{u}_+$ vacuum. Note that the roton mode can occur on either branches of the dispersion relation.

The only remaining vacuum to consider is that of the local minimum $u^2 = \tilde{u}_-^2$ that exists for $v^2 > 0$ and $v^4/24 > \kappa \rho_c > 0$, namely for the case (4). Although it is a local minimum of the effective energy density $\tilde{\mathcal{E}}$, it is always unstable. Indeed, as we vary $\kappa \rho_c$ within this range, $\tilde{u}_-^2$ interpolates between $v^2/6$ and 0, and Eq. (23) then implies that $\omega^2_\pm$ is always negative while $p^2_\pm$ remains positive. Hence, this particular local minimum is classically unstable against an inhomogeneous fluctuation (of length scale $\sim \kappa / v \tilde{u}_-$), unlike any other case we considered before. At this point it is not clear whether such instability will lead to excitations on the homogeneous self-dual vacuum $\tilde{u}^2 = \tilde{u}_+^2$.

## 4 Self-Dual Solitons

Having studied the homogeneous vacuum structure in great detail, we are now in position to ask what are possible self-dual soliton solutions in the symmetric and the asymmetric phases. Previous works shows that in the models without the background charge density there can be q-balls in the symmetric phase and vortices in the asymmetric phase [2]. Do they persist in our case? If so, how does the background deform them? Are there other kind of solitons? Specifically, we must investigate possible self-dual configurations that asymptotically approach a given homogeneous vacuum.

Similarly to what we have seen in the study of homogeneous asymmetric vacua, the answer depends on the parameters of the theory and also on which homogeneous vacuum we consider. For instance, in $\tilde{u}_\pm$ vacua, the second self-dual equation (12) implies a divergent total magnetic flux, thus making a self-dual soliton impossible. Another case where a self-dual soliton does not exist is the symmetric phase in theories with negative $\rho_c$, as pointed out earlier.
Are there self-dual solitons in \( u_\pm \) vacua? While a generic self-dual configuration would not be rotationally symmetric, let us focus on the rotationally symmetric solutions to gain an insight on the qualitative features of the self-dual solutions. The ansatz for a rotationally symmetric configuration around the origin is made of \( f(r), \theta = -u \varphi, A_i = A(r) \hat{\varphi}_i \). Rewriting the second order form of the self-dual equation in terms of a new field \( y = \log(f^2/v^2) \) and the dimensionless radial coordinate \( s = r|v^2|/\kappa \), we find a second order dissipative equation for rotationally symmetric self-dual configurations:

\[
\frac{d^2y}{ds^2} + \frac{1}{s} \frac{dy}{ds} + \frac{\partial V(y)}{\partial y} = 2\pi \frac{\delta(s)}{s},
\]  

(24)

where \( y = \ln(f^2/v^2) \) and \( V(y) = -e^{3y}/2 + e^y + 2\kappa \rho_0 y/v^4 \), \( \pm 1 \) being the sign of \( v^2 \).

Let us first consider the \( u_+ \) vacuum. Note that \( y_+ = \ln(u_+^2/v^2) \) is always a local maximum of \( V(y) \). In Eq. (24), a self-dual anti-vortex in the \( u_+ \) vacuum starts at the asymptotic value \( y_+ = \log(u_+^2/v^2) \) and smoothly evolves to \( y = -\infty \), as \( r \) decreases from \( \infty \) to 0. The dissipative term tends to increase the total “energy” as \( r \) approaches the origin, and thus lets \( f^2 = |v^2|e^y \) reach zero at the center of the vortex even when \( V \) has a linearly increasing potential barrier toward \( y = -\infty (\rho_c < 0) \).

However, if the system starts out at the other possible asymptotic value \( y_- = \log(u_-^2/v^2) \) that exists in the case (5), the only nontrivial solutions at large distances exhibit the following damped oscillatory behaviour:

\[
(y - y_-) \propto \frac{1}{\sqrt{s}} e^{\pm is \sqrt{V(y_-)}},
\]  

(25)

Although this lets \( f^2 \) settle down to \( u_-^2 \) at large spatial distances, the resulting configuration does not have a convergent total magnetic flux,

\[
\int_0^r dr r F_{12} \sim \sqrt{r} e^{\pm is \sqrt{V(y_-)}},
\]  

(26)

If we are considering a rotationally nonsymmetric soliton instead, that will simply introduce an extra term \( \partial_t f/s^2 \) in the equation above that cannot change this asymptotic behaviour. There is simply no nontrivial self-dual configuration that asymptotes to the homogeneous vacuum \( u_- \). This is clearly related to the fact that there are massless roton modes in this vacuum.

