K₈,₉ FORM FACTORS

REVIEW OF THE THEORY
M.K. Gaillard

REVIEW OF EXPERIMENTAL RESULTS
L.M. Chouneț
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K$_{\Xi_3}$ FORM FACTORS

REVIEW OF THE THEORY
M.K. Gaillard*

REVIEW OF EXPERIMENTAL RESULTS
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GENEVA
1970

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PREFACE

The purpose of this report is to review and digest the abundant literature on $K_L$ decay, in order to present a coherent picture of the theoretical and experimental situations.

In Sections 1, 2, and 4, by M.K. Gaillard, the theoretical literature is reviewed and predictions are compared with experiment. In Section 3, by L.M. Choumet, the experimental results are presented and a fit to the decay parameters is made.

The preparation of the theoretical review was stimulated by encouraging and fruitful discussions with J.S. Bell and J. Prentki, and the experimental study received stimulation and encouragement from A. Lagarrigue and B. Aubert.

One of us (M.K.G.) is grateful to J.S. Bell for a critical reading of Sections 1 and 2, and to M. Jacob, M. Le Bellac and A. Martin for helpful conversations.

L.M.C. takes pleasure in thanking all of his co-workers from the X2 experiment; many of the methods used here to clarify the experimental situation in $K_L$ decay were developed in the course of the work of this collaboration.
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APPENDIX B: DISCUSSION OF RESULTS ON $\Gamma_{\mu 3}(K^+)$

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1. INTRODUCTION

Over the past few years there has been a proliferation of theoretical literature on the subject of $K_{L3}$ form factors, mainly based on current algebra and related calculational techniques. The purpose of this review is to attempt to separate the principal assumptions which determine the general behaviour of the form factors from calculational details and specific approximations which predict precise values for the relevant parameters. In Section 3, L.M. Choumet will discuss the present experimental picture to see whether it is possible to confirm or infirm the basic assumptions of current algebra.

1.1 Definition of the form-factors

We shall consider the following properties of $K_{L3}$ decays as established to within sufficient accuracy for our discussion:

i) the $|\Delta I| = \frac{1}{2}$ rule;

ii) CP conservation;

iii) a pure vector nature of the interaction.

Experimental limits on possible violations of the above postulates are discussed in the report by Rubbia\textsuperscript{1).} We further assume the validity of the Cabibbo theory\textsuperscript{2)} of semi-leptonic interactions. The transition matrix element for $K_{L3}$ decay is then given by:

$$
M = \frac{G}{\sqrt{2}} \sin \theta \, \bar{u}_{\nu} \, \gamma_\mu (1 + \gamma_5) \, u_\mu \, \langle \pi^- | V_\mu (0) | K^- \rangle ,
$$

(1.1)

where $G$ is the Fermi coupling constant and $\theta$ is the Cabibbo angle. Equation (1.1) serves to define the normalization of the strangeness changing vector currents $V_\mu^\pi$. Using the $|\Delta I| = \frac{1}{2}$ assumption it is sufficient to consider one decay mode. We define the form factors by:

$$
\begin{align*}
(2\pi)^3 \sqrt{4E_\pi E_\nu} \langle \, \pi^- (q) | V_\mu^\pi (0) | K^0 (k) \rangle &= \langle k + q \rangle \, f_+(t) + \langle k - q \rangle \, f_-(t) \\
&= f_+(t) + f_-(t)
\end{align*}
$$

(1.2)

$$
t = (k - q)^2 .
$$

For $t$-values in the decay region the form factors are usually approximated linearly by the expressions:

$$
f_\mu (t) = f_\mu \left( 1 + \lambda_\mu \, \frac{t}{\mu^2} \right) ,
$$

(1.3)

where $\mu$ is the pion mass. The ratio of the form factors $\xi(t)$ is also approximated by a linear form:

$$
\xi(t) = f_+(t)/f_-(t) = \xi(0) + \Lambda t/\mu^2 .
$$

(1.4)

In addition to the form factors defined in Eq. (1.2) we shall consider the combination:

$$
f(t) = f_+(t) + \frac{t}{M^2 - \mu^2} \, f_-(t) ,
$$

(1.5)
where $M$ is the kaon mass. The amplitude (1.5) is proportional to the matrix element of the current divergence:

$$d(t) = (M^2 - v^2) f(t) = (2\pi)^6 \sqrt{4E_K E_\ell} \langle \pi^+ | \bar{\psi}_\mu V^*_\mu(0) | K^0 \rangle.$$  \hfill (1.6)

Most theoretical predictions are in fact directly concerned with the functions $f_+$ and $f$, and predictions for $\xi(t)$ are extracted via the relation:

$$f_+(t) = \left(\frac{M^2 - v^2}{t}\right) \left[f(t) - f_+(t)\right]$$
$$f_+(0) = f(0). \hfill (1.7)$$

Precise predictions for $\xi(0)$ and $\lambda_+$ are very sensitive to the value assumed for $\lambda_+$ and to the detailed theoretical approximations, while the general behaviour of $d(t)$ depends more on the basic assumptions and less on the details of calculation. The situation is analogous for the experimental determination of the form factors. The value extracted for $\xi$ in many experiments is more sensitive to the value assumed for $\lambda_+$ than is, say, the slope of $f(t)$. Therefore, we shall often express theoretical results in terms of the predicted behaviour of $f(t)$ as a function of $t$, rather than in terms of $\xi$. We may also parametrize $f(t)$ linearly, \textit{via}:

$$f(t) = f_+(0) \left[1 + \lambda_+ t/v^2\right]. \hfill (1.8)$$

1.2 Dispersion relations and polology

We shall very briefly describe some of the dynamical assumptions that are often used in conjunction with current algebra to estimate the $K_{K\ell}$ form factors.

It is usually assumed that the form factors $f_+$ and $f_-$ or $f_+$ and $f$ satisfy at most once subtracted dispersion relations. The absorptive part of $f_+(t)$ is determined by spin one intermediate states and the absorptive part of $f$ is determined by spin zero intermediate states; the dispersion integrals for these amplitudes are therefore uncoupled. If only $K\pi$ intermediate states are taken into account, the once subtracted dispersion relations reduce to Omnès equations\textsuperscript{3}) with solutions:

$$f_+(t) = f_+(0) \exp \left[ \frac{1}{\pi} \int_0^\infty \frac{t}{t'} \frac{\delta_1(t')}{(t' - t - i\epsilon)} \right] \hfill (1.9a)$$
$$f(t) = f_+(0) \exp \left[ \frac{1}{\pi} \int_0^\infty \frac{t}{t'} \frac{\delta_0(t')}{(t' - t - i\epsilon)} \right], \hfill (1.9b)$$

where $\delta_1$ and $\delta_0$ are respectively the $p$-wave and $s$-wave phase shifts in the $\Delta I = \frac{1}{2}$ $K\pi$ scattering channel. The integrals (1.9) have been evaluated\textsuperscript{4}) using resonance or scattering length approximations. The immediate feature they present for values of $t$ in the decay region $[0 < t < (M - \nu)^2]$ is that the slope of the form factor will be positive or negative, depending on whether positive or negative phase shifts dominate the integral for small $t$. 


Unfortunately, the solutions (1.9), although generally used, are not always unique. They may be multiplied\(^3\) by a polynomial of degree \(\leq L + 1\) (or \(\leq L\) if \(L\) is an integer) where

\[
\lim_{t \to -\infty} \delta(t) = L^0 ,
\]

and with \(P(t = 0) = 1\). Unless it is known that \(L = 0\) nothing can be said about the behaviour of the form factors in the decay region.

A more common procedure is to approximate the dispersion integral by poles (zero-width resonances). For example, if pure \(K^*\) dominance is assumed, \(f(t)\) has no singularities; if \(f(t)\) is assumed finite for \(t \to \infty\), then it must be constant. One obtains for the form factors\(^7\):

\[
\begin{align*}
    f_+(t) &= f_+(0) \frac{M_*^2}{M_*^2 - t} \\
    f(t) &= f_+(0) \\
    f_-(t) &= -f_+(0) \frac{[M^2 - \mu^2]}{M_*^2} \frac{M_*^2}{M_*^2 - t} ,
\end{align*}
\]

where \(M_*\) is the \(K^*\) mass.

On the other hand, the notion of "asymptotic current conservation" is sometimes evoked to justify the assumption that the matrix element of the divergence should vanish for \(t \to \infty\), i.e. that \(f(t)\) is unsubtracted. This can be accomplished by assuming that the divergence is dominated by a single scalar pole (\(\pi\)-meson), giving

\[
    f(t) = f_+(0) \frac{M_*^2}{M_*^2 - t} .
\]

Note that pole dominance gives slopes which are positive and determined by the masses of the exchanged particles:

\[
\begin{align*}
    \lambda_+ &= \mu^2/M_*^2 \quad (1.12a) \\
    \lambda_0 &= \mu^2/M_\pi^2 \quad (1.12b)
\end{align*}
\]

If instead one assumes once subtracted dispersion relations and saturates the integrals with poles there is one additional parameter in each case:

\[
\begin{align*}
    f_+(t) &= f_+(0) + \frac{t h_\pi}{M_*^2 (M_*^2 - t)} \quad (1.13a) \\
    f(t) &= f_+(0) + \frac{t}{M_*^2} \frac{h_\pi}{(M_*^2 - t)} \quad (1.13b) \\
    f_-(t) &= (M^2 - \mu^2) \left[ \frac{h_\pi/M_*^2}{(M_*^2 - t)} - \frac{h_\pi/M_\pi^2}{(M_*^2 - t)} \right] , \quad (1.13c)
\end{align*}
\]
where \( h_*(h_\kappa) \) is the product of the \( K^*(\kappa) \) coupling constants to the current (divergence) and to the \( K\pi \) system. The \( \lambda \) parameters are still inversely proportional to the squares of the exchanged masses, but with unknown coefficients:

\[
f_+(0)\lambda_+ = \frac{h_\kappa}{M_\kappa^2} \frac{\mu^2}{M_\kappa^2} \quad (1.14a)
\]
\[
f(0)\lambda_0 = \frac{h_\kappa}{M_\kappa^2} \frac{\mu^2}{M_\kappa^2} \quad . \quad (1.14b)
\]

If both \( f_+(t) \) and \( f(t) \) are at most once subtracted, \( f_+(t) \) must be unsubtracted. If this is not the case, there is an additional subtraction constant. In the simple two pole model of Eqs. (1.12) one obtains:

\[
f(t) = f_+(0) + \left( f_+(0) + \frac{h_\kappa}{M_\kappa^2} \right) + \frac{h_\kappa}{M_\kappa^2} \frac{t^2}{(M_\kappa^2 - t)} \quad (1.15a)
\]
\[
f_+(t) = f_+(0) - \frac{h_\kappa(M^2 - \mu^2)}{M_\kappa^2(M_\kappa^2 - t)} + \frac{h_\kappa t(M^2 - \mu^2)}{M_\kappa^2(M_\kappa^2 - t)} \quad . \quad (1.15b)
\]

In this case \( \lambda_0 \) depends on the subtraction constant for \( f_+ \) rather than on the exchanged scalar mass.

If a \( \kappa \)-meson exists, it is probably heavier than the \( K^* \); therefore, if \( f_+ \) and \( f \) are at most once subtracted one would expect \textit{a priori} that \( \lambda_0 < \lambda_+ \). This implies that \( \xi(t) \) should be neither too large nor too rapidly varying [i.e. the slope \( \xi(0)\lambda_+ \) should never be too large]. In any case there is no \textit{a priori} reason for believing that the linear approximation for \( f(t) \) as well as for \( f_+(t) \) should not be valid. In this approximation the parameter \( \xi \) derived from Eq. (1.7) is constant and presumably represents an average value in the decay region. Since the above estimates are at best gross approximations, the values of say \( \xi(0) \) and \( \lambda_+ \), which are extracted by retaining the quadratic terms, are probably not too meaningful.

Two final remarks:

i) Some authors argue that the use of subtracted dispersion relations allows one to take into account the average effect of high mass states which are neglected in the pole approximation; expressions (1.13) or (1.15) are thus considered valid for small \( t \) but the \( t \to \infty \) behaviour may be less singular than these expressions indicate.

ii) Some authors have conjectured the existence of a \( K\pi \) scalar bound state with \( M < M_\kappa < M + \mu \). If this hitherto unobserved particle should exist one might have a more rapid variation of \( f(t) \). [The solution (1.9b) of the Omnes equation would also be modified in this case.]
2. THE ALGEBRA OF CURRENTS APPLIED TO \(K_{\ell\ell} \) DECAYS

The charged strangeness changing vector currents in Eq. (1.1):

\[ V_{\mu}^{\pm} = V_{\mu}^{q} \pm iV_{\mu}^{s} \]  

(2.1)

are assumed to belong to an SU(3) octet of vector currents \(V_{\mu}^{i} \) (\(i = 1, \ldots, 8\)) of which the strangeness conserving members \((i = 1, 2, 3, 8)\) are conserved in the absence of electromagnetic interactions. They are assumed to satisfy the commutation relations:

\[ [V_{\mu}^{i}(x), V_{\nu}^{j}(0)] \delta(x_{0}) = i f_{ijk} V_{\nu}^{k} \delta^{i}(x) + S.T. \]  

(2.2)

where \(f_{ijk}\) are the SU(3) structure constants and S.T. represents the possible presence of Schwinger terms \(^{3}\), i.e., terms containing space derivatives of \(\delta\)-functions. A charge operator is associated with each current:

\[ g_{i}(x_{0}) = \int d^{3}x \ V_{\mu}^{i}(x) \]  

(2.3)

which is assumed to be a generator of infinitesimal SU(3) transformations, and would be time-independent in the limit of SU(3) invariance. However, the algebra of charges \(^{3}\):

\[ [g_{i}(x_{0}), g_{j}(x_{0})] = i f_{ijk} g_{k}(x_{0}) \]  

(2.4)

is assumed to be valid independently of the extent to which SU(3) is broken.

The group which defines the current or charge algebra is enlarged to chiral SU(3) \(\otimes\) SU(3) by the inclusion of an octet of axial vector currents \(A_{\mu}^{i}\), of which the charged members are assumed to be responsible for axial transitions in semi-leptonic hadron decays. If we define the corresponding axial charges

\[ g_{i}^{n}(x_{0}) = \int d^{3}x \ A_{\mu}^{i}(x) \]  

(2.5)

the extended charge algebra includes the commutation relations \(^{3}\):

\[ [g_{i}(x_{0}), g_{j}^{n}(x_{0})] = i f_{ijk} g_{k}^{n}(x_{0}) \]  

(2.6)

\[ [g_{i}^{n}(x_{0}), g_{j}^{n}(x_{0})] = i f_{ijk} g_{k}(x_{0}) \]  

(2.7)

Similarly, the full algebra of current densities includes the relations:

\[ [V_{\mu}^{i}(x), A_{\nu}^{j}(0)] \delta(x_{0}) = [A_{\mu}^{i}(x), V_{\nu}^{j}(0)] \delta(x_{0}) = i f_{ijk} A_{\nu}^{k}(x) \delta^{i}(x) + S.T. \]  

(2.8a)

\[ [A_{\mu}^{i}(x), A_{\nu}^{n}(0)] \delta(x_{0}) = i f_{ijk} V_{\nu}^{k}(x) \delta^{i}(x) + S.T. \]  

(2.8b)

Note that if the Schwinger terms are \(c\)-numbers, they will not contribute to the commutator of a charge with a current:

\[ [g^{i}(x_{0}), V_{\mu}^{j}(x)] = i f_{ijk} V_{\nu}^{k}(x), \text{ etc.} \]  

(2.9)

Again, the above commutators are assumed correct independently of the validity of SU(3) \(\otimes\) SU(3) as a symmetry of the Lagrangian or of the S-matrix. On the other hand, some
authors consider the possibility that the Lagrangian is nearly invariant under the full chiral group. We shall attempt to make a clear distinction between results depending on the postulated current or charge algebra and results which depend on the approximate invariance of the Lagrangian under transformations of the chiral group or one of its subgroups \([SU(3) \text{ or chiral } SU(2) \otimes SU(2)]\).

### 2.1 SU(3) charge and current algebra

It was first noted by Fubini and Furlan\(^{10}\) that the commutator of the strangeness changing charges:

\[
S^+(x_0) = \int dx \ V_\psi(x) ,
\]

as given by Eq. (2.4), is a time-independent operator:

\[
[S^+(x_0), \ S^-(x_0)] = \mathcal{E} = \sqrt{3} \mathcal{E}_5 = Q + Y ,
\]

where \(Q\) and \(Y\) are, respectively, the charge and hypercharge operators. Sandwiching Eq. (2.11) between \(n^+\) states of momenta \(\vec{q}, \vec{q}'\), and isolating the contribution of the \(K_0^\pm\) intermediate state in the commutator, one obtains the following sum rule:

\[
G^2(\hat{q}) \delta(\vec{q} - \vec{q}') = \delta(\vec{q} - \vec{q}') \left[ 1 - \delta G^2(\hat{q}) \right] ,
\]

where

\[
G(\hat{q}) \delta(\vec{q} - \vec{k}) = (\pi^+(\hat{q}) | S^+(0) | K^0(\vec{k}))
\]

and

\[
\delta G^2(\hat{q}) \delta(\vec{q} - \vec{q}^*) = \sum_{\alpha = K^0} \left| (\pi^+(\hat{q}) | S^+ | \alpha) \right|^2 - \sum_{\beta} \left| (\pi^+ | S^- | \beta) \right|^2 .
\]

Since three-momentum is conserved, the momentum transfer is related to the external momentum \(\hat{q}\) by:

\[
t = \left[ E_{K}(\hat{q}) - E_{\pi}(\hat{q}) \right]^2
\]

with

\[
E_{K}^2 - E_{\pi}^2 = M^2 - \mu^2 .
\]

Then using the definitions of the form factors (1.2) and (1.5), the sum rule (2.12) may be written:

\[
\left[ 1 - t^2/(M^2 - \mu^2) \right]^{1/2} f(t) = 1 - \delta G^2 .
\]

This relation is purely a consequence of the current algebra. However, it is difficult to verify as such -- the matrix elements figuring in Eq. (2.14) are not easily accessible experimentally. Therefore Eq. (2.16) must be supplemented with dynamical assumptions, examples of which we discuss in the following.

#### 2.1.1 Approximate SU(3) symmetry

In the limit of SU(3) invariance, the generator \(S^\pm\) connects the pion only to kaon states and \(\delta G^2 = 0\). Furthermore, since \(\delta G^2\) involves the squares of "off-diagonal" or
"leakage" matrix elements, the correction is second order in the symmetry breaking. Evaluating Eq. (1.16) at \( t = 0 \), one obtains the Ademollo-Gatto theorem\textsuperscript{10,11}:

\[ f(0) = f_s(0) = 1 + O(\varepsilon_s^2) \]  \hspace{1cm} (2.17)

which states that the form factor \( f_s(0) \) is unrenormalized in the first order of the symmetry breaking (which we characterize by the parameter \( \varepsilon_s \)).

In the exact symmetry limit we would have \( M = \mu \) and \( t = 0 \) [cf. Eq. (2.15)]. In the broken symmetry situation we have a sum rule for every value of \( t \) in the interval \( 0 \leq t \leq (M - \mu)^2 \), (corresponding to \( q^2 \geq 0 \)), which essentially coincides with the decay region. Fubini and Furlan\textsuperscript{10} argued that \( \delta G^2 \) should be smallest in the limit \( \bar{q} \to \infty \) \( (t = 0) \) on the grounds that (a) the kinematic factors \( (E_p + E_{q-\beta})/E_p E_{q+\beta} \) in the matrix elements are smallest in that limit, and (b) the corresponding form factors are expected to increase with momentum transfer from \( t = 0 \) as the resonance region is approached.

It is nevertheless interesting to see what behaviour is predicted for the form factors if \( \delta G^2 \) is small \( (\delta G^2 \ll 1) \) for all values of \( \bar{q} \). From Eq. (2.16) we obtain\textsuperscript{12}

\[ f(t) = \left[ 1 - t^2/(M^2 - \mu^2) \right]^{1/2} \left[ 1 + O(\varepsilon_s^2) \right] \quad 0 \leq t \leq (M - \mu)^2 . \]  \hspace{1cm} (2.18)

It should be emphasized that Eq. (1.18) has meaning only for the indicated range of \( t \)-values where \( f(t) \) can be directly related to the matrix element of the generator. Furthermore, the corrections \( O(\varepsilon_s^2) \) cannot be zero; the resulting expression would violate the requirement\textsuperscript{13} that \( f(t) \) have singularities only on the cut \( (M + \mu)^2 \leq t \leq \infty \). However, if the corrections are of the order of a few per cent throughout the decay region one obtains the prediction that \( f(t) \) decreases with \( t \) in that region; in the linear approximation:

\[ \lambda_0 \approx \frac{1}{2}(M + \mu)^2 \approx -0.02 \]

\[ \xi \approx \left( [t^2/(M^2 - \mu^2)] \lambda_s/\mu^2 + 0.28 \right) . \]  \hspace{1cm} (2.19)

It should be noted that the kinematic quantity \( t^2/(M^2 - \mu^2) \) is itself of second order in the symmetry breaking and becomes as large as 30% in the decay region. One may, therefore, question the assumption that \( \delta G^2 \) is small. In any case, comparison of Eqs. (2.18) or (2.19) with experiment provides a direct measurement of \( \delta G^2(\bar{q}) \).

\textbf{2.1.2 Dominance of octet intermediate states}

In view of the empirical fact that mesonic resonances occur only in SU(3) octet and singlet states, it has been argued\textsuperscript{14,15} that octet intermediate states\textsuperscript{*} should also dominate the sum (2.14). Since octet states contribute only to the first term in the sum, one obtains with this argument\textsuperscript{16}:

\[ \delta C^2(\bar{q}) > 0 \]

\[ f(t) < \left[ 1 - t^2/(M^2 - \mu^2) \right]^{1/2} \quad 0 \leq t \leq (M - \mu)^2 . \]  \hspace{1cm} (2.20)

Under this assumption (2.19) represents upper limits for \( \xi \) and \( \lambda_0 \).

\textsuperscript{*} By "intermediate states" one really means states arising from connected diagrams in the commutator. The contributions from vacuum loops are cancelled out in the commutator.
However, for every direct diagram (Fig. 1) which contributes to the first term in the sum (1.14), there is a Z-diagram (Fig. 2) which contributes to the second term (with \( \beta = \hat{\alpha} + 2n^+ \)). In the diagram of Fig. 1, \( \hat{q} \to \infty \) corresponds to \( t_\alpha = (p_\alpha - q)^2 \to 0 \), whereas the momentum transfer at the vertices in Fig. 2 is

\[
\hat{t}_\alpha = (p_\alpha + q)^2 = (E_\alpha + E_q)^2 \xrightarrow{q \to \infty} 0 .
\]

The contribution of this diagram is

\[
\left( \frac{E_\alpha - E_q}{4E_n} \right)^2 \left( f_\alpha(\hat{t}_\alpha) \right)^2 \xrightarrow{\hat{q} \to \infty} \frac{1}{t} f_\alpha(\hat{t}_\alpha) \quad (2.21)
\]

where

\[
(2n)^3 \sqrt{\frac{E_n}{4E_n}} (\tau^+(\hat{q}) | a_\mu V_\mu^+ | \alpha) \equiv (\lambda_{\alpha}^2 - \mu_\alpha^2) f_\alpha(t_\alpha) .
\]

The Z-diagrams vanish in the \( \hat{q} \to \infty \) limit if \( f_\alpha(\hat{t}_\alpha) \) is at most once subtracted. In this case one obtains from the above assumptions

\[
f(0) < 1 . \quad (2.22)
\]

Since the above argument does not depend on the idea that the charges \( S^\pm \) are approximately conserved, it can be extended to values of \( t \) outside the decay region by using the current algebra, Eq. (2.2). Taking the commutator of time components (where Schwinger terms are presumably absent), and sandwiching the commutator between pion states of equal momentum, one obtains:

\[
\int d^4x \epsilon^{\mu\nu\rho\sigma} (\tau^+(q) | V_\rho^+(x) V_\nu^- (0) | \pi^+(q)) \delta(x_0) = \langle \pi^+(q) | V_0^+(0) | \pi^+(q) \rangle = 1 . \quad (2.23)
\]

Bjorken and Quinn\(^{13}\) have studied this sum rule setting \( \vec{p} \cdot \hat{q} = 0 \), and in the limit \( \hat{q} \to \infty \), where the Z-diagrams hopefully vanish. Note that for, say, a scalar state \( \alpha \) in the sum, one now needs the separate conditions that \( f_\alpha^{(s)}(t) \) be at most once subtracted and that \( f_\alpha^{(s)}(t) \) be unsubtracted in order that the Z-diagrams vanish for any value of \( \vec{p} \). In the limit \( \hat{q} \to \infty \) with \( \vec{p} \) finite, we have

\[
E_K - E_e \to 0 \\
t \to -|\vec{p}|^2 .
\]

Separating out the kaon intermediate state, one obtains the sum rule:
\[ (4E_k E_r)^{-1} \left[ (E_k + E_r) f_+(t) + (E_k - E_r) f_-(t) \right]^2 = 1 - \sum_{\alpha = K, P} \left| \langle \bar{\psi}(\bar{q}) | V_\alpha(0) | \sigma(\bar{q} + \bar{p}) \rangle \right|^2 \]
\[ + \sum_{\beta} \left| \langle \bar{\psi}(\bar{q}) | V_\beta(0) | \delta(\bar{q} - \bar{p}) \rangle \right|^2 \]
\[ E_k^2 - E_r^2 = M^2 - \mu^2 - \bar{p}^2 \]
\[ t = (E_k - E_r)^2 - \bar{p}^2 \]

(2.24)

which reduces to Eq. (2.12) for \( \bar{p} = 0 \). With the assumption that octet states dominate the sum on the right-hand side, we obtain the inequality:

\[ f_+(t) < 1 \quad t \leq 0 \quad . \]

(2.25)

Horn\textsuperscript{15} further exploited the sum rule (2.24) by assuming that the right-hand side < 1 for finite values of \( \bar{q} \) and \( \bar{p} \). Setting \( t = 0 \) with \( p^2 = (E_k - E_r)^2 \neq 0 \), he obtained the additional inequality:

\[ \xi(0) \leq 0.91/f_+(0) - 1.35 \quad , \]

(2.26)

which favours negative \( \xi \) values if \( f_+(0) \) is not too small, that is if the non-renormalization of Eq. (2.17) is not too badly violated. Horn's results also include Eq. (2.20) above, giving \( \lambda_0 < 0 \). However, it is not obvious that the \( Z \)-diagrams should be negligible for \( \bar{q} \) finite (\( \bar{t}_\alpha \) finite). Notice that if one takes the limit \( \bar{p} \to \infty \) with \( \bar{q} \) finite (\( t \to \infty \)), the leading term from the \( Z \)-diagram corresponding to a kaon intermediate state just cancels the leading term in the direct diagram, provided that

\[ \begin{aligned} 
\lim_{t \to \infty} f_+(t) &= \lim_{t \to \infty} f_-(t) 
\end{aligned} \]

so that nothing meaningful can be inferred from octet dominance concerning the limit \( |t| \to \infty \) for the form factors.

2.1.3 Saturation of the sum rule

Some authors have attempted to calculate explicitly the correction \( \delta G^2(\bar{q}) \) in Eq. (2.12) by saturating the sum with a particular choice of intermediate states\textsuperscript{17-19}. This was first done by Furlan et al.\textsuperscript{17}, who took the \( \bar{q} \to \bar{q} \) limit and saturated the commutator with pseudoscalar plus vector meson states, finding \( |f_+(0)|^2 \approx 1.06 \).

