AN INSTRUMENT FOR MEASURING
THE RATE OF CHANGE OF THE RF FREQUENCY
IN PARTICLE ACCELERATORS

F. Blandamura, R. Giannini, A. Susini and C. Zanaschi
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F. Blandamura*, R. Giannini, A. Susini and C. Zanaschi

GENEVA
1970

*) Summer student from the Institute of Physics, University of Turin, Turin, Italy.
SUMMARY

An electronic device is described which measures the time derivative of a time-dependent frequency function.

There are systematic and random errors; the latter can be decreased significantly if \( f(t) \) is periodic as in the case of particle accelerators.

The device has been constructed to monitor the accelerating frequency program of the CERN Synchro-cyclotron (SC) and it allows a relative accuracy better than 2.5\%. 
CONTENTS

1. INTRODUCTION 1
2. PRINCIPLE OF OPERATION 1
3. ERRORS 4
   3.1 Systematic errors 4
   3.2 Counting errors 6
4. THE TRIGGER GENERATOR 6
5. THE CIRCUIT 11
REFERENCES 15
APPENDIX: THE CALCULATION OF THE SYSTEMATIC ERROR 17
1. **INTRODUCTION**

In order to operate a frequency-modulated accelerating machine, one needs to know the derivative $\dot{f}$ of the accelerating frequency $f(t)$, for the following purposes:

a) To contribute to checking the theory of the capture process. For the CERN SC machine, a knowledge of $\dot{f}$ to within some percent at the capture frequency is required.

b) To check the stability of the machine, and also to ensure the reproducibility of the accelerating conditions. For this case, the systematic errors can be tolerated provided the random ones and the drifts introduced by the measuring device are kept well below 1%.

And also:

c) as a diagnostic tool, for example, in case of beam loss; and

d) ultimately, for automatic control. One can think of using the measurement of $f$, $\dot{f}$, and the knowledge of $K$ to adjust the voltage gain per turn so as to obtain the required bucket area.

It will be shown that in the instrument we propose, the random errors are proportional to $\dot{f}$, and that they can be sufficiently reduced when making use of the periodicity of the $f(t)$ function; the systematic errors depend on $\ddot{f}$ and can be well computed when performing more than one measurement in the neighbourhood of the point of interest. This argument will be discussed in detail in the Appendix.

2. **PRINCIPLE OF OPERATION**

As far as we know, the time derivative of a frequency function $f(t)$ is obtained by measuring the time interval $\Delta T$ between two pulses, corresponding to a given frequency interval centred around a given frequency $f_X$. In the apparatus constructed by Gabillard\(^1\) (which is still in operation) for the CERN Proton Synchrotron (PS), the pulses are produced by means of tuned circuits.

The main disadvantage appears when $f_X$ has to be changed; the operation is time-consuming even for a skilled operator. The errors can be fairly well estimated, but at present there is no way of guaranteeing the long-term reproducibility. For these reasons, this apparatus is now being replaced by a more sophisticated one constructed by Boussard\(^2\).

The apparatus we propose has the advantage of utilizing digital techniques, which allows the errors to be computed exactly and the result to be displayed in a numerical form. Ordinary integrated electronics are used throughout without special high-speed components.

The frequency $f_X$ is changed by turning a knob, which allows the apparatus to be easily incorporated in a general automatic survey system. All these characteristics have been achieved with a special pulse generator, which is of a new conception (at least to our knowledge) and is described in detail in Section 4.

The general block-diagram is shown in Fig. 1 and the waveforms appearing at the key-points are shown in Fig. 2. To measure $\dot{f}(f_X)$ [i.e., in the neighbourhood of a given value $f_X$ at the time $t_X(f_X)$], the incoming $f(t)$ signal (No. 1 in Fig. 2) is beaten against a
reference frequency \( f_x \) (which is generated and measured in \( C_1 \) and frequency counter 1) by means of a first mixer (MXR1) of the suppressed carrier type. The output of MXR1 (No. 2 in Fig. 2) is again beaten with MXR2 against a fixed reference frequency \( f_0 \), which in our case is equal to 1 MHz. In this way, the output (No. 6 in Fig. 2) of the MXR2 has zero frequency at the instants \( t_1, t_2 \) implicitly defined as:

\[
|f(t_{1,2}) - f_x| = f_0.
\]

