HAMILTONIAN MECHANICS WITH A SPACE COORDINATE AS INDEPENDENT VARIABLE

CANONICAL THIN LENS APPROXIMATION FOR AN ACCELERATING GAP

B. Schnizer
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PART I: HAMILTONIAN MECHANICS WITH A SPACE COORDINATE AS INDEPENDENT VARIABLE

Summary

For application in linear accelerators a kind of Hamiltonian mechanics is developed where the longitudinal space coordinate, \( z \), is the independent variable in place of time, \( t \). This is done by changing in Hamilton's principle the variable of integration from \( t \) to \( z \) and by employing afterwards the usual Legendre transformation. The corresponding Lagrangian, Hamiltonian, canonically conjugate variables and canonical transformations are discussed. Poisson and Lagrange brackets and the Poincaré invariants are listed.

The general symmetry inherent to Hamiltonian mechanics is demonstrated in an eight-dimensional version treating on an equal footing the four canonically conjugate pairs \( x, p_x \); \( y, p_y \); \( z, p_z \); \( t, p_t \). \( E = -E \) (\( E \) = total energy). Any among these variables may be chosen as the independent one; in certain (but not all) cases its canonically conjugate partner is the Hamiltonian and the three remaining pairs give the set of dependent variables.

PART II: CANONICAL THIN LENS APPROXIMATION FOR AN ACCELERATING GAP

Summary

In the so-called thin lens approximation the real trajectory of a single proton crossing a gap of a linac Alvarez structure is replaced by a fictitious one consisting of step functions as if the gap were reduced to its median plane. A thin lens Hamiltonian is derived from which these increments can be derived by partial derivations. By canonical transformations a complete set of reduced variables is introduced. Reduced variables are constant in the absence of fields acting on the particle. The transformed Hamiltonian is solved by iterations starting from free particle motion. To first order it depends only on the constant mid-gap values of the reduced variables and on the independent variable, the longitudinal space coordinate. Integrating with respect to it across the gap gives the thin lens Hamiltonian. By this method are derived the modified Fano-Fesenko equations for the change of kinetic energy and phase across a linac gap in the one-dimensional problem. If transversal motion is also taken into account it turns out that the radial reduced variables differ from those used in the usual linac beam dynamics difference equations. From this and from the coupling between longitudinal and radial motion it follows that radial phase space area is not conserved.
PART I: HAMILTONIAN MECHANICS WITH A SPACE COORDINATE AS INDEPENDENT VARIABLE

1 INTRODUCTION

It is intended to present in this paper a modified version of the formalism of Hamiltonian mechanics suitable for applications in linear accelerators. Here the longitudinal space coordinate, $z$, is the independent variable in place of time, $t$.

In common Hamiltonian mechanics where time, $t$, is the independent variable, the solutions of the equations of motion (Newton's or Hamilton's equations) are functions of time, $t$, e.g. $z = z(t)$. But in linear accelerators the geometry and the physics of the problem single out the longitudinal space coordinate, $z$, measured along the axis of the machine.

(The situation is similar in a synchrotron where this coordinate is the azimuthal angle as measured from the machine's centre.) For example, it is necessary to know the phase of the accelerating radio frequency field at a certain instant, $t_0$, when the particle passes at a certain point, $z = z_0$; or the kinetic energy of a particle leaving a gap is needed. In all these cases the equation $z_0 = z(t_0)$ with a given $z_0$ must be solved for $t_0$. In general, in view of the harmonic time-dependence of the accelerating field this is a complicated transcendental equation whose solution is difficult. In addition, the equations of motion themselves cannot be solved exactly. Therefore it is more convenient to start right from the beginning with equations of motions where $z$ is the independent variable. They are complicated in both cases and must be treated by approximate or numerical methods.

Such an approach has been used before by several authors, e.g. in the theory of linear accelerators by PANOFSKY\textsuperscript{1)}, J S BELL\textsuperscript{2)}, PROME\textsuperscript{3)}, CARNE, LAPOSTOLLE and PROME\textsuperscript{4);} in the theory of circular accelerators by SCHOC\textsuperscript{5)}. However, to the knowledge of the present author this has been done by ad-hoc procedures and never in a systematic way starting from first principles.

A convenient starting point for the derivation of ordinary Hamiltonian mechanics where time, $t$, is the independent variable, is Hamilton's principle\textsuperscript{6),7)}. By a Legendre-transformation canonical momenta are introduced and Hamilton's principle is transformed into a variational principle in canonical form. The Euler equations of this variational problem are Hamilton's equations. These may be further transformed by canonical transformations. Now, by substitution a new variable, say $z$, may be introduced as the variable of integration in Hamilton's principle and it becomes the independent variable of the corresponding Euler equations. This transformed principle can be brought again to canonical form by a Legendre transformation. To the generalized velocities in the ordinary ($t$-) scheme correspond now the derivatives with respect to $z$: $x'$, $y'$, $t'$; the generalized momenta are the partial derivatives of the Lagrangian contained in the transformed version of Hamilton's principle with respect to $x'$, $y'$, $t'$. Such a Legendre transformation can be performed in any variational problem, provided its integrand does not depend on higher
derivatives than the first one and is not of degree one in these derivatives 8). In this way the whole game usually played for time as independent variable may be performed as well for the space coordinate z as independent variable. The abstract mathematical structure is the same, but the analytical form and the physical meaning of some quantities is different.

While the pairs of canonically conjugate transversal variables are the same in both schemes, time, \( t \), and negative total energy, \( p_t = -E \), form the third pair in the \( z \)-scheme; and the Hamiltonian equals \( \pm p_z \). This gives an indication of a certain symmetry inherent to Hamiltonian mechanics which is also investigated: Any variable contained in the four pairs of canonically conjugate variables \( x, p_x; y, p_y; z, p_z; t, p_t = -E \) may be chosen as the independent one. Its partner is the Hamiltonian, and the three other pairs represent the canonically conjugate dependent variables, provided the Lagrangian has been chosen properly.

In chapter 2 are given the basic equations of Hamiltonian mechanics in the common form of it. In chapter 3 the change to \( z \) as the independent variable is performed. The new LaGrangian and Hamiltonian in this scheme are discussed as well as canonical transformations, Poisson and Lagrange brackets and some of the Poincaré invariants are listed.

This is specialized to axial symmetry and to circularly cylindrical coordinates in chapter 4. In chapter 5 the four- (or eight-) dimensional version of Hamiltonian mechanics is described where all variables are treated on an equal footing, and the symmetry described above is demonstrated. Applications of the scheme presented here, dealing with the thin lens approximation for an accelerating gap, are given in part II of this report.

2. ORDINARY HAMILTONIAN MECHANICS

In this chapter are summarized the basis equations of ordinary Hamiltonian mechanics where time, \( t \), is the independent variable. This is necessarily short and incomplete, and the reader is referred to standard texts 6) 7) for more details. In this scheme particle trajectories are described by generalized coordinates, \( q_k(t) \), and generalized velocities, \( \dot{q}_k(t) = \frac{dq_k}{dt} \). There are \( f \) of each where \( f \) is the number of degrees of freedom, and in general \( f = 3 \).

A convenient starting point for the derivation of the various equations of motion is Hamilton's principle:

\[
\delta \int_{t_1}^{t_2} L(q_k, \dot{q}_k; t) \, dt = 0
\]

(2.1)

The non-relativistic Lagrangian for the motion of a charged particle with charge \( e \) and mass \( m \) is:
\[ L = \frac{mv^2}{2} + e(\mathbf{v} \cdot \mathbf{A}) - eU \quad (2.2) \]

\( mv^2/2 = (m/2)(ds/dt)^2 \) is kinetic energy, \( ds \) is arc length in the curvilinear system of coordinates \( q_k \). The external electromagnetic field \( \mathbf{E}, \mathbf{H} \) is described by the vector potential \( \mathbf{A} \) and the scalar potential \( U \):

\[
\mathbf{E} = -\nabla U - \partial \mathbf{A}/\partial t
\]

\[
\mathbf{B} = \mu_0 \mathbf{H} = \nabla \times \mathbf{A}
\quad (2.3)
\]

These potentials are uniquely determined if restricted by a side condition. Here the Lorentz-invariant Lorentz gauge is prescribed:

\[
\mathbf{v} \cdot \mathbf{A} + \varepsilon_{\mu\nu} \partial U / \partial t = 0
\quad (2.4)
\]

A TM-field, \((\varepsilon_z \mathbf{H}) = 0\), as often occurs in accelerators, may be described by an electric Hertz vector \( \mathbf{H} = \mathbf{e}_z V \) with:

\[
U = -\mathbf{v} \cdot \mathbf{H} = -\partial V / \partial z
\]

\[
\mathbf{A} = \varepsilon_{\mu\nu} \partial \mathbf{A} / \partial t = \mathbf{e}_z \varepsilon_{\mu\nu} \partial V / \partial t
\quad (2.5)
\]

The Euler equations of the variational problem (1) give the equations of motion called Lagrange's equations of the second kind:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \quad k = 1, 2, \ldots, f
\quad (2.6)
\]

With the help of canonical momenta defined as:

\[
p_k = \partial L / \partial \dot{q}_k
\quad (2.7)
\]

the variational problem (1) is transformed into a canonical one:

\[
\delta \int_{t_1}^{t_2} \left[ \sum_{k=1}^{f} \frac{p_k}{2} \dot{q}_k - H(p_k, q_k; t) \right] dt = 0
\quad (2.8)
\]

The Hamiltonian is defined as:

\[
H(p_k, q_k; t) = \frac{p_k}{2} \dot{q}_k - L(q_k, \dot{q}_k; t)
\quad (2.9)
\]
The Euler equations of (8) are Hamilton's equations:

$$\dot{p}_k = \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \quad (2.10)$$

They are a system of 2f first order equations displaying a remarkable symmetry. Transformations of the dependent variables $p_k, q_k$ which preserve this symmetry are called canonical. They are not discussed here, since the corresponding transformations in the $z$-scheme are given in chapter 3.

3. HAMILTONIAN MECHANICS WITH z AS INDEPENDENT VARIABLE

3.1. Lagrangian and Hamiltonian

As explained in the introduction it is the purpose of this paper to give a systematic derivation of a version of Hamiltonian mechanics suitable for linear accelerators where $z$, the longitudinal space coordinate, is the independent variable. Consequently, $t$, must become a dependent one. In place of $t$ the dimensionless phase-angle:

$$\phi(z) = \omega t(z) \quad \omega = \text{const} \quad (3.1)$$

is preferred. $\omega$ is the (constant) angular frequency of the accelerating radio frequency field.

This aim can be achieved by substituting in Hamilton's principle, eq. (2.1), the coordinate $z$ as new variable of integration:

$$\phi = \omega t = \phi(z) \quad dt = (dt/dz) \, dz = \omega \, dz = (\phi'/\omega) \, dz \quad (3.2)$$

The dependence of the transversal coordinates and velocities is changed accordingly:

$$x = x(t) = x(t(z)) \rightarrow x(z) \quad y = y(t) = y(t(z)) \rightarrow y(z)$$

$$\dot{x} = \frac{dx}{dt} = x'/t' = \omega x'/\phi' \quad \dot{y} = \frac{dy}{dt} = y'/t' = \omega y'/\phi' \quad (3.3)$$

The same letters are used to denote the new functions $x(z)$ and $y(z)$ which differ from $x(t)$ and $y(t)$. With these definitions is rewritten Hamilton's principle, eq.(2.1):

\[\text{\footnotesize{\cite{3}\textsuperscript{\textdagger}}}}\quad \text{In this and the following chapter the prime denotes derivation with respect to } z, \quad = d/dz, \text{ while dots still refer to time } t, = d/dt.\]
\[
\delta \int_{t_2}^{t_1} L(x, y, z; \dot{x}, \dot{y}, \dot{z}, t) \, dt = \delta \int_{z_1}^{z_2} L(x, y, z; \dot{x}/t; \dot{y}/t; \dot{z}/t; t) \, t' \, dz
\]

\[= \delta \int_{z_1}^{z_2} L(x, y, t; \dot{x}; \dot{y}; t; z) \, dz = 0 \quad (3.4)\]

It is obvious that the transformed integral in the second line has the same mathematical form as the very first one. Only notations and physical meaning have been changed. It seems therefore appropriate to retain in the new scheme (z-scheme) the names used for the corresponding quantities in the old one (t-scheme). The Lagrangian, \( \bar{L} \), for a particle in an electro-magnetic field depending on the new coordinates, \( q^*_k \) (\( x, y, t \)) and derivatives, \( \dot{q}^*_k \) (\( \dot{x}, \dot{y}, \dot{t} \)) (corresponding to the generalized velocities) is:

\[
\bar{L} = m(\dot{x}^2 + \dot{y}^2 + 1)/(2t^2) + e(x' \dot{A}_x + y' \dot{A}_y + A_z) - eU \dot{t}
\]

\[= m\omega(\dot{x}^2 + \dot{y}^2 + 1)/(2\omega^2) + e(x' \dot{A}_x + y' \dot{A}_y + A_z) - eU \omega/\omega \quad (3.5)\]

The new canonical momenta:

\[
p_k = \sqrt{\bar{L}/\dot{q}^*_k} \quad (3.6)
\]

are derived from (5):

\[
p_x = \frac{\delta \bar{L}}{\delta \dot{x}} = mx'/t' + eA_x \quad p_y = m\omega'/t' + eA_y
\]

\[= \frac{m}{2t^2} (\dot{x}^2 + \dot{y}^2 + 1) - eU = -E \quad (3.7)\]

\[
p_t = \frac{\delta \bar{L}}{\delta \dot{t}} = -\frac{m}{2\omega^2} (\dot{x}^2 + \dot{y}^2 + 1) - eU
\]

\[= \frac{m}{2\omega^2} \left[ \frac{m}{2\omega^2} (\dot{x}^2 + \dot{y}^2 + 1) - eU \right] \quad (3.8)\]

The new transversal canonical momenta agree with those of the t-scheme. The momentum canonically conjugate to time \( t \) (phase-angle \( \phi = \omega t \)) is negative total energy (divided by \( \omega \))(7) and (8) are solved for \( t' \):

\[
t' = \sqrt{m/2} \left[ -p_t - eU - \frac{1}{2m} ((p_x - eA_x)^2 + (p_y - eA_y)^2) \right]^{1/2} \quad (3.10)
\]

Some care must be taken since the square root in eq (10) is a two-valued function. For the present applications where particles move in the positive z-direction and never stop:

\[\uparrow \quad \text{In this and the following chapter the prime denotes derivation with respect to } z, \quad \dot{z} = d/dz.\]
\( t' > 0 \). From the expressions above the Hamiltonian is found:

\[
\tilde{H}(p_x, p_y, p_z; x, y, z) = x' p_x + y' p_y + t' p_t - \tilde{L}
\]

\[
= - \sqrt{2m} \left[ - p_t - eU - \frac{1}{2m} \left( (p_x + eA_x)^2 + (p_y + eA_y)^2 \right) \right] - eA_z
\]

(3.11)

From this expression \( \tilde{H}(p_x, p_y, p_z; x, y, z) \) is found by simply replacing \( p_t \) with \( \omega p_\phi \). With the help of the canonical momenta and the Hamiltonian the variational problem (4) is transformed into the equivalent one:

\[
\delta \int_{z_1}^{z_2} \left[ \sum_{k=1}^{\ell} p_k \dot{q}_k - \tilde{H}(p_k, q_k; z) \right] dz = 0
\]

(3.12)

The transition from (4) to (12) has been treated in a rather superficial way. A very careful discussion is given in ref. 7. Of course, in this reference \( t \) is the independent variable, but this does not concern the mathematical nature of the proof. Equation (12) is the variational problem of the simplest form possible. It involves only the derivatives of one of the two sets of dependent variables, and it is even linear in these. The Euler equations of (12) are Hamilton's equations:

\[
p_k' = \frac{dp_k}{dz} = - \frac{\partial \tilde{H}}{\partial q_k}, \quad q_k' = \frac{dq_k}{dz} = \frac{\partial \tilde{H}}{\partial p_k}
\]

(3.13)

The physical meaning of \( \tilde{H} \), eq. (11), is found by comparing it with eq (10):

\[
\tilde{H} = - m/t' - eA_z = - m\dot{z} - eA_z = - p_z
\]

(3.14)

This is already a hint to the four- (or eight-) dimensional symmetry inherent to Hamiltonian mechanics and discussed in Chapter 5.