Subsequently, Fubini, Furlan and Rossetti\textsuperscript{20} developed a covariant technique for evaluating the sum rule. Their method exploits the charge-current, rather than the charge-charge algebra, and therefore contains more information. It depends, however, on the assumption that certain invariant amplitudes satisfy unsubtracted dispersion relations. They consider the amplitude:

\[ m_\pi(p) = (2\pi)^3 \sqrt{4E_1 E_2} \int d^4x \ e^{ipx} \epsilon^\mu(x) \langle \pi^+(q) | \left[ \nabla_\mu V_\nu(0), \ V_\nu^-(-q) \right] | \pi^+(q_2) \rangle \quad . \]

(2.27)
Integrating by parts, one obtains the relation:

$$\lim_{p\nu \rightarrow 0} m_{\nu}(p) = (2\pi)^3 \frac{\sqrt{4E_1 E_2}}{E_1} \left( \pi^+(q_1) \left[ S^+(0) , \ V^+_\nu(0) \right] \pi^+(q_2) \right)$$

$$= (2\pi)^3 \frac{\sqrt{4E_1 E_2}}{E_1} \left( \pi^+(q_1) \left[ V^+_\nu(0) \right] \pi^+(q_2) \right) = F_\nu(\Delta^2) (q_1 + q_2)_\mu ,$$

where $\Delta = (q_1 - q_2)$ and $F_\nu$ is the electromagnetic form factor of the pion; we have used the commutation relations of Eq. (2.9). Writing:

$$m_{\nu}(p) = m_1(\nu, \Delta^2)(q_1 + q_2)_\mu + m_2(\nu, \Delta^2)(q_1 - q_2)_\mu + m_3(\nu, \Delta^2) \ p_\mu$$

$$\nu = p(q_1 + q_2)$$

one obtains from Eq. (2.28):

$$m_1(\nu, \Delta^2) = F_\nu(\Delta^2)$$

$$m_2(\nu, \Delta^2) = 0 .$$

Fubini et al. then assume that the amplitudes $m_1$ and $m_2$ satisfy unsubtracted dispersion relations in $\nu$ for fixed $\Delta^2$, $p^2$ and $(\Delta - p)^2$. The absorptive part of $m_{\nu}$ is given by:

$$(2\pi)^3 \frac{\sqrt{4E_1 E_2}}{E_1} \frac{1}{4} \int e^{ipx} \left( \pi^+(q_1) \left[ S, V^+_\nu(x) \right] \pi^+(q_2) \right) d^4x .$$

The contribution of the single kaon intermediate state is:

$$\pi^+(q_1 + p)^2 - M^2 \left( (q_2 + p)^2 - M^2 \right) f(p) [f(\Delta + p)^2](q_1 + q_2 + p)_\mu + f(\Delta + p)^2 (\Delta + p)_\mu .$$

Fixing the variables $p^2 = 0$ and $(\Delta + p)^2 = \Delta^2$, we have $p \cdot \Delta = 0$, $(q_1 + p)^2 = \nu + \mu^2$. Then extracting the kaon pole term from the dispersion integral, one obtains using Eq. (2.30):

$$f_+(0) f_+(\Delta^2) = F_\nu(\Delta^2) - \frac{1}{\pi} \int d\nu^' \frac{\text{Im} \ m_3(\nu^', \Delta^2)}{\nu^'}$$

$$f_-(0) f_-(\Delta^2) = - \frac{1}{\pi} \int d\nu^' \frac{\text{Im} \ m_3(\nu^', \Delta^2)}{\nu^'} .$$

Since the integrands involve matrix elements of the divergences, they vanish in the SU(3) limit, giving $f_+(\Delta^2) = F_\nu(\Delta^2)$, and $f_-(\Delta^2) = 0$, as expected.

The above formalism has been applied by Bartl and Altes\textsuperscript{14} who saturate the commutator (2.31) with two particle states (vector meson plus pseudoscalar). They find:

$$f_+(0) = 0.96$$

$$f_-(0) = 0.11$$

(2.34)
In a similar calculation, Goni and Pascual retained terms linear in \( \Delta^2 \); their results (except for \( \lambda_+ \)) depend on a cut-off, giving, for particular values of the cut-off parameter:

\[
\begin{align*}
\lambda_+ &= 0.05 \\
\lambda_+ &= 0.05-0.97 \\
\xi(0) &= 0.024-0.065 \\
\lambda_- &= -(2-1.3).
\end{align*}
\]

Although these results have been derived for \( \Delta^2 \leq 0 \), it should be reasonable to extrapolate them to small positive \( \Delta^2 = t \). In spite of the large value for \( \lambda_+ \), \( \xi \) is small for small \( t \) \( [|\xi| < (0.1-0.2) \) for \( t < (M - \mu)^2 \). With the relatively high value found for \( \lambda_+ \) [about twice the value corresponding to \( K^0 \) dominance, Eq. (1.12a)], a positive slope is predicted for \( f(t) \) [\( \lambda_0 = 0.03-0.04 \)].

The results of Eqs. (2.34) and (2.35) agree with the Bjorken-Quinn inequality (2.25), but not with the Horn inequalities (2.20) and (2.26).

### 2.2 Chiral SU(3) ⊗ SU(3) algebra and the soft pion limit

Let us consider the following amplitude:

\[
m_\mu(q) = i(2\pi)^{\frac{1}{2}} \frac{\mu^2 - q^2}{(2\pi)^3} \sqrt{2E_k} \int d^4x \ e^{i q x} \langle T\{\bar{\psi}_u A^\mu(x), \ V_\mu^n(0)\}|\bar{K}^0(k)\rangle
\]

\[
= G_\mu(q^2, t) [k + q]_\mu + G_\mu(q^2, t) [k - q]_\mu, \quad t = [k - q]^2
\]

where

\[
A^-_\mu = A^-_6 - iA_\mu^2
\]

and \( f_\pi \) is the pion decay constant:

\[
\mu^2 f_\pi = (2\pi)^{\frac{3}{2}} \sqrt{2E_k} \langle \bar{\psi}_\mu \psi_\mu^- | \pi^- \rangle .
\]

For \( q^2 = \mu^2 \), the factor \( (\mu^2 - q^2) \) in Eq. (2.36) annihilates every contribution to the amplitude except the pion pole contribution (Fig. 3), so that \( m_\mu \) is just the transition amplitude (1.2) in this limit:

\[
G_\mu(q^2, t) = f_\pi(t) .
\]

On the other hand, upon integrating Eq. (2.36) by parts, we obtain in the limit \( q_\mu \to 0 \):

\[
\lim_{q_\mu \to 0} \left( \frac{m_\mu(q)}{(2\pi)^{\frac{3}{2}} \sqrt{2E_k}} \right) = -i \int d^4x \langle [[\bar{A}_\mu^i(x), V_\mu^n(0)]|\bar{K}^0(k)\rangle = -i \langle V_\mu^n(0) + iV_\mu^n(0)|\bar{K}^0(k)\rangle,
\]

\[
= k_\mu f_\pi/(2\pi)^{\frac{3}{2}} / \sqrt{2E_k}
\]

(2.39)
where $f_k$ is the $K_{-2}$ decay constant, and we have used the chiral charge algebra, Eqs. (2.8) and (2.9). From the definition (2.36), we have:

$$G_s(0, M^2) + G_s(0, M^2) = f_k/f_n \quad \text{(2.40)}$$

This result follows from current algebra alone; however, it is a statement about the amplitude at unphysical values of the kinematic variables: $q^2 = 0$, $t = M^2$. If we make the additional assumption that $G_s(q^2, t)$ is slowly varying in $q^2$ for fixed $t$ and small $q^2$ ($q^2 \ll \mu^2$), or, equivalently that $m(q)/q^2 - \mu^2$ is dominated by the pion pole (PCAC hypothesis):

$$G_s(q^2, t) \approx G_s(0, t) \quad \text{(2.41)}$$

Then we obtain the Callan-Treiman relation:\)

$$f_+(M^2) + f_-(M^2) = f_k/f_n \quad \text{(2.42)}$$

Let us first remark that (2.42) is essentially a statement about the divergence form factor $f(t)$:

$$f(M^2) = f_+(M^2) + \frac{M^2}{M^2 - \mu^2} f_-(M^2) = f_+(M^2) + f_-(M^2) \quad \text{(2.43)}$$

since in the spirit of the approximation (2.41), $\mu^2/M^2$ is negligible. This result can be understood from the fact that at $q^2 = 0$ only a scalar $K\pi$ transition can take place. Now if we assume the Cabibbo theory, with a single angle defining the weakly coupled currents, experiment tells us that:\)

$$f(0) = f_+(0) = f_k/f_n(1.28 \pm 0.06)^{-1} \quad \text{(2.44)}$$

Therefore, if $f(t)$ is assumed to be a smooth function, it is essentially determined by Eqs. (2.42) and (2.44); in a linear approximation:

$$\lambda_0 M^2/\mu^2 = f(M^2)/f(0) - 1$$

$$\lambda_0 \approx 0.28 \mu^2/\mu^2$$

$$\xi \approx [M^2 - \mu^2] (\lambda_0 - \lambda_0)/\mu^2 \quad \text{(2.45)}$$

### 2.8.1 PCAC supplemented with dispersion relations

As we have said, relations (2.43) and (2.44) are sufficient to determine the behaviour of $f(t)$ if it is a smooth function. Therefore, with these equations as a starting point, the use of dispersion relations will give us information about the dispersion parameters.
For example, the single $K^*$ dominance model of Eqs. (1.10) [with $f(t) = \text{constant}$] are
incompatible with the above results if Eq. (2.44) is to be considered valid to better than
25%.

If one assumes $\kappa$ dominance\textsuperscript{23-26} for the divergence form factor, the $\kappa$ mass is pre-
dicted from Eqs. (1.11), (2.42), and (2.44):

$$M_\kappa^2/[M_\kappa^2 - M^2] \approx 1.28$$

$$M_\kappa \approx 1.06 \text{ GeV}$$

(2.46)

Similarly, assuming once subtracted dispersion relations, the constraint (2.42) can
be used to determine a subtraction constant. For example, Nieh\textsuperscript{27} has assumed that the
combination $f_+ + f_-$ is once subtracted, and that $f_+ - f_-$ is unsubtracted. He also saturates
the integrals with a $K^*$ so that the form factors are given by Eqs. (1.13a) and (1.15) with
$h_K = 0$. The Callan-Treiman relation (2.42) gives one relation between the unknowns $f_+(0)$
and $f_-$; to determine the remaining unknown Nieh assumes $f_+(0) = 1$ [then $f_+/f_-$ is given
via (2.44)]; however, since $f_+(0)$ is not directly measurable, for comparison with experi-
ment it is sufficient to use (2.42) and (2.44). These two constraints along with $K^*$
saturation completely determine the variation of the form factors.

Similar considerations have been made by Pati and Sebastian\textsuperscript{26} who use $K^*$ and $\kappa$
saturation and assume $f_+(t)$ once subtracted; they consider the cases in which $f_-(t)$ is
unsubtracted and once subtracted. All these calculations necessarily give approximately
the results (2.45). When the variation of $\xi$ with $t$ is studied, it is found that if $f_-(t)$ is
not dominated by a single pole, $\lambda_\kappa$ may be an order of magnitude greater than $\lambda_+$, but in
this case $\xi(0)$ is very small so that $\xi(t)$ is always small and negative [which reflects the
fact that $f_+(t)$ rises slightly faster than $f(t)$, $\lambda_+ > \lambda_\kappa$; cf. Eq. (2.45)]. The values
found for $\lambda_\kappa$ are always close to the $K^*$ dominance value, $\lambda_+ \approx 0.023$. Kummer\textsuperscript{28}
has pointed out that if $f_+$ and $f$ are at most once subtracted, there is one relation between $\lambda_+$, $\lambda_\kappa$, and
$\xi(0)$ which directly tests $K^*$ and $\kappa$ saturation independently of the current algebra constraint.

Rather than assume the single angle Cabibbo theory, some authors\textsuperscript{6,24,26,28} use independent
estimates of the dispersion parameters to relate $f_+(0)$ and $f(0^2) = f_K/f_\pi$. The ratio of
the axial and vector angles is then determined from the empirical relation:

$$\tan \theta_v f_+(0) = (1.28 \pm 0.06)^{-1} \tan \theta_A f_K/f_\pi.$$  \hspace{1cm} (2.47)

Thus, for example, pure vector dominance $[f_+(t) = \text{constant}]$ gives $\theta_A/\tan \theta_v \approx 1.28$.
However, to abandon the elegance of the single angle theory on the grounds of rather crude
dynamical approximations does not seem very appealing to this reviewer.

\subsection{Dispersion relations supplemented with FDDAC}

Rather than assume PCAC [Eq. (2.41)] \textit{a priori}, some authors assume dispersion relations
in $q^2$ and in $t$, or in some combination of these variables, for the invariant amplitudes
defined in Eq. (2.36). Since the pion pole in $q^2$ has as residue functions the $K_{\pi\rho}$ form
factors, one obtains sum rules involving $f_\pm$ or $f$. 

For example, Fuchs\textsuperscript{29,41} assumes that $G_{\pm}(\mu^2 - q^2)$, as well as the amplitude

$$
\frac{f_{\eta}}{\mu^2} \frac{S(t, q^2)}{[\mu^2 - q^2]} = \left(2\pi\right)^{\frac{3}{2}} \sqrt{2k_0} \int d^4x \ e^{-ipx} \langle \{\mathcal{T}[\partial_\mu A_\alpha(0), \ \partial_\nu A_\mu(-x)]\} | \mathcal{R}(k) \rangle
$$

$$
p_\mu = [k - q]_\mu
$$

(2.48)

satisfy at most once subtracted dispersion relations in $q^2$ for fixed $t$ and in $t$ for fixed $q^2$; Srivastava\textsuperscript{30} assumes unsubtracted dispersion relations for the same amplitudes with the variable $\alpha q^2 + (1 - \alpha)t$ ($0 < \alpha < 1$) fixed. When these assumptions are supplemented with pole saturation ($\pi$ for $q^2$, $K^*$ and $\kappa$ for $t$) of the dispersion integrals, they give identical results. Without assuming pion pole dominance, these authors obtain a value for $f_{\eta}(0)$ in terms of $f_K/f_\pi$ and the dispersion parameters. With the additional assumption that matrix elements of $\partial_\mu A_\mu$ between physical states are dominated by the pion pole (PDDAC hypothesis)\textsuperscript{32}:

$$
\langle B | \partial_\mu A_\mu | A \rangle = \frac{\mu^2 f_{\eta}}{\mu^2 - q^2} \langle B | j_\mu | A \rangle \quad 0 \leq q^2 \leq \mu^2
$$

(2.49)

they recover the Callan-Treiman relation, and their results reduce to those discussed in the preceding section. Matrix elements of the type (2.49) contribute to the integral over the absorptive parts of $G_{\pm}$ for the dispersion relation in $t$.

Following Bjorken\textsuperscript{31}, Ademollo et al.\textsuperscript{33} assume unsubtracted dispersion relations in $q_0$ with $\mathbf{q}$ fixed for the amplitudes $\mu_\mu/(\mu^2 - \mu^2)$ and (2.48); specifically they set $\mathbf{p} = \mathbf{q} = \mathbf{k} = 0$, $p_0 + q_0 = M$. The dispersion integral in $q_0$ picks up contributions from both the transitions $K \rightarrow \pi$ and $K \rightarrow \pi \rightarrow \text{vacuum}$, so that they obtain a sum rule\textsuperscript{33} relating $\frac{\mu^2 f_{\eta}}{\mu^2 - q^2}$, $\frac{\mu^2 f_{K}}{\mu^2 - q^2}$, $f_K/f_\pi$ and other contributions such as $\pi K$ axial transitions (along with $K \rightarrow \pi \rightarrow \text{vacuum}$). However, in order to establish contact with experiment, they must make an $a \text{ priori}$ assumption about the behaviour of $f(t)$ with $t$, so that the direct connection with current algebra disappears.

2.2.3 Other extrapolation procedures

Rather than use dispersion relations to extrapolate the Callan-Treiman relation to the physical region for $t$, Mamm and Primakoff\textsuperscript{36} used the SU(3) sum rule (2.16), and parametrized $6G^2$ linearly in $t$, using (2.42) as a constraint. They find $\xi(t)$ large and rapidly varying. However, since agreement with the Callan-Treiman relation ($\lambda_0 > 0$) requires large "leakage" contributions (large $6G^2$), this calculation is very sensitive to the parametrization. The true function $6G^2$ must be such as to restore the correct analytic properties to $f(t)$, and neglect of this property is probably more serious if the contribution of $6G^2$ is important.

2.2.4 Relation to $K_{\ell 4}$ form factors

In a manner analogous to the derivation of the Callan-Treiman relation which relates $K_{\ell 3}$ to $K_{\ell 2}$ decay, $K_{\ell 4}$ can be related to $K_{\ell 2}$ decay in the limit of vanishing momentum for

\textsuperscript{*} For convenience, we refer to the assumption (2.49) as PDDAC (pole dominated divergence of the axial current), as opposed to PCAC (partially conserved axial current), by which we mean the assumption that any time-ordered product, such as (2.36), involving $\partial_\mu A_\mu$ is dominated by the pion pole. The Goldberger-Treiman relation\textsuperscript{12} follows from PDDAC, and the Adler-Weisberger relation\textsuperscript{12} may be derived using PDDAC alone.
one of the final state pions in $K_{2\pi}$. The $K_{3\pi}$ form factors for the axial part of the hadronic transition are defined by:

$$
\sqrt{2\pi}g_5q_\mu q_{-\mu}
\langle \pi^+(q_+) \pi^-(q_-) | K^0 | K^-(k) \rangle =

i\left[ (q_+ + q_-)_\mu F_1 + (q_+ - q_-)_\mu F_2 + (k - q_+ - q_-)_\mu F_3 \right],
$$

(2.50)

where $K^- = A^\mu - i A^5_\mu$, and the form factors $F_i$ are functions of the invariant variables

$$
t_+ = (k - q_+)^2; \quad s_\pi = (q_+ + q_-)^2; \quad s_\pi' = (k - q_+ - q_-)^2,
$$

three of which are independent:

$$
t_+ + t_- + s_\pi' = s_\mu + M^2 + 2\mu^2.
$$

Form factors with one pion off-mass-shell ($q_+ = \mu^2$, $q_- \neq \mu^2$) may be defined by means of a time ordered product of $K^-_\mu$ with $F_\mu A^5_\mu$, analogous to Eq. (2.56). Taking alternately $q_+ = 0$ with $q_+ = \mu^2$, and $q_- = 0$ with $q_- = \mu^2$, and using the charge-current algebra, one obtains the soft pion limits: \(^{\text{21,17}}\):

$$
q_+ = 0: \quad F_1 = F_2 = F_3 = 0 \quad \text{for} \quad s_\mu = \mu^2, t_+ = M^2, s_\pi = t_-
$$

(2.51)

$$
q_- = 0: \quad \left\{ F_1 + F_2 = \frac{-2f_\pi(s_\pi)}{f_\pi} \right\}
\quad \text{for} \quad s_\mu = \mu^2, t_- = M^2, s_\pi = t_+

(2.52)

where it has been implicitly assumed that the variation with $q^2$ is negligible. The variation of $F_3$ with the other variables can be accounted for by the presence of a pole in the $K_{3\pi}$ amplitude at $s_\pi = M^2$ (Fig. 4). The residue of the pole is given by the $K\pi$ scattering amplitude; for small $q_\pi$ the part of the pole term which is antisymmetric in $q_+$ and $q_-$ is determined by current algebra to be \(^{\text{38}}\)

$$
- \frac{f_\pi}{2f_\pi} \frac{t_+ - t_-}{s_\pi - M^2}.
$$

(2.53)

Since $f_\pi(s_\pi)$ contains no pole at $M^2$, Eqs. (2.52) and (2.53) require that the symmetric part of the residue vanish in either soft pion limit; the form factor $F_3$ may be parametrized as

$$
F_3 = F_{\text{pole}} + R = (t_+ - t_-) \frac{G(t_+, t_-)}{s_\pi - M^2} + \frac{G'(t_+, t_-)}{s_\pi - M^2} + R(t_+, t_-, s_\pi),
$$

(2.54)

where $G(M^2, M^2) = -(f_\pi/2f_\pi)$; $G'(M^2, s_\pi) = G'(s_\pi^2, M^2) = 0$. The function $R$ contains no pole contribution; Eq. (2.51) implies:

$$
R = -f_\pi/2f_\pi^2, \quad t_+ = t_- = s_\pi = M^2
$$

(2.55)
and from Eq. (2.52) we obtain:
\[ f_+(s_\ell) + f_-(s_\ell) = f_+ \left[ G(s_\ell, M^2) + R(s_\ell, M^2, s_\ell) \right] \xrightarrow{s_\ell \to M^2} f_K/f_n, \]  
(2.56)
in agreement with the Callan-Treiman relation.

It is usually assumed that \( F_1, F_2, G \) and \( R \) are slowly varying, and a small value for \( f_- \) is predicted\(^a\). On the other hand, one may use a model for the behaviour of the \( K_{\ell n} \) form factors to obtain predictions about the behaviour of the \( K_{\ell n} \) form factors. Thus in the Veneziano model of Roberts and Wagner\(^b\):
\[ F_1 = F_2 = -\frac{f_+(0)}{f_K} \frac{V[a_{K^*(t_2)}, a_{s}(s_\ell)]]}{V[a_{K^*(0), \frac{1}{2}}]} \]  
(2.57)
In \( F_3 \), only the K-exchange term of Fig. 4 is retained, and the residue function is given by the Veneziano \( K \pi \) scattering amplitude:
\[ F_3 = \frac{a_{K^*(t_2)} + a_{s}(s_\ell) - 1}{f_K^2} \frac{f_+}{f_2} \frac{V[a_{K^*(t_2)}, a_{s}(s_\ell)]]}{V[\frac{1}{2}, \frac{1}{2}]} \]  
(2.58)
The normalization of \( F_3 \), along with the trajectory constraints, \( a_{K^*}(M^2) = a_{\rho}(\mu^2) = \frac{1}{2} \), insures that Eqs. (2.53) to (2.56) are satisfied. These authors obtain \( \xi \) constant and small; the slopes of \( f_+ \) and \( f_- \) are essentially those of \( K^* \) dominance: \( \lambda_{\pm} = 0.023 \).

Harada\(^b\) instead uses for \( F_1 = F_2 \) a five point Veneziano amplitude in which the dependence on all but three variables \( (t_+, s_\pi, s_\ell) \) is suppressed. He obtains \( K^* \) dominance for \( f_+(t) \). The form factor \( F_3 \) is parametrized by:
\[ F_3 = \left( F_3 \right)_{RW} + \lambda \left[ a_{K^*(t_2)} + a_{s}(s_\ell) - 1 \right] F_1, \]  
(2.59)
where \( \left( F_3 \right)_{RW} \) is the kaon exchange term of Eq. (2.58), and \( \lambda \) is an arbitrary parameter.

Since \( F_1 \) has no pole at \( s_\ell = M^2 \), the second term in Eq. (2.59) vanishes at the double soft pion point: \( s_\ell = t_\pi = t_\ell = M^2; \ s_\pi = 0 \), and Eqs. (2.53) to (2.56) are still satisfied. For large \( \lambda \) (\( \lambda \sim 1 \)), this parametrization amounts to allowing a large variation for \( R \) in Eq. (2.54). It is clear that, independently of the model used, by allowing a large variation in \( R \) (and therefore in \( f_- \), if \( f_+ \) varies slowly) one may fit both the Callan-Treiman constraint \( [\xi (M^2) \text{ small}] \) and the experimental data which indicate a large negative \( \xi \) \( [\xi = -(0.5 - 1)] \) in the decay region. However, the somewhat pathological behaviour thus obtained for \( f(t) \) (see Fig. 5) remains to be explained in terms of the singularities of \( f(t) \).

![Fig. 5](image-url)
2.2.6 Can the Callan-Treiman relation be wrong?

Since there is some experimental evidence that the Callan-Treiman relation is violated, it is worthwhile to examine the assumptions from which it follows, namely:

a) SU(3) ⊗ SU(3) charge-current algebra

b) Pion pole dominance (PCAC).

No explicit assumption of strong interaction symmetry (other than isospin) has been used. However, assumption (b) is in fact related to approximate SU(2) ⊗ SU(2) invariance.

Consider the two amplitudes \( m_\mu(q) \) and \( S(q^2,t) \) defined in Eqs. (2.36) and (2.48). For \( q^2 = \mu^2 \), we have

\[
S(t,\mu^2) = p\mu m_\mu(q^2 = \mu^2) = (M^2 - \mu^2) f(t) \ .
\] (2.60)

However, for \( q^2 \neq \mu^2 \), we have the general relation:

\[
S(t, q^2) = p\mu m_\mu \frac{1}{\mu^2 f_\eta} \int d^4x \ e^{ip\cdot x} \langle \bar{q}_\mu | t^\nu q_\mu \rangle \langle [V^\nu_q(0)]^i | q_\nu \rangle, \ \ a_\mu A^\mu_n(0) \rangle [R^i] \]

\[= p\mu m_\mu (M^2 - q^2) G_+(q^2, t) + t G_-(q^2, t) \ .
\] (2.61)

In the limit of chiral SU(2) ⊗ SU(2) symmetry, \( a_\mu A^\mu_n = 0 \), so that \( S(t, q^2) = p\mu m_\mu \) for all \( q^2 \). However, in the limit of SU(3) symmetry, the axial divergence transforms according to the octet representation; setting \( \bar{p} = 0 \) in Eq. (2.61), the charge-divergence commutator gives:

\[
S(t, q^2) - p\mu m_\mu = \frac{1}{\mu^2 f_\eta} M^2 f_k / f_\eta \ .
\] (2.62)

In this case the zero pion mass limits of \( S \) and \( p\mu m_\mu \) are significantly different; it is clear that PCAC (dominance of the pion pole) cannot hold for any amplitudes at \( q^2 = 0 \). Explicitly the assumption

\[
S(t, q^2) \approx S(t, \mu^2) \approx (M^2 - \mu^2) f(t), \ \text{for} \ 0 \leq q^2 \leq \mu^2
\] (2.63)
gives in the SU(3) limit:

\[
f(M^2) \approx 0; \ \ \ g(M^2) \approx -1
\] (2.64)
as opposed to expressions (2.42) and (2.43), which follow from the assumption that \( p\mu m_\mu \) (or equivalently \( G_\pm \)) the explicit \( q^2 \) dependence in \( p\mu m_\mu \) is negligible) is a slowly varying function of \( q^2 \). Equation (2.64) is simply a statement of the fact that the divergence \( \delta_{\mu \nu} V^\nu_q \) vanishes in the SU(3) symmetric limit.

If, in fact, SU(2) ⊗ SU(2) is not a good symmetry, and the last term in Eq. (2.61) is not negligible, one must attempt to determine which, if either, of the amplitudes evaluated at \( q^2 = 0 \) gives a good approximation to the physical amplitude (2.60). First note that \( S \) and \( p\mu m_\mu \), considered as functions of \( t \) for fixed \( q^2 \), have the same absorptive part.
there is no contradiction\(^*\)) in using the PDDAC assumption of Eq. (2.49), from which it follows that:

\[
\text{Im } S(t, q^2) = \text{Im } p_\mu m_\mu \approx \left(M^2 - \mu^2\right) \text{Im } f(t), \quad 0 \leq q^2 \leq \mu^2.
\]  

(2.65)

Secondly, through the use of the Bjorken limit\(^*\)) \( p \to 0, \ p_0 \to \infty \) for studying the high \( t \) behaviour of these amplitudes, it has been shown\(^\text{**} \) that if certain amplitudes are not too singular at infinity, then

\[
\lim_{t \to -\infty} S(t, q^2) = \left(M^2 - \mu^2\right) \lim_{t \to -\infty} f(t)
\]

(2.66)

\[
\lim_{t \to -\infty} p_\mu m_\mu(q) = \left(M^2 - \mu^2\right) \lim_{t \to -\infty} f(t) - \frac{\left(\mu^2 - q^2\right)}{\mu^2 f_n} \sqrt{(2\pi)^2 k_0} \left[S^*(0, \beta_n \gamma_n\gamma(0))\right][R^3]
\]

for \( 0 \leq q^2 \leq \mu^2 \). The condition necessary for Eqs. (2.66) to hold is that amplitudes of the type:

\[
d_n[k - p_n q^2] = (2\pi)^3 4E_n k_0 \langle n | \beta_n \gamma_n(0) | \bar{R}^3 \rangle
\]

\[
d_n[k - p_n q^2] = (2\pi)^3 4E_n k_0 \langle n | \beta_n \bar{V}_n(0) | \bar{R}^3 \rangle
\]

(2.67)

are at most once subtracted:

\[
d_n(t)/t, \quad D_n(t)/t \xrightarrow{t \to -\infty} 0.
\]

(2.68)

If this condition is not satisfied, no conclusion can be drawn concerning the high \( t \) behaviour of the amplitudes \( S \) and \( p_\mu m_\mu \). Note that the condition on \( d_n(t) \) is that which is needed to eliminate 2-graphs in the \( q \to \infty \) limit of the \( SU(3) \) sum rule discussed in Section 2.1; if it is not satisfied, the Fubini-Furlan relation (2.17) is not very meaningful. The condition on \( D_n \) is implicit in the PDDAC assumption.