(1)

The output waveform excites a trigger generator which delivers two (negative) triggers at the instants \( t_1, t_2 \) (No. 7 in Fig. 2). The average value of \( \bar{f} \) in the interval \( \Delta T = t_2 - t_1 \) is given by

\[
\bar{f} = \frac{f(t_1) - f(t_2)}{t_2 - t_1} = \frac{2f_0}{\Delta T}.
\]

(2)

By means of a gate generator (No. 8 in Fig. 2), a train of pulses with a frequency \( f_c \) is used to measure \( \Delta T \) through a zero XNG generator and a scaler \( R \).

Provision exists for adding on the scaler the pulses of \( N \) successive gates in order to reduce the random errors produced in the trigger generator. The train of gates is gated by means of a pre-selected counter actuated by a "main pulse" in synchronism with the operating cycles of the SC machine. Usually, \( N \) is made equal to 10 or 100. There are two modes of obtaining \( \bar{f} \). In the first one, the \( f_c \) frequency is generated in a quartz oscillator; then

\[
\bar{f} = \frac{2f_0 f_c N}{R} = \frac{\text{const}}{R},
\]

(3)

the reading \( R \) being the reading at the same scaler.

The constant is chosen such that

\[
\bar{f} = \frac{1}{R}.
\]

(4)

The reading \( R \) can be stored into an automatic survey system or can be directly utilized by the operator. To facilitate this task, one can replace the \( f_c \) generator by a variable-frequency generator \( C_3 \). As the process is repeated every few seconds by means of an automatic relay, the operator will adjust \( C_3 \) in order to read on the scaler \( R \) as a multiple of ten. Then the value seen at the frequency counter \( L \) gives \( \bar{f} \). If \( f(t) \) changes slowly, there is a corresponding shift on \( R \); this shift allows us to read the percentage difference directly (see Fig. 3).

3. ERRORS

From the principle of operation synthetized by Eq. (2), two types of errors appear.

3.1 Systematic errors

A systematic error \( \epsilon_1 \) exists, as the average value given by Eq. (2) is different from the true value \( \bar{f} \) at the centre of the interval \( \Delta T = 2 f_0 / \bar{f} \). Following a series development
The apparatus measures $\dot{\bar{f}} = 2.712 \text{ MHz/msec}$ at $f = 24.16 \text{ MHz}$ with an error of $3\%$. 
of $f(t)$, it can be shown that

$$e_i = |\hat{f} - \bar{f}| \approx \frac{\Delta f}{3!} \left( \frac{\Delta T}{2} \right)^2$$

(5)

For the SC machine, a rough evaluation of $\bar{f}$ leads to values such as those shown in Table 1. In order to know $e_i$ with sufficient exactitude, one is forced to evaluate $\bar{f}$. Two procedures are discussed in the Appendix. To compute $\bar{f}$ one must measure three quantities: one can perform either three measurements in the neighbourhood of $f_x$, or two only, with the third one being replaced by an hypothesis on the smoothness of the $f(t)$ curve. Such an hypothesis is very well verified when the $f(t)$ curves do not show abrupt changes in the upper time derivatives, as in those cases when the curve is generated by means of mechanically variable condensers.