3.2. Canonical Transformations

Not every non-singular transformation of the dependent variables:

\[
P_i = P_i(p_k, q_k; z) \quad p_k = p_k(P_i, Q_i; z)
\]

\[
Q_i = Q_i(p_k, q_k; z) \quad q_k = q_k(P_i, Q_i; z)
\]

(3.15)

preserves the canonical form (12) of the variational principle, therewith the symmetrical form (13) of the equations of motion. Transformations preserving this symmetry are called canonical. They are defined to lead from (12) to the new variational problem:
\[
\delta \int_{z_1}^{z_2} \left[ \sum_{i=1}^{\infty} P_i^* Q_i - K(P_i^*, Q_i^*; z) \right] \, dz = 0 \quad (3.16)
\]

with the Euler equations:

\[
\frac{dP_i}{dz} = -\frac{\partial K}{\partial Q_i}, \quad \frac{dQ_i}{dz} = \frac{\partial K}{\partial P_i} \quad (3.17)
\]

This does not mean that the integral (16) in the new variables must be identical with that of (12); but only that both assume simultaneously their extremes, i.e. if (12) assumes its extreme for the functions \( P_i(z), Q_i(z) \), then the same should happen in (16) for those functions \( P_i(z), Q_i(z) \) which arise from the \( p_k(z), q_k(z) \) by the substitutions (15). A necessary and sufficient condition is that both integrands only differ by the total derivative of an otherwise arbitrary function \( \phi(q_k, Q_k; z) \), because for fixed limits the integral

\[
\int_{z_1}^{z_2} \frac{d\phi}{dz} \, dz = \phi(z_2) - \phi(z_1)
\]

is a constant which does not influence the extreme of the integral. \( z_1 \)

\[
\frac{1}{k} \sum_{k=1}^{\infty} p_k q_k^* - \bar{H} = \frac{1}{k} \sum_{k=1}^{\infty} p_k Q_k^* - K + \frac{d\phi}{dz} \quad (3.18)
\]

together with:

\[
\frac{d\phi}{dz} = \frac{1}{k} \sum_{k=1}^{\infty} \left( \frac{\partial \phi}{\partial q_k} q_k^* + \frac{\partial \phi}{\partial Q_k} Q_k^* \right) + \frac{\partial \phi}{\partial z} \quad (3.19)
\]

gives by equating coefficients of \( q_k^* \) and \( Q_k^* \):

\[
p_k = \frac{\partial \phi}{\partial q_k}, \quad p_k = -\frac{\partial \phi}{\partial q_k}, \quad K = \bar{H} + \frac{\partial \phi}{\partial z} \quad (3.20)
\]

The second set of equations is solved for \( q_k = q_k(P_i^*, Q_i^*; z) \); these are inserted in the first set to give \( p_k = p_k(P_i^*, Q_i^*; z) \); with both the new Hamiltonian \( K(P_i^*, Q_i^*; z) \) is found.

There is much greater freedom to choose on which variables the generating function \( \phi \) of the canonical transformations should depend. In fact, the most general result is:

Let \( \phi(x_k^*, X_k; z) \) be an arbitrary function of the \( 2f + 1 \) variables \( x_k^*, X_k, z : x_k (k = 1, 2, \ldots, f) \) being any of the \( p_k, p_k \); \( X_k \) being any of the \( P_k, Q_k \). Then:

\[
y_k = -\frac{\partial \phi}{\partial x_k^*}, \quad Y_k = \frac{\partial \phi}{\partial X_k} \quad (3.21)
\]

is a canonical transformation, \( y_k \) is canonically conjugate to \( x_k^* \), \( Y_k \) to \( X_k \). Take the upper (lower) sign when deriving with respect to a coordinate (momentum). Note that this definition of canonical transformations is more general than those given generally in textbooks;
there are listed only restricted definitions involving only two complete sets among the following four sets: old and new coordinates, old and new momenta. Whereas according to the above definition the generating function \( \phi \) may depend on some of the old and some of the new coordinates, and on some of the old and some of the new momenta. There is only one interdiction: \( \phi \) must not depend simultaneously on a pair of canonically conjugate variables, say, for example, on \( p_1 \) and \( q_1 \) (or \( P_1 \) and \( Q_1 \)).

There are two other important sets of conditions which permit to check if a set of transformations (15) is canonical. The first one uses the Poisson brackets defined as:

\[
[u,v] = \sum_{k=1}^{f} \left( \frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right)
\]

\( u,v \) may be identified with any of the new variables \( Q_i = Q_i(p_i,q_i;z) \), \( P_i = P_i(p_i,q_i;z) \). \( q_k \) and \( p_k \) are the canonically conjugate variables of eqs (7) and (9). The above definition differs from the corresponding one in the \( t \)-scheme inasmuch as the partial derivations with respect to \( z \) and \( p_z \) are replaced with derivations with respect to \( t \) and \( p_t \) (or \( \phi \) and \( p_{\phi} \)).

A set of transformation is canonical if and only if the following conditions are fulfilled:

\[
\begin{align*}
\left[ Q_i, Q_j \right] &= \left[ P_i, P_j \right] = 0 \\
\left[ Q_i, P_j \right] &= \delta_{ij}
\end{align*}
\]

\( \delta_{ij} \) is the Kronecker symbol

With

\[
(u,v) = \sum_{k=1}^{f} \left( \frac{\partial q_k}{\partial u} \frac{\partial p_k}{\partial v} - \frac{\partial q_k}{\partial v} \frac{\partial p_k}{\partial u} \right)
\]

(with the \( u,v \) again of any of the \( Q_i \)'s and \( P_i \)'s) the second set of conditions for canonical transformations may be stated which read as (23) except that the square brackets must be exchanged against the above curly ones.

The three sets of conditions: i) existence of a generating function, ii) The Poisson brackets, iii) the Lagrange brackets imply each other. If any one set is fulfilled, the two remaining ones are also guaranteed. The bracket relations are very useful if a certain canonical transformation rendering a wanted effect is looked for. There is no general rule how to find such a transformation; it must be found by trial and error, or by educated guesses. If there are guesses for some (or even all) new variables, the bracket relations may be used to check whether they are canonical and compatible. Unfortunately, this procedure may be very tedious if \( f \), the number of degrees of freedom is greater than unity. It is more convenient to assume a generating function, but it may be rather difficult to set up a function generating canonical transformations producing a wanted result.
The Lagrange and Poisson brackets are invariant against canonical transformations. For example:

\[
(u,v)_{p,q} = \sum_{k=1}^{f} \left( \frac{\partial q_k}{\partial u} \frac{\partial p_k}{\partial v} - \frac{\partial p_k}{\partial u} \frac{\partial q_k}{\partial v} \right) = (3.25)
\]

\[
(u,v)_{p,Q} = \sum_{i=1}^{f} \left( \frac{\partial Q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial Q_i}{\partial v} \right)
\]

for arbitrary u and v, provided the \( p_k, q_k \)'s and \( P_i, Q_i \)'s are connected by a canonical transformation. The brackets are also constant along the trajectory of a particle, since the change in coordinates generated by the motion may be described by a canonical transformation: With the independent variable z increasing say from \( z_1 \) to \( z_2 \), the particle advances in its motion. The dependent particle coordinates corresponding to the states at \( z_1, z_2 \) resp. are related by a canonical transformation and may be identified with the \( p, q \)'s and \( P, Q \)'s in eq. (25).

The above brackets are called differential invariants. The Poincaré invariants are integrals which are invariant. There are f of them. Here the discussions are restricted to the two most important ones. The first is:

\[
J_1 = \int_{S} \left( \sum_{k=1}^{f} \frac{\partial p_k}{\partial q_k} \right) \, dq_k = \int_{S} \left( \sum_{i=1}^{f} \frac{\partial P_i}{\partial Q_i} \right) \, dQ_i = (3.26)
\]

S is an arbitrary surface in the 2f-dimensional \( p, q \) phase space. On account of this arbitrariness of S these integrals are called absolute integral invariants. The meaning of \( J_1 \) becomes clearer if two parameters \( u, v \) are introduced which describe the surface S:

\[
p_k = p_k(u,v;z) \quad q_k = q_k(u,v;z)
\]

The variables arising from the canonical transformations are:

\[
P_i = P_i(u,v;z) \quad Q_i = Q_i(u,v;z)
\]

Introducing these parameters into eq. (26) gives:

\[
J_1 = \int_{S} \left( \sum_{k=1}^{f} \frac{\partial (p_k, q_k)}{\partial (u,v)} \right) \, du \, dv = \int_{S} (u,v)_{p,q} \, du \, dv = (3.27)
\]

\[
= \int_{S} (u,v)_{p,Q} \, du \, dv
\]

The Jacobian \( \frac{\partial (p_k, q_k)}{\partial (u,v)} \) equals the Lagrange bracket \( (u,v)_{p,q} \), eq. (24). The Lagrange
bracket on the right hand side contains the derivatives of the P's and Q's with respect to u and v. The invariance of the Poincaré invariants follows from that of the Lagrange brackets. The discussion of these integral invariants is continued in chapter 11 where they are employed in an investigation on the conservation of phase space area.

The invariance of the integral:

$$J_f = \int \ldots \int \frac{dp_1 \ldots dp_f}{V} dp_1 \ldots dp_f$$  \hspace{1cm} (3.28)

where V is an arbitrary volume of the 2f-dimensional phase space, is equivalent to Liouville’s theorem.

4. SPECIALIZATION TO AXIAL SYMMETRY

In most investigations of beam dynamics in a linac gap deviations from axial symmetry are neglected. Then there are no forces acting in the azimuthal direction, and planar motion in the r,z-plane can be assumed. Results of the preceding chapter are specialized to this symmetry and expressed in circularly cylindrical coordinates r, θ, z:

The Lagrangian is:

$$L = \frac{m}{2} \dot{r}^2 + \frac{1}{2} \omega^2 \dot{\phi}^2 + eA_z - eU \dot{r}$$  \hspace{1cm} (4.1)

In view of the axial symmetry A_r and A_θ are zero; and the azimuth θ and its canonically conjugate momentum can be neglected. The other canonical momenta are:

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{3L}{2} = m \frac{\dot{r}}{\dot{r}} = mr^2/\dot{r}$$  \hspace{1cm} (4.2)

$$p_t = \frac{\partial L}{\partial \dot{t}} = -E = -\frac{m}{2} \left[ \frac{r^2}{\dot{r}^2} + \frac{1}{\dot{r}^2} \right] - eU$$  \hspace{1cm} (4.3)

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = -\frac{E}{\omega} = -\frac{m}{2} \left[ \frac{r^2}{\omega^2} + \frac{1}{\dot{r}^2} \right] - \frac{eU}{\omega}$$  \hspace{1cm} (4.4)

These equations are solved for t^* and ϕ^* respectively:

$$t^* = \sqrt{m/2} \left[ -p_t - eU - p_r^2/(2m) \right]^{-\frac{1}{2}}$$  \hspace{1cm} (4.5)

$$\phi^* = \omega \sqrt{m/2} \left[ -\omega p_\phi - eU - p_r^2/(2m) \right]^{-\frac{1}{2}}$$  \hspace{1cm} (4.6)

Again the branch of the square root belonging to motion in the positive z-direction has been selected.
The Hamiltonian is:

\[ \mathcal{H}(p_\phi, p_\tau, q_\phi, q_\tau, z) = p_\phi \dot{q}_\phi + p_\tau \dot{q}_\tau - \mathcal{L} \]

\[ = -\sqrt{2m} \left[ -\omega p_\phi - eU(z, r, \phi) - p_\phi^2/(2m) \right]^{1/2} - eA_z \]  

(4.7)

The special form of the Poisson brackets, eq. (3.22), is:

\[ [u, v] = \frac{\partial u}{\partial \phi} \frac{\partial v}{\partial p_\tau} - \frac{\partial u}{\partial p_\tau} \frac{\partial v}{\partial \phi} + \frac{\partial u}{\partial \phi} \frac{\partial v}{\partial \phi} - \frac{\partial u}{\partial q_\phi} \frac{\partial v}{\partial q_\phi} \]  

(4.8)

The first Poincaré invariant, eq. (3.26), is:

\[ J_1 = \iint_S (dp_\phi \, dq_\phi + dp_\tau \, dq_\tau) = \iint_S (dp_\phi \, dq_\phi + dp_\tau \, dq_\tau) \]  

(4.9)

5. EIGHT-DIMENSIONAL DESCRIPTION OF HAMILTONIAN MECHANICS

In the preceding chapters various dynamical equations have been presented for either time, \( t \), or the longitudinal space coordinate, \( z \), as the independent variable. There emerged certain hints to a rather symmetrical structure of Hamiltonian mechanics. This can be displayed more clearly if all coordinates, the three space coordinates and time and their canonically conjugate momenta, are treated on an equal footing. It will turn out that among the four pairs \( x, p_x \); \( y, p_y \); \( z, p_z \); \( t, p_t \) = \(-E (E \) is total energy) any variable can be chosen as the independent one; its canonically conjugate partner is the Hamiltonian; the three remaining pairs are the set of canonically conjugate dependent variables. However, this symmetry appears only if the Lagrangian has been chosen properly.

This eight-dimensional version of Hamiltonian mechanics has been developed by mathematicians for a rather abstract discussion of the general mathematical structure of mechanics\(^9\). It is hoped that this presentation is somewhat more accessible to accelerator physicists.

5.1. Non relativistic case

Hamilton's principle, eq. (2.1), is again the starting point. The Lagrangian is assumed as:

\[ L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z, t) \]

(5.1)

\( V \) is potential energy which must not depend on the velocities. It is assumed that time, \( t \), too, is a dependent variable, and a new independent variable, \( \lambda \), is introduced by the substitution:

\[ \text{...} \]
\[ \dot{x}_i = x_i(t) = x_i(t(\lambda)) \quad x_i = x_i(\lambda) \quad \dot{x}_i = \frac{dx_i}{dt} = \frac{(dx_i/d\lambda)/(d\lambda/dt)}{t'} = x_i'/t' \]

\( \lambda \) may be identified with any of the variables \( t, x_i \ldots \) or with any function of them, as for example arc length along the trajectory; but its special meaning is of no importance in the present discussion. Hamilton's principle, eq. (2.1), is transformed into:

\[ \delta \int \Lambda(x_\alpha, x'_\alpha) \, d\lambda = 0 \]

with the Lagrangian:

\[ \Lambda(x_\alpha, x'_\alpha) = \frac{m}{2}(x'^2 + y'^2 + z'^2)/t' - V(x,y,z,t) \quad t' \]

Four canonical momenta are defined by:

\[ p_i = \frac{\partial \Lambda}{\partial x'_i} = mx'_i/t' = mx'_i = \frac{3L}{9} \]

\[ p_0 = \frac{\partial \Lambda}{\partial t'} = - \frac{m}{2}(x'^2 + y'^2 + z'^2)/t'^2 - V = - E \]

The spatial momenta, \( p_i \), agree with the three components of common momentum, \( \vec{p} \), negative energy is the momentum canonically conjugate to time, \( t \).