If we assume the validity of Eqs. (2.66), we may write once-subtracted dispersion relations in \( t \) for \( S \) and \( p_\mu m_\mu \) at fixed \( q^2 \) using the PDDAC relation (2.65). A similar dispersion relation may be written for the physical amplitude \( f(t) \); since the integral over the absorptive part is the same in all cases, we obtain from the asymptotic conditions (2.66):

\[
S(t, q^2) = \left(M^2 - \mu^2\right) f(t)
\]

\[
p_\mu m_\mu = \left(M^2 - \mu^2\right) f(t) + \left(\mu^2 - q^2\right) C/\mu^2 f_n
\]

(2.69)

---

\(^*\) It has been argued\(^1\)\(^2\) that even this assumption requires approximate \( SU(2) \otimes SU(2) \) invariance, because the contribution of higher mass states \( n \) to the dispersion relation in \( q^2 \) is proportional to \( M_n^2 \), cancelling the \( M_n^2 - q^2 \) suppression factor in the denominator; thus a symmetry principle is needed to suppress these contributions. However, there are also phase space effects for high mass (multi-particle) states which tend to reduce the \( M_n^2 \) dependence of the matrix element. For example an explicit calculation\(^1\)\(^2\) of the correction to the Goldberger-Treiman relation, in which no suppression mechanism was used, gave a theoretical correction to PDDAC much smaller than the observed discrepancy.
where $C$ is the commutator in Eqs. (2.66). To illustrate, let us assume that $SU(3) \otimes SU(3)$ chiral symmetry is broken by a term which transforms according to the representation $(3, \bar{3}) + (\bar{3}, 3)$:

$$ H = H_{\text{inv}} + \epsilon_0 \mu_0 + \epsilon_0 \mu_0. $$

(2.70)

In a quark model these symmetry-breaking terms correspond to the quark masses. The charge-divergence commutation relations given by Gell-Mann\(^5\) for this model give:

$$ C = g f_k M^2 = f_k M^2 \left( \frac{1 + \epsilon_0/\sqrt{2} \epsilon_0}{1 - \epsilon_0/2\sqrt{2} \epsilon_0} \right). $$

(2.71)

Using Eq. (2.69) and the soft pion limit for either $S$ or $p_\mu \cdot m_\mu$, we obtain:

$$ f(y^2) \approx \left[ 1 - g \right] f_k/f_\pi, $$

(2.72)

which reduces to the Callan-Treiman relation in the limit of chiral $SU(2) \otimes SU(2)$ invariance ($\epsilon_0 \approx -1/\sqrt{2} \epsilon_0$, $g \approx 0$); and vanishes [cf. Eq. (2.64)] in the $SU(3)$ limit ($\epsilon_0 \approx 0$, $g \approx 1$).

More generally, if $g < 1$ ($\epsilon_0 < \epsilon_0$), an appreciable negative value for $\epsilon$ can be understood within the context of current algebra and PDDAC.

Finally, let us comment briefly on assumption (a) above. We have emphasized (cf. Eq. (2.43) and the preceding discussion) that the Callan-Treiman relation imposes a constraint on the matrix element of the divergence $\bar{a}_\mu V_\mu$. Therefore, if we modify the strangeness changing current:

$$ V_\mu \rightarrow V_\mu + V_\mu^{(i)} $$

so as to modify the charge-current algebra, the behaviour of $f(t)$ required by the Callan-Treiman relation will not be changed unless $\bar{a}_\mu V_\mu^{(i)} \neq 0$. However, it can be shown that if a current contributes to the matrix element of the divergence for $t$-values in the decay region, it must also contribute to the charge matrix element. Such a modification could have unpleasant consequences for the Cabibbo theory of semi-leptonic decays. On the other hand, a modification of the axial charge which is needed in the soft pion limit would invalidate the successful predictions of the Adler-Weisberger relation.

2.3 Phenomenological Lagrangians

A method has been developed\(^5\) for reproducing the results of current algebra and PCAC through the construction of effective Lagrangians which implicitly satisfy these conditions. Exact $SU(2) \otimes SU(2)$ may be achieved either through the existence of parity doublets (an axial charge operator has non-vanishing matrix elements between states of opposite parity and equal mass), or through the existence of zero-mass pions (an axial charge operator connects a state to the same state plus one zero-energy pion). In the second case a Lagrangian which is non-linear in the pion field may be constructed in such a way as to be invariant under chiral $SU(2)$ transformations. The explicit expressions for the Heisenberg currents and associated charges will also be non-linear in the pion field. Strong processes as well as current matrix elements involving any number of pions may be
calculated in lowest order (defined as the sum of "tree diagrams", i.e. diagrams with no internal loops), and will necessarily satisfy the soft pion and current algebra constraints.

As an illustration, Zumino\(^{6}\) wrote down a simple model involving only the pion and kaon field, and no SU(3) symmetry. He obtained the following expression for the vector strangeness-changing current:

\[
V_\mu = i f_\pi (f_\pi f_K) \partial_\mu \left( \frac{1}{1 + \eta^2/f_\pi^2} \right) \vec{\pi} \cdot \vec{K} - i G/f_\pi (1 + \eta^2/f_\pi^2) \frac{1}{2} \partial_\mu \vec{\pi} \cdot \vec{K} \\
= i f_\pi (f_\pi f_K) \partial_\mu \left( \vec{\pi} \cdot \vec{K} \right) - i \left( G/f_\pi \right) \partial_\mu \vec{\pi} \cdot \vec{K} + O(\eta^3),
\]

(2.73)

where \(G\) is an arbitrary constant. For \(K_{K^*}\) form factors only the lowest order term in the pion field is relevant; one obtains immediately:

\[
f_\pi^+ = G/f_\pi; \quad f_\pi^-(t) + f_\pi^+(t) = f_K/f_\pi.
\]

Lee\(^{7}\) has developed a more sophisticated model incorporating the \(K^*\), and a residual SU(3) invariance. His Lagrangian is derived from one which is initially SU(3) \(\otimes\) SU(3) invariant. The symmetry breaking is imposed in such a way that PCAC [near chiral SU(2) invariance] is preserved. By imposing the field-current identity\(^{8}\), \(V_\mu^\tau = K_\mu^\tau\), he obtains essentially \(K^*\) dominance for \(f_\pi^+ (t)\), \(\lambda_+ \approx 0.018\), with

\[
f_\pi^+(0) = \frac{1}{2} \left( \frac{f_K}{f_\pi} + \frac{f_\pi}{f_K} \right) = 1 + \frac{(f_\pi^2 - f_K^2)}{2f_\pi f_K}.
\]

(2.74)

Since the last expression on the right is second order in SU(3) breaking, the Ademollo- Gatto theorem is satisfied. The Callan-Treiman relation is also satisfied:

\[
f_\pi(t) \approx 1 + \frac{t}{M^2 - M^2} \left( \frac{f_K}{f_\pi} - 1 \right).
\]

(2.75)

This result is identical to that obtained using \(K^*\)-saturated dispersion relations with the Callan-Treiman relation as a constraint. As discussed in Section 2.2.1, Lee finds \(\lambda_- > \lambda_+\), but \(\xi(0) \ll 1\) so that \(\xi(t)\) is small throughout the physical region.

2.4 Soft kaons; chiral SU(3) \(\otimes\) SU(3) as an approximate symmetry

In analogy with the off-shell pion amplitudes discussed in Section 2.2, one may define \(K_{K^*}\) form factors with the kaon off its mass shell:

\[
m_\mu^+ = i (2\pi)^3 \frac{(M^2 - k^2)}{2M^3 f_K} \sqrt{2q_0} \int d^4x \ e^{-ikx} \langle \pi^+(t) | a, \bar{K}_0^0(x), V_\mu^+(0) \rangle | 0 \rangle
\]

(2.76)

where \(\bar{K}_0^0 = A^6_μ + i A^7_μ\). For \(k^2 = M^2\), Eq. (2.76) is again the physical \(K_{K^*}\) amplitude:

\[
m_+ (M^2, t) = f_\pi^+.
\]

In the limit \(k_\mu \to 0\), one obtains from the charge-current algebra:

\[
m_-^+(0, \mu^2) - m_-^+(0, \mu^2) = f_\pi/f_K.
\]

(2.77)
Now, the extrapolation in $k^2$ from $0 \leq q^2 \leq M^2$ is rather large (or in dispersion relation language, an a priori estimate of the relative contribution from the cut starting at $k^2 = (M + 2\mu)^2$ is $\leq 0.4$, compared with $\leq 0.4$ for the cut starting at $(3\mu)^2$ in the soft pion case). If one nevertheless assumes kaon pole dominance for the amplitude (2.76), one obtains

$$f_+(\mu^2) - f_-(\mu^2) = f_{\pi}/f_K \ .$$

(2.78)

We may also consider the amplitudes

$$p_{\mu} \, m_{\mu} = (k^2 - \mu^2) \, G_\pi + t G_\pi'$$

(2.79)

and

$$S'(k^2, t) = - (2\pi)^2 \frac{(M^2 - k^2)}{M^2 f_K} \sqrt{2q_0} \int d^4x \, e^{-ikx} \langle \pi'(q)|T(\partial_\alpha \partial_\beta(x), \partial_\beta \partial_\alpha(0))|\pi'\rangle \ .$$

(2.80)

If we assume a symmetry breaking of the form (2.70), we have the relation, analogous to (2.61):

$$p_{\mu} \, m_{\mu} = S' - \frac{(M^2 - k^2)}{M^2 f_K} \frac{1}{g} \, \mu^2 f_{\pi} \ .$$

(2.81)

However, in this case, since we want to extrapolate from $0 \leq k^2 \leq M^2$, the explicit $k^2$ dependence in Eq. (2.79) is not negligible. Since

$$S'(M^2, \mu^2) = (M^2 - \mu^2) \, f_+(\mu^2) \sim M^2 \quad \left[ f_0(0) \sim 1 \right]$$

a self-consistent picture emerges if one assumes that $G_\pi'$ and $S'$ do not vary much with $k^2$, and

$$g \sim \mu^2/M^2 \ ,$$

(2.82)

where $g$ is defined in Eq. (2.71). This means that the violation of current conservation for the axial currents is measured roughly by the squared mass of the associated pseudoscalar meson. It also means that the conservation of the strangeness changing currents is equally good (or bad) for the axial and vector currents.

In order to extract any further information from soft meson limits one must have some consistent way of treating the symmetry breaking. If we assume $\pi$ and $K$ pole dominance for all the invariant amplitudes:

$$G_\pi(q^2, t) = G_\pi(k^2, t) = f_+(t)$$

$$S(q^2, t) = S(k^2, t) = (M^2 - \mu^2) \, f(t) \ ,$$

we obtain by comparing Eqs. (2.61) and (2.81):

$$g = \frac{\mu^2}{M^2} \frac{f_+}{f_K} \ ; \quad f_+(t) = 1 \ .$$

(2.83)
This corresponds to near SU(2) \( \otimes \) SU(2) symmetry as discussed in Section 2.2.5. However, strict pole dominance must be relaxed if we want to account for the variation of \( f_\pi(t) \) with \( t \).

### 2.4.1 Phenomenological Lagrangians

In the full symmetry limit it is not difficult to extend the phenomenological Lagrangian technique considered in Section 2.3 to SU(3) \( \otimes \) SU(3) by making the Lagrangian non-linear in the full octet of pseudoscalar mesons. All soft meson limits will be satisfied for matrix elements derived from the Lagrangian. However, in this approach the symmetry limit requires zero-mass mesons, which may be a good approximation for the pion, but certainly not for the kaon. Furthermore, it will not do to treat all symmetry breaking effects as small, since results are sensitive to the ratios of the unknown symmetry breaking parameters (such as \( \varepsilon_3/\varepsilon_8 \) or \( g \)). Note that in the full symmetry limit \( (M^2 = \mu^2, f_\pi = f_\pi) \), the soft meson results reduce to identities already required by vector current conservation: \( f_{\pi}(0) = 1, f_{\pi}(0) = 0 \).

In the framework of phenomenological Lagrangians with broken chiral symmetry, one can impose the operator form of PCAC

\[
\lambda_\mu^i = c_i \phi_i / \sqrt{2} \quad \text{for } i = 1, \ldots, 8 \tag{2.84}
\]

where \( \phi_i \) is the field corresponding to the \( i \)th member of the octet: and \( c_i \) is a constant \( (c_1, c_3, c_8 = \mu^2 f_\pi; c_4, c_5, c_6, c_7 = M^2 f_\pi) \). However, Eq. (2.84) does not insure pole dominance of the amplitudes involving \( \lambda_\mu^i \) in the dispersion relation sense (where the residue function is evaluated on the mass shell) unless the vertex functions involving \( \phi_i \) are constant (or at least "slowly varying") with respect to the momentum carried by the field. We have seen that strict pole dominance in all channels \( (k^2, q^2, t) \) is not self-consistent; for example, Eq. (2.83) says that \( \pi \) and \( K \) dominance precludes \( K^* \) dominance for \( f_\pi(t) \). However, once one relaxes pole dominance, almost any result can be obtained.

This is illustrated by the work of Arnowitt et al., who impose Eq. (2.84) as well as "PCVC" for the strangeness changing vector current

\[
i\lambda_\mu^i = f_\pi \lambda_\mu^i \phi_i \quad \text{for } i = 4, 5, 6, 7 \tag{2.85}
\]

where \( \phi_\pi \) is the field associated with the \( \pi \)-meson. They also use field-current identities\(^{51}\) for the vector and axial vector mesons \((K^*, A_1, K_A)\). They are able to fit a large negative \( \xi \)-value \((\lambda_0 < 0)\) by allowing a strong momentum dependence in the vertex functions for pseudoscalar and scalar coupling; explicitly, they allow derivative couplings with coupling constants proportional to the inverse masses. Thus in the above-mentioned fit they have neither pion pole dominance for the amplitude (2.36) nor \( \kappa \) dominance of the divergence of \( \lambda_\mu^i \) (either of which requires \( \lambda_0 > 0 \)) in spite of the operator equations (2.84) and (2.85).

### 2.4.2 Hard meson treatment of the divergences

A technique for treating the vertex functions involving currents and their divergences has been developed by several authors\(^{52,53}\) who use a principle of "maximum smoothness". They separate out the propagators which give rise to poles in the various channels (and
assume one particle dominance of the propagator for each set of quantum numbers), and
assume that the residue functions are as slowly varying as is consistent with constraints of the type (2.61) and (2.81).

The divergences are treated systematically by assuming a chiral symmetry breaking Hamiltonian of the form (2.70), as well as operator PCAC and PCVC, Eqs. (2.84) and (2.85). Using the transformation properties of the scalar densities $u_j$ and the pseudoscalar densities $v_j$:

\[
\begin{align*}
[\mathcal{S}_j, u_j] &= -i\delta_{j,k} v_k \\
[\mathcal{S}_j, v_j] &= i\delta_{j,k} u_k \\
[\mathcal{S}_j, u_k] &= if_{ijk} u_k \\
[\mathcal{S}_j, v_k] &= if_{ijk} v_k
\end{align*}
\]  
(2.86)

one obtains the following expressions for the relevant current divergences:

\[
\begin{align*}
\delta_\phi A^-_u &= -\frac{1}{\sqrt{3}} (\sqrt{2} \epsilon_\phi + \epsilon_\pi) (v_1 - iv_2) = \mu^2 f_\pi \phi_+ \\
\delta_\phi K^0 &= -\frac{1}{\sqrt{3}} (\sqrt{2} \epsilon_\phi - \epsilon_\pi/2) (v_3 - iv_4) = M^2 f_K \phi_0 \\
i\delta_\phi V^+_u &= -\frac{\sqrt{3}}{2} \epsilon_\pi (u_4 - iu_3) = M^2 f_\pi \phi_-
\end{align*}
\]  
(2.87)

The "renormalization constants" $Z^\pm_1$ are defined by:

\[
\begin{align*}
\phi_i &= Z_i^{-1} v_i & i = 1, \ldots, 8 \\
\phi_+^i &= Z_i^{-1} u_i & i = 4, \ldots, 7
\end{align*}
\]  
(2.88)

Thus $Z^\pm_1 = \sqrt{3} \mu^2 f_\pi/(\sqrt{2} \epsilon_\phi + \epsilon_\pi)$, etc. Note that if $Z^\pi_\pi = Z^K_\pi$, since $f^K_\pi = f_\pi$ and $M^2 >> \mu^2$, we must have $\epsilon_\pi = \sqrt{2} \epsilon_\pi [\text{approximate chiral SU(2) symmetry}]$, while the SU(3) limit for the Lagrangian ($\epsilon_\pi << \epsilon_\pi$) requires $Z^K_\pi/Z^K_\pi = \mu^2/M^2 << 1$.

There are two unknown symmetry breaking parameters $\epsilon_\pi$ and $\epsilon_\phi$, so we obtain from Eqs. (2.87) one relation among the $Z_i$'s in terms of the physical masses and decay constants:

\[
f^\pi \mu^2 Z^-_\pi = f^K_\pi M^2 Z^-_\pi + f^K_\pi M^2 Z^-_K
\]  
(2.89)

A second relation is obtained by considering the propagator functions:

\[
\Delta_n(q) = i \int d^4x \ e^{i\mathbf{x} \cdot \mathbf{q}} \langle T(\phi_n(x), \phi_n(0)) \rangle, \text{ etc.}
\]  
(2.90)

Using the divergence conditions (2.87) and the commutation relations (2.86), one obtains in the limit $q \mu \to 0$

\[
\Delta_n(0) = -\frac{1}{\mu^2 f_\pi} \langle [\phi_+, \phi_n^-] \rangle = \frac{Z^-_n}{\mu^2 f_\pi} \sqrt{2} \langle u_n \rangle - \langle u_n \rangle
\]  
(2.91a)
Similarly:

$$\Delta_k(0) = \frac{Z_k^{-1}}{M_k f_k} \frac{2}{\sqrt{3}} (\sqrt{2} \langle u_k \rangle - \langle u_k \rangle / 2) \quad (2.91b)$$

$$\Delta_k(0) = \frac{Z_k^{-1}}{M_k f_k} \sqrt{3} \langle u_k \rangle . \quad (2.91c)$$

If we assume that the propagators are dominated by a single particle: \( \Delta_1 = (p^2 - M_1^2)^{-1} \), we obtain upon elimination of the vacuum expectation values \( \langle u_k \rangle \):

$$f_k Z_n^{1/2} = f_k Z_k^{1/2} + f_n Z_n^{1/2} \quad (2.92)$$

Eliminating \( Z_k \) from Eqs. (2.89) and (2.92) gives the relation

$$M_k^2 f_k^2 = \nu^2 f_n^2 + M_k^2 f_k^2 - f_k f_n (\frac{M_k^2}{Y} + \nu^2) = \nu^2 f_n^2 \left( 1 - \frac{1}{g} \right) + M_k^2 f_k^2 (1 - g) \quad (2.93)$$

where

$$y = Z_n^{-1} / Z_k^{-1} = g M_k f_k / \nu^2 f_n .$$

Note that \( y \) is not uniquely determined by Eq. (2.93); if \( y \approx 1 \) (near chiral symmetry) is a solution, so is \( y \approx M^2 / \nu^2 \) [near SU(3) symmetry]. Independently of \( y \), Eq. (2.93) gives an inequality for the \( \kappa \) mass:\n
$$\left| M_k f_k \right| \leq \left| \mu f_n \right| - \left| M f_k \right| \quad g > 0 \quad (2.94)$$

$$\geq \left| \mu f_n \right| + \left| M f_k \right| \quad g < 0 .$$

The solution for \( g \) [Eq. (2.83)] and that given by Gell-Mann et al. correspond to the first inequality.

Next we consider the three-point function:

$$\left\langle \phi_n(x) \phi_k(y) \phi_{k'}(0) \right\rangle \quad (2.95)$$

which is related to \( K_{k3} \) form factors by:

$$\left( \mu^2 - q^2 \right) f(t) = G(M^2, \mu^2, t) f_k M_k^2 / (M_k^2 - t) . \quad (2.96)$$

In, for example, the limit \( q \mu \to 0, t \to \infty \) identifications (2.87) and the commutators (2.86) give:

$$G_0(t, 0, t) / (\mu^2 - t) (M_k^2 - t) = - \frac{1}{f_x} \left[ \frac{1}{x} \Delta_k(t) - x \Delta_k(t) \right], \quad x = \sqrt{z_k / z_x} . \quad (2.97)$$

Assuming that the propagators are dominated by a single pole, Eq. (2.97) becomes:

$$q \mu \to 0: \quad G_0(t, 0, t) = \frac{1}{f_x} \left[ \frac{1}{x} \left( \mu^2 - t \right) - x (M_k^2 - t) \right] . \quad (2.98a)$$
Similarly, taking alternatively $k_\mu$ and $p_\mu = k_\mu - q_\mu$ to zero, we get:

\[
\begin{align*}
k_\mu \to 0: & \quad G_{(0,0,0)} = \frac{1}{f_\pi} \left[ \frac{X}{X} (\mu^2 - \tau) - \frac{X}{Y} (M_\pi^2 - \tau) \right] \\
p_\mu \to 0: & \quad G_{(0,0,0)} = \frac{1}{f_\pi} \left[ -X(\mu^2 - \tau) + \frac{1}{X} (M^2 - \tau) \right].
\end{align*}
\] (2.98)

By setting $t = 0$ in Eqs. (2.98) and comparing the three expressions for $G(0, 0, 0)$ one obtains two consistency equations, but these are not independent of Eqs. (2.89) and (2.92) above.

The next step is to assume that $G_3$ is at most linear in the squared four-momenta:

\[
G_3 = a + bk^2 + cq^2 + dt.
\]

This is equivalent to assuming that the amplitude (2.95) satisfies once subtracted dispersion relations in each variable, saturated with the appropriate single meson state. The coefficients $a, b, c, d$ are determined by Eqs. (2.98). Then using Eq. (2.96) we obtain an expression for the form factor $f(t)$:

\[
(M^2 - \mu^2) f(t) = (M^2 - \mu^2) f_+(0) + \frac{t}{M_\pi^2 - \tau} \left\{ (M_\pi^2 - M^2) \left[ \frac{f_+}{f_\pi} - f_+(0) \right] + \mu^2 [y - f_+(0)] \right\}.
\] (2.99)

\[
f_+(0) = f(0) = (f_+^2 + f_+^2 - f_+^2)/2f_\pi f_\pi.
\] (2.100)

The last term in Eq. (2.99) is negligible if $y \approx f_+(0) \approx 1$ [near $SU(2) \otimes SU(2)$ symmetry]. In this case the Callan-Treiman relation (2.92) is satisfied to within terms of order $\mu^2$:

\[
(M^2 - \mu^2) f(M^2) = \frac{f_+}{f_\pi} M^2 + \mu^2 \frac{yM^2/M_\pi^2 - f_+(0)}{(1 - M^2/M_\pi^2)}.
\]

Note that at the soft pion point $t = M^2$, the pole term $(1 - M^2/M_\pi^2)^{-1}$ arises only from the $q^2$ dependence of the function $G_3 : c = 1/f_\pi \left[ y - f_+(0) \right]$. For strict PCAC ($c = 0$), $f(M^2)$ remains finite for $M_\pi^2 \to M^2$, because in this case the $K\pi$ coupling is proportional to $(K|\bar{q}_\mu A_\mu_\pi|K) = M_\pi^2 - M_K^2$. If one breaks chiral symmetry in these models what we call PCAC [pole dominance of, say, Eqs. (2.95)] and PDDAC [Eq. (2.49)] are simultaneously violated (as in the work of Arnovitt et al. described in Section 2.4.1); thus the considerations of Section 2.2.5, where we assumed PDDAC independently of chiral SU(2) $\otimes$ SU(2), are not valid here. Finally, we remark that the $K$ meson may be removed from the theory by letting $M_K^2 \to \infty$, $f_+ \to 0$ with $f_+ M_K^2$ finite. Then for $y \approx 1$, Eqs. (2.99) and (2.100) reduce to expressions (2.74) and (2.75), derived from the phenomenological Lagrangian model of Lee. In general, the techniques of this section are completely equivalent to a phenomenological Lagrangian formalism in which the divergence conditions (2.87) are imposed.

One may attempt to evaluate explicitly the form factors $f_\pm(t)$, by considering the amplitude:

\[
- (\mu^2 - q^2) (M^2 - k^2) \int d^4x \, d^4y \, e^{iKx} \, e^{-iky} \langle T(\phi_+(x) \phi_-(y) \phi_+(0)) \rangle
\]

\[
= (k + \bar{q})_\mu G_\mu(k^2, q^2, t) + (k - \bar{q})_\mu G_\mu(k^2, q^2, t)
\] (2.101)
which satisfies the relation:

\[
(k^2 - q^2) G_\pi^\prime + tG_\pi^\prime = f_\pi M_\pi^2 G_\pi / (M_\pi^2 - t) - \frac{1}{g} \frac{(M^2 - k^2)}{M^2 f_\pi} \mu^2 f_\pi + \frac{g(u^2 - q^2)}{\mu^2 f_\pi} M^2 f_\pi
\]  

(2.102)

obtained upon partial integration with respect to \( x \), using the commutators (2.86) and the assumption that the \( \Delta_i \) [Eq. (2.90)] are pole dominated. For \( k^2 = M^2 \) or \( q^2 = \mu^2 \), Eq. (2.102) reduces to (2.61) or (2.81), respectively. Isolating the \( K^* \) and \( \kappa \) poles in \( G_\mu^\prime \) one obtains:

\[
G_\pi^\prime = h_{\pi}(k^2, q^2, t) / (M_\pi^2 - t), \quad G_\kappa^\prime = h_{\kappa}(k^2, q^2, t) / (M_\kappa^2 (M_\pi^2 - t) + \frac{h_{\kappa}(k^2, q^2, t)}{M_\kappa^2 - t}) \]  

(2.103)

Eq. (2.102) one sees) that \( h_{\pi} \) must be independent of \( k^2 \) and \( q^2 \) and that \( h_{\kappa} \) and \( h_{\kappa}' \) must be independent of \( t \). This is completely equivalent to the dispersion relation considerations of Section 2.2.2. For example, Fuchs indicated that \( G_\pi^\prime \) is independent of \( q^2 \) and \( G_\kappa^\prime \) unsubtracted in \( t \), by assuming the relevant invariant amplitudes to be at most once subtracted in either variable. [Srivastava also obtained expression (2.100) for \( f_\pi(0) \) by considering the amplitudes with the \( K \) off its mass shell and using dispersion relations similar to those of Fuchs.] Thus for the physical form factors one obtains an unsubtracted dispersion relation for \( f_\pi(t) \), and a once subtracted dispersion relation for \( f_\pi(t) \) and \( f(t) \) as given by Eqs. (1.13), with

\[
h_{\pi}(k, q, t) / (M_\pi^2 - t) = g_{\pi \pi KK*} \frac{(M^2 - \mu^2)}{M_\pi^2}, \quad h_{\kappa}(k, q, t) / (M_\kappa^2 (M_\pi^2 - t)) = g_{\kappa KK} f_{\pi}
\]

where \( g_{\pi \pi KK*} = (k^+ / q^+). \)

The coupling constant \( g_{\pi \pi KK*} \) is known from experiment, and \( g_{\pi \kappa KK} = G_\pi(M^2, \mu^2, M_\pi^2) \) is given by Eqs. (2.98). One may attempt to determine the dispersion parameters further by assuming, for example, that \( f(t) \) is unsubtracted in \( t \). Then \( d = 0 \) in the expression for \( G_\pi \) giving more constraint on the parameters. This determines the \( \kappa \) mass as discussed in Section 2.2.1 (\( M_\kappa \approx 1 \) GeV if \( y \approx 1 \)).

Other authors have used spectral function sum rules to determine \( g_{\pi} \) and \( f_{\pi} \). This involves saturating spectral functions of the type \( J_\mu^A(s) \) with the appropriate spin 1 and spin 0 states, and using current algebra constraints to relate the unknown couplings \( (g_{\pi}, f_{\pi}) \) to the known \( (g_\mu, f_\mu) \). This gives \( f_{\pi} / f_\pi = 0.58 \), and from Eqs. (2.94) \( M_\kappa < 670 \) MeV if \( g > 0 \).

In any case the conclusions of Section 2.2 remain valid here; if the Callan-Treiman relation is satisfied, all these calculations give a form factor \( f(t) \) which increases with \( t \) from \( f(0) = f_\pi(0) \) to \( f(M^2) = f_\kappa / f_\pi = 1.28 f_\pi(0) \).