This leaves us with only two choices for the asymptotic state of self-dual solitons: either the homogeneous asymmetric vacuum \( f^2 = u_+^2 \) that exists when \( 8\kappa \rho_c + v^4 > 0 \) or the the homogeneous symmetric vacuum \( f^2 = 0 \) with \( \rho_c \geq 0 \). The second half of this section is mainly devoted to understanding novel solitonic states that arise in these vacua when the background \( \rho_c \) is nontrivial. As we will see shortly, not all of the self-dual solitons here survive the limit \( \rho_c \rightarrow 0 \).

First of all, in the asymmetric \( u_+ \) phase with \( v^2 > 0 \), the self-dual solitons are rather similar to the zero background case. Figure 1. shows the self-dual anti-vortex configuration for \( n = 1, v^2 > 0 \) and \( k \rho_c = v^4 \).
Figure 1: Plots of a typical rotationally symmetric anti-vortex in the asymmetric phase that saturates the self-dual bound. The first graph shows $f^2/|v^2|$ (solid line) and $\kappa F_{12}/\rho_x$ (broken line) as functions of the distance from the center. The second graph is a plot of the flux (divided by $2\pi$) within finite distances from the center. The radial distance is measured in terms of $\kappa/|v^2|$.

These self-dual anti-vortices do survive the limit $\rho_x \to 0$, and become the ordinary topological solitons of the background-free theory. However, if $v^2$ happens to be negative, an unexpected new type of solitons arises, as we discuss below.

One interesting aspect of the self-dual bound here is that the energy bound is proportional to the magnetic flux rather than to the absolute value thereof. As a result, a unit self-dual anti-vortex in the $a_+$ vacuum of case (1) actually has a negative energy $-\pi|v^2|$. Of course, this does not mean that the homogeneous vacuum of case (1) is unstable against the proliferation of anti-vortices, for whenever an anti-vortex is created, a vortex must be also created to conserve the total magnetic flux. In fact, as mentioned before Eq. (13), and due to the broken CTP, a vortex does not saturate its self-dual bound $\pi|v^2| > 0$, for it cannot solve the self-dual equations at the vortex center. Hence, a vortex-anti-vortex pair always costs more energy than $\pi|v^2| - \pi|v^2| = 0$, and the homogeneous asymmetric vacuum is stable against this particular process. Even though anti-vortices have negative rest energy, we see their kinetic mass must be positive since the self-dual energy bound implies that slowly moving anti-vortices should have a positive kinetic energy.

Still, one may wonder what happens to such negative energy solitons as we take the limit $\rho_x \to 0$ where the self-dual bound of the soliton energy is clearly positive. A useful equation to look at in order to understand what happens, is the second self-dual equation (12) that can be solved for $f^2$ algebraically:

$$0 \leq 2f^2 = -|v^2| + \sqrt{v^4 + 8\kappa\rho_x - 8\kappa^2 F_{12}}.$$  \hspace{1cm} (27)
The flux density $F_{12}$ must be bounded above by $\rho_+/\kappa$ to maintain real $f$, while the total flux of the anti-vortex is quantized at a positive value $\int F_{12} = 2\pi$. Therefore, as $\rho_+/\kappa \to 0$, the soliton core must be spread out to infinity, leaving behind a symmetric vacuum, $f^2 = 0$.

What about the symmetric phase? Are there any negative energy q-balls, possibly with an anti-vortex embedded inside. Again using the second self-dual equation, we can express the energy bound as follows,

$$E - E_0 \geq \frac{v^2}{2} (\Psi - \Psi_0) = \frac{v^2}{2} \int d^2 x \left( F_{12} - \frac{\rho_+}{\kappa} \right) = \frac{v^2}{4\kappa^2} \int d^2 x f^2 (v^2 - f^2). \quad (28)$$

For negative $v^2$, the right-hand-side is manifestly positive, while the same could be true for positive $v^2$ provided that $f^2$ remains smaller than $v^2$.