It might be tempting to try to arrive at a generalization of the canonical equations of motion, eqs. (2.10), by replacing in the variational principle (2.9) the sum \[ \sum_{k=1}^3 p_k q_k \]

with \[ \sum_{\alpha=0}^3 p_\alpha x_\alpha \] and by introducing a generalized Hamiltonian \[ H = \sum_{\alpha=0}^3 p_\alpha x'_\alpha - \Lambda. \]

However, such a transformation is principally impossible in variational problems where the Lagrangian is a homogeneous function of degree one in the derivatives \( x'_{\alpha} \):

\[ \Lambda(x_\alpha, kx'_\alpha) = k \Lambda(x_\alpha, x'_\alpha) \]

as is the case with the Lagrangian (4). This follows simply from the identity:

\[ \sum_{\alpha=0}^3 p_\alpha x'_\alpha - \Lambda = \sum_{\alpha=0}^3 \frac{3\Lambda}{9x'_\alpha} x'_\alpha - \Lambda \equiv 0 \]

---

In Chapter 5 the prime denotes derivation with respect to the parameter \( \lambda \). In section 5.1 Latin subscripts may assume the values 1,2,3; Greek ones the values 0,1,2,3.
which is a consequence of the homogeneity of \( \Lambda \), eq (6) and may be verified from (4) and (5). The number of variables is greater than that of the degrees of freedom; they cannot be independent. This is the source of the present trouble.

A different approach must be adopted. The parameter \( \lambda \) is introduced as the independent variable into equation (2 8):

\[
\delta \left[ \sum_{k=1}^{3} p_k \dot{x}_k - H(p_k, x_k; t) \right] \quad \delta \lambda =
\]

\[
\delta \left[ \sum_{k=1}^{3} p_k x'_k - H(p_k, x'_k; t') \right] \quad \delta \lambda =
\]

(5 8)

The spatial canonical momenta (5a) agree with those defined in (2 7). The Hamiltonian \( H \) is defined in eq (2 9). The form of the integrand in eq (8) suggests to introduce:

\[
t = x \quad t' = x' \quad - H(p_k, x_k; t) = - E = p_o
\]

(5 9)

This definition is consistent with (5b), since the Hamiltonian defined by (2 9) and belonging to the Lagrangian (1) equals total energy. Eqs (8) and (9) are rewritten as:

\[
\delta \left[ \sum_{\alpha=0}^{3} p_\alpha x'_\alpha \right] \quad \delta \lambda = \delta \left[ \sum_{\alpha=0}^{3} p_\alpha \right] \quad \delta \lambda = 0
\]

(5 8a)

\[
0 \equiv H(p_\alpha, x_\alpha) \equiv p_o + H(p_k, x_k; t)
\]

(5 9a)

\[
= p_o + \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(x, y, z, t)
\]

This is now a variational problem with a constraint which is called the energy equation. This side condition may be taken into account with the help of a Lagrangian multiplier \( \mu \). The Euler equations are:

\[
\frac{1}{\mu} \frac{dp_\alpha}{d\lambda} = - \frac{\partial H}{\partial x_\alpha} \quad \frac{1}{\mu} \frac{dx_\alpha}{d\lambda} = \frac{\partial H}{\partial p_\alpha}
\]

(5 10)

In view of (9a) these equations are rewritten:

\[
\frac{1}{\mu} \frac{dx_o}{d\lambda} = \frac{1}{\mu} \frac{dt}{d\lambda} = \frac{\partial H}{\partial p_o} = 1 \rightarrow \frac{d\lambda}{dt} = \frac{dt}{\mu}
\]

(5 11a)

\[
\frac{dp_o}{dt} = - \frac{dH}{dt} = - \frac{\partial H}{\partial x_o} = - \frac{\partial H}{\partial t}
\]

(5 11b)

\[
\frac{dp_k}{dt} = - \frac{\partial H}{\partial q_k} = - \frac{\partial H}{\partial q_k} \quad \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} = \frac{\partial H}{\partial p_k}
\]

(5 12)
From \( \alpha \) as given in eq (9a) it follows that time, \( t \), is the independent variable and (12) are Hamilton's equations in the \( t \)-scheme. It is not at all surprising that they come out again, since the variational principle (2 8) has been the starting point for the present considerations.

But \( \alpha \) may be cast into another form which is equivalent to eq (9a) as far as physics is concerned but which gives a different analytical description. Solving eq (9a) for \( p_z = p_5 \) gives:

\[
0 \equiv \alpha^* (p_\alpha, x_\alpha) \equiv p_3 \sqrt{2m(p_0 + V(x,y,z,t))} - \frac{p_1^2}{2} - p_z^2
\]

\[
= p_z^* \tilde{H}(x_\alpha, p_\alpha, p_5 t + p_0 = -E; x, y, z; t) \quad (5.13)
\]

The Euler equations of the variational problem (8a) with the constraint (13) give:

\[
\frac{1}{\mu} \frac{d\alpha_3}{d\lambda} = \frac{1}{\mu} \frac{dz}{d\lambda} = \frac{\partial \alpha}{\partial p_5} = 1 \rightarrow \frac{d\lambda}{dz} = 1 \quad (5.14a)
\]

\[
\frac{1}{\mu} \frac{dp_3}{d\lambda} = \frac{dp_z}{dz} = \frac{\partial \alpha}{\partial x_5} = \frac{\partial H}{\partial z} \quad (5.14b)
\]

\[
\frac{dp_k}{dz} = -\frac{\partial \alpha}{\partial q_k} = -\frac{\partial H}{\partial p_k} \quad \frac{dx_k}{dz} = \frac{\partial \alpha^*}{\partial p_k} = \frac{\partial H}{\partial p_k} \quad (5.15)
\]

\[k = 0, 1, 2\]

By the change from \( \alpha \) to \( \alpha^* \) it has been achieved that now \( z \) is the independent variable in Hamilton's equations; eqs (15) agree with eqs (3 13). The Hamiltonian, \( \tilde{H} \), as defined by (13) corresponds to that given in eq (3 11), provided the difference in the Lagrangian (cf eqs (1) and (2 2)) are taken into account.

The symmetry of the theory in the eight variables stated at the beginning of the chapter is now fairly obvious. If the energy equation, \( \alpha = 0 \), is linear in any one of the variables \( x_\alpha \), \( p_\alpha \) then this is equal to the Hamiltonian, cf eqs (9a) and (13), and its canonically conjugate partner becomes the independent variable. Of course, there is no rule, that the independent variable must coincide with any one of the above variables; for example, it may be identified with arc length and the energy equation may be chosen accordingly.

5.2 Relativistic Case

The relativistic theory is discussed by SYNGE \(^9\) and SAUER \(^10\). It may start from Hamilton's principle, eq (2 1), with the Lagrangian:

\[
L = \rho_0 c^2/\gamma + e(V \vec{A}) - eU \quad (5.16)
\]
where $m_0$ is the particle rest mass, $e$ its charge and where

$$\beta = \frac{v}{c} \quad \gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}}$$

$\vec{A}$ and $U$ are vector and scalar potential, see eq (2.3) The Lagrangian is chosen such that the Euler equations of (2.1) (i.e. Lagrange's equations of the second kind, eq (2.6)) give the common relativistic equations of motion:

$$\frac{d(m_0\gamma \vec{v})}{dt} = e\vec{E} + e(\vec{v} \times \vec{B})$$

Minkowskian four-vectors are introduced \(^1\) The position-time vector:

$$\vec{X} = (x_1, x_2, x_3, x_4) \quad x_4 = ix_0 = ict$$

derived with respect to proper time $\tau$,

$$d\tau = dt/\gamma$$

gives the four-velocity:

$$\vec{U} = \frac{d\vec{X}}{d\tau} = (\gamma \vec{v}, ic) \quad u_a u_a = \frac{dx_a}{d\tau} \frac{dx_a}{d\tau} = -c^2$$

The potentials are combined into the four-potential:

$$\vec{A} = (\vec{A}, A_4) \quad A_4 = iA_0 = iU/c$$

With these quantities Hamilton's principle containing the Lagrangian (16) is rewritten:

$$\delta \int dt \left[ -m_0 \gamma \sqrt{-u_a u_a} + eA_a u_a \right] = 0$$

Now the unspecified parameter $\lambda$ is introduced as the independent variable:

$$\tau = \tau(\lambda) \quad d\tau = \tau' d\lambda$$

With this Hamilton's principle, eq (22), assumes the shape given in eq (3) with the Lagrangian:

$$\mathcal{L} = -m_0 \gamma \sqrt{-\dot{X}_a \dot{X}_a} + eA_a \dot{X}_a$$

\(^1\) Repeated Latin subscripts must be summed over 1,2,3; Greek ones from 1 to 4
\[ p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{x}_\alpha} = \frac{m_0 c \ u_\alpha}{\sqrt{u_\alpha^2}} + e \ A_\alpha \]  
\[ p_k = m_0 c \ v_k + e \ A_k = \frac{\partial \mathcal{L}}{\partial \dot{x}_k} \]  
\[ p_4 = \frac{i}{c} \left[ m_0 c^2 \gamma + eU \right] = \frac{i}{c} E/c \]  

The fourth component, \( p_4 \), equals total energy (including the particle rest mass \( m_0 c^2 \)) apart from the unessential factor \( i/c \)

\( \lambda \) is homogeneous of degree one in the \( x_\alpha \) and it fulfills eqs (6) and (7) Therefore the derivation of Hamiltonian's equations in the extended (four-dimensional) form must follow the lines of eqs (8) and (9) From (24a) and (29) the Hamiltonian is found:

\[ H = c/\sqrt{m_0^2 c^2 + (p_k - eA_k)^2} + eU = E \]  

Relativistic mechanics is determined by Hamilton's principle eqs (8) and (8a) with the constraint:

\[ 0 \equiv \nabla(p_\alpha, x_\alpha) = ip_4 + H/c \]  

\[ = \sqrt{m_0^2 c^2 + (p_k - eA_k)^2} + i(p_4 - eA_4) \]  

By the derivation time, \( \tau \), is the independent variable This can be proved as in eq (11) The energy equation, (26), may be written in a more symmetrical way as:

\[ 0 \equiv 2 \ \Omega^+(p_\alpha, x_\alpha) = m_0^2 c^2 + (p_\alpha - eA_\alpha)(p_\alpha - eA_\alpha) \]  

For this \( \Omega^+ \) the parameter \( \lambda \) is proportional to proper time, \( \tau \) From the one set of Hamilton's equations:

\[ \frac{1}{\mu} \frac{dx_\alpha}{d\lambda} = \frac{\partial \Omega^+}{\partial p_\alpha} = p_\alpha - eA_\alpha \]  

and from (27):

\[ \frac{1}{\mu^2} \ \frac{dx_\alpha}{d\lambda} \ \frac{dx_\alpha}{d\lambda} = (p_\alpha - eA_\alpha)(p_\alpha - eA_\alpha) = -m_0^2 c^2 \]  

it follows by comparison with (20):

\[ m_0 \mu \ dl = d\tau \]  

(5 24)

(5 24a)

(5 24b)

(5 25)

(5 26)

(5 27)

(5 28)

(5 29)

(5 30)
The energy equation, (27) may be brought to a form where it is linear in a spatial momentum so that a space coordinate, e.g. \( z \), is the independent variable, and its canonical momentum is the Hamiltonian. Again there is realized the general symmetry of coordinates described at the beginning.

5.3 The Symmetry arising from the Lagrangian in Chapter 3

The Lagrangian (2.2) is somewhat asymmetrical, inasmuch as it contains the vector potential which is a typically relativistic quantity, and the non-relativistic expression for kinetic energy, \( mv^2/2 \). This unifying combination is the reason why the symmetry of the four pairs of canonically conjugate variables is not completely realized in this case. In the \( t \)-scheme the Hamiltonian, \( H \), eq (2.9):

\[
H = \frac{1}{2m} (p - eA)^2 + eU
\]

differs from \( p_t = -E \), eq (3.8). But in the \( z \)-scheme the Hamiltonian, \( \tilde{H} \), equals (up to the sign) the momentum \( p_z \).
PART II: CANONICAL THIN LENS APPROXIMATION FOR AN ACCELERATING GAP

6 INTRODUCTION

A linear accelerator of the Alvarez structure type consists of a cavity along whose axis drift tubes are aligned. They are designed to screen the protons against the radio frequency field in the cavity during about three-quarters of each period. In the remaining quarter protons pass through the gaps separating the drift tubes and are accelerated. Considering the successive gaps as really independent, and separated by drift tube space where the radio frequency field is practically zero, PANOFSKY 1) derived an expression for energy gain of a particle in such a gap as if it were reduced to its median plane. The corresponding equation for phase has been discussed by J. S. BELL 2). The method has been extended to include transverse motion by SWENSON 11) and by LAPOSTOLLE 12,14). According to this method called the thin lens approximation, the motion can be computed in a way similar to beam optics with thin lenses, i.e. at each gap centre the particle receives a longitudinal and radial kick (and displacement), and thereafter it drifts freely to the next gap centre.

Difference equations giving the increments of the particle coordinates across this thin lens have been derived 4) 12,13) The question has been raised whether there exists a thin lens Hamiltonian depending only upon the particle coordinates at the gap centre and on quantities characterizing the gap (the transit time factor) from which these increments can be derived. In this paper it is shown how such a thin lens Hamiltonian can be found. For this purpose a version of Hamiltonian mechanics presented in part I is used where the longitudinal space coordinate, z, is the independent variable in place of time, t. This has been worked out in Part I. The corresponding pairs of canonically conjugate dependent variables describing particle motion are transversal position and momentum, phase angle $\phi = \omega t$ and $p_\phi = -E/\omega$. $\omega$ is the angular frequency of the accelerating radio frequency field and E is total energy. The essential step in the derivation of the thin lens Hamiltonian is the introduction of new canonically conjugate variables, P and Q which are all reduced ones. By definition, reduced variables are constant if there is no field acting on the particle. Their change across the gap equals the integrated effect of the gap field on a particle crossing it. The corresponding Hamiltonian, $K(P,Q;z)$, is zero if there is no field. The equations of motion (Hamilton's equations) are solved by iterations starting from free particle motion where the $P$'s and $Q$'s are constant, $P_0$ and $Q_0$, and equal to their mid-gap values. In the first iteration the arguments of the Hamiltonian, K, are replaced with the constants $P_0$ and $Q_0$; and similarly the partial derivations:

$$\frac{dQ(1)}{dz} = -\frac{\partial K(P,Q;z)}{\partial P} = -\frac{\partial K(P_0,Q_0;z)}{\partial P_0}$$

$$\frac{dP(1)}{dz} = \frac{\partial K(P,Q;z)}{\partial Q} = \frac{\partial K(P_0,Q_0;z)}{\partial Q_0}$$

(6.1)
Integrating these equations with respect to \( z \) across the gap gives the increments of particle coordinates across the thin lens. On the right hand side this may be interchanged with the partial derivations and the integral \( \int K(p_0, q_0; z) \, dz \) gives the wanted thin lens Hamiltonian \( H_{TL} \).