2.4.3 Expansion in the symmetry breaking parameters

Dashen and Weinstein have taken the point of view that both \( \epsilon_0 \) and \( \epsilon_8 \) are small parameters, i.e. that the Lagrangian is nearly SU(3) \( \otimes \) SU(3) symmetric -- and it is therefore legitimate to make a perturbation expansion in terms of these parameters. They do

---

*) Glashow and Weinberg assumed that \( G_\pi''(k^2, q^2, 0) = G_{\pi'}(0, 0, 0) \) and deduced from relation (2.102) pole dominance for the propagators \( \Delta_1^\prime \). We have done the inverse here.
not assume a specific symmetry breaking interaction; here $\epsilon_0$ measures the general SU(3) $\otimes$ SU(3) breaking (which gives rise to a common mass for the pseudoscalar mesons) and $\epsilon_8$ measures the SU(3) breaking (which splits the masses). Then writing:

\[(M^2 - \mu^2) f(t) = a_0 + a_1 t + a_2 t^2 + \ldots \]  

(2.104)

they find

\[a_0 = (M^2 - \mu^2) + O(\epsilon^3)\]  

(2.105)

\[a_1 = \frac{1}{2} \left( \frac{f_K}{f_\pi} - \frac{f_\pi}{f_K} \right) + O(\epsilon^3),\]  

(2.106)

where $\epsilon$ is $\epsilon_0$ or $\epsilon_8$. Equation (2.105) is just the Ademollo-Gatto theorem, $f_\pi(0) = 1 + O(\epsilon_8^0)$, since $(M^2 - \mu^2)$ is $O(\epsilon_8)$). Equation (2.106) is obtained by noting that pion and kaon PCAC are exact in the full symmetry limit; therefore, the variation of $G_\pi$ in Eq. (2.95) with $q^2$ and $k^2$ is $O(\epsilon^3)$ (since the leading term is $O(\epsilon)$ because $\frac{\alpha}{\mu} V_{2\mu}$ is $O(\epsilon_8)$) and can be dropped in determining $a_1$. Rather then give the general derivation of Eq. (2.106) (which is independent of the existence of a $\kappa$ meson), we simply note that it is formally satisfied by Eq. (2.99) above.

First we note that $f_K/f_\pi - f_\pi(0)$ and $y - f_\pi(0)$ are $O(\epsilon_8)$ ($y = 1$ in the SU(3) limit $M^2 = \mu_\pi^2$). But in this theory $M^2$ and $\mu^2$ are $O(\epsilon_8)$ so that we may drop the terms containing $M^2$ and $\mu^2$. We then obtain for the term linear in $t$:

\[a_1 = \left[ f_K/f_\pi - f_\pi(0) \right] + O(\epsilon_8^2)\]  

Furthermore, in the expression (2.100) for $f_\pi(0)$, $f_K^2$ is $O(\epsilon_8^0)$ and may be dropped. Equation (2.106) is then obtained. However, the significance of an approximation which neglects terms proportional to $M^2$ may be questioned.

2.5 Discussion and conclusions

We have not reviewed all the theoretical literature on $K_{\ell 3}$ decay. For example, some authors have used techniques similar to those of Section 2.4.2, but have neglected the so-called "$s$-terms" (commutators of charges with divergences) or have treated them in a less systematic fashion. Other authors have used dispersion relations and, rather than use the Callan-Treiman relation as a constraint, have used current algebra to determine the dispersion parameters (coupling constants). Some authors have attempted to extrapolate from the soft pion point to the mass shell by explicitly including non-pion pole terms in the time-ordered product. Others have used "asymptotic SU(3)" constraints on the form factors or a formal solution of the current algebra. We have not been able to study the most recent preprints, and we apologize to authors whose work we have overlooked.

We have described essentially two different applications of current algebra to $K_{\ell 3}$ decays: (i) saturation of SU(3) charge and current commutators, and (ii) soft pion limits evaluated with the help of chiral SU(3) $\otimes$ SU(3) algebra. The second method is by far the more powerful in its predictions. With the use of this technique, a self-consistent
picture emerges, in which soft kaon as well as soft pion results make sense, if the fundamental Lagrangian is nearly symmetric under SU(2) ⊗ SU(2). Specifically, if the symmetry breaking transforms according to the representation (3, 5) ⊗ (3, 3), the ratio of the chiral SU(2) breaking parameter \((c_8 + \sqrt{2} c_7 c_9)\) to the SU(3) breaking parameter \(c_8\) is essentially given by the pseudoscalar meson masses:

\[
\frac{c_8 + \sqrt{2} c_7 c_9}{c_8} \sim \mu^2 / (M^2 - \mu^2)
\]

Note that in terms of quark masses, this means that \(M_N/M_\Lambda \sim \mu^2/M^2\), where \(N\) and \(\Lambda\) are the \((p, n)\) and \(\Lambda\) quarks. In this picture the quarks are considered as mathematical artifices rather than as real particles.

However, the question remains as to whether this picture corresponds to reality. As discussed above, the soft pion results give \(\lambda_0 > 0\) in the physical region if \(f(t)\) is a smooth function. However, there is mounting experimental evidence, to be discussed briefly below, that \(\lambda_0 < 0\) in the decay region. What does this imply? There are several alternatives.

i) Chiral SU(2) ⊗ SU(2) is a good symmetry and the Callan-Treiman relation,

\[
f(M^2) = f_K f_\pi / f_\pi + O(\mu^2/M^2)
\]

is satisfied, but the form factor \(f(t)\) is not smooth. It has a negative slope at \(t = 0\), and a dip somewhere in the region \(0 < t < M^2\) so that the slope becomes positive as \(t \to M^2\). This could be achieved if there is a \(k\)-meson nearly degenerate with the kaon. If PDGAC is valid for the \(K\kappa\) vertex, the pole contribution is proportional to \((M_K^2 - M_{\kappa}^2)/(M_{\kappa}^2 - t)\) and contributes very little if \(M_{\kappa}^2 \approx M^2\), except precisely at the soft pion point \(t = M^2\). This possibility has been considered particularly by Berman and Roy\(^{65}\). Note that the required behaviour does not occur in the model of Section 2.4.2, even for \(M_{\kappa}^2 \approx M^2\). In addition to the \(\kappa\)-meson pole there must be a background integral which dominates in the decay region and gives \(\lambda_0 < 0\) for small \(t\).

ii) Chiral SU(2) ⊗ SU(2) is not a good symmetry, and PCAC is not always valid. If PDGAC (which is sufficient for understanding the success of the Goldberger-Treiman and Adler-Weisberger relations) is nevertheless valid, one can understand the result \(\lambda_0 < 0\) if SU(3) is a good symmetry, as discussed in Section 2.2.5. This means that the symmetry breaking is more like what one would expect from a naive model of heavy quarks.

Alternatively, one can break chiral symmetry by breaking simultaneously PDGAC and PCAC with the phenomenological Lagrangian or hard meson techniques of Sections 2.4.1 and 2.4.2. However, in this case the results are model dependent and one must violate PCAC \textit{a posteriori} at different vertices depending on whether its predictions are good or bad.

iii) One can instead put the blame on the current algebra. That is, one can argue that the SU(3) ⊗ SU(3) algebra and SU(2) ⊗ SU(2) symmetry are both good, but that the hadronic currents which satisfy the algebra are not the same as those which are coupled to leptons. However, as discussed in Section 2.2.5, this possibility would have other unpleasant consequences.
iv) A final possibility is to abandon the single angle Cabibbo theory. If we impose $f(M^2) = f_K/f_\pi$ and $\lambda_0 < 0$ we must have $f(0) > f_K/f_\pi$, so that [cf. Eq. (2.47)]
\[
\tan \theta_V < \tan \theta_A / 1.28 .
\]
This possibility is not very appealing.

3. REVIEW OF EXPERIMENTAL RESULTS ON $K_{3\pi}$ FORM FACTORS

3.1 Introduction. Parametrization of form factors

This section will be devoted to a compilation of all experimental data on $K_{3\pi}$ form factors.

When one analyses the $K_{3\pi}$ decay one can try to fit the most general coupling in $K_{3\pi}$ as a mixture of scalar, vector and tensor terms. One usually defines the scalar, vector and tensor matrix elements by:

\[
\begin{align*}
    m_S &= M_f \bar{u}_s (1 + \gamma_5) u_v, \\
    m_V &= \frac{\kappa}{2} [f_s (k + q)_u + f_\pi (k - q)_u] \bar{u}_v \gamma_\mu (1 + \gamma_\nu) u_v, \\
    m_T &= \frac{f_T}{M} (k_u q_v - q_u k_v) \bar{u}_v \sigma_{\mu\nu} (1 + \gamma_5) u_v.
\end{align*}
\]

(3.1)

where

- $k$ and $q$ are the $K$ and $\pi$ four-momentum
- $f_s$ is the scalar form factor
- $f_\pi, f_\pi$ are the vector form factors
- $f_T$ is the tensor form factor.

So far all experiments have shown consistency with pure vector coupling\(^1\). The most accurate limits on the contribution of scalar and tensor terms are:

a) in $K_{e3}^+$ [see Ref. 1]:

\[
\begin{align*}
    f_T &= 0; \left| f_S / f_\pi \right| < 0.18 & \text{at 90% confidence level (see Ref. 84)} \\
    f_S &= 0; \left| f_T / f_\pi \right| < 0.58 & \text{at 90% confidence level (see Ref. 88)}
\end{align*}
\]

b) in $K_{\mu3}^+$:

\[
\begin{align*}
    f_T &= 0; \left| f_S / f_\pi \right| < 0.15 & \text{at 68% confidence level (see Ref. 94)} \\
    f_S &= 0; \left| f_T / f_\pi \right| < 0.22 & \text{at 68% confidence level (see Ref. 94)}
\end{align*}
\]

In a second step, one keeps only the vector contribution, which yields the following transition probability\(^1\):
\[ p = Af_+^2 + Bf_+f_- + Cf_-^2 \]

where

\[ A = M(2E_v E_\nu - ME_\nu^2) + m^2 \left( \frac{E_v^2}{4} - E_\nu \right) \]

\[ B = m^2 \left( E_\nu - \frac{E_\nu^2}{2} \right) \]

\[ C = m^2 \frac{E_\nu}{4} \]

\[ E_\nu' = E_\nu^\text{max} - E_\nu = \frac{M^2 + \mu^2 - m^2}{2M} - E_\nu \quad . \]

For \( K_{e3} \), \( B \) and \( C \) are completely negligible. The form factors \( f_+ \) and \( f_- \) are functions only of the invariant momentum transfer \( t \):

\[ t = (k - q)^2 = M^2 + \mu^2 - 2ME_\nu \quad . \]

One usually introduces the parameter \( \xi \):

\[ \xi = \frac{f_-}{f_+} = \xi(t) \quad . \]

The experiments provide four means of studying \( f_+ \) and \( f_- \) (see Table 1).

**TABLE 1**

<table>
<thead>
<tr>
<th>Means of analysis</th>
<th>Information</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Dalitz plot in ( K_{e3} )</td>
<td>Variation of ( f_+^2 ) with ( t )</td>
</tr>
<tr>
<td>2 Dalitz plot in ( K_{\mu3} )</td>
<td>Variation of ( f_+^2 ) with ( t ) Determination of ( \xi )</td>
</tr>
<tr>
<td>3 Polarization of ( \mu^+ ) in ( K_{\mu3} )</td>
<td>Determination of ( \xi )</td>
</tr>
<tr>
<td>4 Ratio ( \frac{f_+}{f_-} )</td>
<td>Relation between ( f_+^2 ) and ( \xi ) (under hypothesis on the variation of ( f_+ ) and ( f_- ) with ( t ))</td>
</tr>
</tbody>
</table>

Due to the limited statistics one has to assume some simple model for the variation of \( f_+ \) and \( f_- \) with \( t \). The simplest is to assume linearity, so that:

\[ f_+(t) = f_+(0) \left( 1 + \lambda_+ \frac{t}{\mu^2} \right) \quad (3.3) \]
Then we have:

\[ \xi(t) = \xi(0) \frac{1 + \lambda_- t/\mu^2}{1 + \lambda_+ t/\mu^2} \]

\[ \xi(0) = f_-(0)/f_+(0) \]  

(3.4)

and for small values of \( \lambda_+ \) and \( \lambda_- \), one can write:

\[ \xi(t) = \xi(0) + \Lambda t/\mu^2 \]

\[ \Lambda \approx \xi(0) (\lambda_- - \lambda_+) \]  

(3.5)

As \( f_+(0) \) \( \sin \theta \) is determined by the rate \( \Gamma_{e^3} \), we have only three parameters in \( K_{\ell^3} \) analysis with this approximation:

\( \xi(0), \lambda_+, \lambda_- \)

or alternatively:

\( \xi(0), \Lambda, \lambda_+ \)

With these parameters, we obtain Table 2.

**TABLE 2**

Information provided by the different means of analysis of \( K_{\ell^3} \) form factors when these are assumed to be linear.\(^{12}\)

<table>
<thead>
<tr>
<th>Means of analysis</th>
<th>Information</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Dalitz plot in ( K_{e^3} )</td>
<td>( \lambda_+ )</td>
</tr>
<tr>
<td>2 Dalitz plot in ( K_{\mu^3} )</td>
<td>( \xi(0), \lambda, \lambda_+ )</td>
</tr>
<tr>
<td>3 Polarization of ( \mu^* ) in ( K_{\mu^3} )</td>
<td>( \xi(0), \Lambda )</td>
</tr>
<tr>
<td>4 Ratio ( \Gamma_{\mu^3}/\Gamma_{e^3} )</td>
<td>Quadratic relation between ( \xi(0), \lambda, \lambda_+ )</td>
</tr>
</tbody>
</table>

In the following sections we shall first test the hypothesis of linearity of the form factors; secondly, we shall present the results of all experiments, for each of the four methods above, in terms of \( \xi(0), \lambda_+, \lambda_- \). In conclusion (Section 3.9) we shall return to the problem of parametrization.
3.2 \( K_{L}^+ \) data

In Table 3 we list the experiments on \( K_{L}^+ \) with the methods of analysis used in each case. Our purpose is to extract the results of all experiments under the same hypotheses and in terms of the same parameters, i.e. \( \xi(0), \lambda_+, \lambda_- \).

3.2.1 Real and imaginary parts of the form factors

A T non-invariance can lead to a phase difference between \( f_+ \) and \( f_- \), which introduces an imaginary component for \( \xi \): \( \xi = \text{Re} \xi + i \text{Im} \xi \). The most direct way to test possible T non-invariance in \( K_{L}^+ \) decay is to look for a component of \( \mu^+ \) polarization normal to the decay plane\(^{129}\)), which is proportional to \( \text{Im} \xi \) and vanishes if T invariance holds. So far, the experiments are compatible with T invariance:

Callahan et al.\(^{89}\), longitudinal polarization: \( |\text{Im} \xi| = 0.5^{+1.4}_{-0.5} \)

" total polarization: \( \text{Im} \xi = 1.6 \pm 1.5 \)

Experiment X2 \(^{90}\), total polarization: \( \text{Im} \xi = -0.1 \pm 0.3 \)

We shall see in Section 3.8 that even lower limits on \( \text{Im} \xi \) are observed with \( K_{L}^+ \). Hence we shall limit ourselves, in the following analysis, to real form factors.

3.2.2 Linearity of form factors

Our first task is to check the validity of the hypothesis of linear form factors.

a) Linearity of \( f_+ \)

The \( K_{L}^+ \) experiments are, in the limits of present statistics, compatible with a linear variation. See for example, Fig. 3c in the paper of the Oxford group\(^{88}\).

b) Linearity of \( f_- \)

As the coefficient \( \lambda_+ \) of linear variation of \( f_+ \) is small (see Section 3.3) we need only to check the linearity of \( \xi \). For this test we need the value of \( \xi \) in small intervals of the momentum transfer \( t \); we have found this information in three papers only:

i) Cutts et al.\(^{89}\): analysis of \( \mu^+ \) polarization by bins of \( t \).

ii) Experiment X2 \(^{90}\): "

iii) Experiment X2 \(^{90}\): analysis of \( K_{\mu3} \) Dalitz plot by bins of \( t \).

The different points have been combined in Fig. 6. We have fitted a line,

\[ \xi = \xi(0) + \Lambda t/\mu^2 \]

The best fit gives

\[ \xi(0) = -1.95 \pm 0.55 \]
\[ \Lambda = 0.21 \pm 0.18 \]

with \( \chi^2 = 14.1 \) for 14 degrees of freedom (= 45% probability). These values correspond to the straight line on Fig. 6. But \( \xi(0) \) and \( \Lambda \) are strongly correlated; from the contour at one standard deviation in the \( [\xi(0), \Lambda] \) plane, we deduce in the same figure the envelopes of all linear fits to \( \xi \) compatible with data at a 68% confidence level. From this fit one concludes that a linear variation of \( \xi \) (and hence of \( f_- \)) is fully compatible with the available data.
3.2.3 Three-parameter fit and two-parameter fit

If we try to perform a three-parameter fit we shall end up with a three-dimensional likelihood function, which is something difficult to handle and to display, particularly for the reason that the parameters are strongly correlated. So we shall adopt for the analysis of experiments a two-step procedure:

i) To express the results of all experiments in terms of $\xi(0)$ and $\lambda_+$, with $\lambda_-$ fixed at the value $\lambda_- = 0$ (Sections 3.3, 3.4, 3.5, 3.6).

ii) Afterwards, to study the variation of the over-all results with $\lambda_-$ (Section 3.7).

We have many reasons for adopting such a procedure:

a) The experiments which have studied most precisely this three-parameter fit show that $\xi(0)$ is very strongly correlated with $\lambda_+$, but $\xi(0)$, $\lambda_+$ are much less correlated with $\lambda_-$ which is found compatible with 0 though poorly determined. As an after-the-fact argument, our analysis will show the same effects and our combined results will yield also a value of $\lambda_-$ compatible with zero, with large errors.

b) Due to the limited information given in most papers, the only way to combine all data under the same assumptions is to take $\lambda_- = 0$.

c) We shall see in Section 3.9 that the weak correlation of $\xi(0)$ and $\lambda_+$ with $\lambda_-$ [weak in comparison with the correlation between $\xi(0)$ and $\lambda_+$] and the poor sensitivity of the data to this parameter parallels the fact that $\lambda_+$ and $\lambda_-$ are related to terms of different order in $t$, in the physical amplitudes.

So, in effect, for each experiment, we shall express the results as a likelihood plot in the plane $[\xi(0), \lambda_+]$ for $\lambda_- = 0$, and combine the results under this assumption before varying $\lambda_-$. 

![Fig. 6 Determination of $\xi$ by bins of momentum transfer $t$; best linear fit and envelopes of the lines fitting the data at a 68% confidence level (e^{-1.15} level for the likelihood function).](image)
TABLE 3

Experiments on $K^{*}_{0}$ form factors and the information provided; the publications referring to the same experiment are assembled.

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Authors and year</th>
<th>Reference number</th>
<th>Technique</th>
<th>Dalitz plot of $K^{*}_{0}$</th>
<th>$T_{E1}$</th>
<th>$T_{M1}$</th>
<th>$T_{M1}/T_{E1}$ only</th>
<th>Dalitz plot of $K^{*}_{0}$</th>
<th>Polarization of $u^{<em>}$ in $K^{</em>}_{0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Birge et al.</td>
<td>1956</td>
<td>Em.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Alexander et al.</td>
<td>1957</td>
<td>Em.</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Taylor et al.</td>
<td>1958</td>
<td>Em.</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Ruggild et al.</td>
<td>1961</td>
<td>Em.</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Roe et al.</td>
<td>1961</td>
<td>XeNC</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Brown et al.</td>
<td>1962</td>
<td>XeNC</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Boyarski et al.</td>
<td>1962</td>
<td>Counter</td>
<td></td>
<td>*</td>
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</tr>
<tr>
<td>8</td>
<td>Jensen et al.</td>
<td>1964</td>
<td>XeNC</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
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<tr>
<td>9</td>
<td>Shaklee et al.</td>
<td>1964</td>
<td>HBC</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Bisi et al.</td>
<td>1965</td>
<td>HBC &amp; HJBC</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Cotts et al.</td>
<td>1965</td>
<td>Spark</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Gallahan et al.</td>
<td>1966</td>
<td>HJBC</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Cester et al.</td>
<td>1966</td>
<td>Spark</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>14</td>
<td>Amado et al.</td>
<td>1967</td>
<td>HJBC</td>
<td></td>
<td>*</td>
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<tr>
<td>15</td>
<td>Young et al.</td>
<td>1967</td>
<td>Em.</td>
<td></td>
<td>*</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>Belloti et al.</td>
<td>1967</td>
<td>HJBC</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>17</td>
<td>Kalmus et al.</td>
<td>1967</td>
<td>HJBC</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>Imlay et al.</td>
<td>1967</td>
<td>HJBC</td>
<td></td>
<td>*</td>
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<tr>
<td>19</td>
<td>Rutter et al.</td>
<td>1968</td>
<td>Spark</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>20</td>
<td>Cott et al.</td>
<td>1968</td>
<td>Spark</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>Experiment X2</td>
<td>1968</td>
<td>HJBC</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
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<tr>
<td>22</td>
<td>Garland et al.</td>
<td>1968</td>
<td>Spark</td>
<td></td>
<td>*</td>
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<td></td>
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<tr>
<td>23</td>
<td>Eschtruch et al.</td>
<td>1968</td>
<td>Spark</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Eisler et al.</td>
<td>1968</td>
<td>Spark</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>25</td>
<td>Zeller et al.</td>
<td>1969</td>
<td>Spark</td>
<td></td>
<td>*</td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>
3.3 $K_{e3}^+$ : determination of $\lambda_+$

In Table 4 we have collected the different values of $\lambda_+$ determined by the analysis of the Dalitz plot in $K_{e3}^+$. These results are displayed in Fig. 7.

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Reference</th>
<th>Method</th>
<th>$\lambda_+$</th>
<th>Radiative corrections</th>
</tr>
</thead>
</table>
| 1     | Brown et al. 
|        |        | Pion spectrum | 0.036 ± 0.045       | No                    |
| 2     | Jensen et al. 
|        |        | Pion spectrum | -0.010 ± 0.029      | No                    |
| 3     | Borreani et al. 
|        |        | Electron spectrum | -0.040 ± 0.050  | No                    |
| 4     | Bellotti et al. 
|        |        | Dalitz plot   | 0.045 ± 0.017       | Yes                   |
| 5     | Kalmus et al. 
|        |        | Pion spectrum | 0.028 ± 0.014       | No                    |
| 6     | Inlay et al. 
|        |        | Dalitz plot   | 0.016 ± 0.016       | No                    |
| 7     | Botterill et al. 
|        |        | Electron spectrum | 0.080 ± 0.040 a) | Yes                   |
| 8     | Eisler et al. 
|        |        | Pion spectrum | -0.029 ± 0.080      | No                    |
| 9     | Botterill et al. 
|        |        | Dalitz plot   | 0.045 ± 0.015 a)    | Yes                   |

Mean value

$\chi^2 = 8.3$ for 8 d.f.  
[$\sim 40\%$ probability]

0.030 ± 0.007

a) These two results can be combined (see page 74 of Ref. 99).

---

Fig. 7 Experimental determinations of $\lambda_+$ in $K_{e3}^+$ analysis (the numbers refer to Table 4). Mean value: 0.030 ± 0.007.
In spite of the agreement between these nine results on $\lambda_+$, one can question the reliability of the mean value, for two reasons:

a) Problem of radiative corrections

E.S. Ginsberg\textsuperscript{127} has evaluated the radiative corrections for $K^+_{e3}$ decays; these are especially important for $K^+_{e3}$, where the variation of density of population becomes greater than 5% in a large zone of the Dalitz plot. From the last column in Table 4, one notices that only three publications have taken into account these corrections. It is striking to note that if we average separately results with and without radiative corrections one gets two somewhat incompatible results:

i) Three experiments with radiative corrections : $\lambda_+ = 0.048 \pm 0.021$;

ii) Six experiments without radiative corrections : $\lambda_+ = 0.017 \pm 0.010$.

But when one examines the exact effect of radiative corrections one may question whether the discrepancy between 0.017 and 0.048 comes entirely from the neglect of these corrections. First these corrections depend critically on the zone of the Dalitz plot retained for the analysis; it appears [see Figs. 4 and 5 of C. Rubbia's report\textsuperscript{1} on page 234] that when only the pion or the electron spectrum is used, one has important systematic effects. However, when the Dalitz plot density is used, the effect may be more or less cancelled. We give two examples of this cancellation:

i) From a Monte Carlo study\textsuperscript{128} for the whole Dalitz plot, one estimates that if one neglects the radiative corrections the measured value of $\lambda_+$ is increased by 0.005 (so in this case the effect goes in the opposite direction from the discrepancy mentioned above).

ii) The Oxford group\textsuperscript{98} states that, if they neglect these corrections, the measured $\lambda_+$ is increased only by 0.001.

So, even if it seems that the neglect of the radiative corrections introduced systematic errors, particularly for spectrum analysis, one cannot conclude that this neglect explains all the discrepancy mentioned above; it would be important, however, for future experiments on $K^+_{e3}$ to take them into account. Nevertheless, the preceding remark leads us to draw attention to another possible source of bias in $K^+_{e3}$ study.

b) Problem of $\gamma$-ray detection

In the Dalitz plot analysis on $K^+_{e3}$ (we shall meet the same problem in $K^+_{u3}$ analysis), most of the information comes from the $\pi^0$ energy distribution; so an uncertainty, even if apparently small, in $\gamma$-ray detection yields an important bias in $\lambda_+$ determination. For example, in the Oxford group's work\textsuperscript{88}, when a uniform detection efficiency for low-momentum $\gamma$-rays is replaced by a non-uniform one, the measured value of $\lambda_+$ increases from 0.020 to 0.045; this point is discussed in detail on page 75 of Botterill's thesis\textsuperscript{99} in connection with the results of Imay et al.\textsuperscript{66}, where a uniform detection efficiency was assumed for these low-momentum $\gamma$-rays.

In conclusion, the mean value $\lambda_+ = 0.030 \pm 0.007$ can be taken as a good fit to $K^+_{e3}$ data; but, when one looks at experiments in detail, one cannot exclude a higher value of $\lambda_+$, around 0.045; the discrepancy could come from the neglect of radiative corrections and other possible experimental biases.
3.4 $K_{\mu 3}^+$: polarization of $\mu^+$

In Table 5 we give the results of experiments on $\mu^+$ polarization. This polarization depends on kinematical variables and on the parameter $\xi$, only. We have seen in Section 3.2 that there was no experimental evidence for T violation in $K_{\mu 3}^+$ so that the polarization $\vec{P}$ of $\mu^+$ lies in the decay plane and is defined only by two components: $P_L$ (longitudinal) along the $\mu^+$ direction and $P_T$ (transverse) normal to the $\mu^+$ axis in the half plane of decay where the $\nu^0$ lies. The first experiments measured only the longitudinal component $P_L$ which yields two solutions for $\xi$, whereas the later experiments used the total polarization $(P_L, P_T)$ which gives a unique determination of $\xi$ -- see Fig. 3 in Ref. 90.

In Table 5, we also give the mean value of $P_L$ and $P_T$ for each experiment; but we recall that it is meaningless to average these components from one experiment to the other, since $\vec{P}$ depends on $\xi$ but also on kinematical variables -- see Fig. 1 in Ref. 89. Hence, the averaged polarization depends on cuts and detection efficiency over the Dalitz plot; in particular one gets: $(P_L)^2 + (P_T)^2 < 1$, which does not contradict the fact that the $\mu^+$ is completely polarized, i.e. that $P_L^2 + P_T^2 = 1$, at fixed $E_{\nu^0}, E_{\mu}$.

All results in Table 5, are expressed under the hypothesis that $\xi$ is constant. But we need a result expressed in terms of the three parameters $\xi(0), \lambda_+, \lambda_-$ -- without assuming $\lambda_+ \equiv \lambda_- = 0$ -- or, in other words, the variation of $\xi$ with the momentum transfer $t$.

**TABLE 5**

Results on the polarization of $\mu^+$ in $K_{\mu 3}^+$. $P_L$ and $P_T$ are the averaged longitudinal and transverse polarization in the decay plane (see text). T invariance is assumed (Im $\xi = 0$).