However, it is easy to see that, for a strictly positive $\rho_+$, $f^2$ may grow larger than $v^2$. To see this, again consider Eq. (24), where the “potential” energy $V$ now has a unique maximum at $y_+ = \log(u_+/v^2) > 0$ but no minimum. In this picture, a self-dual q-ball would interpolate smoothly between $y = -\infty$ at $r = \infty$ and some $y_0 \neq -\infty$ at $r = 0$, without the delta function source on the right-hand side. The only restriction on $y_0$ is that it be on the left hand side of the maximum: $y_0 < y_+$, which translates into $f^2(0) < u_+^2$. Then, a q-ball may have a central region ($v^2 < f^2 < u_+^2$) of negative charge density ($F_{12} - \rho_+/\kappa < 0$), surrounded by an outer region ($0 \leq f^2 \leq v^2$) that has a positive charge cloud ($F_{12} - \rho_+/\kappa \geq 0$). In figure 2, two typical shapes of q-balls are depicted, depending on whether $f^2(0)$ is smaller or larger than $v^2$. In all figures in this section, the parameters are chosen such that $\kappa \rho_+/v^4 = 1$.

![Figure 2](image_url): Plots of two q-balls in the symmetric phase with $\kappa \rho_+/v^4 = 1$. Solid lines are $f^2/v^2$, and the broken lines are the excess magnetic field, or equivalently, the Higgs charge density divided by $\kappa$, in the unit of $v^4/\kappa^2$. The first graph corresponds to $f^2(0) = 0.9 \, v^2$ while the second to $f^2(0) = 1.4 \, v^2$. 

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Furthermore, as $f^2(0) \to u_+^2$, the central region of negative charge density becomes larger and larger, so that, in the strict limit $f^2(0) \equiv u_+^2$, it overtakes the entire plane such that the corresponding self-dual configuration is simply that of the homogeneous asymmetric vacuum $f^2(r) \equiv u_+^2$ with $F_{12} \equiv 0$. In other words, there exist self-dual q-balls of arbitrary negative charge $\kappa (\Psi - \Psi_0) < 0$, or equivalently of arbitrary negative energy $E - E_0 = (v^2/2)(\Psi - \Psi_0) < 0$. This is illustrated in figure 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_image.png}
\caption{Plots of $f^2/v^2$ and $\kappa F_{12}/\rho_e$ as the value of $f^2(0)$ approaches $u_+^2 = 2v^2$. As $f^2(0)/u_+^2$ increases toward one from below, the central region of vanishing magnetic field grows steadily (the second graph).}
\end{figure}

In the other limit of $y_0 \to -\infty$, the charge cloud of the soliton remains manifestly positive everywhere, but the strength thereof decreases indefinitely so that the net flux also decreases indefinitely. The upshot is that we have a one-parameter family of rotationally symmetric q-balls such that their total charge is bounded from above by a finite positive quantity. Furthermore, for each net positive charge $\kappa (\Psi - \Psi_0) > 0$ allowed, there exist two different solitons, depending on whether $f^2(0)$ is smaller than or larger than $v^2$.

One interesting consequence of this is the degeneracy of the symmetric vacuum. For a moderate value of $f^2(0)/v^2 > 1$, there must exist a nontrivial configuration of zero total charge, thus of zero net magnetic flux and zero net energy. This is simply because there exists a one-parameter family of solutions that interpolate between the positive charge solution of $f^2(0) = v^2$ and arbitrarily large negative charge solutions that show up as $f^2(0) \to u_+^2$. The zero-charge configuration looks like a small compact island of broken phase enclosed by a cloud of positive charge that exactly cancels the negative charge inside. In figure 4, such a configuration is depicted when $\kappa \rho_e/v^4 = 1$. 

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It is not too difficult to convince ourselves that similar considerations may be applied to q-balls with anti-vortices at the center. After fine-tuning the behaviour of $f^2$ near the origin for each fixed anti-vorticity $n$, we find that there exist rotationally symmetric solutions of arbitrary negative net energy $E - E_0 < 0$, and a unique solution of zero net energy for each $n$. Such zero-energy configurations consist of a thick ring of negative charge sandwiched between a positive charge core and a surrounding outer region of positive charge cloud, such that the total charge is exactly zero.