In chapter 7 it is treated the one-dimensional problem of particle motion in a uniform field. It is easy to find several canonical transformations leading to reduced variables and to a corresponding thin lens Hamiltonian. Among these are \( p = -\dot{T}/\omega \) \((T = \text{kinetic energy})\) and reduced phase \( \dot{\phi} = \dot{\phi} - z \, \ddot{\phi}/dz \) which lead to the modified Panofsky equations as given by PROME 3). A CARNE et al 4) In chapter 8 are discussed the general principles of the derivation of a thin lens Hamiltonian. These are applied to a realistic gap model in chapter 9. This task is more difficult, and the dependence of the field on radius, \( r \), must be approximated by a Taylor's series. If higher powers of \( r \) than the first one are neglected, it is possible to find reduced variables by exact canonical transformations, but the difference equations for transversal coordinates no longer depend on the mid-gap values of these quantities. If also terms quadratic in \( r \) are retained in the Taylor's series of the field, only approximately canonically conjugate reduced variables can be found; they do not fulfill exactly the Poisson brackets, but the deviations are quadratic in the field. The corresponding thin lens Hamiltonian is derived in both cases. The method how the reduced variables have been found by solving the Poisson brackets, is described in chapter 10 and in the Appendix.

It is important to note that the reduced longitudinal coordinates, reduced phase angle, \( \dot{\phi} = \dot{\phi} - z \, \ddot{\phi}/dz \), and \( -\dot{T}/\omega \) \((T = m \dot{z}^2/2 = \text{longitudinal kinetic energy})\) agree with the coordinates used in the practical treatment of proton linac beam dynamics 13). On the contrary, the only reduced radial variables which lead to a thin lens Hamiltonian:

\[
\begin{align*}
\dot{p}_r &= p_r + \\
\dot{q}_r &= q_r - p_r (\phi + \phi_0)/(m\omega) + \tag{6.2}
\end{align*}
\]

\((\phi_0 \text{ is a phase constant, and the dots denote terms involving the field which are not important here}),\) differ from the radial variables, \( \dot{r} = dr/dz \) and \( \ddot{r} = r - z \, dr/dz \), used in the common difference equations. For this reason it is impossible to give a thin lens Hamiltonian for the set of difference equations commonly used, e.g. ref 13), which concern \( \dot{r} \) and \( \ddot{r} \). In chapter 11 it is shown that \( \dot{r} \) and \( r \) are not compatible as canonically conjugate variables, neither among themselves nor with the longitudinal coordinates \( \dot{\phi} \) and \( -\dot{T}/\omega \). The non-conservation of area (phase space density) in the 2-dimensional \( \dot{r}, \ddot{r} \) "phase" space is discussed which results from the fact that \( \dot{r} \) and \( \ddot{r} \) are not canonical variables and that longitudinal and radial motion are coupled. The Jacobian between the radial output and input coordinates across a linac gap is calculated. It differs from unity by terms linear in the field. This is at variance with SWENSON's result 14). The reason for this discrepancy is indicated.
7 ONE-DIMENSIONAL PROBLEM

7.1 Energy and Phase as Dependent Variables

The simplest model of an accelerating gap is the uniform field of a plane condensor whose plates are connected to a source with harmonically alternating voltage. There is no force acting in the transversal direction and transversal variables, $r$ and $p_r$, are neglected in the formulae of chapter 4. The subscript $\phi$ of $p_\phi = p = -E/\omega$ is dropped. Such a uniform time harmonic field can be derived from a scalar potential:

$$E_z = -\frac{\partial U}{\partial z}, \quad U(z, \phi) = -E_1 z \cos(\phi + \phi_0) \quad (7.1)$$

$\phi = \omega t$, $E_1 = V_0/g$ is the peak electric field strength; $V_0$ is the voltage amplitude, $g$ is gap length, $\phi_0$ is a phase constant. The Hamiltonian is found by specializing eq (4.7):

$$H = -\sqrt{2m} \left[ -\omega p - eu \right]^{1/2} \quad (7.2)$$

Hamilton's equations are:

$$p' = \frac{dp}{dz} = -\sqrt{\frac{2m}{\omega}} \left[ -\omega p - eu \right]^{-1/2} e^{\partial U_{\phi}} = -\sqrt{\frac{2m}{\omega}} e^{\partial U_{\phi}} \quad (7.3a)$$

$$\phi' = \frac{d\phi}{dz} = \frac{\omega}{\sqrt{2m}} \left[ -\omega p - eu \right]^{-1/2} = \omega \sqrt{\frac{m}{2\hbar}} \quad (7.3b)$$

$T = m \frac{z^2}{2}$ is (longitudinal) kinetic energy. Initial conditions are given at the centre:

$$z = 0: \begin{cases} p = -E/\omega = -\frac{W}{\omega} = -m\frac{z_0^2}{2\omega} = -m\omega/(2k^2) \\ \phi = 0 \end{cases} \quad (7.4a)$$

$W$ is kinetic energy of the proton in the centre, $z_0$ is the corresponding velocity, and $k = \omega z_0$. It is not possible to give an exact solution of eqs (3) in closed form. This is shown by the following argument: The equation of motion in Newtonian form corresponding to (3) can be solved exactly and gives (13), eq (31 3a); for $\kappa$ see eq. (13)):

$$kz(\phi) = \phi - \kappa \left[ \sin\phi_0 + \cos(\phi + \phi_0) - \cos\phi_0 \right]$$

This transcendental equation cannot be solved in closed form for a given $z$. But such a solution would be an exact solution of eq (3b).

Approximate solutions may be found by iterations starting from free particle motion where the field is zero ($U \equiv 0$). Eq (3a) then gives together with eq. (4a):

$$p'(0) = 0, \quad p(0) = -m\omega/(2k^2) \quad (7.5a)$$
With this the solution of (3b) is found:

$$\phi'(o) = k = \omega/\lambda_0 \quad \phi(o)(z) = kz$$  \hspace{1cm} (7.5b)

Inserting again these solutions into the right hand sides of (3) the procedure may be continued to give higher order solutions. This is not done since the transformed equations derived below are more convenient for this purpose. Note that the equations of motion (3a) and (3b) are asymmetric in the field. This is the reason why the zero order solutions \( p(o) \) is a constant, while \( \phi(o) \) is variable.

7.2 Reduced Phase and Kinetic Energy

The Thin Lens Hamiltonian

Now new variables are introduced by canonical transformations. They are assumed as:

$$P = -\frac{T}{\omega} = p + eU/\omega$$  \hspace{1cm} (7.6a)

$$Q = \phi + \varphi_o - \omega/\sqrt{m}(2\omega P)$$

$$= \phi + \varphi_o - \omega \frac{ds}{dz}$$  \hspace{1cm} (7.6b)

In defining \( Q \) eq. (3b) is used for \( ds/dz \). The main reason for this choice of variables is that similar ones are used in the thin lens approximation and have been used by Prange, 3, 4, to derive a thin lens Hamiltonian. The choice is admissible; \( Q \) and \( P \) fulfill the Poisson bracket relation (cf. eqs (3.22), (3.23) and (4.8)):

$$[Q, P] = 1$$  \hspace{1cm} (7.7)

From \( \dot{\phi} = -\partial\phi/\partial p \) and \( Q = \partial\phi/\partial z \), (cf eq. (3.21)), the generating function is found:

$$\phi(p, P; z) = z/\omega^2 P^2 + \phi_o p + (P-p) \arccos(\omega(P-p)/(eE_1 z))$$

$$+ \frac{1}{2} \left[ \frac{(eE_1 z/\omega)^2}{(P-p)^2} - (P-p)^2 \right]^{1/2}$$  \hspace{1cm} (7.8)

With this the new Hamiltonian is found, \( K = H + \partial\phi/\partial z \):

$$K(P, Q; z) = (eE_1/\omega) \sin( Q + \omega z/\sqrt{m}(2\omega P) )$$  \hspace{1cm} (7.9)

$$\frac{dQ}{dz} = \frac{\partial K(P, Q; z)}{\partial P} \quad \frac{dP}{dz} = -\frac{\partial K(P, Q; z)}{\partial Q}$$  \hspace{1cm} (7.10)

The Hamiltonian (9) and Hamilton's equations (10) are all proportional to the field (-E_1). Therefore they are more suitable than (2) and (3) for approximate solutions which are power series in the field. For free particle motion, \( E_1 = 0 \), the above equations reduce to:
\[ K_{(0)} = 0 \]
\[ Q_{(0)} = 0 \quad Q_{(0)} = Q_o = \phi_o = \text{const} \]  
\[ P_{(0)} = 0 \quad P_{(0)} = P_o = -\frac{W}{\omega} = \text{const} \]  

Inserting \( P_o \) and \( Q_o \) in the right hand sides of (10) and integrating with respect to \( z \) gives the next approximation:

\[ Q_{(1)} = Q_{(0)} + \kappa Q_{(1)} \]
\[ = \phi_o + \kappa \left[ kz \sin(kz + \phi_o) + \cos(kz + \phi_o) - \cos \phi_o \right] \]  
\[ P_{(1)} = P_{(0)} + \kappa P_{(1)} \]
\[ = -\frac{W}{\omega} + \kappa \frac{W}{\omega} Z \left[ \sin(kz + \phi_o) - \sin \phi_o \right] \]  

This method leads to a direct way to the perturbation parameter:

\[ \kappa = \frac{eE_f}{(m_0^2 \omega)} = \frac{(eE_f/\omega)}{(m_0^2 \omega)} \]  

It represents the ratio: impact by the radiofrequency field (of strength \( E_f \)) exerted during one period (~1/\( \omega \)) divided by free particle momentum, \( m_0^2 \omega \kappa < 0 \), in a proton machine above 0.5 MeV. This parameter has also been very useful in the treatment of proton motion in realistic gap fields, cf ref 13), chapter 5.

Eq (1) describes a uniform electric field not limited in its extension. The finite width of the gap is accounted for by integrating equations (10) from \( z = -g/2 \) to \( z = g/2 \). To the first order in the field this gives:

\[ \Delta P = -\frac{1}{\omega} (W_+ - W_-) = -g/2 \]
\[ \Delta Q = \Delta \phi = Q_+ - Q_- = \frac{g}{2} \]

\[ \Delta P = -\frac{\partial H_{TL}}{\partial Q_o} = -\frac{\partial H_{TL}}{\partial \phi_o} \]
\[ \Delta Q = \frac{\partial H_{TL}}{\partial P_o} = \frac{\partial H_{TL}}{\partial \phi_k} \]  

\[ \phi_+ - \phi_- = \frac{\partial H_{TL}}{\partial P_+} = \frac{\partial H_{TL}}{\partial \phi_k} \]

\[ \phi_o = \frac{\partial H_{TL}}{\partial \phi_o} \]

\[ \frac{\partial H_{TL}}{\partial \phi_o} \]

\[ \frac{\partial H_{TL}}{\partial \phi_k} \]

\[ \frac{\partial H_{TL}}{\partial \phi_k} \]

[*) Throughout this paper subscripts indicate the order of iteration. Superscripts denote the power of the perturbation parameter \( \kappa \) preceding the expression.\]
$H_{TL}$ is the thin lens Hamiltonian:

$$H_{TL} = \frac{g/2}{K(Q=\phi_o, P= -\frac{W}{\omega}; z) \, dz = \frac{eV_0}{\omega} T_{oh}(k) \sin \phi_o}$$

(7.16)

$$T_{oh} = \frac{\sin \left( \frac{kg}{2} \right)}{\frac{kg}{2}}$$

(7.17)

is the transit time factor for a uniform field, (13), section 3.1. Since the above equations are limited to terms linear in the field and since $K$, eq (9), is already linear in it, it is permitted to replace in $K(P,Q;z)$ the variables $P$ and $Q$ by their zero order solutions, the constants $P_0$ and $Q_0$. This also applies to the derivatives $\partial/\partial P$ and $\partial/\partial Q$. In consequence, $K(P_0,Q_0;z)$ depends on $z$ only through its explicit dependence on this variable. It can be easily integrated with respect to $z$ and this process may be interchanged with the partial derivatives. At the contrary, all these manipulations are no longer possible in the equations arising from (10) by higher order iterations.

The definitions of $P$ and $Q$, eqs (6), and the first order equations (14) and (15) are valid for any field $U(z,\phi)$; so is eq (16) defining the thin lens Hamiltonian, provided $T_{oh}$ is replaced with the general transit time factor $T_{o}(k)$, eq (9.6). This follows from the treatment of the general case in chapter 9. The general expression for the thin lens Hamiltonian is found from eqs (9.19), (9.21) by putting the radial variables equal to zero. Equations (14) and (15) are then the Modified PANOFSKY-equations for the change of reduced phase and kinetic energy, $\omega$, as derived by PRIME 3, CARNE, LAPOSTOLLE and PRIME 4.

$\Delta Q = \phi_{n+1} - \phi_n$ and $-\omega dP = W_+ - W_- = W_{n+1} - W_n$ give the amounts by which the phase, $\phi_n$, and the kinetic energy $W_n$, of the arriving proton must be increased across the thin lens equivalent to the $n$-th gap. Up to this plane the particle drifts freely, after this plane it drifts again freely with increased velocity.

PANOFSKY 1 put the right hand side of eq (15) equal to zero assuming $\phi_{n+1} = \phi_n$. For this very reason this equation taken together with (14) violated Liouville's theorem. J S BELL 2 then gave a correct set of difference equations for an accelerating gap where the validity of Liouville's theorem was assured. But he used somewhat different variables, (cf (25) to (27) ).

7.3 Further Examples of Canonical Transformations

A detailed discussion of the thin lens approximation is the subject of chapter 8. For this purpose it proves useful to present some further examples.

In principle, the canonical transformations (6) can be performed for all potentials sufficiently well behaved so that eq. (6a) can be solved everywhere for $\phi$:
\[ \omega(P - p) = eU(z, \phi) + \phi = \psi(z, P - p) \quad (7.18) \]

The generating function is:

\[ \phi(P, P; z) = \int_{P-p}^{P+p} \psi(z, \tau) \, d\tau + z \sqrt{-2m\omega P + \phi_0 P} \quad (7.19) \]

and the new Hamiltonian is:

\[ K(P, Q; z) = \int_{P-p}^{P+p} \frac{\partial \psi(z, \tau)}{\partial z} \, d\tau \quad (7.20) \]

In eq (6a) \( \psi \) must be eliminated with the help of (6b). The resulting expression is used to replace \( p \) in eq (20) with a function of the new variables.