<table>
<thead>
<tr>
<th>Reference</th>
<th>Technique and number of events</th>
<th>Method of analysis</th>
<th>Band of $\nu^0$ energy (MeV)</th>
<th>$(P_L)$</th>
<th>$(P_T)$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boyarski et al.\textsuperscript{13}, 1962</td>
<td>Counter 122</td>
<td>$P_L$</td>
<td>$115 &lt; T_{\mu} &lt; 134$ MeV</td>
<td>0.6 $\pm$ 0.2 $\pm$ 0.3</td>
<td>-</td>
<td>?</td>
</tr>
<tr>
<td>Skrinitzki and Weissenberg\textsuperscript{14}, 1964</td>
<td>Emulsion 399</td>
<td>$P_L$</td>
<td>$40 &lt; T_{\mu} &lt; 100$ MeV</td>
<td>0.7 $\pm$ 0.45</td>
<td>-</td>
<td>+2</td>
</tr>
<tr>
<td>Gatts et al.\textsuperscript{15}, 1965</td>
<td>Spark 489</td>
<td>$P_L$</td>
<td>$107 &lt; T_{\mu} &lt; 126.4$ MeV</td>
<td>0.61 $\pm$ 0.30</td>
<td>-</td>
<td>-9</td>
</tr>
<tr>
<td>Borromini et al.\textsuperscript{16}, 1965</td>
<td>HBC 2100</td>
<td>$P_L$</td>
<td>$6 &lt; T_{\mu} &lt; 27$ MeV</td>
<td>0.33 $\pm$ 0.18</td>
<td>-</td>
<td>1.2 $\pm$ 2.4</td>
</tr>
<tr>
<td>Callahan et al.\textsuperscript{17}, 1966</td>
<td>HBC 2950 c)</td>
<td>$P_L$</td>
<td>$38 &lt; T_{\mu} &lt; 95$ MeV</td>
<td>?</td>
<td>-</td>
<td>$-$0.7 $\pm$ 1.0 $\pm$ 1.4 $\pm$ 3.7 $\pm$ 0.8</td>
</tr>
<tr>
<td>397 c)</td>
<td>$P_L P_T$</td>
<td>$40 &lt; T_{\mu} &lt; 90$ MeV</td>
<td>?</td>
<td>?</td>
<td>-0.6 $\pm$ 2.7 $\pm$ 1.2</td>
<td></td>
</tr>
<tr>
<td>Gatts et al.\textsuperscript{18}, 1968</td>
<td>Spark 3113</td>
<td>$P_L$</td>
<td>$55 &lt; T_{\mu} &lt; 90$ MeV</td>
<td>?</td>
<td>?</td>
<td>-0.95 $\pm$ 0.30</td>
</tr>
<tr>
<td>Experiment X2\textsuperscript{19}, 1968</td>
<td>HBC 6000</td>
<td>$P_L P_T$</td>
<td>$47 &lt; T_{\mu} &lt; 94$ MeV and $T_{\mu} &gt; 106$ MeV</td>
<td>0.95 $\pm$ 0.12</td>
<td>-0.31 $\pm$ 0.12</td>
<td>-1.00 $\pm$ 0.30</td>
</tr>
</tbody>
</table>

\textsuperscript{a) $T_{\mu}$: kinetic energy of $\mu^+$.

\textsuperscript{b) $\xi$: there are two solutions for $\xi$ when one uses only $P_L$.

\textsuperscript{c) There is no overlap between these two samples.
This information cannot be obtained directly from Table 5. For further analysis we omit the first four experiments of Table 5, which are in agreement with the last three, but provide poor information in comparison with them. Let us examine carefully these three experiments, from the point of view of $\xi$ variation with $t$.

In the last two experiments, of Cutts et al.\textsuperscript{81)} and of the X2 Collaboration\textsuperscript{90)}, $\xi$ has been determined by bands of $t$, which allows us to fit a linear variation of $\xi$ with $t$ to the experimental points.

For the experiment of Callahan et al.\textsuperscript{81)}, we first add the likelihood functions for the two determinations of $\xi$, one using $P_\perp$ only and the other $P_\perp$ and $P_\parallel$, and this for the assumption $\text{Im } \xi \equiv 0$ (see Figs. 6 and 11 of Ref. 80). We get $\xi = -0.7 \pm 0.9$. Then, we try to estimate the mean value of $t$ which this value of $\xi$ refers to; by comparison with data of the X2 experiment referring to the same zone of the Dalitz plot as for this experiment, we estimate an averaged value of $t$: $t = 5 \mu^2$ for the sample used; it must be recalled that when we speak in $K_{\mu^3}$ analysis of a mean value of $t$, we do not refer to the straight-forward mean value -- which would be $3.2 \mu^2$ in this case -- but to an average weighted by the sensitivity to $\xi$ of the different zones (see, for example, the variation of the sensitivity to $\xi$ of $\mu^+$ polarization in Fig. 21 of Reference 94).

We assemble the information of the three experiments in Table 6.

\begin{table}
\centering
\begin{tabular}{|l|c|c|}
\hline
Reference & $t/\mu^2$ & $\xi$ \\
\hline
Callahan et al.\textsuperscript{81)} & 5 & $-0.7 \pm 0.9$ \\
Cutts et al.\textsuperscript{81)} & 2 & $-0.9 \pm 0.6$ \\
" & 4 & $-1.9 \pm 0.5$ \\
" & 5 & $-0.7 \pm 0.7$ \\
" & 6.3 & $-0.2 \pm 0.7$ \\
Experiment X2 \textsuperscript{90)} & 1.5 & $-2.70 \pm 1.50$ \\
" & 2.5 & $-0.95 \pm 1.20$ \\
" & 3.5 & $0.05 \pm 0.90$ \\
" & 4.5 & $-0.65 \pm 0.75$ \\
" & 5.5 & $-1.40 \pm 0.70$ \\
" & 6.5 & $-0.95 \pm 0.90$ \\
\hline
\end{tabular}
\caption{Determination of $\xi$ by the $\mu^+$ polarization analysis, as a function of the momentum transfer $t$.}
\end{table}
The data of Table 6 are displayed in Fig. 8. We have fitted a linear variation of $\xi$: $\xi = \xi(0) + \Lambda t/\mu^2$. In Fig. 9 we show the likelihood function that we obtained. The best fit gives

$$
\begin{align*}
\xi(0) &= -1.45 \pm 0.70 \\
\Lambda &= 0.11 \pm 0.15
\end{align*}
$$

with a $\chi^2 = 8.0$ for nine degrees of freedom (53% probability). We have drawn in Fig. 8 a straight line corresponding to this best fit and the envelopes of all lines fitting the data at 68% confidence level. This fit shows a good consistency with a linear variation of $\xi$. Figure 9 shows the likelihood contours in the plane $[\xi(0), \Lambda]$.

In a second step we have assumed $\lambda_- = 0$. Instead of the approximate fit: $\xi = \xi(0) + \Lambda t/\mu^2$ [where $\Lambda = \xi(0) (\lambda_- - \lambda_+)$] we have fitted exactly: $\xi = \xi(0)/(1 + \lambda_+ t/\mu^2)$. This last expression leads to the likelihood function plotted in the plane $[\xi(0), \lambda_+]$ in Fig. 10. This function will be combined with the results of the Dalitz plot and branching ratio analyses, obtained under the same assumptions, in Section 3.7.

![Graph](image-url)  

**Fig. 8** Analysis of $\mu^+$ polarization: experimental data as a function of $t$ and the best linear fit with the contours at the $e^{-5,5}$ (dashed lines) and $e^{-1,1}$ (full lines) levels.
Fig. 9 Analysis of $\mu^+$ polarization in terms of $\xi(0)$ and $\Lambda$ 
$[\xi = \xi(0) + \Lambda t/\mu^2]$: optimum value of the likelihood function and the contours at the $e^{-3.16}$ (dashed line) and $e^{-1.1}$ (full line) levels.

Fig. 10 Analysis of $\mu^+$ polarization in terms of $\xi(0)$ and $\lambda^+_+$ for $\lambda^+_+ \equiv 0$ 
$[\xi = \xi(0) / (1 + \lambda^+_+ t/\mu^2)]$: optimum value of the likelihood function and the contours at the $e^{-3.16}$ (dashed line) and $e^{-1.1}$ (full line) levels.
3.5 \( K_{\mu}^+ \) : Dalitz plot analysis

Before discussing the experiments on the Dalitz plot analysis listed in Table 7, we draw attention to the fact that there exists a very strong correlation between \( \xi(0) \) and \( \lambda_+ \), so that a change of 0.04 in \( \lambda_+ \) can induce a variation of -1 in the determination of \( \xi \). One can easily understand this correlation if one considers formula (3.2) which can be written as:

\[
\rho(E_\pi, t) \propto \left[ 1 + \lambda_+ \frac{t}{\mu^2} \right] \left[ A(E_\pi, t) + B(E_\pi, t) \xi + C(t) \xi^2 \right].
\]  

(3.6)

So a variation of \( \lambda_+ \) changes completely the fit to \( \xi \). That is to say that all the results for the Dalitz plot analysis expressed in terms of \( \xi \) without specifying \( \lambda_+ \) are meaningless.

In view of an over-all fit we have divided the results into three classes:

a) Analysis of \( \mu^+ \) spectrum

We have added in Fig. 11 the likelihood functions depending on \( \xi \) for three experiments. The first two\(^{77,86} \) analyse the \( \mu^+ \) spectrum after integration over \( \pi^0 \) energy; this spectrum depends quadratically on \( \xi \). The third one\(^{92} \) analyses the \( \mu^+ \) spectrum at given \( \pi^0 \) energies, which depends linearly on \( \xi \) and, hence, yields a unique solution for \( \xi \). The total likelihood analysis, displayed in the upper part of Fig. 11, gives:

\[
\xi = -0.95^{+0.45}_{-0.30} \text{ (68% confidence level)}.
\]

Fig. 11 Analysis of \( \mu^+ \) spectrum: i) below: logarithm of the likelihood function for different experimental results; ii) above: sum of the preceding. The 1\( \sigma \) and 2\( \sigma \) levels indicate one \((e^{-0.5})\) and two \((e^{-1})\) standard deviations. The origins of the scales are arbitrary.
TABLE 7
Results of the Dalitz plot analysis for $K_{\mu 3}^+$.  

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Reference</th>
<th>Technique</th>
<th>Method and number of events</th>
<th>Ots (units: kinetic energies in MeV)</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Bisi et al. 77, 1965</td>
<td>HLBC &amp; HBC</td>
<td>$\mu^+$ spectrum 2845</td>
<td>$T_{\mu} &lt; 120$</td>
<td>$\xi = -2$</td>
</tr>
<tr>
<td>2</td>
<td>Callahan et al. 87, 1966</td>
<td>HLBC</td>
<td>$\mu^+$ 2648</td>
<td>$42 &lt; T_{\mu} &lt; 94$</td>
<td>$\xi = 0$ for errors, see Fig. 11 since the likelihood functions are not Gaussian-like</td>
</tr>
<tr>
<td>3 a)</td>
<td>Experiment X 291, 1969</td>
<td>HLBC</td>
<td>$\mu^+$ 4347</td>
<td>$50 &lt; T_{\mu} &lt; 92$ ; $20 &lt; T_{e} &lt; 100$</td>
<td>$\xi = -0.6$</td>
</tr>
<tr>
<td>4</td>
<td>Brown et al. 69, 1962</td>
<td>XeNC</td>
<td>$(\mu^+ + e^+) 76$</td>
<td>$0 &lt; T_{\mu} &lt; 70$ b)</td>
<td>$\xi = 1.46 \pm 1.2$ for $\lambda_+ = \lambda_- = 0$</td>
</tr>
<tr>
<td>5</td>
<td>Jensen et al. 72, 1973, 1964</td>
<td>XeNC</td>
<td>$(\mu^+ + e^+) = 120$</td>
<td>$\xi = -0.50 \pm 0.7$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Eisler et al. 77, 1968</td>
<td>HLBC</td>
<td>$\mu^+$ 78</td>
<td>$3 &lt; T_{\mu} &lt; 55$</td>
<td>$\xi = -0.50 \pm 0.9$</td>
</tr>
<tr>
<td>7 c)</td>
<td>Callahan et al. 87, 1966</td>
<td>HLBC</td>
<td>$(\mu^+ + e^+) 444$</td>
<td>$40 &lt; T_{\mu} &lt; 90$ ; $T_e &lt; 110$</td>
<td>$\xi = 0.72 \pm 0.5$ ; $\lambda_+ = 0.000 \pm 0.035$ ; $\frac{d\sigma}{d\xi} = -16$</td>
</tr>
<tr>
<td>8 a)</td>
<td>Experiment X 291, 1969</td>
<td>HLBC</td>
<td>$\mu^+$ 3240</td>
<td>$50 &lt; T_{\mu} &lt; 92$ ; $20 &lt; T_e &lt; 100$</td>
<td>$\xi(0) = -1.1 \pm 0.5$ ; $\lambda_+ = 0.050 \pm 0.019$ ; $\frac{d\sigma(0)}{d\xi} = -30$</td>
</tr>
<tr>
<td>9</td>
<td>Kijewski 111, 1969</td>
<td>Spark</td>
<td>$\mu^+$ 2041</td>
<td>$55 &lt; T_{\mu} &lt; 90$</td>
<td>$\xi(0) = -0.5 \pm 0.8$ ; $\lambda_+ = 0.009 \pm 0.026$ ; $\frac{d\sigma(0)}{d\xi} = -35$</td>
</tr>
</tbody>
</table>

For the three results above (7, 8, 9), the errors are defined by the projections of the $\alpha = 0.5$ contour of likelihood functions, in the plane ($\xi, \lambda_+$).

For the over-all fit c):

$$\xi(0) = -1.00 \pm 0.40, \lambda_+ = 0.043 \pm 0.017; \lambda_- = 0$$

a) The result (3) is almost all included in the result (8), and so is not counted in the over-all fit.
b) The efficiency is high up to 60 MeV, drops to 50% at 70 MeV, and is negligible above 100 MeV.
c) The result (7) is discarded from the over-all fit: see text.
This result is not correlated with \( \lambda_+ \) (this is strictly true at least for the third result). This method of analysis gives much less information than the \((\mu^+ + \pi^0)\) analysis, but it is free of many biases, particularly those connected with \(\gamma\)-ray detection.

b) Analysis of \(\pi^0\) and \(\mu\) spectra at fixed values of \(\lambda_+\)

Three experiments\(^{70,72,97}\), each one with about 100 \((\pi^0 + \mu)\) events, give an estimation of \(\xi\), assuming a fixed value of \(\lambda_+\). The results are:

i) Brown et al.\(^{70}\): \(\xi = 0.7 \pm 1.2\) for \(\lambda_+ = 0.036\) and \(\lambda_- = 0\).

The result: \(B/A = \frac{g_V}{f_V} = 0.4 \pm 0.3\), stated in the conclusion of this paper, comes from the combined Dalitz plot and branching ratio analyses. In order to get the Dalitz plot result separately, we read in Fig. 2 of this paper: \(B = -0.01 \pm 0.04\) for \(\lambda' = 0, \lambda_+ = 0.036\) and \(A = 0.07\); the spectrum analysis yields: \(\frac{g_V}{f_V} = B/A = (-0.01 \pm 0.04)/0.07 = -0.15 \pm 0.6\); whence: \(\xi = 1 + 2 \frac{g_V}{f_V} = 0.7 \pm 1.2\).

The results on \(K_{\mu3}\) and \(K_{e3}\) rates will be taken into account, in the appropriate place, in Section 5.6.

ii) Jensen et al.\(^{72}\) give two points:

\[\xi = -0.08 \pm 0.7 \text{ for } \lambda_+ = -0.020, \lambda_- = 0\]

\[\xi = 0.6 \text{ for } \lambda_+ = -0.052, \lambda_- = 0\].

iii) Eisler et al.\(^{97}\) give:

\[\xi = -0.5 \pm 0.9 \text{ for } \lambda_+ = \lambda_- = 0\].

From the data of Jensen et al., we have: \(\text{d}\xi/\text{d}\lambda_+ = -0.68/0.032 = -21.2\); as the experiment of Brown et al. is performed in the same chamber as that of Jensen et al., and as the experiment of Eisler et al. refers also to the same zone of the Dalitz plot as the first two, we extrapolate the three preceding results to the value \(\lambda_+ = 0\), by taking \(\text{d}\xi/\text{d}\lambda_+ = -21\) for each of them. Whence:

\[
\begin{array}{ll}
\text{Brown et al.} & : \xi = 1.46 \pm 1.2 \\
\text{Jensen et al.} & : \xi = -0.50 \pm 0.7 & \lambda_+ = 0 \\
\text{Eisler et al.} & : \xi = -0.50 \pm 0.9 & \lambda_- = 0 \\
\text{Combined results} & : \xi = -0.15 \pm 0.5
\end{array}
\]

We can, finally, recapitulate these three results under the form:

\[\xi = -0.15 \pm 0.5 \text{ for } \lambda_+ = \lambda_- = 0\]

\(\text{d}\xi/\text{d}\lambda_+ = -21\).

c) Analysis of \(\pi^0\) and \(\mu\) spectra in terms of \(\xi\) and \(\lambda_+\)

Three experiments\(^{85,92,101}\), using the \((\pi^0 + \mu^+)\) information, give an analysis in terms of \([\xi(0), \lambda_]\), for \(\lambda_+ \equiv 0\). The optimum values of the likelihood functions (crosses) and the contours at a 68\% confidence level (approximate ellipses) are drawn in Fig. 12. As to the contour for the experiment of Callahan et al.\(^{80}\), we have drawn it under our responsibility, from Fig. 8 and Table 1 of this publication. Note that, for the same
statistics, one gets about ten times more information with $(\mu^+ + \pi^0)$ analysis than with $\mu^+$ analysis.

We assemble now all the preceding results in Fig. 12. All the results are expressed in terms of the same parameters: $\xi(0)$, $\lambda_+$, under the assumption $\lambda_- = 0$. The full lines represent contours in the plane $[\xi(0), \lambda_+]$ at the same confidence level (68%), corresponding to a factor $e^{-1.14}$ in comparison with the optimum of the two-parameter likelihood function. The results from the $\mu^+$ spectrum analysis give approximately a band parallel to the $\lambda_+$ axis; the three results of point (c) are taken directly from the publications; the results of point (b) give a band with a slope $d\xi/d\lambda_+ = -21$.

Figure 12 displays remarkable agreement between all the experiments except one; this last result, the analysis with $(\pi^0 + \mu)$ of Callahan et al.\textsuperscript{86}, is outside the zone of agreement of the eight other results. We shall discard it for the over-all fit for two

![Diagram](image)

**Fig. 12** Analysis of the density over the Dalitz plot in terms of $\xi(0)$ and $\lambda_+ (\lambda_- = 0)$. All the contours (full lines) correspond to the $e^{-1.14}$ level (68% of probability, for the two-dimensional likelihood functions); dotted contour: experiment discarded.
reasons; first, one can hardly understand how this result with 444 events gives the same precision on \( \xi(0) \) and \( \lambda_+ \) as, for example, the experiment X2 with 3240 events or Kijewski with 2041 events corresponding to almost the same zone in the Dalitz plot. Second, one can try to explain what looks like underestimated errors and perhaps systematic bias. We note that the authors use the method of likelihood but keep in their sample the events along the lower border of the Dalitz plot (low \( \pi^0 \) energy). Now, the sensitivity to \( \xi \) of the Dalitz plot analysis increases considerably in this zone while the density of population tends to zero [see, for example, Fig. 1 in Ref. 92, or discussion and figures in Thomson and Willis's report\(^{186}\)]. But, according to the fundamental theorem of likelihood theory, one cannot use the likelihood method in a zone where the derivatives of the likelihood function with respect to the parameters analysed diverge, as is the case on the lower border of the Dalitz plot. In practice, one has to cut a zone depending on the measurement uncertainties. This inadequate use of likelihood method may be at the origin of the underestimated of the errors in this result; an analogous effect was observed in the X2 experiment\(^{131}\), and suppressed by the cut \( T_{\pi} > 20 \text{ MeV} \). This effect is absent when one uses a \( \chi^2 \) method.

Finally we add the likelihood of the seven results in agreement (we do not add twice the two X2 results which are not independent). The standard deviation contours are drawn in Fig. 13. This over-all fit gives:

\[
\begin{align*}
\xi(0) &= -1.00 \pm 0.40 \\
\lambda_+ &= 0.043 \pm 0.017
\end{align*}
\]

But one has to recall that these two parameters are strongly correlated, as shown in Fig. 13.

Fig. 13 Over-all fit to the Dalitz plot data of Fig. 12 in terms of \( \xi(0) \) and \( \lambda_+ \) (\( \lambda_- \equiv 0 \)). The contours of the likelihood for a one parameter determination (\( e^{-1.5} \) level: 10) and a two-parameter determination (\( e^{-1.15} \) level: 10; \( e^{-3.15} \) level: 20) are displayed.
3.6 $K^+_{e3}, K^+_{\mu3}$: branching ratios

We examine the fourth method of analysis of $K^+_{e3}$ form factors, the ratio of $K^+_{e3}$ and $K^+_{\mu3}$ rates: $R = \Gamma_{\mu3}/\Gamma_{e3}$. Let us rewrite the formula (3.2):

$$ \rho(E_+, E_-) = \frac{A^2 + Bf_+^2 + Cf_+^2}{f_+^2(0)} $$

where $B$ and $C$ are negligible for $K^+_{e3}$, being proportional to $m_K^2$. After integration over the Dalitz plot, one gets the absolute rates:

$$ \Gamma_{\mu3} = K \int \frac{(A^2 + Bf_+ + Cf_+^2)/f_+^2(0)}{\text{DP}_{K_{\mu3}}} $$

$$ \Gamma_{e3} = K \int \frac{A^2}{f_+^2(0)} \text{DP}_{K_{e3}} $$

where the constant $K$ includes the constants $G^2$, $f_+^2(0)$, and $\sin^2 \theta$ ($\theta = \text{Cabibbo angle}$). So if one assumes some model of variation for $f_+$, the ratio $R$ will provide a relation between $f_+$ and $f_-$. Under our assumption of linear form factors we can expand the formula (3.2) (we take $\lambda_- = 0$, as stated before):

$$ \rho = A + 2AT\lambda_+ + AT^2\lambda_+^2 + B\xi(0) + BT\xi(0)\lambda_+ + CT^2\lambda_+^2 $$

(3.7)

where we denote:

$$ T = t/\mu^2 $$

And we get an exact expression for $R$:

$$ R = \frac{\Gamma_{\mu3}}{\Gamma_{e3}} = \int \frac{A + 2AT\lambda_+ + AT^2\lambda_+^2 + B\xi(0) + BT\xi(0)\lambda_+ + CT^2\lambda_+^2}{\text{DP}_{K_{\mu3}}} $$

$$ \int A + 2AT\lambda_+ + AT^2\lambda_+^2 $$

(3.8)

$A, B, C$, defined in formula (3.2) (Section 3.1), depend on kinematical variables. Using the latest mass constants\(^{13}\), we perform the integration and we get (for $K^+$):

$$ R_{K^+} = \frac{0.6457 + 3.8008\lambda_+ + 6.8120\lambda_+^2 + 0.1264\xi(0) + 0.4757\xi(0)\lambda_+ + 0.0192\xi(0)^2}{1.0000 + 3.6995\lambda_+ + 5.4777\lambda_+^2} $$

(3.9)

If we take $\lambda_+ = 0$ in formula (3.8) we get the well known formula for constant form factors:

$$ R_{K^+} = \frac{0.6457 + 0.1264\xi + 0.0192\xi^2}{1.0000 + 3.6995\lambda_+ + 5.4777\lambda_+^2} $$

Let us examine the data; a look at Table 1 (Section 3.2) shows that, among the experiments, some give both relative rates $\Gamma_{\mu3}/\Gamma_{e3}$, others give only one of these data, and one alone gives $\Gamma_{\mu3}$ and $\Gamma_{\mu3}/\Gamma_{e3}$ as direct results. On the other hand, the experimental difficulties are absolutely different for $\Gamma_{\mu3}$ and $\Gamma_{e3}$. These are two solid reasons not to
use the ratio $R$ for the combination of the experiments, but to discuss and average separately the $\Gamma_{\mu_3}$ and $\Gamma_{e_3}$ data, and then to take the ratio of these two mean values to get $R$.

**Note:** As many experiments normalize $\Gamma_{e_3}$ and $\Gamma_{\mu_3}$ by $\gamma^+$, $K_{\mu_2}$ or $K_{\pi_2}$ decays, we have updated these normalizations by taking $^{136}$) the following relative rates, in %:

\[
\begin{align*}
\Gamma_{\mu_2} & = 63.65 \pm 0.29 ; \\
\Gamma_{\pi_2} & = 21.033 \pm 0.301 ; \\
\Gamma_{\gamma^+} & = 5.57 \pm 0.04 \\
\Gamma_{\mu_2} + \Gamma_{\pi_2} & = 84.68 \pm 0.42
\end{align*}
\]

### 3.6.1 $K^*_e$ : relative rate $\Gamma_{e_3}$

For $K^*$, there is no important decay mode involving positrons other than $K_{e_3}^*$. So the problem of background is much less critical in $K_{e_3}^*$ than in $K_{\mu_3}^*$. The main problem is the detection efficiency of the positron.

#### TABLE 8

Measurements of the relative rate $\Gamma_{e_3}$, for $K^*$

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Reference</th>
<th>Technique</th>
<th>Cuts and detection efficiency for $e^*$ spectrum</th>
<th>$\Gamma_{e_3}$ (in %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Birge et al. $^{45}$</td>
<td>Em.</td>
<td>no cuts</td>
<td>3.2 ± 1.3</td>
</tr>
<tr>
<td>2</td>
<td>Alexander et al. $^{45}$</td>
<td>&quot;</td>
<td>&quot;</td>
<td>5.1 ± 1.3</td>
</tr>
<tr>
<td>3</td>
<td>Héggild et al. $^{46}$</td>
<td>&quot;</td>
<td>&quot;</td>
<td>3.4 ± 1.7</td>
</tr>
<tr>
<td>4</td>
<td>Hae et al. $^{61}$</td>
<td>XeBC</td>
<td>$e^*$ detection efficiency : 89%</td>
<td>5.0 ± 0.5</td>
</tr>
<tr>
<td>5</td>
<td>Shaklee et al. $^{43}$</td>
<td>&quot;</td>
<td>$e^*$ detection efficiency : 97.4%</td>
<td>4.7 ± 0.3</td>
</tr>
<tr>
<td>6</td>
<td>Borromini et al. $^{47}$</td>
<td>HBC</td>
<td>$e^*$ detection efficiency = 2%</td>
<td>5.00 ± 0.34</td>
</tr>
<tr>
<td>7</td>
<td>Callahan et al. $^{48}$</td>
<td>HILBC</td>
<td>$e^*$ detection efficiency : 68.1%</td>
<td>4.02 ± 0.21</td>
</tr>
<tr>
<td>8</td>
<td>Young et al. $^{49}$</td>
<td>Em. $^{45}$ $K_{e_3}$ detection efficiency : 94.6%</td>
<td>5.50 ± 0.90</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Belloti et al. $^{50}$</td>
<td>HILBC</td>
<td>$e^*$ detection efficiency = 15%</td>
<td>5.24 ± 0.50</td>
</tr>
<tr>
<td>10</td>
<td>Auerbach et al. $^{51}$</td>
<td>Spectr. &amp; Spark</td>
<td>$130 &lt; p_{e} &lt; 190$ MeV/c</td>
<td>4.93 ± 0.16</td>
</tr>
<tr>
<td>11</td>
<td>Eshchennik et al. $^{43}$</td>
<td>&quot;</td>
<td>80 MeV/c &lt; $p_{e}$</td>
<td>5.17 ± 0.19</td>
</tr>
<tr>
<td>12</td>
<td>Garland et al. $^{52}$</td>
<td>&quot;</td>
<td>80 MeV/c &lt; $p_{e}$</td>
<td>4.35 ± 0.40</td>
</tr>
<tr>
<td>13</td>
<td>Bottoni et al. $^{53}$</td>
<td>&quot;</td>
<td>90 MeV/c &lt; $p_{e}$</td>
<td>4.92 ± 0.21</td>
</tr>
<tr>
<td>14</td>
<td>Experiment X2 $^{11}$</td>
<td>HILBC</td>
<td>no cuts; efficiency &gt; 99%</td>
<td>4.75 ± 0.11</td>
</tr>
<tr>
<td>15</td>
<td>Zeller et al. $^{10}$</td>
<td>Spectr. &amp; Spark</td>
<td>120 MeV/c &lt; $p_{e}$</td>
<td>4.31 ± 0.40</td>
</tr>
</tbody>
</table>
In Table 8, we have listed the results on $\Gamma_{e3}$, updated by us when necessary. There are two reasons for updating the original results given in the publications:

a) The first one, straightforward, is that some $K^\pm_{e3}$ rates are related to well-determined rates, $\Gamma_{\tau^+}$, for example. We have mentioned above the constants used for these normalizations.

b) The second one, not clearly stated generally in papers, is the dependence on $\lambda_+$ of results obtained with a limited part of the $e^+$ spectrum, usually in experiments with spark chambers or spectrometers. Suppose, for example, that in an experiment one detects only positrons with: $p_{e,\text{min}} < p_e < p_{e,\text{max}}$. One will get the final $K^+_{e3}$ rate after dividing the number of events observed by the fraction of positron spectrum actually measured; we shall call $F$ this fraction. If we return to formula (3.7), we can calculate $F$:

$$
F = \frac{\int_{p_{e,\text{min}}}^{p_{e,\text{max}}}}{\int_{\text{whole spectrum}}} \left( \frac{\alpha^* + \beta^* \lambda_+ + \gamma^* \lambda_+^2}{\alpha + \beta \lambda_+ + \gamma \lambda_+^2} \right).
$$

(3.10)

We have estimated $F$ for the different cuts $(p_{e,\text{min}}, p_{e,\text{max}})$ used in the experiments and for different values of $\lambda_+$; the results are in Table 9. This table shows that, in such

### Table 9

<table>
<thead>
<tr>
<th>Limits in $e^+$ spectrum (units: MeV/c)</th>
<th>$F$</th>
<th>$\lambda_+ = 0$</th>
<th>$\lambda_+ = 0.050$</th>
<th>$\lambda_+ = 0.045$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$60 &lt; p_e$</td>
<td>0.933</td>
<td>0.937</td>
<td>0.939</td>
<td></td>
</tr>
<tr>
<td>$130 &lt; p_e &lt; 190$</td>
<td>0.438</td>
<td>0.448</td>
<td>0.452</td>
<td></td>
</tr>
<tr>
<td>$80 &lt; p_e$</td>
<td>0.856</td>
<td>0.863</td>
<td>0.8665</td>
<td></td>
</tr>
<tr>
<td>$90 &lt; p_e$</td>
<td>0.804</td>
<td>0.813</td>
<td>0.8175</td>
<td></td>
</tr>
<tr>
<td>$120 &lt; p_e$</td>
<td>0.608</td>
<td>0.619</td>
<td>0.624</td>
<td></td>
</tr>
</tbody>
</table>

experiments, $\Gamma_{e3}$ decreases slightly when $\lambda_+$ increases. As, in Section 3.3, we have obtained $\lambda_+ = 0.030 \pm 0.007$ in $K^\pm_{e3}$ analysis, we have adopted this value to calculate $F$ for each experiment involving cuts in the $e^+$ spectrum. For a value $\lambda_+ = 0.045$, which was found possible in Section 3.3, $\Gamma_{e3}$ would further decrease by about 0.5% (see Table 9).