Hence, whenever both $v^2$ and $\rho_c$ are strictly positive, the homogeneous symmetric vacuum is degenerate with infinite number of rotationally symmetric self-dual configurations. Are there other, more complicated, states of vanishing net charge that are also degenerate with the homogeneous symmetric vacuum? One possible suggestion is that these zero-energy “solitons” do not interact with each other so that we may create many of them at once without costing any energy. This possibility along with many interesting questions concerning general multi-soliton configurations are postponed to a future study.

5 Angular Momentum

One of the more important aspects of Chern-Simon theories is the fractional spin and statistics carried by the solitons. In view of the fact that the background charge density alters the behaviour of the theory rather significantly, and in particular of the explicitly broken CTP, it would be very interesting to find out whether spin and statistics are modified. As a first step along this direction, let us consider the conserved angular (and linear) momenta.
The naïve angular momentum density obtained by the Noether theorem contains a term, proportional to \( \rho_e A_j \), that not only cause divergences in all phases but is gauge-invariant only up to a total derivative. Explicitly, the Noether angular momentum is

\[
J_{\text{Noether}} = -\int d^2 x \, \epsilon_{ij} x^i \left\{ \int \partial_j f + f^2 (\theta + A_0)(\partial_j \theta + A_j) - \rho_e A_j \right\}.
\]

A conventional way of dealing with such problems is to isolate a finite part by adding an appropriate manifestly conserved current, and then justify that the final expression is a physically sensible one [5]. However, we want to present an alternate and certainly more elegant procedure by considering a similar system on a two-sphere of radius \( R \) in the limit \( R \to \infty \).

A two-sphere has isometries in the form of \( SO(3) \) rotations, and there exist the corresponding conserved angular momentum 3-vector \( \mathbf{L} \). Although the naïve Noether expression for the density contains a similarly problematic term as above, we may now perform a partial integration to recover a manifestly gauge-invariant expression for the density, since there is no longer a boundary to worry about:

\[
\mathbf{L} = -\int R^2 d\Omega \left\{ \int (r \times \nabla) f + f^2 (\theta + A_0) r \times (\nabla \theta + A) - \rho_e \, r \cdot (\nabla \times A) \right\}.
\]

This satisfies the usual \( so(3) \) commutation relations. Here, \( r \) denotes the coordinate vector of the three dimensional Euclidean space where the spatial sphere is imbedded. In this notation, the actual spatial manifold corresponds to \( |r| = R \). The 3-vector \( A \) must be tangent to this two-sphere, since it represents the spatial gauge field on the two-sphere. Finally, the total magnetic flux \( \Psi = \int R^2 d\Omega \, \hat{r} \cdot \nabla \times A \) should be quantized in the unit of \( 2\pi \) to ensure the gauge invariance.

Already, we can see the advantage of working on the sphere instead of the plane: the last term is now proportional to \( \nabla \times A \), thus manifestly gauge-invariant. Furthermore, given a uniform radial magnetic field strength \( B \), there is no \( BR^4 \) divergence in the total angular momentum \( \mathbf{L} \), due to cancellations between two hemispheres, unlike \( J_{\text{Noether}} \) that has such a nasty divergence in the symmetric phase. In order to take the plane limit \( R \to \infty \), we imagine all interesting activities of the system occurs within finite distances from the north pole.

First, let us consider the asymmetric phase where we have a uniform Higgs charge density \(-f^2 (\theta + A_0) = -\rho_e \) at large spatial distances from the north pole. The vortices and the antivortices of total flux \( \Psi = 2\pi \), if distributed near the north pole, cause a quadratic divergence \( \Psi \rho_e R^2 \) in the third component \( L_3 \). However, this should not be surprising at all. A similar divergence arises if we consider an electrically charged particle on a large two-sphere threaded by a uniform magnetic field \( B \) of fixed strength. The extra angular momentum \( \sim e\Omega \hat{r} \), familiar from the studies of charge-monopole system [9], would diverges quadratically simply because the total magnetic charge \( q = BR^2 \) diverges quadratically. In the asymmetric phase, there exists the so-called Magnus force, between the uniform Higgs charge density \(-\rho_e \) and the localized fluxes, which simulates the Lorentz force.