A completely different canonical transformation starting from (2) is introduced by the generating function:

\[ \phi(q = \phi, \bar{P}; z) = (\phi + \phi_0) \bar{P} + z \sqrt{-2m\omega \bar{P}} \quad (7.21) \]

It leads again to reduced variables:

\[ \bar{Q} = \partial \phi/\partial \bar{P} = \phi + \phi_0 - z \sqrt{m/(2\omega \bar{P})} \quad (7.22a) \]

\[ \bar{P} = P = -E/\omega \quad (7.22b) \]

Eq (22b) follows from \( p = \partial \phi/\partial q = \partial \bar{P}/\partial \phi \) (cf eq (3.21)). The Hamiltonian:

\[ \tilde{K}(\bar{P}, \bar{Q}; z) = \sqrt{(-2m\omega \bar{P})} - \sqrt{(2m(-\omega \bar{P} - eU))} \quad (7.23) \]

is zero if the field is switched off, \( U = 0 \). The zero order solutions \( P_0 \) and \( Q_0 \) are constants. A thin lens Hamiltonian can be derived in the same way as in eqs (14) to (16):

\[ \tilde{H}_{\text{TL}} = \int \frac{g/2}{z} \tilde{K}(P_0, Q_0; z) \, dz = \frac{eV_0}{\omega} \left[ T_{th}(k) - \cos \left( \frac{kg}{2} \right) \sin \phi_0 \right] \quad (7.24) \]

Yet another set of variables has been used by J S Bell\(^2\), namely time, \( t \), and relativistic kinetic energy. In close analogy phase-angle, \( \phi \), and non-relativistic kinetic energy, \( T \), are used for \( Q \) and \( P \):
Q = \phi  \quad P = -T/\omega = p + eU(z,\phi)/\omega  \quad (7.25)

\phi(q = \phi, P; z) = \phi P - \frac{e}{\omega} \int_0^\phi U(z, r) \, dr  \quad (7.26)

K(P, Q; z) = -\sqrt{2m_eP} - \frac{e}{\omega} \int_0^Q \frac{\partial U(z, r)}{\partial z} \, dr  \quad (7.27)

Q = \phi \quad \text{is not a reduced variable} \quad \text{The Hamiltonian (27) does not vanish even in the absence of an external field} \quad \text{It is impossible to perform with it the manipulations leading to a thin lens Hamiltonian}

8  \text{PRINCIPLES OF THE CANONICAL THIN LENS APPROXIMATION}

The thin lens approximation as used in the theory of proton motion in an Alvarez structure linac has been proposed by PANOFSKY 1\textsuperscript{)}, extended and modified by SWENSON 11\textsuperscript{)}, LAPOSTOLLE 4\textsuperscript{)} 12\textsuperscript{)} and other authors. This name covers a mathematical prescription but not a physically realizable device. The real trajectory crossing a gap (or a whole cell) is replaced with a fictitious one consisting of three parts. For the first one the trajectory of the entering particle is extended up to the gap centre, z = 0, as if the particle moved freely. The third one starts after the plane z = 0 and describes again free particle motion determined by the real dynamical state of the particle leaving the gap (or cell). The second part consists of increments of the particle coordinates across the plane z = 0 chosen in such a way that both the total fictitious and the real trajectory give the same change of particle coordinates across the whole gap (or cell).

The term thin lens has a different meaning in optics. In ray optics a physically thin lens may be replaced with one principal plane with two focal points assigned to it. The fictitious ray across it is changed abruptly but continuously. If the lens is too thick, two principal planes must be introduced; to each one belongs one focus. The fictitious ray through the whole system is continuous. An accelerating gap is a thick lens in the sense of the preceding sentence. In opposition to the ray optical approach, the action of the accelerating time-dependent field distributed throughout the gap is described by only one plane acting upon the fictitious trajectory. In consequence, this trajectory cannot be continuous at this place. This question has been dismissed during some time but seems to be settled.\textsuperscript{)}) The thin lens increments of momentum-like co-ordinates, as e.g. \( \Delta w, \Delta p_r, \Delta r^* = \Delta (dr/dz) \) (of longitudinal kinetic energy, radial momentum and radial slope), equal the real total change of these quantities across the whole gap, (see Fig 1). Coordinates, as e.g. phase-angle, \( \phi(z) = \omega t(z) \), or transverse radius \( r \), increase linearly if a particle moves freely. Therefore the thin lens increments \( \Delta \phi, \Delta r^* \) of these variables differ from the differences \( \Delta \phi, \Delta r \) of the real trajectory across the whole

\textsuperscript{*)} A list of pertinent work may be found in 13\textsuperscript{)} Chapter 4 2
gap. They are only the differences of reduced phase, \( \delta \phi = \phi - z \frac{d\phi}{dz} \), see Fig. 2, and reduced radius, \( \delta r = r - z \frac{dr}{dz} \). Correct and fairly accurate difference equations giving \( \delta W, \delta \phi, \delta r, \delta r' \) as functions of the mid-gap \( z = 0 \) values \( W, \phi, r, r' \) have been derived 13.

The canonical thin lens approximation starts from these results. There arises the question whether a thin lens Hamiltonian depending only on the mid-gap values of the dependent variables exists from which the increments of particle coordinates across the thin lens can be derived by the (typically Hamiltonian) partial derivations with respect to the mid-gap values. The general discussion by which methods such a Hamiltonian can be found, is based upon the examples given in chapter 7; the results of chapter 9 where the thin lens Hamiltonian for a realistic gap model is derived, have also been taken into account.

The derivation of a thin lens Hamiltonian essentially involves two steps:

a) First of all, "reduced variables", \( P_i, Q_i \) must be introduced by canonical transformations starting from the variables \( p_k, q_k \) defined in chapter 3. By definition reduced variables are constant if there is no field acting on the particle. The change of \( P_i \) and \( Q_i \) across the gap then characterizes the integrated effect on the particle crossing it. The thin lens approximation consists in replacing the continuous change of the reduced variables through the gap with a step function with equal initial and final values. The step of it at the gap centre, \( z = 0 \), equals the total increment of the corresponding reduced variable across the gap. Momentum-like variables, as e.g. kinetic energy are reduced ones, (see Fig 1). Phase-angle, \( \phi = \omega t \), is not a reduced variable, it increases linearly in the absence of forces. This increase is cancelled by the term \( - z \frac{d\phi}{dz} \) in reduced phase-angle, eq. (7.6a), (see Fig. 5). A similar transformation must be applied to radius, \( r \). This is worked out in chap. 9.

b) In the second step Hamilton's equations with the Hamiltonian, \( K(P_i, Q_i; z) \), expressed in the new variables are solved by iteration starting from free particle motion. As zero order solution free particle motion is assumed. Then the reduced variables are constant. Therefore the Hamiltonian must be zero (or a constant) to that degree of approximation:

\[
P_i = P_{10} = \text{const.} \quad Q_i = Q_{10} = \text{const}
\]

\[
P_i' = 0 = - \frac{\partial K(0)}{\partial Q} \quad Q_i' = 0 = \frac{\partial K(0)}{\partial P}
\]

In fact, the Hamiltonians \( K \), eq. (7.9), and \( \bar{K} \), eq. (7.23) are zero if the field is put equal to zero. - The values \( P_{10} \) and \( Q_{10} \) are determined by the initial conditions; they are equal to the initial values prescribed for \( P_i \) and \( Q_i \) at the gap centre which in turn may be found from the initial values of the \( p_k \)'s and \( q_k \)'s with the help of the canonical transformations. In the first iteration the constants \( P_{10} \) and \( Q_{10} \) are inserted for the \( P_i \)'s and \( Q_i \)'s in the right hand sides of Hamilton's equations:
\[
\frac{dP_i(z)}{dz} = -\frac{\partial K(P_i, Q_i; z)}{\partial Q_i} \bigg|_{P = P_0, Q = Q_0} = -\frac{\partial K(P_{i0}, Q_{i0}; z)}{\partial Q_{i0}} \\
\frac{dQ_i(z)}{dz} = \frac{\partial K(P_i, Q_i; z)}{\partial P_i} \bigg|_{P = P_0, Q = Q_0} = \frac{\partial K(P_{i0}, Q_{i0}; z)}{\partial P_{i0}} 
\]

(8.2)

As indicated, to the first order in the field derivations with respect to the canonical variables, \(P_i, Q_i\), are equivalent to those with respect to their zero order constants, \(P_{i0}, Q_{i0}\). The only quantity on the right hand sides of eqs (2) which is still variable, is the explicit dependence of \(K\) on \(z\). To the first order the equations (2) can be solved by simple integration with respect to \(z\), and this process may be interchanged with the partial derivations. In this way the thin lens Hamiltonian is found:

\[
H_{TL}(P_{i0}, Q_{i0}) = \int_{-L}^{L} dz \ K_{lin}(P_{i0}, Q_{i0}; z) 
\]

(8.3)

The thin lens increments are given by:

\[
\Delta P_i = -\frac{\partial H_{TL}}{\partial Q_{i0}} \quad \Delta Q_i = \frac{\partial H_{TL}}{\partial P_{i0}}
\]

(8.4)

In the integral (3) the limits, \(z = -L\) and \(z = L\) give the borders of the region filled by the accelerating field. They may be equal to infinity provided the field decreases sufficiently fast. This condition is fulfilled for an accelerating gap confined by drift tubes extending to infinity. A further approximation is indicated by the subscript "lin" of the Hamiltonian \(K\) in (3). It is necessary to neglect in the Hamiltonian:

\[
K = K_{lin} + K_{n.1.}
\]

(8.5)

the terms non-linear in the field (or potential) since no satisfactory method has been found for their evaluation. From general perturbation theoretic considerations \(^{13}\) where the influence of the accelerating field is regarded as a perturbation on free particle motion and which uses expansions in powers of the parameter \(\kappa\), eq. (7.13), it can be expected that contributions of the terms \(K_{n.1.}(P_{i0}, Q_{i0}; z)\) are at least of the order \(eV_0\kappa\), therefore appreciably smaller than those due to \(K_{lin}(P_{i0}, Q_{i0}; z)\) which are of the order \(eV_0\). \((\kappa < 0.1\) in a proton linac over 500 keV). But they are of the same order, \(eV_0\kappa\), as the terms \(K_{lin}(P_{i1}(z), Q_{i1}(z); z)\) arising from the second iteration in \(K_{lin}\). These second iteration terms are again non-linear in the potential, since the \(P_i\)'s and \(Q_i\)'s themselves are at least linear in it. In addition, in a Hamiltonian containing first order solutions a step corresponding to that performed in eqs. (2) where the partial derivations \(\partial / \partial Q_i\) and
\( \partial / \partial P_1 \) have been replaced with \( \partial / \partial Q_{10} \) and \( \partial / \partial P_{10} \) is not longer permitted. But this step is one of the essential ingredients of the canonical thin lens approximation. For the same reason it is also necessary that the new Hamiltonian \( H_{\text{lin}}(P_{1},Q_{1};z) \) does no longer contain explicitly any one of the constants \( P_{10}, Q_{10} \). This must be achieved by choosing the canonical transformations properly. This especially applies to the phase constant \( \phi_0 \) which appears in the Hamiltonian \( \tilde{H}(p_{1},q_{1};z) \) (cf. eqs. (7.1) and (9.7)) and is the mid-gap value, \( Q_{\phi_0} = \phi_0 \) of \( Q_{\phi} \).

The use of reduced variables is always favourable (even if the variables employed are not canonical), whenever the equations of motion shall be solved by iteration (or perturbation theory) starting with free particle motion as the zero order solution. The equations of motion become simpler and can be more easily integrated \(^{19}\).

9. THIN LENS APPROXIMATION FOR A REALISTIC GAP MODEL

9.1. The field representations

The accelerating gap is confined by drift tubes which are assumed to extend from \( z = -\infty \) to \( z = \infty \). Their radius is so small that the cut-off frequencies of all wave guide modes are much higher than the frequency of the radio frequency field filling the gap. Therefore the field in the drift tubes decreases exponentially with increasing distance from the gap. The structure and the field are axially symmetrical and symmetrical about \( z = 0 \), the gap centre.

The radio frequency field is of the TM-type and can be derived from an electric Hertz vector:

\[
\mathbf{H} = \varepsilon_0 c \mathbf{E}(z,r,\phi) = \varepsilon \mathbf{E}_z(z,r) \cos(\omega t + \phi_0)
\]  

(9.1)

where \( \mathbf{E}(z,r) \) is a solution of the scalar Helmholtz equation, \( \Delta \mathbf{E} + k_0^2 \mathbf{E} = 0 \) \( \mathbf{E} \) may be expressed by a Fourier integral:

\[
\mathbf{E}(z,r) = \frac{k_0}{2\pi} \int_{-\infty}^{\infty} dk_z b(k_z) e^{ik_z z} \frac{J_0(\gamma r)}{\gamma J_0(\gamma a)}
\]  

(9.2)

with

\[
k_z^2 = k_0^2 - \gamma^2
\]

(9.3)

Expressions for the amplitude \( b(k_z) \) have been given elsewhere \(^{13}\). It is an even function if \( \mathbf{E}(z,r) \) is even, \( \mathbf{E}(-z,r) = \mathbf{E}(z,r) \). \( a \) is inner radius of the drift tubes. The non-zero components of the four-potential are:
\( U(z,r,\phi) = -\frac{\phi^2}{\varepsilon_0} = -\frac{3V(z,r)}{3z} \cos(\phi + \phi_0) \)  

\[ A_z(z,r,\phi) = e^{i\omega t} = \frac{k^2}{\omega} V(z,r) \sin(\phi + \phi_0) \]  

(9.4)

The field is:

\[ E = \cos(\phi + \phi_0) \left( E_z \left( k_0^2 + \frac{3^2}{3z^2} \right) V(z,r) + E_t \frac{3^2 V(z,r)}{3z^2} \right) \]

(9.5)

\[ B_\theta = \sin(\phi + \phi_0) e^{j\omega t} \frac{3V(z,r)}{3z} \]

An important quantity characterizing the field in the gap is the transit time factor defined as the Fourier transform of the longitudinal electrical field on the axis

\[ T_0(k) = \frac{1}{V_0} \int_{-\infty}^{\infty} E_z(z,0) \cos(kz) \, dz = E_1 b(k)/J_0(\gamma a) \]

(9.6)

where

\[ V_0 = \int_{-\infty}^{\infty} E_z(z,0) \, dz \]

is the voltage along the axis. \( \gamma^2 = k_0^2 - k^2 \). The transit time factor is a measure of the field distribution in the longitudinal direction. It appears in beam dynamics difference equations there describing the influence of the field on the particle crossing the gap.

It has not been possible to find canonical transformations leading to reduced variables for a Hamiltonian, \( \hat{H} \), eq. (4.7), containing the potential \( U \) in its general form (2) and (4). For this reason the Hertzian potential \( V(z,r) \) (and with it the other potentials) is expanded in a Taylor's series around \( r = r_1 \):

\[ V(z,r) = V(z,r_1) + (r - r_1) \frac{3V(z,r_1)}{3r_1} + \frac{1}{2} (r - r_1)^2 \frac{3^2 V(z,r_1)}{3r_1^2} + \ldots \]  

(9.7a)

\[ U(z,r,\phi) = \cos(\phi + \phi_0) \left[ C + (r - r_1) D + (r - r_1)^2 \frac{B}{2} + \ldots \right] \]

(9.7b)

\[ C = \frac{3V(z,r_1)}{3z}, \quad D = \frac{3^2 V(z,r_1)}{3z^2 r_1}, \quad B = \frac{3^3 V(z,r_1)}{3z^3 r_1^2} \]

(9.7c)

For \( r_1 \) a convenient value may be assumed, including \( r_1 = 0 \) and \( r_1 = r_0 \) where \( r_0 \) is the real radial position of the proton in the gap centre, \( z = 0 \). But in order to avoid confusion, it is very useful to distinguish in the course of the derivations which will
follow, between this expansion parameter, $r_1$, and the initial value, $r_o$, prescribed for $r$, even if their numerical values are equal. Approximations in the radial variable similar to those above are common practice in the treatment of particle dynamics in linear accelerators. (cf., 13), eq. (5.1.12)).