We discuss the updating of data in detail for each case, in Appendix A. The results, assembled in Table 8, are displayed in Fig. 14a (page 51). When we average the fifteen results we get:
\[ \Gamma_{e3} = 4.77 \pm 0.07 \]

\[ \chi^2 = 25 \text{ for 14 degrees of freedom (4\% probability).} \]

But one notices that this high \( \chi^2 \) is due to a unique result [Callahan et al.\textsuperscript{8b}) experiment] which is at four standard deviations from the mean value; in that experiment, performed in a heavy liquid bubble chamber, a large amount of e\(^+\) tracks go out of the chamber and are difficult to identify. Once this result is removed, we get

\[
\begin{align*}
\Gamma_{e3} & = 4.85 \pm 0.07 \\
\chi^2 & = 10.9 \text{ for 13 d.f. (60\% probability)}
\end{align*}
\]

This fit yields a remarkable agreement between experiments. We shall keep it as definitive in further discussions.

### 3.6.2 \( \kappa^+_{\mu3} : \text{relative rate } \Gamma_{\mu3} \)

At first glance the situation is not as clear for \( \kappa^+_{\mu3} \) as for \( \kappa^+_{\mu1} \). The reason is that the sources of background for this mode are very important, since 90\% of all \( \kappa \) decays involve a single pion or muon and about 1 in 30 of such decays are \( \kappa^+_{\mu3} \). Let us mention especially the \( \pi^+ \) mode, seven times more abundant than the \( \kappa^+_{\mu3} \) mode. Now the \( \pi \) of \( \kappa^+_{\mu3} \) has a range which falls into the range spectrum of the \( \mu \) of \( \kappa^+_{\mu3} \). So, in many cases, \( \pi \) and \( \mu \) may be confused and an ambiguity appears between both modes; the fit procedure (when it can be used) cannot solve entirely this ambiguity. As to the \( \pi \) of \( \tau' \), its range spectrum covers the low part of the range spectrum for \( \mu \) of \( \kappa^+_{\mu3} \). Further, there are other abundant sources of background in \( \kappa^+_{\mu3} \) samples still more difficult to discriminate, such as \( \kappa^+_{\mu3} \) with \( \pi \) decaying in flight into \( \mu + \nu \), \( \kappa^+_{\mu2} \) or \( \pi \) decaying in flight, radiative decay modes \( \kappa^+ + \mu^+ \gamma \) or \( \kappa^+ + \pi^+ \gamma \), etc. The percentage of such events in a \( \kappa^+_{\mu3} \) sample is very difficult to estimate accurately. Particularly we think that to introduce theoretical matrix elements of such and such a background into a Monte Carlo calculation, in order to get its contribution, is not always completely reliable. However, the experiments with large statistics and detection of all particles (as in heavy liquid bubble chambers) allow also direct experimental studies of the background events which can be used as cross checks of the Monte Carlo calculations and so improve the reliability of these.

As a consequence of the important contribution of pseudo-\( \kappa^+_{\mu3} \) events, most of the experiments make cuts in the \( \mu^+ \) spectrum in order to remove the most contaminated zones, particularly the \( \kappa^+_{\mu3} \) contamination. In this way, we come to the same problem as in \( \kappa^+_{\mu3} \).

Let us denote by \( F \) the fraction of \( \mu^+ \) spectrum actually observed; from formula (3.6) we get:

\[
F = \frac{\int A + 2AT\lambda_+ + AT^2\lambda_+^2 + B \xi(0) + B \xi(0) + C \xi(0)}{\int A + 2AT\lambda_+ + AT^2\lambda_+^2 + B \xi(0) + B \xi(0) + C \xi(0)}
\]  \hspace{1cm} (3.10)
### Table 10

Measurements of the relative rate $\Gamma_{\pi^+}^{K^+}$ for $K^+$.  
Mean value of the last eleven results: $\langle 3.04 \pm 0.08 \rangle$%.

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Reference</th>
<th>Technique</th>
<th>No. of evts.</th>
<th>Cuts and background (see Appendix B for more details)</th>
<th>$\Gamma_{\pi^+}^{K^+}$ (in %)</th>
</tr>
</thead>
</table>
| 1     | Davies et al. 115, 1955 | Em.        | 5            | Whole $\mu^+$ spectrum + maximal background (as important as $K_{\mu^+}$) | 2.5 ± 1.2  
|       | Ritson et al. 113, 1956 | ”          | 7            | ”                                                      | 1.3 ± 0.7  
|       | Crussard et al. 113, 1956 | ”          | 2            | ”                                                      | 5.8 ± 2.3  
|       | Birge et al. 114, 1956   | ”          | 16 b)        | ”                                                      | 2.8 ± 1.0  
|       | Alexander et al. 44, 1957| ”          | 20 b)        | ”                                                      | 5.9 ± 1.3  
|       | Ñeggild et al. 44, 1961  | ”          | 8 b)         | ”                                                      | 4.3 ± 1.8  
|       | Young et al. 44, 1967    | ”          | 39           | ”                                                      | 5.3 ± 0.9  |
| 2     | Cutts et al. 79, 1965    | Spark      | 489          | 107.1 < $E_{\pi}$ < 126 MeV; background as important as $K_{\mu^+}$ (see Appendix B) | 4.12 ± 0.6  
| 3     | Birge et al. 65, 1956    | Em.        | 7 b)         | $T_{\pi} < 80$ MeV; below main $K_{\pi^+}$ and $K_{\mu^+}$ contaminations but no estimation of background | 3.7 ± 0.9  
|       | Alexander et al. 44, 1957| ”          | 12 b)        | ”                                                      |                           
|       | Ñeggild et al. 44, 1961  | ”          | 6 b)         | ”                                                      |                           
| 4     | Taylor et al. 17, 1959   | Em.        | 37           | $T_{\pi} < 42$ MeV; below main $K_{\pi^+}$ and $K_{\mu^+}$ contaminations but no estimation of background | 2.7 ± 0.5  
| 5     | Giacomelli et al. 71, 1964| Em.        | 87           | $T_{\pi} < 28$ MeV; below main $K_{\pi^+}$ and $K_{\mu^+}$ contaminations but no estimation of background | 3.5 ± 0.5  
| 6     | Shaklee et al. 74, 1964  | XeBC       | 120          | The detection falls off for $T_{\pi} > 70$ MeV, i.e. in the most contaminated zone | 3.0 ± 0.5  
| 7     | Risi et al. 77, 1965     | HLRBC & HBC| 2100         | Combination of data from three experiments (1 in HLRBC, 2 in HLRBC): 5 < $T_{\pi} < 95$ MeV and 105 < $T_{\pi} < 120$ MeV. Some sources of background omitted in analysis. | 3.52 ± 0.40  
| 8     | Callahan et al. 44, 1966 | HLRBC      | 656          | Zone of $K_{\mu^+}$ cut; background subtracted: = 304 $K_{\mu^+}$ | 2.83 ± 0.19  
| 9     | Auerbach et al. 44, 1967 | Spark      | 310          | 130 < $p_{\pi} < 190$ MeV/c; background subtracted: = 404 $K_{\mu^+}$ | 3.14 ± 0.28  
| 10    | Botterill et al. 47, 1968| Spark & Spectr. | 5600 $K_{\mu^+}$ & (7770 $K_{\pi^+}$) | 110 < $p_{\pi} < 350$ MeV/c; background subtracted: = 203 $K_{\mu^+}$ | $\Gamma_{\pi^+}^{K^+} / \Gamma_{\pi^+}^{K^+} = 0.667 ± 0.019$  
| 11    | Garland et al. 44, 1968  | Spark & Spectr. | 350          | 90 < $p_{\pi} < 190$ MeV/c; background subtracted: = 404 $K_{\mu^+}$ | 3.05 ± 0.24  
| 12    | X2 experiment 44, 1968   | HLRBC      | 1505         | 46.4 < $T_\pi < 72.7$ MeV; background subtracted: = 204 $K_{\mu^+}$ | 2.80 ± 0.11  
| 13    | Zeller et al. 44, 1969   | Spark & Spectr. | 30           | 120 MeV/c < $p_{\mu}$; background (?) | 3.44 ± 0.06  

a) These data are taken from the publication of Birge et al. 114.

b) The results of Birge et al. 114, Alexander et al. 44, 115, and Ñeggild et al. 44, are stated twice; first for the results (discarded) deduced from the whole spectrum; second for the results (retained in fit) deduced from a limited zone of spectrum.

c) The enlarged error is due to the omission of background sources (see Appendix B).
So, F and the measured $\Gamma_{\mu 3}$ depend on $\lambda_+$ and $\xi(0)$. In the discussion of the experiments (in Appendix B), we have calculated F whenever the information provided by the publications made this possible. In accordance with the results of Section 3.4 and 3.5 we have assumed that $\xi(0)$ and $\lambda_+$ lie in the zones $\xi(0) = -1.0 \pm 1.0$, $\lambda_+ = 0.03 \pm 0.03$, which cover the entire range of experimental results; from which, in each case, a numerical value $F \pm \delta F$ is obtained.

In Appendix B, we give a discussion of all available results and we explain how we have updated the original numbers when necessary. The results are displayed in Table 10. As to the over-all fit, we have adopted it as a criterion of reliability to discard the results where the background appears as important as the $K_{\mu 3}$ sample. Only some older results based on few events are discarded by this criterion. This leaves eleven data for the over-all fit; the results are displayed in the lower part of Fig. 14 and are in quite good agreement. The best fit yields:

$$\Gamma_{\mu 3} = (3.04 \pm 0.08)\%$$

$$\chi^2 = 13.8 \text{ for 10 d.f. (18\% probability)}$$

---

Fig. 14 Experimental determination of $\Gamma_{e3}$ (above) and $\Gamma_{\mu 3}$ (below). (The numbers refer respectively to Tables 9 and 10.)
A scale factor 1.2 was taken into account for the estimate of the error. This slightly too high $\chi^2$ can be related to the importance of the background in two ways:

a) The experimental errors are a combination of statistical errors and of systematic errors which are mainly due to the subtraction of the background. But the latter does not decrease as automatically as the former when the statistics increase. It seems, however, that they have been underestimated in some cases.

b) The sources of background are so numerous that it may happen that some of them are omitted in the analyses. Of course, these omissions yield systematically rates of $\Gamma_{\mu_3}$ which are too high, and the highest experimental values of $\Gamma_{\mu_3}$, in Table 10, correspond to experiments where many sources of background were not studied. So one cannot exclude a true value of $\Gamma_{\mu_3}$ lower by one standard deviation than the mean value stated here.

### 3.6.3 Ratio $\Gamma_{\mu_3}/\Gamma_{e_3}$ and $\xi$ determination

Let us now return to the ratio $R = \Gamma_{\mu_3}/\Gamma_{e_3}$. With the preceding averages we get

$$R = \frac{3.04 \pm 0.08}{4.85 \pm 0.07} = 0.626 \pm 0.019$$

---

**Fig. 15** Relation between $\xi(0)$ and $\lambda_+$ ($\lambda_- = 0$) corresponding to different values of $R = \Gamma_{\mu_3}/\Gamma_{e_3}$. The heavy dashed line is the best overall fit, and the heavy full lines the contour at the e^{-1.14} level.
We insert this numerical value into the formula (3.9), and thus we get a relation between $\xi(0)$ and $\lambda_+$. We have drawn, in Fig. 15, the curves (in fact quasi-linear) corresponding to different values of $R$, the heavy lines correspond to our best fit and its one standard deviation contour (for a two-parameter fit). For a given value of $\lambda_+$ we get a value of $\xi$; for example:

$$
\lambda_+ = 0.000 \frac{\xi(0)}{d\xi(0)/d\lambda_+} = -0.17 \pm 0.15
$$
$$
\lambda_+ = 0.030 \frac{\xi(0)}{d\xi(0)/d\lambda_+} = -0.33 \pm 0.18
$$
$$
\lambda_+ = 0.045 \frac{\xi(0)}{d\xi(0)/d\lambda_+} = -0.71 \pm 0.20
$$

(Here the errors correspond to a one parameter fit.)

As, in Section 3.6.2, we have obtained some rates of $\Gamma_{\mu_3}$ by means of the factor $F$ which depends slightly on the exact numerical values of $\xi(0)$ and $\lambda_+$, we have performed a more sophisticated fit taking into account the quoted partial rates when they were the original data. Thus we are free of any a priori assumption on the value of $\xi(0)$ and $\lambda_+$. This second fit yields actually the same curves in the plane $[\xi(0), \lambda_+]$ as the preceding one.

**Note:** In Section 3.6.1 we have combined separately 14 data on $\Gamma_{\mu_3}$ and in Section 3.6.2 11 data on $\Gamma_{e_3}$. In both cases we got a good fit and this allows us to divide the two averages to get $R$. It happens that some experiments give contributions for both averages. So some compilations (see for example Ref. 135, reprinted in Ref. 1) divide the $\Gamma_{\mu_3}$ of Peter by the $\Gamma_{e_3}$ of Peter, the $\Gamma_{\mu_3}$ of Paul by the $\Gamma_{e_3}$ of Paul and so on, and the "unpaired" $\Gamma_{\mu_3}$ by the world average of $\Gamma_{e_3}$ (which includes of course the $\Gamma_{e_3}$ of Peter, Paul, ... and the unpaired $\Gamma_{e_3}$) instead of dividing each $\Gamma_{\mu_3}$ rate by the world average $\Gamma_{e_3}$, as we do. Then one deduces the $\xi$ of Peter, the $\xi$ of Paul, etc., and one obtains quite a considerable spread of $\xi$ values. This procedure is irrelevant for many reasons:

i) From a statistical point of view it is not consistent to take into account twice some $\Gamma_{e_3}$ rates.

ii) Except one experiment (Ref. 87), the original experimental results are $\Gamma_{\mu_3}$ and $\Gamma_{e_3}$ and not $R = \Gamma_{\mu_3}/\Gamma_{e_3}$ (for $K^0_L$ it is the reverse). So one keeps the maximum information and the maximum chance to compensate well the experimental uncertainties when averaging separately $\Gamma_{\mu_3}$ and $\Gamma_{e_3}$ instead of averaging $R$: if one of these two rates is biased, $R$ will be biased as well and the good information from the other rate is lost.

iii) This procedure reduces 14 + 11 independent results ($\Gamma_{\mu_3}/\Gamma_{e_3}$) to 11 results ($R$) by assuming a specific correlation between the two groups (i.e. $\Gamma_{\mu_3}$ of Peter with $\Gamma_{e_3}$ of Peter, etc.). This introduces a bias for the reason that the experiments which are well adapted to $K_{\mu_3}$ detection are not necessarily well adapted to $K_{e_3}$ detection and vice versa. For example, the HLBC used by Callahan et al. is quite well adapted for $K^0_L$ analysis, but yields a result for $\Gamma_{e_3}$ too low by four standard deviations, for the reason that many $\pi^+$ tracks go out of the chamber; on the other hand, the emulsion experiments give acceptable $\Gamma_{e_3}$ rates but $\Gamma_{\mu_3}$ rates biased by an important contamination.
Now, the experimental biases generally give too high $\Gamma_{\mu 3}$ rates (unsubtracted background) or too low $\Gamma_{e 3}$ rates ($e^+$ tracks missed); thus this procedure yields quite systematically too high ratios $\Gamma_{\mu}/\Gamma_{e}$.

So the more pedestrian way (to average separately $\Gamma_{\mu 3}$ and $\Gamma_{e 3}$ rates) is certainly the more reliable.

### 3.7 Over-all fit to $K^+_{\mu 3}$ data

We compare now the results obtained by the four methods of analysis of $K^+_{\mu 3}$ form factors, under the same assumptions.

#### 3.7.1 $K^+_{\mu 3}$ analysis : $\bar{\xi}(0)$, $\lambda_+$

We have displayed in Fig. 16 the optimum values of $\bar{\xi}(0)$ and $\lambda_+$ with the contours at a 68% confidence level, obtained by the three methods of analysis:

---

**Fig. 16** Comparison of the one standard deviation contours ($\sigma = 1.1 \sqrt{\lambda}$ level) for the $\mu^+$ polarization, the Dalitz plot and the $\Gamma_{\mu 3}/\Gamma_{e 3}$ analyses in terms of $\bar{\xi}(0)$ and $\lambda_+$ ($\lambda_- = 0$). Full line: over-all fit to the three results.
i) the polarization of $\mu^+$ (Section 3.4)

ii) the Dalitz plot analysis (Section 3.5)

iii) the ratio $\Gamma_{\mu^3}/\Gamma_{e^3}$ (Section 3.6).

There is a fairly good agreement between the three results and the three contours overlap. This agreement allows us to combine the three results which provide an over-all fit:

\[
\begin{align*}
\zeta(0) &= -0.85 \pm 0.20 \\
\lambda_+ &= 0.045 \pm 0.012 \quad \lambda_- = 0
\end{align*}
\]

The best fit and its contours are displayed in Fig. 17. Note the correlation $d\zeta(0)/d\lambda_+ \approx -17$.

Thus we get two independent results on $\lambda_+$:

- $K_{e^3}$ analysis (Section 3.3): $\lambda_+ = 0.030 \pm 0.007$
- $K_{\mu^3}$ analysis (Section 3.7.1): $\lambda_+ = 0.045 \pm 0.012$

(with $\lambda_- \equiv 0$).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig17}
\caption{Over-all fit to all the $K_{e^3}^+$ data (see Fig. 16). The contours of the likelihood function for a one parameter determination ($e^{-0.5 \times 2}$ level: 1o) and a two parameter determination ($e^{-1.14 \times 2}$ level: 1o, $e^{-3.10 \times 2}$ level: 2o) are displayed.}
\end{figure}
These two results are also in good agreement; both combined give:

\[ \lambda_+ = 0.034 \pm 0.06 \]

Note that for \( \lambda_+ = 0.034 \) the best value of \( \xi(0) \) becomes \(-0.65 \pm 0.20\).

3.7.3 Discussion

In spite of the fairly good agreement between preceding data, one notices two slight discrepancies:

i) The Dalitz plot and polarization analyses are in very good agreement, but not the Dalitz plot and branching ratio analyses. This slight discrepancy is surprising since, in fact, once we assume \( \mu - e \) universality, the branching ratio provides the same information as the Dalitz plot analyses after integration. A ratio \( R = 0.600 \) (instead of \( R = 0.626 \)) would give very good agreement between both methods; if we consider the average \( \Gamma_{e3} = 4.85 \) as solid, this yields \( \Gamma_{\mu3} = 2.91 \) instead of the actual average 3.04. But we have emphasized in Section 3.6.2 that we cannot exclude values of \( \Gamma_{\mu3} \) lower than the actual average, because of the important and complex background which must be subtracted from \( K_{\mu3} \) samples.

ii) The optimum value \( \lambda_+ \) is lower for \( K_{e3} \) than for \( K_{\mu3} \). But we must still recall the discussion in Section 3.3 of the experimental difficulties, the conclusion of which stated that one cannot exclude in \( K_{e3} \) analyses the possibility of values of \( \lambda_+ \) around 0.4.

So, with respect to this discussion, one needs further experimental information on \( \Gamma_{\mu3} \) and \( \lambda_+ \) (in \( K_{e3} \)) taking into account all the experimental difficulties emphasized.

3.7.4 Estimation of \( \lambda_+ \)

However, one can think also that these slight discrepancies come from the assumption made at the beginning: \( \lambda_+ \equiv 0 \).

i) The determination of \( \lambda_- \) is not so easy. First, the experimental analyses provide much less information on it than on \( \lambda_+ \). Secondly, one notices that the parameter actually determined in analyses is \( \lambda \), i.e. the slope of \( \xi = \xi(0) + \lambda t/\mu^2 \). For small \( \lambda_+ \), \( \lambda_- \) we have \( \lambda = \xi(0) (\lambda_+ - \lambda_-) \). As \( \lambda_+ \) is quite well determined independently of \( \lambda \), the three parameters on which one gets information when one combines the three methods of analysis of \( K_{\mu3} \) are in fact \( \xi(0), \lambda_+, \xi(0)\lambda_- \). Thus if \( \xi(0) \) tends to zero, \( \lambda_- \) becomes undetermined (\( \lambda_+ \) remains well determined).

ii) The experimental data on \( \lambda_- \) are rare.

- The experiment X29\(^{\text{a,1}}\) gives as a result of an over-all fit:

\[ \xi(0) = -1.00 \pm 0.30, \lambda_+ = 0.060 \pm 0.019, \lambda = 0.01 \pm 0.11 \]
From this one obtains
\[ \lambda_- = 0.070 \pm 0.102 \]

As the same data gives also:
\[ \xi(0) = -0.98^{+0.20}_{-0.25} \quad \lambda_+ = 0.049^{+0.012}_{-0.010}, \quad \lambda_- = 0 \]

we deduce:
\[ \frac{d\xi(0)}{d\lambda_-} \approx 3, \quad \frac{d\lambda_+}{d\lambda_-} \approx 0.07 \]

Kijewski gives for a Dalitz plot analysis only:
\[ \xi(0) = -0.9^{+3.5}_{-2.9}, \quad \xi(0)\lambda_- = 0.00 \pm 0.5, \quad \lambda_+ = 0.023 \]

From this one obtains
\[ \lambda_- = 0.00 \]

But no error can be quoted since \( \xi(0) \) becomes zero inside the one standard deviation limits. This analysis confirms, as the preceding experiment shows, that \( \xi(0) \) is much less correlated with \( \lambda_- \) than with \( \lambda_+ \) and that \( \lambda_+ \) is almost independent of \( \lambda_- \). These two results, which are the only significant results on \( \lambda_- \) as yet published, show that the hypothesis \( \lambda_- = 0 \), that we have assumed when combining the experimental data, is fully consistent with experiment.

iii) Nevertheless, we have tried to see how our global fit could vary with \( \lambda_- \) in Fig. 16:

- In the polarization one measures only \( \xi(0) \) and \( \lambda_- \), so a value \( \lambda_- \neq 0 \) gives a translation of the contour parallel to the \( \lambda_+ \) axis.

- For the branching ratio, we calculate from formula (3.9) that, for fixed values of \( \lambda_+ \) and of \( R \) we have:
\[ \delta \lambda_+ [0.306 \delta(0) - 0.32] + 0.108 \delta(0) = 0 \]

- As to the Dalitz plot analysis, we deduce from Refs. 94 and 101 that in first approximation \( \lambda_+ \) does not change with \( \lambda_- \) and that one gets:
\[ \frac{d\xi(0)}{d[\xi(0)\lambda_-]} = -7 \]

Then we have taken successive numerical values for \( \lambda_- \) and examined the agreement of the three methods of analysis of \( K^{+\mu\nu} \). The results are displayed in Fig. 18. From this figure one sees that the best fit occurs around \( \lambda_- = 0.05 \), and that there is no longer agreement for \( \lambda_- = -0.05 \) or 0.015. From this we deduce a rough estimate of \( \lambda_- \):

\[ \lambda_- = 0.05 \pm 0.10 \]
3.7.5 Conclusion

In conclusion, we cannot decide whether a better agreement between the three methods would come from a slightly positive value of $\lambda_-$ or from the elimination of some possible slight experimental biases when one determines $F_{\bar{\mu}3}$ and $\lambda_+$ (in $K_{e3}$).

3.8 $K_{e3}$ form factors for $K_L^0$

We shall be briefer in the discussion of $K_L^0$ data than we were for the $K^+$ data for two reasons. First the information is rarer; secondly we shall see that the results of the different experiments are, so far, quite incoherent in the frame of the parameters used for $K^+$ data; this does not allow any over-all fit.
## TABLE 11

Experiments on $K_L^0$ form factors and the information provided; the publications referring to the same experiment are assembled.

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Authors and year</th>
<th>Reference number</th>
<th>Technique</th>
<th>Methods of analysis in $K_{L3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Dalitz plot of $K_{e3}$</td>
</tr>
<tr>
<td>1</td>
<td>Adair et al. 1964</td>
<td>102</td>
<td>HBC</td>
<td>*</td>
</tr>
<tr>
<td>2</td>
<td>Luers et al. 1964</td>
<td>103</td>
<td>HBC</td>
<td>*</td>
</tr>
<tr>
<td>3</td>
<td>Astbury et al. 1965</td>
<td>104</td>
<td>Cloud</td>
<td>*</td>
</tr>
<tr>
<td>4</td>
<td>Kulyukina et al. 1965</td>
<td>106</td>
<td>Cloud</td>
<td>*</td>
</tr>
<tr>
<td>5</td>
<td>Bartlett et al. 1966</td>
<td>107</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td>6</td>
<td>Auerbach et al. 1966</td>
<td>108</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td>7</td>
<td>Fisher et al. 1965</td>
<td>105</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>Carpenter et al. 1966</td>
<td>109</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td>8</td>
<td>Firestone et al. 1967</td>
<td>110</td>
<td>HBC</td>
<td>*</td>
</tr>
<tr>
<td>9</td>
<td>Helland, Longo and Young 1967, 1969</td>
<td>111, 118, 126</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td>10</td>
<td>Hopkins et al. 1967</td>
<td>112</td>
<td>HBC</td>
<td>*</td>
</tr>
<tr>
<td>11</td>
<td>Kadyck et al. 1967</td>
<td>113</td>
<td>HBC</td>
<td>*</td>
</tr>
<tr>
<td>12</td>
<td>Lowys et al. 1967</td>
<td>114</td>
<td>HILBC</td>
<td>*</td>
</tr>
<tr>
<td>13</td>
<td>de Bouard et al. 1967</td>
<td>115</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td>14</td>
<td>Hawkins et al. 1967</td>
<td>116</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td>15</td>
<td>Aronson et al. 1967</td>
<td>117</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td>16</td>
<td>Abrams et al. 1968</td>
<td>121</td>
<td>Spark</td>
<td>*</td>
</tr>
<tr>
<td>17</td>
<td>Badagov et al. 1968</td>
<td>119</td>
<td>HILBC</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>Evans et al. 1969</td>
<td>122</td>
<td>HILBC</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>Beillière et al. 1969</td>
<td>123</td>
<td>HILBC</td>
<td>*</td>
</tr>
<tr>
<td>18</td>
<td>Basile et al. 1969</td>
<td>120</td>
<td>Spark</td>
<td>*</td>
</tr>
</tbody>
</table>
In this type of experiment, one can seldom detect the secondary tracks with a uniform efficiency, which is usually possible for K⁺ experiments (when K⁺ is at rest), at least for a restricted zone of the Dalitz plot. This makes the study of the Dalitz plot (K_e3, K_µ3) particularly difficult: one generally has to divide an observed distribution by a non-uniform detection efficiency evaluated by some Monte Carlo calculation, and then to look for effects of a few per cent in the density of population. Another feature, particularly important for the Dalitz plot analyses, is the non-zero momentum of the K which generally leads to more uncertainties in kinematic reconstruction than in comparable K⁺ experiments. Finally, when one combines different experiments, it is more difficult than in K⁺ to relate the results to well-defined zones in the Dalitz plot and thus to study their variations with the momentum transfer. These are some of the experimental difficulties in the K⁺_e3 experiments. These experiments are listed in Table 11, as well as the information that they provide on form factors.