**TABLE 1**

Typical values of the SC-machine accelerating cycle

<table>
<thead>
<tr>
<th>$\tau$ (msec)</th>
<th>$f$ (MHz)</th>
<th>$\dot{f}$ (MHz/msec)</th>
<th>$\ddot{f}$ (MHz/msec$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>29.8</td>
<td>0.78</td>
<td>0.16</td>
</tr>
<tr>
<td>1</td>
<td>29.4</td>
<td>0.97</td>
<td>0.14</td>
</tr>
<tr>
<td>1.5</td>
<td>29.07</td>
<td>1.23</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td>28.15</td>
<td>1.43</td>
<td>0.11</td>
</tr>
<tr>
<td>2.5</td>
<td>27.1</td>
<td>1.76</td>
<td>0.11</td>
</tr>
<tr>
<td>3</td>
<td>26.4</td>
<td>2.08</td>
<td>0.06</td>
</tr>
<tr>
<td>3.5</td>
<td>25.3</td>
<td>2.44</td>
<td>0.07</td>
</tr>
<tr>
<td>4</td>
<td>23.85</td>
<td>2.75</td>
<td>0.3</td>
</tr>
<tr>
<td>4.5</td>
<td>22.3</td>
<td>3.02</td>
<td>0.62</td>
</tr>
<tr>
<td>5</td>
<td>20.9</td>
<td>3.09</td>
<td>0.96</td>
</tr>
<tr>
<td>5.5</td>
<td>19.3</td>
<td>2.91</td>
<td>1.55</td>
</tr>
<tr>
<td>6</td>
<td>17.95</td>
<td>2.46</td>
<td></td>
</tr>
</tbody>
</table>

3.2 Counting errors

The time interval is affected by a counting error of the order of $2/f_c$. This can be sufficiently reduced by increasing the clock frequency $f_c$ and, correspondingly, the speed of the scaler $R$.

4. THE TRIGGER GENERATOR

Whereas the errors mentioned in Section 3 are common to all $\dot{f}$ meters that utilize the principle of Eq. (2), other errors of a random nature are introduced by the digital type of trigger generator such as the one described below.

The basic principle of functioning is as follows.
When a sinusoidal signal with variable frequency $f(t) = f_T + ft$ is beaten, by means of a suppressed carrier mixer $A_1$, against a reference frequency $f_T$, then the output voltage $V_A$ has the shape:

$$V_A = \cos \left( \frac{\pi f t^2}{2} - \alpha_1 \right),$$

where $\alpha_1$ depends on the relative phase at the time $t = 0$ when $f(t)_{t=0} = f_x$. The waveform is sent on a zero crossing discriminator which gives a train of A pulses. In a second channel $B$, the signal $f_x$ is dephased by $\pi/2$ and beaten successively in a mixer $B_1$ with $f(t)$. One will have also

$$V_B = \cos \left( \frac{\pi f t^2}{2} - \alpha_1 - \frac{\pi}{2} \right),$$

with the corresponding train of B pulses.

To simplify the representation, Eqs. (6) and (7), as well as the train of pulses, are plotted as a function of $t^2$, more precisely of $t^2/|t|$, which leads to Figs. 4 and 5. The sequences A and B are symmetrical with respect to the time $t = 0$; therefore, in the neighbourhoud of $t = 0$ two successive pulses of the same sequence will appear. At the time $\tau_1$, defined as the arrival of the first pulse of the sequence after $t = 0$, a sequence comparator allows one, and only one, pulse to pass through.

Now we want to calculate the value of $\tau_1$. From Figs. 4a and 4b and Figs. 5a and 5b, we can see that, qualitatively, four situations can occur: in Figs. 4a and 4b the first pulse after $t = 0$ is $B$; in Figs. 5a and 5b it is $A$.

If $V_A = 0$,

$$\pi \tau_1^2 \frac{\omega}{f} = \alpha_1 + \frac{\pi}{2} \quad \text{where} \quad -\frac{\pi}{2} < \alpha_1 < 0,$$

and if $V_B = 0$,

$$\pi \tau_1^2 \frac{\omega}{f} = \alpha_1 \quad \text{where} \quad \frac{\pi}{2} > \alpha_1 > 0.$$  

This gives for $V_A = 0$:

$$\tau_1 = \frac{1}{\sqrt{\frac{\omega}{f}}} \sqrt{\frac{\alpha_1}{\pi} + \frac{1}{2}} \quad \text{where} \quad -\frac{\pi}{2} < \alpha_1 < 0,$$

and for $V_B = 0$:

$$\tau_1 = \frac{1}{\sqrt{\frac{\omega}{f}}} \sqrt{\frac{\alpha_1}{\pi}} \quad \text{where} \quad \frac{\pi}{2} > \alpha_1 > 0.$$  