Somewhat different methods must be used for the derivation of the reduced radial variables depending on the degree of the polynomial approximating the radial dependence of the potentials. For clarity, the canonical transformations leading to reduced variables are just stated in the following sections, and the thin lens Hamiltonian belonging to the field of a real linac gap is derived from them. The method how the canonical transformations have been found, is explained in the next chapter.

5.2. Reduced variables linear in $r$

At first the term proportional to $(r - r_1)^2$ in the potentials, eqs. (9.7) is neglected, ($B = 0$). The radial field is then constant in the radial direction. New canonically conjugate coordinates of reduced character are introduced by the canonical transformation:

$$p_\phi = p_\phi + \frac{p_r^2}{m \omega} + \frac{e}{\omega} U(z, r, \phi) = - \frac{m}{2 \omega} \left( \frac{dz}{dt} \right)^2$$

$$= p_\phi + \frac{p_r^2}{m \omega} + \frac{e}{\omega} \left[ C + (r - r_1)D + \ldots \right] \cos(\phi + \phi_0)$$

$$Q_\phi = \phi_0 + \phi - z \sqrt{\frac{m \omega}{2p_\phi}} = \phi_0 + \phi - z \frac{d\phi}{dz}$$ (9.8)

$$p_\rho^I = p_\rho + \frac{eD}{\omega} \sin(\phi + \phi_0)$$

$$Q_\rho^I = r - \frac{p_\rho}{m} (\phi + \phi_0)$$

$$= \frac{eD}{m \omega^2} \left[ (\phi + \phi_0) \sin(\phi + \phi_0) + \cos(\phi + \phi_0) \right]$$

The superscript I of $p_\rho^I$ and $Q_\rho^I$ indicates that they are only linear approximations in the old transversal variables $p_\rho$ and $r$. The expression for reduced phase, $Q_\phi$, has been found with the help of the equation : $dz/dt = \dot{z} / \dot{p_\rho}$, with the Hamiltonian $\tilde{H}$ of (4.7), (cf., eqs. (7.3) and (7.6)). $Q_\phi$, reduced phase, and $p_\rho$ have been taken over from the one-dimensional problem. $Q_\rho^I$ and $P_\rho^I$ have been found with the help of the Poisson brackets, eqs. (3.22), (3.23), (4.8). They too are reduced variables : $D$ is zero, if there is no field as e.g. at $|z| = \infty$. Therefore $P_\rho$ is a reduced variable as well as the momentum $p_\rho$. In $Q_\rho$, the linear increase of $r$ in field-free regions is cancelled by the second term.

Together with the variables the initial conditions must be transformed. In the old variables the particle coordinates at the gap centre serve as initial conditions :
\[ \phi = 0 \]
\[ z = 0 : \]
\[ p_\phi = -E/\omega = -\frac{m}{2\omega} \left( \phi_o^2 + \phi_o'^2 \right) + ... \]
\[ r = r_0 \]
\[ p_r = m \dot{r}_0 \]

(9.9)

This gives for the transformed variables:
\[ Q_\phi = \phi_o \]
\[ p_\phi = -W/\omega = -\frac{m}{2} \frac{\dot{\phi}_o}{\omega} + \sqrt{-m\omega/(2P)} = k = \omega/2 \]
\[ Q_r = Q_{r0} = r_0 - \frac{r_0}{\omega} \phi_o/\omega + ... \]
\[ p_r = m \phi_o + ... \]

(9.10)

The dots indicate that terms proportional to the field have been omitted. This is permitted, since the initial values of the \( P_i \)'s and \( Q_i \)'s will be inserted into the new Hamiltonian, \( K \), which is already proportional to the field.

The new Hamiltonian, \( K^I \), eq. (5.21) is found with the help of the function \( \Theta^I \), eq. (10.4), generating the canonical transformations (8). In it are separated the terms which are linear in the field, from the non-linear ones:

\[ K^I(p_\phi, p_r; Q_\phi, Q_r; z) = K^I_{\text{lin}} + K^I_{\text{nl}}. \]

(9.11)

\[ K^I_{\text{lin}} = \frac{c}{\omega} \sin(\phi + \phi_o) \left[ 1 + (Q_r - r_1 + (\phi + \phi_o) \frac{p_r}{m\omega^2} \frac{3}{\omega^2} \frac{3}{\omega^2} + k^2 \right] \]
\[ + \frac{c}{\omega} \cos(\phi + \phi_o) \frac{p_r}{m\omega^2} \frac{3}{\omega^2} \frac{3}{\omega^2} \]

(9.12)

For convenience, \((\phi + \phi_o)\) has been written in place of the more complicated expression in the new variables:

\[ (\phi + \phi_o) = Q_\phi + z \sqrt{-m\omega/(2P)} \]

(9.13)

As explained in chapter 8, \( K_{\text{nl}} \), (given in eq. (10.5)) must be neglected in the computation of the thin lens Hamiltonian. This computation is performed in section 9.4, after \( K^I_{\text{lin}} \) has been found.

A slightly different definition of \( Q_r \) has been used in ref. 15:

\[ Q_r^I = r - \frac{p_r}{m} \phi - \frac{\epsilon d}{m\omega^2} \left[ \phi \sin(\phi + \phi_o) + \cos(\phi + \phi_o) \right] \]

(9.14)
Though $q^I_T$ is canonically conjugate to $p^I_T$, $Q^I$ and $P^I_\phi$ of eqs. (8), it is less suitable since the phase constant $\phi^0_o$ still appears in the new Hamiltonian, \( H^I_{lin} \). In place of \( (\phi + \phi^0_o)P^I_P + z\sqrt{\mu_0/(2P^I_\phi)} \) the new Hamiltonian, \( H^I_{lin} \), contains:

\[
\phi^0_o P^I_P = (Q^I + z\sqrt{\mu_0/(2P^I_\phi)} - \phi^0_o)P^I_P.
\]

Then it is no longer possible to interchange in Hamilton's equations (8, 2), the derivation \( \partial/\partial Q^I \) with the replacement \( Q^I = Q^I_{\phi^0} = \phi^0_o \) and the subsequent derivation with respect to \( \phi^0_o \). In order to make possible this step essential to the thin lens approximation, the \( \phi^0_o \) in eq. (15) has been distinguished from \( Q^I_{\phi^0} = \phi^0_o \) by attaching to it another subscript. But this procedure is artificial and unsatisfactory.

5.3. Reduced variables quadratic in \( r \)

In order to get a still better approximation, the term quadratic in \( (r - r^I) \) is now taken into account in the Taylor's series of the potentials, eqs. (7). The new canonical variables are:

\[
P^I_\phi = P^I_\phi + \frac{r^2}{2\mu_0} + \frac{e}{\omega} \left[ C + (r - r^I)D + (r - r^I)^2 \frac{B}{2} + \cdots \right] \cos(\phi + \phi^0_o)
\]

\[
Q^I_\phi = \phi^0_o + \phi - z\sqrt{\mu_0/(2P^I_\phi)}
\]

\[
起^I = \frac{eB}{\omega} \left[ (r - r^I) \sin(\phi + \phi^0_o) + \frac{P^I_r}{\mu_0} \cos(\phi + \phi^0_o) \right]
\]

\[
Q^I_r = Q^I_r - \frac{eB}{\mu_0} \left[ (r - r^I) \left( \cos(\phi + \phi^0_o) \cos\phi^0_o \right) \right]
\]

Since the terms which have been added, are proportional to the field, these new variables are reduced as well as the \( Q^I_T \)'s and \( P^I_\phi \)'s of eqs. (8). The above variables (16) are canonical up to terms linear in the field. The transformations are not exactly canonical; they fulfill the Poisson brackets, (3.22), (3.23) and (4.8), only approximately:
\[
\begin{align*}
[Q_\phi, P_\phi] &= 1 \\
[Q_r, P_\phi] &= \frac{eB}{MC^2} \left[ D + (r-r_1)B \right] \left( \phi + \phi_0 \right) \cos(\phi_0) - 2 \sin(\phi + \phi_0) \\
[Q_r, Q_\phi] &= -z \sqrt{\mu \omega/(2P_\phi)} \left[ Q_r, P_\phi \right] \\
[P_r, P_\phi] &= -\frac{eB}{MC^2} \left[ D + (r-r_1)B \right] \cos(\phi + \phi_0) \\
[P_r, Q_\phi] &= -z \sqrt{\mu \omega/(2P_\phi)} \left[ P_r, P_\phi \right] \\
[Q_r, P_r] &= 1 - \left( \frac{eB}{MC^2} \right)^2 \left[ 1 + \sin^2(\phi + \phi_0) \right]
\end{align*}
\]

The terms by which the above Poisson brackets differ from the exact ones, eqs. (3.23), are quadratic in the potential. Therefore they fit into the frame of approximations used where terms non-linear in the potential are neglected. There is no exact generating function for these transformations. The additional term of the Hamiltonian which is linear in the potential, has been found with the help of the equations of motion in Newtonian form, see section 10.2. It is:

\[
K_{\text{lin}}^{II}(P_\phi, P_r; Q_\phi, Q_r; z) =
\]

\[
+ \frac{e}{\omega} k_z^2 \sin(\phi + \phi_0) \frac{1}{2} (Q_r - r_1 + \frac{P_r}{MC} (\phi + \phi_0))^2 \frac{\partial^2 V}{\partial r^2} (z, r_1)
\]

\[
+ \frac{e}{\omega} \left[ \sin(\phi + \phi_0) \left( \frac{1}{2} (Q_r - r_1 + \frac{P_r}{MC} (\phi + \phi_0))^2 - \left( \frac{P_r}{MC} \right)^2 \right) + \cos(\phi + \phi_0) \frac{P_r}{MC} (Q_r - r_1 + \frac{P_r}{MC} (\phi + \phi_0)) \right] \frac{\partial^2 V}{\partial r_1^2}
\]

(9.18)

with:

\[
(\phi + \phi_0) = Q_\phi + z \sqrt{\mu \omega/(2P_\phi)}
\]

9.4. Computation of the Thin Lens Hamiltonian

The computation of the thin lens Hamiltonian:

\[
H_{\text{TL}} = \lim_{L \to \infty} \int_{-L}^{L} K_{\text{lin}}(P_{\phi0}, Q_{\phi0}; z) \, dz
\]

(9.19)

presents no special difficulties. \(K_{\text{lin}}\) consists of the expressions (12) and (18). When the zero order solutions, (10), are inserted for the P's and Q's, \((\phi + \phi_0)\) becomes:

\[
(\phi + \phi_0) = Q_{\phi0} + z \sqrt{\mu \omega/(2P_{\phi0})} = \phi_0 + k z
\]

(9.20)
with \( k = \omega / t_0 \). The integral representation (2) for \( V(z, r_1) \) is inserted. The path of integration \( C \) in the complex \( k_z \)-plane is indented upwards (downwards) at \( k_z = -k \) (+k). Then the integration with respect to \( z \) from \(-L\) to \(+L\) is performed. For \( z = \pm L \) (-L) the path \( C \) is completed to a closed contour by semi-circles (of infinite radius) \( C_U \) \( C_L \) situated in the upper (lower) half-plane. The remaining integrals in \( k_z \) are afterwards evaluated with the help of Cauchy's residue theorem. The arguments \( k_z = \pm i \eta \) \( k_z = \mp \omega / t_0 \) for which the Bessel function in the denominator of the integral (2), 
\[ J_\nu (a \sqrt{k_z^2 - k_0^2}) = J_\nu (a \sqrt{k_z^2 - k_0^2}) = J_\nu (i \eta) = 0, \]
are purely imaginary, since the frequency \( \nu \) is lower than the lowest cut-off frequency, \( c(j_1/a) \) of the wave guide of radius \( a \). In consequence the residues due to the simple poles \( k_z = \pm i \eta \) are proportional to \( \exp (-\eta \sqrt{L}) \) and vanish in the limit \( L \to \infty \). There remain only the contributions arising from the poles \( k_z = \pm k = \pm \omega / t_0 \). Finally the amplitude function \( b(k) \) is replaced with the transit time factor, \( T_0(k) \), eq. (6). This method is described in more detail in ref. 13. Some of the above integrals may be combined so that \( (k^2 + \omega^2 / \sin^2) V = E_z \) can be inserted. The arising integrals may be identified by comparing them with the transit time factor, eq. (6), and its derivatives, \( dT_0(k) / dk \), ...

In this way the thin lens Hamiltonian \( H_{TL} \) is derived from \( K_{lin}^I \) eq. (12), and \( K_{lin}^{II} \), eq. (18):

\[
H_{TL} = H_{TL}^I + H_{TL}^{II}
\]

\[
H_{TL}^I = \frac{eV}{\omega} T_0 I_0 \sin \phi_0 - \frac{eV}{\omega} \frac{t_0}{\omega} \frac{d}{dk} (T_0 I_0) \cos \phi_0
\]

\[+ \frac{eV}{\omega} \left[ Q_{r_0} - r_1 + \frac{t_0}{\omega} \phi_0 \right] T_0 k_1 I_1 \sin \phi_0
\]

\[
H_{TL}^{II} = \frac{eV}{\omega} \frac{t_0}{\omega} \left[ Q_{r_0} - r_1 + \frac{t_0}{\omega} \phi_0 \right] T_0 k_1^2 I_1 \sin \phi_0
\]

\[ - \frac{eV}{\omega} \frac{t_0}{\omega} \left[ Q_{r_0} - r_1 + \frac{t_0}{\omega} \phi_0 \right] \frac{k_0}{r} \frac{d}{dr} (T_0 k_1 I_1) \cos \phi_0
\]

\[ - \frac{eV}{\omega} \left( \frac{t_0}{\omega} \right)^2 \left( \frac{d^2}{dr^2} (T_0 k I_1) - \frac{k_0^2}{k^2} \frac{d^2}{dr^2} (T_0 I_1) \right) \sin \phi_0
\]

The arguments \( k \) of \( T_0(k) \) and \( k r_0 = \sqrt{k^2 - k_0^2} r_0 \) of the modified Bessel functions have been omitted. Hamilton's equations give the increments of particle coordinates across the thin lens situated at the gap centre, \( z = 0 \):
\[ \Delta P_\phi = P_{\phi^+} - P_{\phi^-} = \frac{1}{\omega} (W_+ - W_-) = - \frac{\partial H}{\partial Q_{\phi^0}} = - \frac{\partial H}{\partial \phi_0} \]

\[ \Delta Q_\phi = Q_{\phi^+} - Q_{\phi^-} = \Delta \phi = \frac{\partial H}{\partial P_{\phi^0}} = \frac{k^3}{m\omega} \frac{\partial H}{\partial k} \]

\[ \Delta P_r = P_{r^+} - P_{r^-} = P_{r^+} - P_{r^-} = - \frac{\partial H}{\partial Q_{r^0}} = - \frac{\partial H}{\partial (r_0 - \frac{\phi_0}{\omega})} \]

\[ \Delta Q_r = Q_{r^+} - Q_{r^-} = \bar{r} = r - \phi \frac{dr}{d\phi} = \frac{\partial H}{\partial P_{r^0}} = \frac{1}{m} \frac{\partial H}{\partial r_0} \]

\[ (9.23) \]

In forming the difference of the transversal variables use has been made of the fact that \( V \) (and with it \( D \) and \( B \)) vanish for \(|z| \to \infty\).

\[ \bar{r} = r - \phi \frac{dr}{d\phi} \]

(9.24) is a kind of reduced radius but differs slightly from :

\[ \bar{r} = r - z \frac{dr}{dz} \]

(9.25) After the partial derivation with respect to \( Q_{\phi^0} \) has been performed in eq. (25), \( Q_{r_0} = r_0 - \frac{\phi_0}{\omega} \) may be inserted into \( H_{TL} \) and \( r_1 \) may be identified with \( r\phi^0 \), the real position of the proton in the gap centre. The arising difference equations are similar to those currently employed in the treatment of particle motion in a linac gap (cf. ref. 13, Table III). The equations for the increments of kinetic energy and phase differ inasmuch as in the present derivation the magnetic field force has been taken into account while in ref. 13 it is regarded as a relativistic effect and neglected in the non-relativistic equations. The transversal variables differ in several respects. In practical computations it is preferred to use the change in radial slope :

\[ r' = \frac{dr}{dz} \]

(9.26) which can be directly measured by slits; and that in reduced radius, \( \bar{r} = r - z \frac{dr}{dz} \), eq(25). The set of difference equations (23) is not written down explicitly. It is easy to derive it from (21) and (22). This system is more interesting from the theoretical point of view since it is the proof that a thin lens Hamiltonian for an accelerating gap can be given.