8.8.1 Determination of λ⁺ in K⁰_e3

We have listed in Table 12 the seven results on λ⁺ available for the K⁰_e3 analysis which are displayed in Fig. 19. We have mentioned above some of the experimental difficulties that one has to overcome generally in this analysis. As to the radiative corrections, they have not been taken into account in most of the results stated here; further, these corrections are different from those for K⁺, since here we have electromagnetic final state interactions. The average value is:

\[
\lambda_+ = 0.017 \pm 0.008
\]

with \(\chi^2 = 6.5\) for 7 d.f. (50% probability)

This value is lower than in K⁺ analysis; but, because of the experimental difficulties, more studies are needed on the K⁰_e3 before concluding a significant discrepancy with K⁺_e3.

![Fig. 19 Experimental determinations of \(\lambda_+\) in K⁰_e3 analysis (the numbers refer to Table 12); mean value: 0.017 ± 0.008.](image)
TABLE 12

Determination of $\lambda_+$ by the Dalitz plot analysis in $K^0_{e3}$

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Reference</th>
<th>Technique</th>
<th>$\lambda_+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Luers et al. $^{111}$, 1964</td>
<td>HBC</td>
<td>0.07 ± 0.06</td>
</tr>
<tr>
<td>2</td>
<td>Fisher et al. $^{123}$, 1965</td>
<td>Spark</td>
<td>0.15 ± 0.08</td>
</tr>
<tr>
<td>3</td>
<td>Firestone et al. $^{116}$, 1967</td>
<td>HBC</td>
<td>-0.01 ± 0.02</td>
</tr>
<tr>
<td>4</td>
<td>Kadyck et al. $^{113}$, 1967</td>
<td>HBC</td>
<td>0.01 ± 0.015</td>
</tr>
<tr>
<td>5</td>
<td>Lowys et al. $^{114}$, 1967</td>
<td>HLC</td>
<td>0.08 ± 0.10</td>
</tr>
<tr>
<td>6</td>
<td>Aronson et al. $^{117}$, 1968</td>
<td>Spark</td>
<td>0.020 ± 0.015</td>
</tr>
<tr>
<td>7</td>
<td>Basile et al. $^{120}$, 1968</td>
<td>Spark</td>
<td>0.023 ± 0.012</td>
</tr>
</tbody>
</table>

Mean value
$\chi^2 = 6.5$ for 7 d.f. (50% probability)
$0.017 ± 0.007$

3.8.2 Polarization of $\mu^+$ in $K^0_{\mu3}$

a) Three experiments have looked for a component $P_N$ of $\mu^+$ polarization normal to the decay plane; this component, if non-zero, is related to a $T$ violation which is expressed in terms of an imaginary part of $\xi$: $\text{Im} \, \xi$. The results are:

Bartlett et al. $^{117}$) : $\text{Im} \, \xi = 0.11 ± 0.35$ (Re $\xi = 0$)

Abrams et al. $^{121}$) : $\text{Im} \, \xi = -0.20 ± 0.60$ (Re $\xi = -1.6$)

Longo et al. $^{126}$) : $\text{Im} \, \xi = -0.02 ± 0.08$ (Re $\xi = -1.81$)

b) We have listed in Table 13 the three results on the components of $\mu$ polarization in the decay plane, which provide an estimate of Re $\xi$. As to the third experiment, the

TABLE 13

Results on the polarization of $\mu^+$ in $K^0_{\mu3}$

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Reference</th>
<th>Technique</th>
<th>No. events</th>
<th>Method</th>
<th>$t$</th>
<th>Re $\xi$ (Im $\xi = 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Auerbach et al. $^{109}$, 1966</td>
<td>Spark</td>
<td>2600</td>
<td>$P_{\text{transverse}}$</td>
<td>-1.2 ± 0.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Abrams et al. $^{121}$, 1968</td>
<td>Spark</td>
<td>665</td>
<td>$P_{\text{total}}$</td>
<td>-1.6 ± 0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t &lt; 3.3 \mu^2$ \quad -2.4 ± 1.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t &gt; 3.3 \mu^2$ \quad -1.4 ± 0.4</td>
</tr>
<tr>
<td>3</td>
<td>Longo et al. $^{126}$, 1969</td>
<td>Counter</td>
<td>P_{total}</td>
<td>(t) = 3.2 $\mu^2$</td>
<td>-1.81 ± 0.50 \quad -0.26</td>
<td></td>
</tr>
</tbody>
</table>
authors state a mean value of \( t = 2.65 \, \mu^2 \), in their publication\textsuperscript{126}). But, as already mentioned in Section 3.4, in order to get a mean value useful for the polarization analysis, one has to weigh the events with the sensitivity of \( \mu^+ \) polarization to \( \xi \) for the different kinematical configurations. As this sensitivity increases with \( t \), the effective mean value of \( t \) is higher than the straightforward mean value. Table 1 of their publication\textsuperscript{126}) allows one to estimate the effective \( t \) averaged for polarization analysis: \( t = 3.2 \, \mu^2 \).

The first experiment (Re \( \xi = -1.2 \pm 0.5 \)) and the second one for the events with \( t > 3.3 \, \mu^2 \) (Re \( \xi = -1.4 \pm 0.4 \)) give results compatible with the average for \( K^+ \) data. But the second experiment, for the events with \( t < 3.3 \, \mu^2 \) (Re \( \xi = -2.4 \pm 1.4 \)), and the third one, which has most of the events with small \( t \) (Re \( \xi = -1.81 \pm 0.50 \)), give lower values of \( \xi \). Nevertheless, these last data should not be compared with a constant \( \xi \), but with the best linear fit for \( K^+ \) data (for \( \mu^+ \) polarization) displayed in Fig. 8: \( \xi = -1.45 \pm 0.11 \, t/\mu^2 \), this gives a better agreement with \( K_\mu^0 \) results. So, one cannot conclude whether this discrepancy is only statistical, or whether this corresponds to a drop of \( \xi \) for small \( t \).

Whatever the case may be, the discussion would be much clearer if the future experiments on \( \mu \) polarization for \( K_\mu^0 \) would give the values of \( \xi \) by small bins of \( t \), instead of averaged values of \( \xi \).

### 3.8.3 Dalitz plot analysis in \( K_\mu^0 \)

There are only two results on the Dalitz plot analysis in \( K_\mu^0 \). But before discussing these experiments we recall that, in this analysis, the sensitivity to \( \xi \) of the density of population increases considerably with \( t \), while the density itself tends to zero. So few events with a high value of \( t \) give more information on \( \xi \) than a lot with a low value of \( t \).

On the other hand, the uncertainty on the \( K_\mu^0 \) momentum induces in such experiments kinematical uncertainties which make the analysis still more difficult.

a) Carpenter et al.\textsuperscript{109}) have analysed 1371 \( K_{\mu3}^0 \) events in a spark chamber experiment. They get:

i) \( \xi = 1.2 \pm 0.8 \) for \( \lambda_+ = \lambda_- = 0 \).

ii) From Fig. 9 of their publication we get roughly:

\( \xi(0) = -2.4 \pm 1.0 \) for \( \lambda_+ = 0.2, \lambda_- = 0 \).

So we notice the typical strong \( \left[ \xi(0), \lambda_+ \right] \) correlation already observed; here it is \( d\xi(0)/d\lambda_+ = -18 \). Their values yield approximately:

\( \xi(0) = 0.65 \pm 0.8 \) for \( \lambda_+ = 0.030, \lambda_- = 0 \),

a value not compatible with polarization results in \( K_\mu^0 \), nor with Dalitz plot results in \( K^+ \).

We notice that the authors have used all the Dalitz plot without any cut, and particularly the zone where the sensitivity to \( \xi \) diverges and where small kinematical uncertainties or small losses can induce a strong bias in the estimation of \( \xi \).
b) Basile et al.\textsuperscript{124}), in a spark chamber experiment, have a detection efficiency which decreases quickly when \( t \) increases, i.e. in the region containing most of the information on \( \xi \). Thus, for \( \lambda_+ = \lambda_- = 0 \), they find two solutions: \( \xi = -3.9 \pm 0.1 \) (best solution) and \( \xi = 1.52 \pm 0.2 \). In view of their number of events (\( \sim 3000 \)) this small error is surprising. We have met the same problem in Section 3.5 for the \( K^0 \) experiment of Callahan et al.\textsuperscript{80}). The same source of bias as already mentioned may be suspected: in this experiment the detection efficiency drops to less than 0.1 in the region of high sensitivity to \( \xi \), so that a small systematic error in the estimate of this efficiency by a Monte Carlo calculation induces a large shift of the result in terms of \( \xi \).

Finally, no conclusion can be drawn for the Dalitz plot analyses in \( K^0_L \), and new experiments avoiding the possible sources of biases mentioned above are needed.

### 3.8.4 Branching ratio \( \Gamma_{\mu^3} / \Gamma_{e^3} \)

The experimental situation for \( K^0_L \) is totally different from \( K^+ \). The modes \( \Gamma_{e^3} \) and \( \Gamma_{\mu^3} \) are the most abundant and almost free of background contamination from other modes. As a consequence, contrary to the case of \( K^+ \), the direct experimental result is the ratio \( R = \Gamma_{\mu^3} / \Gamma_{e^3} \). We have listed in Table 14 the different results (the results 1, 3, 4 were originally expressed in terms of \( \Gamma_{\mu^3} / \Gamma_{\text{charged}} \) and \( \Gamma_{e^3} / \Gamma_{\text{charged}} \); the result 11 is originally expressed in the form: \( \xi(0) = -0.5 \pm 0.5 \) for \( \lambda_+ = 0.02, \lambda_- = 0.00 \), which yields \( R = 0.62 \pm 0.05 \)).

#### Table 14

<table>
<thead>
<tr>
<th>Expt.</th>
<th>Reference</th>
<th>Technique</th>
<th>( R = \Gamma_{\mu^3} / \Gamma_{e^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Adair et al.\textsuperscript{102}, 1964</td>
<td>HBC</td>
<td>0.81 ± 0.19</td>
</tr>
<tr>
<td>2</td>
<td>Luers et al.\textsuperscript{103}, 1964</td>
<td>HBC</td>
<td>0.73 ± 0.15</td>
</tr>
<tr>
<td>3</td>
<td>Astbury et al.\textsuperscript{114}, 1965</td>
<td>Counter</td>
<td>0.85 ± 0.25</td>
</tr>
<tr>
<td>4</td>
<td>Kulyukina et al.\textsuperscript{114}, 1966</td>
<td>Counter</td>
<td>0.65 ± 0.16</td>
</tr>
<tr>
<td>5</td>
<td>de Bouard et al.\textsuperscript{115}, 1967</td>
<td>Spark</td>
<td>0.82 ± 0.10</td>
</tr>
<tr>
<td>6</td>
<td>Hawkins\textsuperscript{114}, 1967</td>
<td>HBC</td>
<td>0.70 ± 0.20</td>
</tr>
<tr>
<td>7</td>
<td>Hopkins et al.\textsuperscript{112}, 1967</td>
<td>HBC</td>
<td>0.81 ± 0.08</td>
</tr>
<tr>
<td>8</td>
<td>Budagov et al.\textsuperscript{119}, 1968</td>
<td>HLBCC</td>
<td>0.71 ± 0.05</td>
</tr>
<tr>
<td>9</td>
<td>Reillière et al.\textsuperscript{114}, 1969</td>
<td>HLBCC</td>
<td>0.71 ± 0.04</td>
</tr>
<tr>
<td>10</td>
<td>Evans et al.\textsuperscript{112}, 1969</td>
<td>HLBCC</td>
<td>0.648 ± 0.03</td>
</tr>
<tr>
<td>11</td>
<td>Basile et al.\textsuperscript{124}, 1969</td>
<td>Spark</td>
<td>0.620 ± 0.050</td>
</tr>
</tbody>
</table>

**Mean value**

\( \chi^2 = 9.2 \) for 10 d.f. (50% probability) \[0.684 ± 0.018\]
So we get a mean value of $R$, much higher than in $K^*$. It is difficult to say if this discrepancy is significant or not. When one looks at Fig. 20 where the data are displayed, it is striking to note that the results decrease quite regularly with the date of experiment! This may be due to experimental difficulties not really overcome. Because of the non-zero momentum of $K$ and detection problems for secondary tracks, these ratios are obtained more roughly than for $K^*$ data. Particularly, with these data one cannot check separately the $K_\mu$ and $K_e$ rates or study partial rates as in $K^*$. New experiments are necessary before arriving at a definite conclusion on this discrepancy.

![Graph](image)

Fig. 20 Experimental determinations of $\Gamma_{\mu3}/\Gamma_{e3}$ for $K^0$. The numbers refer to Table 14 and their order is chronological. Mean value: $0.684 \pm 0.018$.

Nevertheless, if we take the mean value $0.684 \pm 0.018$ as solid, we can deduce a relation between $\xi(0)$ and $\lambda_+$ for $\lambda_- = 0$. If we perform in formula (3.8) the integration over the $K^0_{\mu3}$ and $K^0_e3$ Dalitz plot, we get:

$$R_{K^0_L} = \frac{0.6452 + 5.5467\lambda_+ + 5.9320\lambda_+^2 + 0.1245\xi(0) + 0.4571\xi(0)\lambda_+ + 0.0187\xi(0)^2}{1.0000 + 3.4571\lambda_+ + 4.7791\lambda_+^2}.$$  \hspace{1cm} (3.11)

This relation, together with the preceding numerical value of $R$, yields:

$$\xi(0) = 0.30 \pm 0.15 \quad \lambda_+ = \lambda_- = 0 \quad .$$

3.8.5 Conclusions on $K^0_{\mu3}$ results

The experimental situation is not clear at all for $K^0_{\mu3}$ form factors.

i) For the estimate of $\lambda_+$ in $K^0_e3$, the most recent experiments, which give values of $\lambda_+$ compatible with $K^*$ results, should be confirmed by future experiments.
ii) For the polarization analysis, we need experiments giving the variation of $\xi$ with $t$, before concluding on a value of $\xi$ lower than that for $K^+$. 

iii) For the Dalitz plot, the experiments done so far have mainly shown the big difficulties of this analysis in comparison with $K^+$, particularly because of problems of detection efficiency.

iv) As to the branching ratio $\Gamma_{\mu_3}/\Gamma_{e_3}$, in spite of quite a large list of experiments, one can wonder why the results decrease regularly with time; certainly many experimental problems remain to be solved in this field.

All this underlines the experimental difficulties of any $K^0_L$ experiment; we hope that they will be overcome in the future. Nevertheless, so far, one cannot perform an overall fit for $K^0_L$ data nor make a significant comparison with $K^+$ results.

3.9 Conclusions

We have seen in Section 3.7 that a good fit to all $K^{+}_{e_3}$ data was obtained with the following set of values:

$$\xi(0) = -0.85 \pm 0.20; \quad \lambda_+ = 0.045 \pm 0.012; \quad \lambda_- = 0$$

$$\frac{d\xi(0)}{d\lambda_+} = -17$$

and that $\lambda_-$ was found compatible with zero. For the $K^0_{e_3}$ the confused experimental situation does not allow such an over-all fit. As to $K^0_{e_3}$ results, they yield the following values:

$$\lambda_+ = 0.030 \pm 0.007 \quad \text{for} \quad K^+$$

$$\lambda_+ = 0.017 \pm 0.008 \quad \text{for} \quad K^0_L$$

but the discussion of some difficulties in the analysis as well as the most recent results show that one cannot exclude values up to 0.045. We shall now state some peculiar features of the $K^0_L$ form factor analysis that we have met all through the discussion of experimental results and which, if neglected, make difficult the understanding of these results.

5.8.1 Linearity of $f_+^{K^{+}}$ and $\xi$

We recall that in all our analysis we have assumed the linearity of $f_+$ and of $\xi$. As to the linearity of $\xi$, Fig. 6 has shown this hypothesis to be compatible with the available results for $K^+$ (no information exists in this respect for $K^0_L$). On the other hand we have noticed, first, that in Fig. 6 the two measured values of $\xi$ drop below -2 for $t < 2 \mu^2$, though they remain compatible with -1; secondly, that the two polarization results from $K^0_L$ which yield values of $\xi$ around -2, correspond to low values of $t$ (between 0 and 3). So far, one cannot say whether this drop of $\xi$ at small momentum transfer $t$ is really significant or not.
3.3.3 Problems of parametrization

There are some difficulties when combining the data on \( K_{\ell 3} \) form factors, because, first the authors use different sets of parameters from one publication to another, and secondly it is difficult to display results in terms of three parameters when they are correlated with one another. We shall discuss this problem of parametrization.

a) Choice of parameters

The most natural set of parameters is:

\[ \xi(0), \Lambda, \lambda_+ \]

where

\[ \xi = \xi(0) + \Lambda \frac{t}{\mu^2} \quad (t \text{ is the momentum transfer}) \]

\[ f_+ = f_+(0)[1 + \lambda_+ \frac{t}{\mu^2}] \]

since these three parameters are those actually measured in experiments (when one assumes linearity of \( \xi \) and \( f_+ \)):

i) In polarization analysis one gets: \( \xi(0) \) and \( \Lambda \).

ii) In Dalitz plot analysis of \( K_{\mu 3} \): \( \lambda_+, \xi(0) \) and \( \Lambda \).

iii) From \( \Gamma_{\mu 3}/\Gamma_{e 3} \) one gets a relation between: \( \lambda_+, \xi(0) \) and \( \Lambda \).

iv) From Dalitz plot analysis of \( K_{e 3} \) one gets: \( \lambda_+ \).

Another possible set is:

\[ \xi(0), \lambda_+, \lambda_- \]

related to the first set by the relation \( \Lambda = \xi(0) (\lambda_- - \lambda_+) \) valid for small \( \lambda_+ \), \( \lambda_- \). From an experimental point of view one must notice that, since one measures actually \( \Lambda \), this choice is meaningful only if \( \xi(0) \neq 0 \). [As to \( \lambda_+ \) there is no problem for \( \xi(0) = 0 \) since \( \lambda_+ \) is determined independently of \( \Lambda \) in the \( K_{\mu 3} \) and \( K_{e 3} \) Dalitz plot analyses.] So, some authors use a compromise between both sets, by taking as parameters:

\[ \xi(0), \lambda_+, \xi(0)\lambda_- \]

As \( \xi(0) \) and \( \lambda_+ \) appear to be less correlated with \( \lambda_- \) than \( \xi(0) \) with \( \lambda_+ \), and as \( \lambda_- \) is so far poorly determined and compatible with zero, we have adopted a two-parameter analysis:

\[ \xi(0), \lambda_+ \quad \text{with} \quad \lambda_- \equiv 0 \quad [\text{i.e.} \quad \Lambda \simeq \xi(0)\lambda_+] \]

in the fit to \( K^+ \) data of this publication.

b) Correlation between parameters

We examine the problem of correlation of the parameters mentioned above. Typical correlations observed between these parameters, during this work, were:
\[ \frac{d\xi(0)}{d\Lambda} \approx -3 \] in the polarization analysis (see Fig. 9),
\[ \frac{d\xi(0)}{d\lambda_+} \approx -24 \] in the Dalitz plot analysis (see Fig. 13),
\[ \frac{d\xi(0)}{d\lambda_-} \approx -11 \] in the \( \Gamma_{\mu\nu}/\Gamma_{e3} \) analysis for \( K^+ \) (see Fig. 15),
\[ \frac{d\xi(0)}{d\lambda_+} \approx -10 \] in the \( \Gamma_{\mu\nu}/\Gamma_{e3} \) analysis for \( K^0 \),
\[ \frac{d\xi(0)}{d(\xi(0)\lambda_-)} \approx -3 \] in the over-all fit for \( K^+ \) (see Figs. 16 and 17),
\[ \frac{d\lambda_-}{d\Lambda} \approx 0 \] in the over-all fit for \( K^+ \).

Except the \( \Gamma_{\mu\nu}/\Gamma_{e3} \) results, these values depend on the zones of the Dalitz plot analysed, but we think that they give typical orders of magnitude.

So a problem appears: how to quote clearly results for quantities which are so strongly correlated? One can, for example, fix different values of one of the parameters and display the results for the other two in the form of a two-dimensional plot, just as we do by fixing \( \lambda_- = 0, 0.05, -0.05, \) etc. On the other hand, in order to quote results in terms of \( \xi(0) = a \pm \delta a, \lambda_+ = b \pm \delta b, \) etc., we have projected on the axis the likelihood contours corresponding to a factor \( e^{-0.5} \) in comparison with the optimum (see Appendix D in Ref. 94).

We have also encountered, throughout the discussion, the problem of the averaged value of the momentum transfer \( t \). For a sample of \( K_{\mu3} \) or \( K_{e3} \) one can define, of course, a simple averaged value of \( t \) that we shall denote \( \bar{t} \). But, according to the type of analysis performed (polarization of \( \mu \) or Dalitz plot analysis), this value will make more sense if the events are weighted by the sensitivity to \( \xi \) determination, which depends on kinematical variables. So we shall get an averaged value for the polarization: \( \bar{t}_p \), and one for the Dalitz plot: \( \bar{t}_{DP} \). As the sensitivity to \( \xi \) increases with \( t \) for the polarization analysis, and still more for the Dalitz plot, we have the typical relation: \( \bar{t}_{DP} > \bar{t}_p > \bar{t} \).

Now, suppose that the result of one of the preceding analyses gives a result in terms of the correlated parameters, \( \xi(0), \Lambda \). [The likelihood contours are thus approximate ellipses in the plane \( \xi(0), \Lambda \).] One can find a set of uncorrelated parameters, \( \xi(0) \) and \( X = a \xi(0) + b \Lambda \). It can be shown easily (see, for example, Appendix C of Ref. 94, or Ref. 101) that \( X \) can be expressed as a \( \xi(t^*) \), where \( t^* \) is in first approximation one of the averaged values which we discussed above and is equal to \(-d\xi(0)/d\Lambda \). Thus there exists a means of quoting the correlated result \( \xi(0), \Lambda \) in terms of uncorrelated (i.e. independent) parameters \( \xi(0), \xi(t^*) \). But we have not this possibility when the analysis depends on \( \xi(0) \) and \( \lambda_+ \) (and this corresponds to the strongest correlations); so both results on \( \xi(0) \) and \( \lambda_+ \) must be quoted together and not averaged from one experiment to another as if they were independent.

c) Scalar and vector amplitudes: \( \lambda_0 \) and \( \lambda_+ \)

In most of the publications that we have analysed for this work, people present the parameters \( \lambda_+ \) and \( \lambda_- \) (and \( f_+, f_- \)) as if they were of the same nature. We emphasize that this is incorrect both from a theoretical point of view, and from an experimental one.

As to the experiment, \( \lambda_+ \) plays the same role as \( \xi(0) \); it is as well determined as this parameter and strongly correlated with it, which is not the case for \( \lambda_- \). We have already discussed this point.
This apparently paradoxical statement has its parallel in theory. In fact, the physical amplitudes for \( K_{L3} \) are proportional not to \( f_+ \) and \( f_- \) but to \( f_+ \) and \( f \) (see Section 1.1 or Ref. 139):

\[
\text{amplitude } l^+ \propto f_+(t),
\]
\[
\text{amplitude } 0^+ \propto f(t),
\]
where

\[
f(t) = f_+(t) + f_-(t) \frac{t}{M^2 - \mu^2}.
\]

If we assume that \( f_+ \) and \( f \) are linear we get:

\[
f_+(t) = f_+(0) [1 + \lambda_+ t/\mu^2]
\]
\[
f(t) = f(0) [1 + \lambda_+ t/\mu^2]
\]

Further, if we assume that \( \xi \) does not diverge for \( t = 0 \), which seems in accordance with experiment, we have \( f_+(0) = f(0) \). So we get:

\[
\lambda_- = 0
\]
\[
\xi(0) = (\lambda_0 - \lambda_+) \frac{M^2 - \mu^2}{\mu^2}; \quad \left( \frac{M^2 - \mu^2}{\mu^2} = 12.4 \right).
\]

Hence, first \( \lambda_- \) depends only on terms of second order in the development of the physical amplitudes in terms of \( t \). Secondly, \( \xi(0) \) is proportional to \( (\lambda_0 - \lambda_+) \); that is to say that one cannot consider \( \lambda_+ \) as a term of second order with respect to \( \xi(0) \), but exactly at the same level. So the theory parallels the experimental situation.

As to the comparison with theoretical predictions it is interesting to handle directly \( \lambda_0 \) and \( \lambda_+ \) instead of \( \xi(0) \) and \( \lambda_+ \). The over-all fit on \( K_{L3}^+ \) data in Section 3.7 yields:

\[
\lambda_0 = -0.024 \pm 0.020
\]

### 3.3.3 Comparison of this compilation with other works

When one reads the compilation works on \( K_{L3} \) results one gets quite depressed in view of the abundant, but apparently contradictory results. However, two recent experiments, one in an HLBC \(^{93,94}\), the other in a spark chamber\(^{89,101}\), have shown that the different means of analysis applied to their own data give coherent results. In this compilation, applying the same methods as in these two experimental analyses, we have shown that the bulk of \( K^0 \) results is consistent and that, for \( K^0_{L3} \), more studies are needed. So one may wonder if the "incoherence" usually stated for \( K \) form factors is not due to an incorrect combination of the original results coming from different experiments. We shall not discuss here all the literature on the subject but restrict ourselves to a significant example, "The Review of Particle Properties\(^{137}\)" of the Particle Data Group of the UCRL;
first, this review paper, so useful and remarkable in many respects, is widely distributed; secondly, as to the K form factors, it contains a typical sample of the errors to be avoided when combining the data. We shall limit ourselves to the discussion of the K\(^4\) form factors (pages 15 and 16 of Ref. 157).

i) In the section devoted to \(\lambda_+\), the authors give in parentheses the result of a Dalitz plot K\(\mu_3\) analysis with the comment "not averaged because indirect measurement", which is incorrect. In fact the estimation of \(\lambda_+\) in K\(\mu_3\) analysis is as direct as in K\(e_3\) analysis and obtained in exactly the same way: the study of the density of population over the Dalitz plot, which is: \(\rho = (1 + \lambda_+ t/\mu^2)\) for K\(e_3\) and \(\rho = (1 + \lambda_+ t/\mu^2) \times (A + BE + CE\xi^2)\) for K\(\mu_3\). The neglect of \(\lambda_+\) from K\(\mu_3\) induces a second error:

ii) The authors explain in a comment that \(\lambda_+\), \(\lambda_-\) are not found strictly equal to zero (in fact \(\lambda_-\) is, so far, found compatible with zero), so that \(\xi\) is not strictly constant in the physical range of \(t\): \(\xi = \xi(0) + \Lambda t/\mu^2\). Nevertheless they state that they will assume \(\xi\) constant in their compilation. We agree with this hypothesis since \(\Lambda\) is found so far to be small and compatible with zero. But this approximation, clearly stated, is of second order in comparison with a feature neglected by the authors: the Dalitz plot density depends on both parameters \(\xi\) and \(\lambda_+\) which appear to be strongly correlated, with typical values of \(d\xi/d\lambda_+\) around -25. Thus a change of \(\pm 0.05\) in \(\lambda_+\) can give a change of 1.3 in the quoted value of \(\xi\), which is ten times the final error on \(\xi\) quoted by the authors. However, they have combined results on \(\xi\) which were obtained assuming values of \(\lambda_+\) ranging from -0.020 (Jensen) to 0.030; such an average is meaningless.

iii) Now let us examine the branching ratio results. In a first section of the compilation that we discussed, a very nice and statistically consistent over-all fit to all the different K\(^4\) rates is performed. We quote their results and ours:

a) Rev. of Particle Properties
(Fit to all K\(^4\) rates)
: \(\Gamma_{e_3} = (4.85 \pm 0.07)\%\), \(\Gamma_{\mu_3} = (3.18 \pm 0.11)\%\);

b) This work
(separated fit for K\(\mu_3\) and K\(e_3\))
: \(\Gamma_{e_3} = (4.85 \pm 0.07)\%\), \(\Gamma_{\mu_3} = (3.04 \pm 0.08)\%\).