As long as we are concerned with the plane limit, we may as well remove this divergence in each superselection sector of fixed \( \Psi \). Then, we find the following modified angular momentum, valid
for the symmetric phase on the plane:

\[ J_{\text{asym}} = - \int d^2x \left\{ \epsilon_{ij}x^i \left[ \frac{1}{2} \partial_j f + f^2 \left( \partial + A_0 \right)(\partial_j \theta + A_j) \right] + \frac{\rho_\epsilon}{2} x^2 F_{ij} \right\}. \tag{31} \]

Note that the vestige of the Magnus force contribution still remains in the last term. To see this more clearly, consider a generic \( \theta \) field configuration,

\[ \theta = - \sum_a (-1)^a \text{Arg}(\hat{r} - \hat{q}_a(t)) + \eta, \tag{32} \]

where \( \eta \) is, as usual, single-valued. Then, the above angular momentum can be rewritten, after an integration by part, as

\[ J_{\text{asym}} = - \int d^2x \, x^i \epsilon_{ij} \left\{ \frac{1}{2} \partial_j f + \left[ f^2 (\partial + A_0) - \rho_\epsilon (\partial_j \theta + A_j) \right] - \pi \rho \sum_a (-1)^a |\hat{q}_a|^2. \tag{33} \]

The last term that sums over the locations of vortices, is similar to what one would get for a charged particle moving under a uniform magnetic field. Vortices in our case feel the aforementioned Magnus force instead, and this term represents the corresponding modification. For a similar expression in the pure Maxwell-Higgs systems, see Ref. [5]

In the large sphere limit, the linear momenta \( P^i \) on the plane are found from its relation to the \( SO(3) \) angular momentum via \( L^1 = -RP^2 \) and \( L^2 = RP^1 \). Note that as long as the nontrivial configurations are concentrated near the north pole there will be no quadratic divergences in \( L^1 \) and \( L^2 \). Alternately, we may extract \( P^i \)'s from the plane angular momentum, for the latter transforms like \( J \rightarrow J + \epsilon_{ij} a^i P^j \) under an infinitesimal translation \( x^i \rightarrow x^i + a^i \). The two resulting expressions coincide with each other. From the quadratically divergent part of the commutation relation \([L^1, L^2] = iL^3 \) in the plane limit, we find \([P^1, P^2] = i\rho \Psi \). This modified form of the commutator is again due to the Magnus force exerted by the Higgs charge density \(-\rho_\epsilon \). Here we have to use the commutation relation \([A_1(x, t), A_2(y, t)] = i\delta^2(x - y)/\kappa \).

In the symmetric phase, the field \( J \) will be nonzero only near the north pole so that at large distances from the north pole, there is a uniform magnetic field \( \mathbf{r} \cdot \nabla \times \mathbf{A} = \rho_\epsilon / \kappa \). Physically interesting configurations are then those with finite net magnetic flux, \( \Psi - \Psi_0 < \infty \) where \( \Psi_0 = 4\pi R^2 \rho / \kappa \). Again nontrivial configurations are assumed to be within finite distance from the north pole as \( R \rightarrow \infty \). In order to take the plane limit, it is sensible to subtract \( 0 = R^3 \int d\Omega \rho_\epsilon \mathbf{r} \), and make the angular momentum density to be concentrated near the north pole. This effectively replaces \( \mathbf{r} \cdot \nabla \times \mathbf{A} \) by \( \mathbf{r} \cdot \nabla \times \mathbf{A} - \rho_\epsilon / \kappa \) in the last term in Eq. (30).