Equations (9a) and (9b) define the function $\tau_1(\alpha_1)$ in the range

$$-\frac{\pi}{2} < \alpha_1 < \frac{\pi}{2}.$$
We can see immediately that \( \tau_1(\alpha_1) \) is periodic in (9c) with period \( \pi/2 \), and then to study this function we can limit ourselves to the range \( 0 < \alpha_1 < \pi/2 \) or alternatively \( -\pi/2 < \alpha_1 < 0 \). We shall choose the second possibility that corresponds to Eq. (9a). Therefore the information regarding the instant at which \( f(t) = f_\star \), is delivered with the delay expressed in Eq. (9a).

The most unfavourable case occurs when \( \alpha_1 \to 0^- \) (see Fig. 5a), which gives

\[
\tau_{1,\text{max}} = \frac{0.707}{\sqrt{f}}.
\]

(10)

The most favourable case occurs (see Fig. 5b) when \( \alpha_1 \to (-\pi/2)^+ \), thus leading to \( \tau \approx 0 \).

Since \( \alpha_1 \) is equiprobable between 0 and \( -\pi/2 \) [as the frequency program \( f(t) \) is generated from a frequency- (not phase-) controlled oscillator], we can see from Eq. (9a) that the probability on \( \tau_1 \) will not be uniformly distributed. The average value of the delay will be obtained by putting \( \alpha = -\pi/4 \) in Eq. (9a), which gives:

\[
\tau_{1,\text{av}} = \frac{0.5}{\sqrt{f}}.
\]

(11)

With regard to Fig. 5a, it may happen that for sufficiently small values of \( \tau \), the zero XNG generator is unable to generate two distinct pulses. In this case, the sequences do not invert and no trigger comes out. With the present electronics, this occurs with a probability of roughly 1/2000.

From the preceding, one sees that the leading edge of the gate 8 entering \( M \) is affected by the jitter \( \tau_1(\alpha_1) \) coming from the trigger generator.

The trailing edge of the gate is affected by a jitter \( \tau_2(\alpha_2) \). The phase between the incoming signal frequency \( f(t_2) - f_\chi \) and \( f_\chi \) is given by \( \alpha_1 + \phi \), with

\[
\phi = \int_0^{\Delta T} f(t) \, dt,
\]

with \( \Delta T \) given by Eq. (2).

Therefore the angle \( \alpha_2 \) which determines the jitter \( \tau_2 \) is given by

\[
\alpha_2 = \alpha_1 + \phi - \left[ \frac{\alpha_1 + \phi}{\pi/2} \right] \cdot \frac{\pi}{2},
\]

where the symbol \( (\cdot) \) means "integral part of". If the stability of the machine was perfect, it would be possible to compute \( \phi \), and therefore \( \alpha_2 \), from \( \alpha_1 \). This cannot actually be observed, which means that the trailing edge is almost uncorrelated.

Considering only one gate, the worst case occurs when the gate is opened with zero delay and then closed with the maximum delay given by Eq. (9) or vice-versa. Then for this case:

\[
\frac{\Delta T}{\sqrt{f}} = \frac{\tau_{\text{max}}}{\Delta T} = \frac{0.707}{\Delta T \sqrt{f}} = \frac{0.707 \sqrt{f}}{2f_\star},
\]

(12a)
where $\Delta T$ is the gate length as defined in Eq. (2).

Of course, averaging over successive measurements might lead to more accurate results. The statistics shown in Fig. 6 have been obtained by a computer simulation of 1000 measurements.