As to \(\Gamma_{e_3}\), after taking into account the dependence on \(\lambda_+\) of some \(\Gamma_{e_3}\) results (see Appendix A of this section) we get the same mean value with only the \(\Gamma_{e_3}\) data as is obtained from the general fit to all K\(^4\) rates; this seems to confirm the validity of our corrections. As to \(\Gamma_{\mu_3}\) we get a result somewhat lower and more precise than the result (a) for two reasons; first we perform a cut-off on the data based on a physical argument (the percentage of background), while they adopt a cut-off related to the age of the publication (in fact we end up with more results); secondly, we have made the necessary corrections for the data based on partial \(\Gamma_{\mu_3}\) rates (see Appendix B). These corrections are more important than for K\(e_3\); they lead to improved agreement among data and thus a smaller error on the average value. However, we retain, for the following, their averaged \(\Gamma_{\mu_3}\) value which yields \(\Gamma_{\mu_3}/\Gamma_{e_3} = 0.656 \pm 0.023\). From Fig. 15 in Section 3.6 we obtain roughly with this ratio:
\[ \xi(0) = 0.1 \pm 0.20 \quad \text{for} \quad \lambda_+ = 0 \]
\[ \xi(0) = -0.25 \pm 0.20 \quad \text{for} \quad \lambda_+ = 0.03 \]
\[ \xi(0) = -0.60 \pm 0.20 \quad \text{for} \quad \lambda_+ = 0.06 . \]

[One can use the formula (3.9) for more accuracy.]

Now, for the estimation of \( \xi \) using \( \Gamma_{\mu_3} / \Gamma_{e_3} \), the authors do not use their preceding averaged ratio. They have simply taken the \( \xi \) values quoted in the publications, which are calculated assuming different values of \( \lambda_+ \). (We recall the typical correlation \( d\xi/d\lambda_+ = -10 \) that one gets in the \( \Gamma_{\mu_3} / \Gamma_{e_3} \) analysis; see Section 3.6 and Fig. 15.) Then, they reject values extracted assuming \( \lambda_+ = 0 \), but retain three values corresponding to \( \lambda_+ \) ranging from 0.023 to 0.045. Thus, in the experiments retained, the variation of \( \lambda_+ ; \delta \lambda_+ = 0.022 \) corresponds to a systematic variation of \( \xi , \delta \xi = -0.22 \), as large as the experimental errors. Furthermore, most of the experimental information is unnecessarily lost.

In fact, there are two ways of combining the \( \Gamma_{\mu_3} / \Gamma_{e_3} \) results: (a) to deduce directly \( \xi \) from each experiment where this ratio is determined, after adjustment to the same value of \( \lambda_+ \) (the world average, for example); this way is not completely reliable (see the note in Section 3.6.3); (b) to deduce \( \xi \) directly from the value of \( \Gamma_{\mu_3} / \Gamma_{e_3} \) deduced from an over-all fit to all data, which is simpler and reliable.

Our conclusion is that the available data on \( K^+ \) form factors are consistent, as shown in this section, and allow a comparison with theoretical predictions which are presented in the following section.

4. COMPARISON OF THE DATA WITH THEORY

We conclude this review with a brief comparison of the theoretical predictions with the data presented in Section 3.

4.1 The parameter \( \lambda_+ \)

Most of the theoretical calculations described above, which use dispersion relations or hard meson techniques in conjunction with the Callan-Treiman relation find values for \( \lambda_+ \) very near the value required by \( K^+ \) dominance: \( \lambda_+ = 0.023 \). This is compatible with the \( (K_\ell^0)_{e_3} \) data and with the average value from \( K^+_0 \) data. The value extracted from the \( K_{\mu_3} \) analysis (which agrees with the most recent \( K^+_0 \) results) is somewhat higher. The situation is therefore still uncertain. It should perhaps not be too surprising if \( K^+ \) dominance for \( f_+ \) turns out to fail since \( \rho \) dominance is known to be inadequate to describe the nucleon electromagnetic isospin form factor.

4.2 The variation of \( f(t) \) with \( t \)

If one assumes a value for \( \lambda_+ \), one can extract the variation of \( f(t) \) from the variation of \( \xi(t) \) as determined from the polarization and Dalitz plot data (Fig. 6, Section 3). We have used the value \( \lambda_+ = 0.034 \pm 0.006 \), obtained from an average over \( K_{e_3}^+ \) and \( K_{\mu_3}^+ \) data, to calculate \( f(t) \) from the relation
\[ f(t) = f_0(0) \left( 1 + \lambda_+ t/\mu^2 \right) \left[ 1 + \left[ t/(M^2 - \nu^2) \right] \right] \left[ e(0) + \Lambda t/\mu^2 \right] \]  

(4.1)

The result is shown in Fig. 21. The curve through the data points corresponds to the best linear fit to \( \xi(t) \) and the outer curves correspond to the envelopes of all linear fits within the one standard deviation contour of Fig. 9.

The Callan-Treiman point (see Section 2.2) at \( t = M^2 \) is plotted with allowance for a 15% theoretical error. The dashed lines indicate the region in which \( f(t) \) should lie if it is a monotonic function and the Callan-Treiman relation is approximately satisfied. The data allow for the possibility that \( f(t) \) has a dip and rises to meet the Callan-Treiman point as proposed by Berman and Roy (Section 2.5) and Hara (Section 2.2.4). They are also compatible with a linear form for \( f(t) \).

Note that if we had used a lower value for \( \lambda_+ \) (e.g. \( K^+ \) dominance of \( f_+ \)), the negative slope of \( f(t) \) would be even more pronounced. \( K^+ \) dominance of both \( f_+ \) and \( f_- \) requires \( f_+(t) = \text{constant} = f_+(0) \).

The dotted line in Fig. 21 represents the function (see Section 2.1.1):

\[ f_d(t) = \left[ 1 - t^2/(M^2 - \nu^2) \right]^{1/2} \]  

(4.2)
obtained by setting $\delta G^2 = 0$ in the sum rule Eq. (2.16). The hatched area corresponds to "leakage" contributions ($\delta G^2$) less than 0.05. Horn's octet dominance hypothesis (Section 2.1.2) predicts that $f(t)$ should lie in or below this area if $f_4(0) > 0.95$, that is, if the Ademollo-Gatto theorem is not too badly violated.

4.3 Linear fit to $f(t)$

In a linear approximation for $f(t)$, $\lambda_-$ is neglected. The $K^+\mu^3$ data have been fitted setting $\lambda_- = 0$, and this fit is compared with theory in Figs. 22 to 25. The thin solid lines represent lines of constant $\lambda_0$. The heavy solid lines show contours of one standard deviation, and the heavy dashed lines contours of two standard deviations obtained from the experimental fit.

The horizontally hatched regions correspond to the Callan-Treiman requirement $\lambda_0 = 0.021 \pm 0.014$, where a 15% theoretical error has again been allowed for. Most of the calculations described in Sections 2.2.1, 2.2.2, 2.3 and 2.4.2 predict a point near the intercept of the dashed line (Callan-Treiman result) and the dotted line ($K^*$ dominance of $f_4$), as indicated by a small circle. This region is ruled out in the over-all fit of Fig. 25.

$K^*$ dominance of both $f_+^*$ and $f_-^*$ [Eqs. (1.10)] requires $\lambda_0 = 0$, $\lambda_+ = 0.023$, $\xi = \text{constant} \approx 0.3$, as indicated by the heavy dots in Figs. 22 to 25. However, this model also requires $\lambda_- = \lambda_+$. The arrows indicate the shift due to the assumption $\lambda_- = 0$ (cf. Fig. 18, Section 3) if the true value is $\lambda_- = 0.023$. The data should fall at the arrow points in this analysis if $K^*$ dominance is correct.

The vertically hatched areas correspond to small "leakage" [cf. Eq. (4.2)]:

$$ |f(t) - f_0(t)| \leq 0.05$$
(4.3)

the predicted slope $\lambda_0 = 0.025 \pm 0.013$ is obtained by requiring $f_4(0) = 1 \pm 0.05$, $\xi = (M - \mu)^2 = 0.82 \pm 0.05$, so that:

$$ \left( \frac{f((M - \mu)^2)}{f_4(0)} - 1 \right) = \frac{(M - \mu)^2}{\mu^2} \lambda_0 = 0.82 \pm 0.10 - 1$$
(4.4)

Horn's predictions fall within or to the left of these regions.

The polarization data (Fig. 22) do not determine $\lambda_0$ unless some hypothesis is made on $\lambda_+$; note that agreement of these data with the Callan-Treiman relation requires a very high value ($\lambda_+ > 0.08$ for agreement within one standard deviation). On the other hand, the lines of constant $\lambda_0$ in Fig. 24 are nearly parallel to the lines of constant branching ratio; this type of experiment, in principle, measures just $\lambda_0$ for a reasonable range of $\lambda_+$ values (and is very insensitive to $\lambda_-$). Unfortunately, the data of Fig. 24 are not precise enough to be conclusive. They are compatible with $\lambda_0 = 0$. The over-all fit (Fig. 25) definitely favours $\lambda_0 < 0$. 
Fig. 22 Linear fit to \( f(t) \) from \( K^+_{\mu 3} \) polarization data:
\[
\lambda_0 = 0.012_{-0.01}^{+0.018}
\]
--- Callan-Treiman prediction
\( \uparrow \) \( K^* \) dominance (see text)
+ experiment (see Fig. 10).

Fig. 23 Linear fit to \( f(t) \) from \( K^+_{\mu 3} \) Dalitz plot data:
\[
\lambda_0 = -0.038_{-0.017}^{+0.017}
\]
--- Callan-Treiman prediction
\( \leftarrow \) \( K^* \) dominance (see text)
+ Experiment (see Fig. 13).
Fig. 24 Linear fit to $f(t)$ from $K^+_{\mu3}$ branching ratio data:
$\lambda_0 = -0.012 \pm 0.026$
--- Callan-Treiman prediction
$\rightarrow$ $K^*$ dominance (see text)
$\longrightarrow$ experiment (see Fig. 15).

Fig. 25 Linear fit to $f(t)$ from total $K^+_{\mu3}$ data:
$\lambda_0 = -0.025 \pm 0.016$
--- Callan-Treiman prediction
$\rightarrow$ $K^*$ dominance (see text)
$+$ experiment (see Fig. 17)
4.4 Quadratic fit to \( f(t) \)

Figure 26 shows the comparison of theory with experiment for a quadratic fit to \( f(t) \)

\[
\begin{align*}
    f_+(t) &= f_+(0) \left[ 1 + \lambda_+ \frac{t}{\mu^2} \right]
    \tag{4.5}
    \\
    f(t) &= f_+(0) \left[ 1 + \lambda_+ \frac{t}{\mu^2} + \frac{t}{M^2 - \mu^2} \xi(0) \left[ 1 + \lambda_- \frac{t}{\mu^2} \right] \right]
    \tag{4.6}
\end{align*}
\]

with \( \lambda_- = 0.05 \) (see the discussion of Section 3).

In this case the Callan-Treiman relation again gives a linear relation between \( \lambda_+ \) and \( \xi(0) \) for fixed \( \lambda_- \), obtained from Eq. (4.6) and the relation

\[
f(M^2) = \frac{f_1}{f_2} = (1.28 \pm 0.06) f_+(0)
\tag{4.7}
\]

This prediction is plotted in Fig. 26, where the horizontally hatched region again corresponds to a 15% theoretical error for Eq. (4.7), and the small circle corresponds to the predictions of the work discussed in Sections 2.2.1, 2.2.2, 2.3 and 2.4.2.

Similarly, the prediction corresponding to small "leakage", Eq. (4.3), is obtained by using Eq. (4.6) and the relation between the end points of the decay region, Eq. (4.4). This prediction corresponds to the vertically hatched region of Fig. 26. Horn's predictions fall in or to the left of this region.

We have also plotted the line \( \lambda_0 = 0 \) in Fig. 26. \( \lambda_0 \) increases (decreases) to the right (left) of this line. The region of overlap of all the data (diagonally hatched area) corresponds to \( \lambda_0 = 0.005 \pm 0.005 \). However, the quadratic term in Eq. (4.6) gives a negative contribution to \( f(t) \), so that it is still a decreasing function of \( t \). This fit, when
extrapolated to \( t = M^2 \) gives a function \( f(t) \) which curves downward, away from the Callan-Treiman point. This may be compared with the dip behaviour discussed in Section 4.2 in connection with Fig. 21.

However, the quadratic term in \( f(t) \) extracted from these analyses seems to depend strongly on the parametrization. It is, therefore, not possible to conclude from the present data whether or not a dip can occur. On the other hand, the evidence seems strong that \( f(t) \) decreases with \( t \) in some part of the decay region, as anticipated in Section 2. We refer the reader to the conclusions of that Section for a discussion of the theoretical implications of this result.
APPENDIX A

DISCUSSION OF RESULTS ON $\Gamma_{e^3}$ ($K^+$)

We discuss here in detail the $\Gamma_{e^3}$ data; the main lines of this discussion are given in Section 3.6.1. Let us recall that we denote by $F$ the fraction of $e^+$ spectrum observed in an experiment and that $F$ depends on $\lambda_+^0$: $F = F(\lambda_+)$. For each case we shall take $F$ for the value $\lambda_+ = 0.03$; a list of values of $F$ for different cuts and $\lambda_+$ is given in Table 9, Section 3.6.1.

a) Callahan et al.\textsuperscript{80} give $\Gamma_{e^3} = (3.94 \pm 0.21)$\% for $\Gamma_{+} = 5.46\%$. So $\Gamma_{e^3} = (4.02 \pm 0.21)$\% for $\Gamma_{+} = 5.57\%$.

b) Borreani et al.\textsuperscript{74} give $\Gamma_{e^3}/\Gamma_{+} = 0.898 \pm 0.06$; whence $\Gamma_{e^3} = (5.00 \pm 0.34)$\%.

c) Belloti et al.\textsuperscript{83} give $\Gamma_{e^3}/\Gamma_{+} = 0.941 \pm 0.095$; whence $\Gamma_{e^3} = (5.24 \pm 0.5)$\%.

d) The result of Auerbach et al.\textsuperscript{82} combines two measurements, under the assumption $\lambda_+ = 0$:

- one, with Čerenkov counter\textsuperscript{81} [see also note (1) in Ref. 82] gives, with $p_e > 60$ MeV/c, $\Gamma_{e^3}/(\Gamma_{\mu^2} + \Gamma_{\tau^2}) = 0.0589 \pm 0.0021$. Introducing the present $\Gamma_{\mu^2}$ and $\Gamma_{\tau^2}$ and our calculation, $F(0)/F(0.03) = 0.933/0.937$, we get $\Gamma_{e^3} = (4.96 \pm 0.18)$\%.

- a second, without Čerenkov counter\textsuperscript{81}, gives $(472 \pm 30) K_{e^3}$ in the band $130 < p_e < 190$ MeV/c, for $(15843 \pm 152) K_{\mu^2}$. As we have calculated $F(0.03) = 0.448$ for this band, we get $\Gamma_{e^3} = (4.84 \pm 0.30)$\%.

- the two results combined give $\Gamma_{e^3} = (4.95 \pm 0.16)$\%.

e) Eschstruth et al.\textsuperscript{84}, for $p_e > 80$ MeV/c, give $\Gamma_{e^3}/(\Gamma_{\mu^2} + \Gamma_{\tau^2}) = 0.0616 \pm 0.0022$. They have assumed $\lambda_+ = 0$ and found $F(0) = 0.856$ in exact concordance with our calculation. As we get $F(0.03) = 0.863$ for the same band in $e^+$ spectrum, we obtain $\Gamma_{e^3} = (5.17 \pm 0.19)$\%.

f) Garland et al.\textsuperscript{85}, using the same band of $e^+$ momentum as in (e), give (for $\lambda_+ = 0$) $\Gamma_{e^3}/\Gamma_{\mu^2} = 0.069 \pm 0.006$; so multiplying by $\Gamma_{\mu^2}$ and by $F(0)/F(0.03)$ we get $\Gamma_{e^3} = (4.58 \pm 0.40)$\%.

g) Botterill et al.\textsuperscript{87}, limited to $p_e > 90$ MeV/c, give $\Gamma_{e^3} = 4.96\%$ for $\lambda_+ = 0$, 4.92\% for $\lambda_+ = 0.02$, and 4.78 for $\lambda_+ = 0.08$, a variation in concordance with our calculations; they use $\Gamma_{\mu^2} = 65.46\%$. Finally we get $\Gamma_{e^3} = (4.92 \pm 0.21)$\%.

h) Zeller et al.\textsuperscript{100}, for $p_e > 120$ MeV/c, give $\Gamma_{e^3}/\Gamma_{\mu^2} = 0.069 \pm 0.006$. Multiplying this number by $F(0)/F(0.03) = 0.982$ and by $\Gamma_{\mu^2}$, we get $\Gamma_{e^3} = (4.31 \pm 0.40)$\%. 
DISCUSSION OF RESULTS ON $\Gamma_{\mu^3}$ ($K^+$)

We discuss here the $\Gamma_{\mu^3}$ data; an introduction to this discussion is given in Section 3.6.2. Let us recall that we denote by $F$ the fraction of $\mu^+$ spectrum observed in an experiment and that $F$ depends on $\lambda_+$ and $\xi(0)$: $F = F \left[ \xi(0), \lambda_+ \right]$. For each case $F \pm \delta F$ is evaluated for the following zones of variation for $\xi(0)$ and $\lambda_+$: $-2 < \xi(0) < 0$ and $0.00 < \lambda_+ < 0.06$; these zones cover all the experimental results from the polarization and Dalitz plot analyses.

a) Emulsion experiments. First we mention the results where the whole $\mu^+$ spectrum is taken into account:

<table>
<thead>
<tr>
<th>Reference</th>
<th>$\Gamma_{\mu^3}$ (in %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Davies et al.</td>
<td>$2.5 \pm 1.2$</td>
</tr>
<tr>
<td>Ritson et al.</td>
<td>$1.3 \pm 0.7$</td>
</tr>
<tr>
<td>Crussard et al.</td>
<td>$3.8 \pm 2.3$</td>
</tr>
<tr>
<td>Birge et al.</td>
<td>$2.8 \pm 1.3$</td>
</tr>
<tr>
<td>Alexander et al.</td>
<td>$5.9 \pm 1.3$</td>
</tr>
<tr>
<td>Bøggild et al.</td>
<td>$4.3 \pm 1.8$</td>
</tr>
<tr>
<td>Young et al.</td>
<td>$5.3 \pm 0.9$</td>
</tr>
</tbody>
</table>

In such experiments the background contribution is maximum, since one retains the whole $\mu^+$ spectrum and in the same way all possible sources of background and particularly the largest ones, $K_{\pi^2}$ and $K_{\mu^2}$. One must add that the discrimination between $\pi^+$ and $\mu^+$ is based only on the recognition of the secondary charged tracks since there is no fit procedure. We emphasize that, here, the amount of background is at least as important as in the $K_{\mu^3}$ sample and difficult to estimate accurately. This leads to the above observed spread of data where the errors are underestimated. It is striking to note that in these experiments high $\Gamma_{\mu^3}$ rates correspond to low $\Gamma_{K_{\pi^2}}$ rates and vice versa; if we consider for example the more recent emulsion experiment, by Young et al. in 1967, we get (in %) $\Gamma_{\mu^3} = 5.3 \pm 0.9$, $\Gamma_{\pi^2} = 19.3 \pm 1.6$, to be compared to the world averages (for $\Gamma_{\mu^3}$ we take our result)

$\Gamma_{\mu^3} = 3.04 \pm 0.08$, $\Gamma_{\pi^2} = 21.0 \pm 0.3$. So the lack of $K_{\pi^2}$ corresponds to a gain of $K_{\mu^3}$ by about two standard deviations, and the error on $K_{\mu^3}$ is underdetermined. So, we no longer keep these data for the over-all fit.

In a second step, we have tried to extract more reliable information on $\Gamma_{\mu^3}$ from the emulsion experiments. We have looked for experiments where partial $K_{\mu^3}$ rates were stated for zones below the $K_{\pi^2}$ contamination, i.e. approximately $T_{\mu} < 80$ MeV.

i) Birge et al. give $\Gamma'_{\mu^3} = (2.47 \pm 1.0)$% for $T_{\mu} < 80$ MeV. Bøggild et al. give, for both their experiment and that of Alexander et al. combined with it,

$\Gamma'_{\mu^3} = (2.74 \pm 0.7)$% also for $T_{\mu} < 80$ MeV; the three results together yield $\Gamma'_{\mu^3} = (2.65 \pm 0.6)$%. We calculate $F = 0.708 \pm 0.01$, the fraction of $K_{\mu^3}$ with $T_{\mu} < 80$ MeV. So we get $\Gamma_{\mu^3} = (3.7 \pm 0.9)$% for these three experiments combined.
ii) Taylor et al. also used the low part of $\mu^+$ spectrum but with a phase space calculation (in 1950) to estimate their factor $F$. We keep only the events in the zone $T_\mu < 41.4$ MeV where the detection was good enough. So we get $\Gamma_{\mu^3} = 0.75 \pm 0.15$ and $F = 0.280 \pm 0.01$ for $T_\mu < 41.4$ MeV; from this we obtain $\Gamma_{\mu^3} = (2.7 \pm 0.5)\%$.

iii) Giacomelli et al., in the zone $T_\mu < 28$ MeV, find $139.9 \pm 16 K^0_s$ events for $25800 \pm 1400 K^0_s$ decays. For this cut we calculate $F = 0.154 \pm 0.01$; and from this we obtain $\Gamma_{\mu^3} = (3.5 \pm 0.5)\%$.

These three results, though not completely free of background contaminations, are based on purer samples of $K_{\mu^3}$ than the set of results listed above.

b) In the XeBC experiments one meets also the ambiguity between $K_{\mu^3}$ and $K_{\pi^2}$ since the whole $\mu^+$ spectrum is used; but, first the detection efficiency falls off quickly above $T_\mu \approx 70$ MeV, i.e. in the zone of $K_{\pi^2}$ background; secondly one has also the fit procedure to discriminate between both modes. The same group has published two independent and incompatible results (see Ref. 69 and 73); but the second article states that in the first publication some difficulties concerning the fit discrimination between $K_{\mu^3}$ and $K_{\pi^2}$ were not solved. So we keep only the second result, internally consistent with the $K_{\pi^2}$ rate: $\Gamma_{\mu^3} = (5.0 \pm 0.5)\%$.

c) Bisi et al. give $\Gamma_{\mu^3}/\Gamma_{\pi^2} = 0.632 \pm 0.035$; from which they obtain $\Gamma_{\mu^3} = (5.52 \pm 0.20)\%$. This rate is based on three different samples: 670 events from an XeBC experiment in the range $5 < T_\mu < 30$ MeV; 745 + 1430 events from two HLC experiments in the ranges $25 < T_\mu < 95$ MeV and $105 < T_\mu < 120$ MeV. We should like to make two remarks here.

i) In order to extrapolate from the zones observed to the total sample, the authors have not calculated exactly the fraction $F$, as we do, but have used "linear interpolation for the missing intervals" (see page 1071 of Ref. 77). This is quite rough with respect to the error stated for their $\Gamma_{\mu^3}$; the information provided does not allow us to estimate exact corrections for this point.

ii) The contamination due to radiative decays, mainly $K^+ \rightarrow \mu^+\gamma$, has not been taken into account. Now, from the contribution of such events stated in Refs. 78, 87, and 91, we can say roughly that, here, this contamination is at least 5% of $K_{\mu^3}$. Let us note especially that in the band $105 < T_\mu < 120$ MeV this contamination is about 20% of the $K_{\mu^3}$ (this background increases with $T_\mu$); so the linear interpolation for missing intervals based on this band introduces the same percentage of background in the upper part of the spectrum.

That is to say that a reanalysis of the data would decrease $\Gamma_{\mu^3}$ by more than the error stated. As we are not able to do this work here we shall only multiply by two the error in view of the over-all fit $[\Gamma_{\mu^3} = (5.52 \pm 0.4)\%]$.

d) Cutts et al., study only the end of $\mu^+$ spectrum, $107.1 < T_\mu < 126.4$. They get, after corrections, $605.4/0.876 = 691 K_{\mu^3}$ for $166,169 K_{\mu^2}$; we calculate $F = 0.064 \pm 0.003$. We get $\Gamma_{\mu^3} = (4.12 \pm 0.6)\%$. But the authors have to subtract $174.9 K_{\mu^3\gamma}$, $4.3 K_{\mu^2}$ and
195.6 $K_{\pi^+\pi^+}$, to get 489.2 $K_{\mu^3}$, i.e. 77% of background; further, the $K_{\mu^3\nu}$ mode has not yet been studied experimentally, which makes such a large and purely theoretical estimate of background quite unreliable. We do not take this result into the final fit.

e) Callahan et al.\textsuperscript{89)\textsuperscript{2}}), assuming $\Gamma_{\pi^+} = 5.46\%$ give $\Gamma_{\mu^3} = (2.77 \pm 0.19)\%$. So $\Gamma_{\mu^3} = (2.83 \pm 0.19)\%$. The principal zone of $K_{\pi^+\pi^+}$ contamination is cut out.

f) Auerbach et al.\textsuperscript{82)\textsuperscript{2}} have found $309.6 \pm 24 K_{\mu^3}$ in the band $130 < p_\mu < 190$ MeV/c for $15843 \pm 132 K_{\mu^2}$. For this band, we calculate $F_\mu = 0.454 \pm 0.01$. So $\Gamma_{\mu^3} = (3.14 \pm 0.28)\%$. In the article, the authors state that $309.6 K_{\mu^3}$ with $130 < p_\mu < 190$ correspond to 834 for the whole spectrum; this means $F = 0.371$; we disagree with their calculations.

g) Botterill et al.\textsuperscript{87)\textsuperscript{2}}\textsuperscript{2} give $\Gamma_{\mu^3}/\Gamma_{e^3} = 0.667 \pm 0.019$, after a count of $K_{\mu^3}$ and $K_{e^3}$ events in the momentum range: $110$ MeV/c $< p_\mu < 150$ MeV/c. Let us denote by $F_\mu (\lambda_+)$ the fraction of $e^+$ spectrum observed, by $F_\mu (\lambda_+, \xi(0))$ the fraction of $\mu^+$ observed, and by $N_{\mu^3}$, $N_{e^3}$ the number of $K_{\mu^3}$ and $K_{e^3}$ events detected; one must have:

$$R = \Gamma_{\mu^3}/\Gamma_{e^3} = (N_{\mu}/N_{e}) \times [F_\mu (\lambda_+, \xi(0))]/[F_\mu (\lambda_+, \xi(0))]$$

So the result depends on $\xi(0)$ and $\lambda_+$. It seems that the authors have studied this small variation of the exact value of $R$ with $\xi(0)$ and $\lambda_+$ and have taken into account the effect for the estimate of the error. We multiply by the mean value of $\Gamma_{e^3}$, and we get $\Gamma_{\mu^3} = (3.23 \pm 0.10)\%$.

h) Garland et al.\textsuperscript{86)\textsuperscript{2}} have found $424 \pm 33 K_{\mu^3}$ in the band $90 < p_\mu < 190$ MeV/c, for $11666 K_{\mu^2}$; they estimate $F = 0.66$; but by our own calculation we get $F = 0.757 \pm 0.01$. This last number yields $\Gamma_{\mu^3} = (3.05 \pm 0.24)\%$.

i) The experiment X2\textsuperscript{91)\textsuperscript{2}} obtains $\Gamma_{\mu^3}/\Gamma_{+} = 0.503 \pm 0.019$ for the restricted zone $46.4 < T_\mu < 72.7$ MeV, where the percentage of background is minimum (18.5%). We find $F = 0.292 \pm 0.004$, in exact agreement with that stated in this publication. So we get:

$$\Gamma_{\mu^3} = (2.80 \pm 0.11)\%.$$ 

j) Zeller et al.\textsuperscript{190)\textsuperscript{2}} give $\Gamma_{\mu^3}/\Gamma_{\mu^2} = 0.054 \pm 0.009$; and thus $\Gamma_{\mu^3} = 3.44 \pm 0.6$.

An over-all fit and a general discussion are given in Section 3.6.
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**K° data**


K^0 data


Miscellaneous

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