Now, \( L^3 \) diverges quadratically in the plane limit as \( R^3 \rho_\epsilon (\Psi - \Psi_0) \). In the symmetric phase, the origin of this divergence is even more transparent, for the system is basically that of charged particles (q-balls) of total charge \( \kappa (\Psi - \Psi_0) \), moving in a large two-sphere that surrounds total magnetic charge \( R^3 \rho_\epsilon / \kappa \) at the center. In each superselection sector, we may again remove the quadratic divergences to obtain the modified angular momentum, valid for the symmetric phase on
the plane.

\[ J_{\text{sym}} = - \int d^2 x \left\{ \epsilon_{ij} x^i \left[ j \partial_j f + f^2 (\hat{\theta} + A_0)(\partial_j \theta + A_j) \right] + \frac{\rho_c}{2 \kappa} x^2 (\kappa F_{12} - \rho_c) \right\}. \quad (34) \]

We can find the linear momenta similarly as in the asymmetric phase, and can show that they satisfy a nontrivial commutator \([P^1, P^2] = i \rho_c (\Psi - \Psi_0)\). This last expression may be rewritten, using Gauss’s constraint, in terms of the Higgs charge density and the background magnetic field \(\rho_c / \kappa\) of the symmetric phase: \([P^1, P^2] = -i (\rho_c / \kappa) \int d^2 x f^2 (\hat{\theta} + A_0)\). Written this way, the nontrivial commutator can be naturally attributed to the velocity-dependent Lorentz force on the q-balls, exerted by the homogeneous magnetic field \(\rho_c / \kappa\).

Now that we derived the correct angular momenta in the presence of the uniform background fields, it is time to consider how these abstract expressions translates into angular momenta of specific solitons. For simplicity, we shall consider single static solitons.

Let us start with topological anti-vortex in the asymmetric phase. Since we can safely exclude the center of the soliton at \(x^i = \hat{q}^i\) from the space integration of Eq. (33) without affecting the value of the angular momentum, we may introduce a vector potential \(\vec{A}_j = \partial_j \theta + A_j\), and rewrite the \(J_{\text{sym}}\) as an integral over \(R^2_\theta \equiv R^2 - \{\hat{q}\}:

\[ J_{\text{sym}} = - \int_{R^2_\theta} d^2 x \epsilon_{ij} x^i \left\{ j \partial_j f - \kappa \vec{A}_j \epsilon_{kl} \partial_k \vec{A}_l \right\} - n \pi \rho_c |\vec{q}|^2. \quad (35) \]

For the static self-dual solutions (and also for rotationally symmetric solutions), \(\partial_i \vec{A}_i = 0\) and the integrand becomes a total divergence:

\[ J_{\text{sym}} = \kappa \int_{R^2} d^2 x \partial_j \left\{ \frac{1}{2} x^j (\vec{A}_k \hat{A}_k) - \vec{A}_j (x^k \hat{A}_k) \right\} - n \pi \rho_c |\vec{q}|^2. \quad (36) \]

Note that \(\vec{A}\), as a gauge invariant quantity, must be exponentially small at large distances so that the only boundary to speak of is at the center. If the anti-vortex has a topological charge (i.e., if the total magnetic flux is \(2 \pi n > 0\)), \(\vec{A}_j\) near the center is dominated by \(\partial_j \theta \approx n \epsilon_{jk} x^k / r^2\), and the angular momentum is easily obtained as follows,

\[ J_{\text{sym}} = -\pi \kappa n^2 + \pi n (-\rho_c) |\vec{q}|^2. \quad (37) \]

In the limit \(\rho_c \to 0\), this reproduces the expected results found in Ref.[2, 3]. The last term obviously reflects the Magnus force on the anti-vortex exerted by the background Higgs charge density \(-f^2 (\hat{\theta} + A_0) = -\rho_c\). Clearly, the above expression is true also for rotationally symmetric vortices. Thus, we have shown vortices and anti-vortices in Chern-Simons systems with nonzero magnetic flux carry nonzero spins \(-\pi \kappa n^2\) that are independent of \(\rho_c\).

In the self-dual case, we can have static multi-anti-vortex configurations where anti-vortices do not overlap each other. Eq. (36) can then be explored much further to gain an understanding of
slowly moving vortices and thus of their statistics. (See Ref.[3] for the case \( \rho_e = 0 \).) We hope to return to this topic in near future.