There are two reasons why earlier trials to guess a thin lens Hamiltonian for the difference equations of a linac gap (ref. 13, Table III; or their forerunners, ref. 4, Table I; ref. 12, eq. (70)) have been bound to fail. First of all, the radial variables used in these tables are not canonical. This is not so serious as the second point, since \( r' = \frac{dr}{dz} \) and \( \bar{r} = r - \phi \frac{dr}{d\phi} \) resp.,
which are canonical. The second difficulty arises from the fact that the dependence of the field on radius, \( r \), has been expanded into a Taylor's series before the equations of motion are integrated. (cf. eq. (7); ref. 12), eqs. (32) and (33); ref 13, eq. (5.1,12)) By this expansion the radial variable, \( r \), is split into a new dynamical variable, \( (r - r_o) \) (or \( r - r_1 \)) and a parameter, \( r_o \) (or \( r_1 \)). In consequence, the partial derivations on the right-hand side of eqs. (33) typical for Hamiltonian theory no longer act upon the radial argument of the Bessel function (denoted here by \( r_1 \)) but on \( r_0 \) which is in front of the Bessel function. The necessity for this distinction has not been seen before it has been revealed by the analysis presented in ref. 15) and repeated here. Before it has been supposed that the partial derivation acts on \( r_0 \) by which letter two different things have been denoted, the initial value of the particle's radial position and the point around which the dependence of the field on radius is expanded.

10. THE DERIVATION OF THE REDUCED CANONICAL VARIABLES

The longitudinal reduced canonical variables, reduced phase, \( Q_\phi \), eq. (9.8), and \( P_\phi = -T/\omega = -(m/2\omega)(dz/dt)^2 \) are assumed from the start. They represent the best choice in the one-dimensional problem and they are used in practical applications. They are canonically conjugate, since their Poisson bracket equals unity, \([Q_\phi, P_\phi] = 1\). This is valid for any potential \( U \).

Each of the new radial variables, \( P_r \), \( Q_r \), which are subsumed under one symbol, \( X \), must fulfil two Poisson brackets relating it to \( P_\phi \) and \( Q_\phi \):

\[
0 = \left[ X, P_\phi \right] = L(X) = \frac{\partial X}{\partial P_\phi} \frac{\partial}{\partial \phi} - \frac{\partial X}{\partial \phi} \frac{\partial}{\partial P_\phi}
\]

(10.1)

\[
0 = \left[ X, Q_\phi \right] = -z \left( \frac{m \omega}{2P_\phi} \right)^{1/2} \left[ L(X) = \frac{\partial X}{\partial P_\phi} \frac{\partial}{\partial \phi} - \frac{1}{z} \left( \frac{m \omega}{2P_\phi} \right)^{-1/2} \right]
\]

(10.2)

\( L(X) \) is an abbreviation for the linear differential operator:

\[
L(X) = \frac{\partial X}{\partial \phi} + \frac{\partial}{\partial \phi} \frac{\partial X}{\partial P_\phi} - \frac{\partial X}{\partial \phi} \frac{\partial}{\partial P_\phi}
\]

(10.3)

Conditions (1) and (2) are only compatible if the new radial variables are solutions of the partial differential equation: \( L(X) = 0 \), and do not depend on \( P_\phi \), \( \partial X/\partial P_\phi \) = 0. The last condition excludes \( r = r - z \, dr/dz = r + z \, \partial H/\partial P_r \) as a possible candidate for \( Q_r \).

It appears prohibitively difficult to solve the partial differential equation \( L(X)=0 \) containing the general expressions (9.2) and (9.4) for \( U \). Therefore only the Taylor's series (9.7) is inserted. If only the approximation linear in \((r-r_1)\) is used, exact solutions can be found. If the quadratic term, too, is retained, only approximate solutions linear in the potential can be given. All this is done in the Appendix.
In the linear approximation, \( B = 0 \), the solutions \( x^I_1 \) and \( x^I_2 \), eqs. (A.8), (A.9) are used. \( x^I_1 \) is a reduced variable and is suggesting itself as \( P^I_1 \cdot x^I_2 = q^I_1 \) is also a reduced variable, but for the reason explained after eq. (9.14) it is preferred to define \( q^I_1 \) as the linear combination \( q^I_1 = x^I_2 - \phi_0 x^I_1 \). Both \( q^I_1 \) and \( q^I_1 \) fulfil the last Poisson bracket, \( [Q^I_1, P^I_1] = 1 \). The function \( \phi^I \) generating the set of canonical transformation (9.8) is determined by eqs. (3.21) and is found by guesses:

\[
\phi^I(p^I_\phi, p^I_\psi, p^I_{1+z}) = \frac{z \sqrt{2m_0 p_\phi + \phi_0 p_\psi + \frac{e}{\omega} (r^D_1 - C)(y-x) + (p_\phi - p_\psi)\arcsin \omega}}{(ed)^2 \left[ \left( \frac{1}{2} (x + \rho)^2 + \frac{1}{4} \arcsin \omega \right) \right]
\]

(10.4)

\[
x = \omega p_{1+z}/eD \quad y = \omega p^I_{1+z}/eD \quad y - x = \rho
\]

With the help of \( \phi^I \) the new Hamiltonian \( K^I \) is found according to (3.21). \( K^I_{n,1} \) is given in eq (9.12) \( K^I_{n,1} \) is:

\[
K^I_{n,1} = -\frac{e^2}{m_0^3} \frac{a^2 V}{a_2 a^I_1} \frac{a^3 V}{a_2 a^I_1} \left[ \frac{1}{2}(\phi + \phi_0) + \frac{1}{4} \sin(2\phi + 2\phi_0) \right]
\]

(10.5)

\[
K^I_{n,1} = \frac{e^2 k_0^2}{m_0^3} \frac{a^2 V}{a_2 a^I_1} \frac{a^3 V}{a_2 a^I_1} \cos(\phi + \phi_0)
\]

\( K^I_{n,1} \) alone is not unique. In fact, with the help of \( y - x = (ed/\omega) \sin(\phi + \phi_0) \), eq. (9.8) and similar relations it is possible to change terms in \( K^I_{n,1} \); terms non-linear in the potential arising in this process are shifted afterwards to \( K^I_{n,1} \). This ambiguity is a well-known feature of such approximation schemes 16.

In the quadratic approximation, \( \phi \neq 0 \), the solutions (A.13), (A.14) are employed for \( P^I_1 = x^I_1 \) and \( Q^I_1 = x^I_2 - \phi^I_1 \). They are only approximate solutions of \( L(X) = 0 \); or of the Poisson brackets as shown in eqs. (9.17). Therefore an exact generating function \( \phi^I \) does not exist. It is impossible to compute the new Hamiltonian according to (3.21).

The method for finding the part \( K^I_{n,1} \) which is linear in the field (proportional to \( \delta \) and its derivative \( \delta^2 \)), of the Hamiltonian \( K^I \) quadratic in the transversal variables uses Hamilton's equations (8.2). Even one of them is sufficient for the present purpose:

\[
\frac{dP^I_{\phi}(l)}{dz} = \frac{\partial K_{n,1}}{\partial P^I_{\phi}} = -\frac{\partial K_{n,1}}{\partial \phi_0}
\]

(10.6)

The above equation is rewritten in a more detailed way as:
\[
\frac{dP_{\phi}}{dz} (1) + \frac{\partial K_{I}^{I}}{\partial \phi_0} (P_{10}, Q_{10}, z) = - \frac{\partial K_{I}^{II}}{\partial \phi_0} (P_{10}, Q_{10}, z)
\] (1.7)

\(K_{I}^{\text{lin}}\) is known, eq. (9 12). If \(dP_{\phi}/dz\) is determined by another method, the right hand side of eq. (7), i.e. \(\partial K_{I}^{II}/\partial \phi_0\), can be found.

\[dP_{\phi}/dz\text{ can be computed from the equations of motion in their Newtonian form where time, } t, \text{ (or phase-angle } \phi = \omega t) \text{ is the independent variable:}
\]
\[d^2 \phi_0/dz = \left(\frac{m}{2}\right) \frac{d^2 z}{d\phi^2} = m \Psi = m \omega^2 \frac{d^2 z}{d\phi^2} =
\]
\[= eE_z (z, r, \phi) + e\omega B_0 (z, r, \phi) \frac{dr}{d\phi}
\]
This equation is solved by iterations starting from free particle motion:
\[z_0 (t) = \frac{r}{k} \quad r_0 (t) = r_0 + \frac{t}{\omega} \phi_0
\]
\[= Q_{r0} + \phi P_{r0}/(m \omega)
\]
The first iteration gives:
\[dP_{\phi}/dz = eE_z (z (t), r_0, \phi) + e\omega B_0 (r_0, r_0, \phi) dz/d\phi
\]
\(E_z\) and \(B_0\) are expressed by the Hertz potential, \(V\), eqs. (9.5). For \(V\) are inserted the series (9.7). \(K_{I}^{\text{lin}} (P_{10}, Q_{10}, z)/\partial \phi_0\) (with \(K_{I}^{\text{lin}}\) taken from eq. (9 12)) \(k_z\) may be replaced with the zero order expression, \(k_z (o) = \phi\), since \(K_{I}^{\text{lin}}\) is already linear in the field. All this together with eq. (7) gives:
\[\frac{\partial K_{I}^{II}}{\partial \phi_0} (P_{10}, Q_{10}, z) = - \frac{\partial V (z, r_1)}{\partial z^2} \frac{\partial V (z, r_1)}{\partial z} \frac{\partial^2 V (z, r_1)}{\partial z \partial ^2 r_1} \frac{R^2}{2} \cos (\phi + \phi_0) +
\]
\[+ \frac{e \omega}{2} \frac{\partial V (z, r_1)}{\partial z} \frac{\partial^2 V (z, r_1)}{\partial z \partial ^2 r_1} \left[ \frac{R^2}{2} \cos (\phi + \phi_0) + \frac{P_{r0}}{m \omega} R \sin (\phi + \phi_0) \right]
\]
with the abbreviation : \(R = Q_{r0} - r_1 + P_{r0} (\phi + \phi_0)/(m \omega)\). It is easy to integrate the above expression with respect to \(\phi_0\) to get \(K_{I}^{II} (P_{10}, Q_{10}, z)\). Dropping the subscript \(o\) of \(P_{r0}\) and \(Q_{r0}\) gives \(K_{I}^{II} (P_{10}, Q_{10}, z)\) of eq. (9.18).

The equation conjugate to (6), \(dQ_{\phi}/dz = d\phi_0 (1)/dz = - \frac{\partial K_{I}^{II}}{\partial \phi_0} (P_{10}, Q_{10}, z)\) (with \(\phi\) the reduced phase , ref. 13) , eq. (5.1.20)) just reproduces the above result. The two remaining equations involving the derivatives of the transversal variables can be used together with the Newtonian equation for radial motion to check the consistency of \(K_{I}^{II}\) and \(P_{r}, Q_{r}\).
defined in eqs. (9.16). In ref. 15 the method just described has been used to derive $k_{II}$ as well as $p_{r}^{II} - p_{r}^{I}$ and $q_{r}^{II} - q_{r}^{I}$. 

11. CONSERVATION OF PHASE SPACE AREA. COMPARISON WITH SWENSON'S RESULTS

In several paper 3) 4) 14) the canonical character of dynamical equations has been investigated under the aspect of conservation of phase space density. Not all of these discussions are entirely satisfactory. It may be worthwhile to start an investigation using the rigorous Poincaré integral invariants.

If the motion is planar and if transversal (radial) motion is negligible, the integral invariant $J_{1}$, eq. (4.9), becomes:

$$J_{1} = \iiint_{S} dp_{\phi} \, d\phi = \iiint_{S} dP_{\phi} \, dQ_{\phi}$$  \hspace{1cm} (11.1)

In that approximation the total phase space is the 2-dimensional space of the dependent variables $p_{\phi} = -E/\omega$ and $\phi$. (1) is just the expression of Liouville's theorem, eq. (3.28). For example, $p_{\phi}$ and $\phi$ may be the particle input coordinates of a beam cross section at some point $z_{1}$ before the accelerating gap and $P_{\phi}$, $Q_{\phi}$ the particle coordinates at some point $z_{2}$ after the gap. Care must be taken that $S$ is the same surface in both integrals. From the invariance of the integrals in eq. (1) follows that the Jacobian for the two sets of variables is unity. This agrees with the results in ref. 4).

The situation is more complex, if radial motion must not be disregarded. Total phase space embraces the four variables $p_{\phi}$, $p_{r}$, $\phi$, $r$. The integral invariant $J_{1}$, eq. (4.9), consists of two 2-dimensional integrals. Only their sum is an invariant. Only in the exceptional and unrealistic case where there is no coupling between longitudinal and radial motion, each of the integrals may be separately invariant. To prove this statement the Poincaré invariant as written in eq. (3.27) is used, with $u = r_{i}$, $v = p_{r_{i}}$ ($p_{r_{i}}$, $r_{i}$ are the radial input coordinates):

$$\{p_{r_{i}}, r_{i}\}_{p_{r}, q_{i}} = 1$$

$$J_{1} = \iiint_{S} dp_{p_{r_{i}}} \, dr_{i} = \iiint_{S} \left( \frac{\partial Q_{r}}{\partial u} \frac{\partial P_{r}}{\partial v} - \frac{\partial P_{r}}{\partial u} \frac{\partial Q_{r}}{\partial v} + \frac{\partial Q_{\phi}}{\partial u} \frac{\partial P_{\phi}}{\partial v} - \frac{\partial P_{\phi}}{\partial u} \frac{\partial Q_{\phi}}{\partial v} \right) \, dp_{p_{r_{i}}} \, dr_{i}$$  \hspace{1cm} (11.2)

The $P$'s and $Q$'s may be assumed to be the output coordinates. In the absence of longitudinal-radial coupling the last two terms on the right hand side vanish and the remaining expression gives then:
\begin{equation}
J_1 = \iint_S dP_r \, dQ_r = \iint_S dp_{re} \, dp_{re}
\end{equation}

The subscript \( e \) denotes the output (exit) coordinates.