For the CERN SC machine, letting $f_0 = 1$ MHz, $\dot{f} = 3$ MHz/msec (typical value), one obtains from Eq. (12a):

$$\frac{df}{f} = 1.93 \times 10^{-2}. \quad (12b)$$

There are several ways of decreasing the value of Eq. (12a), some of which are mentioned below:

i) **Making use of two distinct trigger stabilizing devices** -- as described elsewhere\(^1\)
the jitter of each trigger can be slowed down to $\pm 0.5$ usec independently of $\dot{f}$.

ii) **Substituting channel B in Fig. 1 by M equal channels**, each one dephased by $\pi/2\pi$ from the previous one; in this way, the jitter is reduced by $\sqrt{M}$.

iii) **Multiplying $f(t)$ by $N$ with a frequency multiplier\(^*\)** -- same as above.

iv) **Averaging over $N$ gates** -- to calculate the improvement that can be obtained, the process was simulated on the computer. Over 1000 runs, for $N = 100$, the time error never exceeded the value $\pm 0.08/\sqrt{f}$. This suggests that for $N \gg 1$ the jitter error is,

$$\tau \leq \frac{0.08}{\sqrt{f}} \cdot \sqrt{\frac{100}{N}}. \quad (13)$$

The method is very simple, and in fact the instrument has run with $N = 100$, which for the CERN SC machine implies a measuring time of 2 sec; but experience has shown that when the SC machine is unstable, this delay may appear excessively long. Then the following method is preferable:

v) **The average is performed over 10 gates**, which reduces the value (9) by a factor $\sqrt{10}$; by means of a pre-selected scaler, the gate is made to start (and to end) in correspondence with the 10th period of the waveform at $XOR2$ succeeding trigger $t_1$ (and $t_2$). The result is the same as in point (iv). For the SC machine, the measuring time is now 0.2 sec.

5. **THE CIRCUIT**

The circuitry which performs the operation (iv) in Section 4 is shown in Fig. 7. The input signals $f(t)$ and $f_x$ are first clipped to avoid amplitude modulation disturbances. Then the $f_x$ signal is sent to the A channel and also to a $\pi/2$ delaying cable; it is then sent to the B channel. The output signals of $XOR1$, 1B are beaten with a frequency $f_0$ ($f = 1$ MHz) into $XOR2$, 2B and then sent to the sequence comparator. To avoid spurious signals that might occur at times very far from $t_1$, $t_2$, the output of $XOR1(2)$ is amplified into a 1 MHz tuned amplifier (3), rectified (4), and squared (5). Therefore, the output (10) of the sequence comparator is gated with the waveform (5).
Fig. 6
Other facilities are

a) an over-all gate (9) to avoid signals in the fly-back time;

b) triggers at times $t_0$, $t_6$ (5) are added to the signal trigger $t_1$, $t_2$ to start or to stop the gate (8) in those cases when $t_1$ or $t_2$ does not appear. Then the measurement of $f$ is wrong by 1% when the instrument works with $N = 10$ and by $1\%$ for $N = 100$.

c) Either gates (11) or (12) are sent to the modulator, not shown on Fig. 7. No particular requirements exist for them; in our case an EG&G model is used.

In the method proposed under (v) in Section 4, the gate (8) is sent to open a pre-selected counter, which counts 10 pulses taken from the point (13). After 10 pulses, a gate is opened. The same procedure is used to close the gate. In order to gain measuring time, one decade may be short-circuited at point (15).

Finally, Fig. 8 shows some $f(t)$ and $\hat{f}(t)$ curves. Curve No. 1 is $f(t)$ as obtained with an old resonance method. Curve No. 2 is obtained by differentiating the first one. Curve No. 3 shows $\hat{f}(t)$ measured with the proposed instrument. The last two curves have different shapes, as they have been obtained at different times and under different operating conditions. The dispersion of the measurements around the curve is substantially different.

Acknowledgement

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* * *

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APPENDIX

THE CALCULATION OF THE SYSTEMATIC ERROR

We have seen that by tuning the local oscillator on \( f_x \), one performs the measurement \( \bar{f}(f_x) \), which is the average value of \( \bar{f} \) in the interval \( f_x \pm f_0 \) (where in this particular apparatus \( f_0 \) has been made equal to 1 MHz). The error \( \bar{f} - \bar{f} \) depends on \( \bar{f}(f_x) \). To get a better value of \( \bar{f}(f_x) \) it is necessary to compute \( \bar{f}'(f_x) \); for this, other measurements in the neighbourhood of \( f_x \) are necessary.