In the symmetric phase, on the other hand, we introduce \( \tilde{A}_i \equiv \partial_i \theta + A_i + \epsilon_{ij} x^j \rho_e / (2\kappa) \). Using Gauss’s constraint, we again find that the angular momentum for a static self-dual configurations can be written as an integral of a total divergence,

\[
J_{\text{sym}} = \kappa \int_{R^2} d^2 x \epsilon_{ij} x^i \dot{\tilde{A}}_j \epsilon_{kl} \partial_k \tilde{A}_l = \kappa \int_{R^2} d^2 x \partial_j \left\{ \frac{1}{2} x^j (\tilde{A}_i \tilde{A}_k - \tilde{A}_j (x^k \tilde{A}_k)) \right\}. \tag{38}
\]

As we have seen earlier in a more general context, the angular momentum in the symmetric phase must also contain a term \( \sim |q|^2 \) that originates from the Lorentz force exerted by the uniform magnetic field \( \rho_e / \kappa \). However, this expression, compared to the analogue in the asymmetric phase, does not seem to contain such a term. The crucial observation that resolves this apparent puzzle is that, unlike \( \tilde{A} \) above in the asymmetric phase, \( \tilde{A} \) exhibits a powerlike behaviour at large distances so that we need to worry about the contribution from the boundary at infinity. In fact, if the soliton does not contain any vortex or anti-vortex at the center and thus is a pure q-ball, the only boundary is at infinity.

\[
J_{\text{sym}} = \kappa \int_{R^\infty} dt \left\{ \frac{1}{2} \frac{r}{|\tilde{A}_i \tilde{A}_k|} - \left( \frac{\rho_e}{2|q|} \right)^2 \right\}. \tag{39}
\]

For a single \( \tilde{A} \), rotationally symmetric q-ball of net magnetic flux \( 2\pi \alpha \), in particular, the self-dual equations (11) and (13) imply the following asymptotic behaviour,

\[
\tilde{A}_i \rightarrow \epsilon_{ij} \left\{ - \alpha \frac{x^j}{|x - q|^2} + \frac{\rho_e}{2\kappa} \frac{q_i}{|q|^2} \right\}, \tag{40}
\]

provided that the soliton is centered at \( x^i = q^i \). After a careful usage of this limiting form, we find the total angular momentum of a self-dual q-ball of net magnetic flux \( 2\pi \alpha \) at \( x^i = q^i \):

\[
J_{\text{sym}} = 2\pi |q|^2 = \frac{\rho_e}{\kappa} |q|^2. \tag{41}
\]

The last term is exactly the position dependent term we need, which again shows that our angular momentum is sensible.

In this section, we derived appropriate expressions of angular momentum in each of the symmetric and asymmetric phases, and evaluated them on self-dual q-balls and anti-vortices. While the angular momentum acquires a new position dependent terms that reflect the interaction between the solitons and uniform fields of the ground states, it also include the usual fractional spin that is independent of the background. Those additional position dependent terms, which are proportional to \( \rho_e \), arise because the slowly moving solitons feel a velocity-dependent force exerted by the uniform fields of the surrounding vacuum.

6 Conclusion

We studied the characteristics of a self-dual Chern-Simons Higgs model coupled to a background electric charge density. We found a rich vacuum structure, and subsequently tested the classical
stability of homogeneous vacua along magnetoroton modes. In certain stable vacua, novel self-dual solitonic configurations are found and studied in detail. Finally, the divergent Noether angular momentum is successfully modified to a sensible and finite expression. From this formula, we find that the soliton angular momentum is corrected by an effective Lorentz force, and acquires a term quadratic in the position as well as the usual (background-independent) fractional spins.

As the structure of the theory is very rich, we were not able to cover even the classical aspects of the theory completely. For instance, there are cases where no self-dual solution exists: Although we still expect to find non-self-dual solitons in such cases, practically nothing is known about them other than the spin. Another important topic which is not covered here is the matter of the spin-statistics. Since our solitons carry nonzero fractional spin, we expect there is nontrivial statistical interaction between solitons. It would be interesting to find out whether the statistical interaction follows the naïve expectation that it is made of the spin contribution and the background Magnus/Lorentz force contribution. The study of the classical dynamics of slowly moving anti-vortices in the asymmetric vacuum may lead to a better understanding in this regard. Finally, it is not clear to us at all how our self-dual model could be extended to have $N = 2$ or $N = 3$ supersymmetry, which is expected from self-dual models in general [10].

Quantum aspect of our theory should be also quite rich. Especially, the understanding of the infinite vacuum degeneracy of the symmetric phase poses a challenge.

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