The same considerations apply to the set of canonically conjugate variables \( p_\phi, p_T, Q_\phi, Q_T \) defined in Chapter 9.

The situation is completely different for a set of variables where the radial variables are represented by \( r' = dr/dz \) and \( \tilde{r} = r - z \frac{dr}{dz} \). They are not canonically conjugate to each other, their Poisson bracket is different from unity:

\[
[r'_r, \tilde{r}_r] \neq 1
\]

Each of them is incompatible as a canonically conjugate variable with longitudinal kinetic energy \( p_\phi \) and reduced phase \( Q_\phi \); the longitudinal variables used in the common difference equation for a linac gap. To show this the following set of variables is assumed and their mutual Poisson brackets are evaluated:

\begin{align}
\hat{P}_r &= \beta \frac{dr}{dz} \\
\hat{Q}_r &= \alpha \tilde{r} = \alpha (r - z \frac{dr}{dz})
\end{align}

\( \tilde{\Pi} \) is given in eq. (4.7). For convenience and without restricting generality the constants \( \alpha, \beta \) are put to:

\[
\alpha = k = \omega^2 / \omega_o \quad \beta = m_o \omega^2 / k \quad \alpha \beta = m_o^2
\]

The above variables fulfil exactly the following Poisson brackets:
\[
\begin{align*}
\left[ Q_{\phi}, P_{\phi} \right] &= 1 \\
\left[ Q_r, P_r \right] &= (1 + (dr/dz)^2) \xi_0 / \xi \\
\left[ Q_r, P_\phi \right] &= f / \xi - (kz/m^2) \varepsilon E_r / \omega = r_o^\prime + ... \\
\left[ P_r, Q_r \right] &= -f_s^2 / \xi^3 - z(e/2\hbar)(\xi_0^b / \xi^b) E_r = -r_o^\prime + ... \\
\left[ Q_r, Q_\phi \right] &= (kz^2 / 2m^2) \phi^\prime \varepsilon E_r / \omega \xi^2 \\
\left[ P_r, P_\phi \right] &= (\phi^\prime / \xi^2) \varepsilon E_r / \omega \\
\end{align*}
\]

\( E_r = \partial U / \partial r \) has been used. Dots indicate that terms proportional to the field have been omitted. It is obvious that these Poisson brackets differ from those for canonical variables eq. (3.23). Therefore the Poincaré invariant cannot be used as above.

Swenson 14) discusses area conservation in \( r^\prime, r \) space by putting:

\[
\int \int dr_1^\prime dr_1 = \int \int \frac{\partial (r_1^\prime, r_1)}{\partial (r_1^e, r_1)} dr_e^\prime dr_e
\]

(11.5)

According to his calculations the above Jacobian differs from unity by terms at least quadratic in the field. However, such a transformation as in (5) makes only sense if the longitudinal variables are kept constant:

\[
\frac{\partial (r_1^\prime, r_1)}{\partial (r_1^e, r_1)} P_\phi = \text{const} , Q_\phi = \text{const}
\]

Obviously, this is not the case in an accelerating gap where the particle is accelerated in the longitudinal direction. In addition, the same Jacobian calculated for the difference equations derived in ref. 13) differs from unity by a term linear in the field.

Table III of ref. 13) gives the increments

\[
\Delta W = W_+ - W_- = W_1 - W_e \\
\Delta \phi = \phi_+ - \phi_- \\
\Delta r^\prime = r_1^\prime - r_1^e = r_1^\prime - r_1^c \\
\Delta \xi = \xi_1 - \xi_e
\]

(11.6)

as functions of the mid-gap values \( W, \phi, \xi \), \( r_1^\prime, r_1^e \). (The subscripts e, i, denote output,
input values resp. (at the cell ends), while +, - resp. denote values just after, before
the thin lens at \( z = 0 \). These mid-gap values differ from \( \tilde{\mathcal{W}}_-, \tilde{\mathcal{R}}_-, \tilde{\mathcal{R}}_- \) by terms linear
in the field and may be replaced by them since they occur in terms proportional to the field.
These substitutions give:

\[
\begin{align*}
\mathcal{W}_+ &= \mathcal{W}_- + eV_0 \left\{ T_{o \ o} T \cos \phi_+ + \frac{eV_0}{d\ell_o} \left( T \frac{d}{d\ell_o} \right) \right\} \\
\mathcal{R}_+ &= \mathcal{R}_- + eV_0 \left\{ \frac{d}{d\ell_o} \left( T \frac{d}{d\ell_o} \right) \sin \phi_+ - \frac{k^2}{d\ell_o} \left( T \frac{d}{d\ell_o} \right) \right\} \\
\mathcal{R}_+ &= \mathcal{R}_- - \frac{eV_0}{d\ell_o} \left\{ \frac{d}{d\ell_o} \left( T \frac{k}{k} \right) \frac{d}{d\ell_o} - \frac{d}{d\ell} \frac{d}{d\ell} \right\} \\
\mathcal{R}_+ &= \mathcal{R}_- - \frac{eV_0}{d\ell_o} \left\{ \frac{d}{d\ell_o} \left( T \frac{k}{k} \right) \frac{d}{d\ell_o} - \frac{d}{d\ell_o} \frac{d}{d\ell_o} \right\}
\end{align*}
\]

(11.9)

The argument of \( T_{o \ o} \) is \( k = \omega \sqrt{2W} \), that of the Bessel functions is \( k_o r_- = \sqrt{k^2 - (\omega/c)^2} r_- \). It is sufficient to calculate the main diagonals of the 2 by 2 Jacobians below, since the other diagonal gives only terms quadratic in the field.

The Jacobian for longitudinal motion is:

\[
\begin{vmatrix}
\frac{\partial \mathcal{W}_+}{\partial \mathcal{W}_-} & \frac{\partial \phi_+}{\partial \phi_-} \\
\frac{\partial \mathcal{W}_-}{\partial \mathcal{W}_-} & \frac{\partial \phi_-}{\partial \phi_-} \\
\frac{\partial \mathcal{W}_+}{\partial \phi_+} & \frac{\partial \phi_+}{\partial \phi_-} \\
\frac{\partial \mathcal{W}_-}{\partial \phi_-} & \frac{\partial \phi_-}{\partial \phi_-}
\end{vmatrix} = 1 - \left[ \frac{eV_0}{d\ell_o} \right]^2 \ k^2 \ \cdots
\]

(11.8)

It differs from unity by terms quadratic in the field. This result is not exact since radial motion is neglected.

The Jacobian for radial motion is:

\[
\begin{vmatrix}
\frac{\partial \mathcal{R}_+}{\partial \mathcal{R}_-} & \frac{\partial \mathcal{R}_+}{\partial \mathcal{R}_-} \\
\frac{\partial \mathcal{R}_-}{\partial \mathcal{R}_-} & \frac{\partial \mathcal{R}_-}{\partial \mathcal{R}_-} \\
\frac{\partial \mathcal{R}_+}{\partial \mathcal{R}_+} & \frac{\partial \mathcal{R}_+}{\partial \mathcal{R}_-} \\
\frac{\partial \mathcal{R}_-}{\partial \mathcal{R}_-} & \frac{\partial \mathcal{R}_-}{\partial \mathcal{R}_-}
\end{vmatrix} = 1 - \frac{eV_0}{d\ell_o} \left\{ T_{o \ o} T_{o \ o} \frac{\partial \phi_+}{\partial \phi_-} + \frac{d}{d\ell_o} \frac{d}{d\ell_o} \right\} \ r^2 \ \sin \phi_0 + \ \cdots
\]

(11.9)
At the end \( \dot{\phi} \) and \( r_- \) have been replaced with \( \dot{\phi}_0 \), \( r'_0 \) resp. Of course, here again applies the objection already raised in discussing eq (5) that longitudinal motion is neglected without justification. The Jacobian (9) contains terms linear in the field. This is at variance with SWENSON's result. The method by which the mid-gap values \( r'_0 \), \( r''_0 \) are introduced in the difference equations of this ref. is slightly ambiguous and this may be the reason for the contradicting results.

Eq (9) gives the Jacobian across the thin lens while that in ref. is the Jacobian across the whole gap. But the Jacobian \( \dot{\phi}(r'_0, r'_E)/\dot{\phi}(r'_1, r'_1) \) differs from \( \dot{\phi}(r'_0, r'_1)/\dot{\phi}(r''_0, r''_1) \) by terms proportional to \( L \), the cell length. This difference cannot cancel the terms linear in the field contained in eq (9) which are independent of \( L \).

This discussion of the work in ref. concerns only conservation of radial phase space density. SWENSON is right in adding to his radial difference equations for radius the term accounting for the fact that the accelerating field is distributed throughout the gap.

Most of the work described in this report has been done while the author worked as a CERN-fellow in the Intersecting Storage Rings Division. He is very indebted to Dr. P. M. LAPOSTOLLE for suggesting this topic and many helpful discussions. He also drew great profit from discussions with Prof. A. SESSLER.
APPENDIX: SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION \( L(X) = 0 \)

Inserting the expansion (9.7) into \( L(X) = 0 \), eq (10.3), gives:

\[
\frac{3X}{3\phi} + \frac{R}{\mu x} \frac{3X}{3r} - \frac{3X}{3p} \frac{e}{\omega} \left[ D + (r-r_1)B \right] \cos(\psi + \phi_0) = 0 \tag{A 1}
\]

According to standard theory \(^{17}\) in order to get the general solution of (1), the system of ordinary differential equations belonging to the characteristic curves must be solved:

\[
\frac{dr}{du} = \frac{p_r}{(m_0)} \tag{A 2}
\]

\[
\frac{dp_r}{du} = - \left( \frac{e}{\omega} \right) \left[ D + (r-r_1)B \right] \cos(\psi + \phi_0) \tag{A 3}
\]

\[
\frac{d\phi}{du} = 1 ; \quad \phi = u + C_3 \tag{A 4}
\]

Eqs (2) and (3) together with the solution of (4) give:

\[
\frac{d^2r}{du^2} = - \frac{e}{m_0} \left[ (r-r_1)B + D \right] \cos(u + u_0) \tag{A 5}
\]

\[u_0 = \phi_0 + C_3 \quad \text{Substituting in Mathieu's equation}^{18}:
\]

\[
d^2y/dz^2 + \left[ a - 2q \cos(2z) \right] y(z) = 0
\]

\[2z = u + \pi + u_0 \quad r(u) = y(z)
\]

gives:

\[
\frac{d^2r}{du^2} + r \left[ \frac{a}{4} + \frac{q}{2} \cos(u + u_0) \right] = 0
\]

Therefore the homogeneous equation belonging to (5), (where \( D = r_1 = 0 \)), is a special case of Mathieu's equation

a) \( B = 0 \):

In this case the system (2) to (4) is simple. Its general solution is:

\[
p_r = - (eD/\omega) \sin(\psi_0 + u + C_3) + C_2 \tag{A 6}
\]

\[
r = (eD/m_0^2) \cos(\psi_0 + u + C_3) + C_2u/\omega + C_1 \tag{A 7}
\]

\( C_1, C_2, C_3 \) are constants of integration. From (4), (6) and (7) are found the two following expressions:
\[ X_1^I = p_r + (eB/\omega) \sin(\phi + \phi_0) = C_2 \]  
\[ X_2^{II} = r - p_r/(m_0) - (eB/m_0^2) \left[ \phi \sin(\phi + \phi_0) + \cos(\phi + \phi_0) \right] \]  
(A 9)

\( X_1 \) and \( X_2 \) regarded as functions of \( r, p_r \) and \( \phi \) are constants for arbitrary values of the parameters \( C_1, C_2 \) and \( C_3 \); so each is constant along any characteristic curve and is a solution of (1) (with \( B = 0 \))

b) \( B \neq 0 \)

Eq (5) is solved by iterations, starting from free particle motion where \( B = D = 0, \)
\[ \frac{d^2 r_0}{du^2} = 0 \]

\[ r_0(u) = \tilde{r}_0 + \tilde{r}_0' u + \tilde{r}_0'' \]
\[ \frac{p_r}{m_0} = \tilde{r}_0' \]

(A 10)

\( \tilde{r}_0' \) and \( \tilde{r}_0'' \) are just arbitrary constant of integration; they are not identical with the particle coordinates in the gap centre \( r_0(u) \) is inserted for \( r \) in the right hand side of (5) Integrating twice the resulting equation gives \( \frac{dr_1}{du} \) and \( r_1(u) \):

\[ \frac{dr_1}{du} = \frac{p_r}{m_0} = - \frac{eB}{m_0^2} \left[ \tilde{r}_0 \sin(u + u_0) + \tilde{r}_0' \left( u \sin(u + u_0) + \cos(u + u_0) \right) \right] \]
\[ + \frac{e}{m_0^2} \left[ r_1 - D \right] \sin(u + u_0) + \tilde{r}_0' \]

(A 11)

\[ r_1(u) = - \frac{eB}{m_0^2} \left[ - \tilde{r}_0 \cos(u + u_0) + \tilde{r}_0' \left( 2 \sin(u_1 + u_0) - u \cos(u + u_0) \right) \right] \]
\[ - \frac{e}{m_0^2} \left[ r_1 - D \right] \cos(u + u_0) + \tilde{r}_0' u + \tilde{r}_0'' \]

(A 12)

In the spirit of the present approximation which is restricted to expressions of the first order in the field, the following substitutions:

\[ \tilde{r}_0 + \tilde{r}_0' u + r \]
\[ \tilde{r}_0' = \frac{p_r}{m_0} \]

may be performed in expressions already multiplied by \( B \) This gives approximate solutions of the differential equations (2) and (3) From these together with (4) are formed the following expressions:
\( x_1^{II} = x_1^I + \frac{eb}{\omega} \left[ (r-r_1) \sin(\phi + \phi_o) + \frac{p_r}{m_o} \cos(\phi + \phi_o) \right] = m_o \hat{\rho}_o \)  \( (A\ 13) \)

\[
x_2^{II} = x_2^I - \frac{eb}{\omega} \left[ (r-r_1) \left[ \sin(\phi + \phi_o) + \cos(\phi + \phi_o) \right]
+ \frac{p_r}{m_o} \left[ \cos(\phi + \phi_o) - 2 \sin(\phi + \phi_o) \right] \right]
= \hat{\rho}_o - \hat{\rho}_o^I C_3
\]

\( x_1^{II} \) and \( x_2^{II} \) are constant along the characteristic curves; they are (approximate) solutions of (1).
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Fig 1  Real (----) and thin lens (---) trajectory for a reduced variable, e.g.
Longitudinal kinetic energy

Fig 2  Real (----) and thin lens (---) trajectory for phase-angle, \( \phi(z) + \phi_o = \omega t + \phi_o \).

Fig 3  Phase, \( \phi(z) + \phi_o \), and reduced phase \( Q_\phi = \phi(z) - z \frac{d\phi}{dz} + \phi_o \).