Two methods for the computation of \( \bar{f} \) are proposed:

1) to perform two more measurements symmetrically with respect to \( f_x \);
2) only one more measurement is necessary.

1) As well as \( \bar{f}(f_x) \), one measures also \( \bar{f}(f_x + 2f_0) \) and \( \bar{f}(f_x - 2f_0) \). With reference to Fig. A.1, we get

\[
\bar{f}(f_x + 2f_0) = \frac{2f_0}{t_u - t_3} \quad (A.1)
\]

\[
\bar{f}(f_x - 2f_0) = \frac{2f_0}{t_2 - t_1}
\]

Together with

\[
\bar{f}(f_x) = \frac{2f_0}{t_3 - t_2}, \quad (A.2)
\]

one obtains \( t_2, t_3, t_4 \) versus \( t_1 \), which can be chosen arbitrarily.

There exists one polynomial of third degree passing through the four points \( (t_1, f_1); (t_2, f_2); (t_3, f_3); (t_4, f_4) \):

\[
f(t) = At^3 + Bt^2 + Ct + D \quad (A.3)
\]

The coefficients can be computed by a matrix inversion, which in our case is performed with the program BESTFI. From Eq. (A.3) one obtains the third derivative:

\[
\dddot{f} = 6A \quad (A.4)
\]

Of course, the value \( (A.4) \) is constant in the interval \( t_1-t_4 \), which is not the case in reality. The discrepancy decreases when the degree of the polynomial, i.e. the number of measurements, is increased.

To get an idea of such an accuracy, we suppose that the frequency program has the shape

\[
f = h + c \sin \omega t.
\]

For the present SC machine, we assume \( \omega = 2\pi \times 55 \text{ rad/sec} \); thus

\[
f(t) / \text{MHz} = 23 - 7 \sin \omega t \quad (A.5)
\]
For the instant $t_x = 0$, we obtain from expression (A.5):

$$ f''(0) = 7w^1 $$

(A.6)

With reference to Fig. (A.1), we have

$$ f_x = 23 \text{ MHz} \; ; \; f_3 = 22 \text{ MHz} \; ; \; f_4 = 20 \text{ MHz} \; ; \; f_2 = 24 \text{ MHz} \; ; \; f_1 = 26 \text{ MHz} \; . $$

Entering these values into (A.5) we have, for $\sin wt$, respectively:

$$ 0 \; ; \; 0.142 \; ; \; 0.428 \; ; \; -0.142 \; ; \; -0.428 \; . $$

The polynomial (A.3) can now be constructed, which gives:

$$ \frac{f''}{f} = 7 \times 0.9892 \; . $$

Making a comparison with Eq. (A.4), one sees that the error over $\frac{f''}{f}$ is about 1%.

2) The measurements are performed:

$$ \frac{f''}{f}(x) = \frac{2f_3}{t_3 - t_2} \; , \; \frac{f''}{f}(y) = \frac{2f_0}{t_A - t_B} $$

(A.7)

where $t_A$, $t_B$ corresponds to the frequency $f_A$, $f_B$ such as

$$ f_x < f_A < f_3 \; ; \; f_3 < f_B < f_2 \; . $$

From Eqs. (A.7) the length of the segments $t_A - t_B$ and $t_3 - t_2$ is known. Their reciprocal position, for instance $t_B - t_2$, is unknown and can be obtained by minimizing the curvature of the $f(t)$ curve.

The second derivatives are replaced by the second differences, which exist only at the intermediate points $f_B$, $f_3$, as the number of points in the curve is only four, namely $f_2$, $f_B$, $f_3$, $f_A$ (see Fig. A.2). Then we minimize the sum of the squares of the second differences as a function of $t_B - t_2$ or $t_B$, if $t_2$ is taken as a reference.
If $D_1(t)$ and $D_2(t)$ are the first and the second difference, respectively, we have

$$D_1(T_{A3}) \equiv \frac{f_A - f_3}{t_A - t_3} ; \quad D_2(T_{3B}) \equiv \frac{f_3 - f_B}{t_3 - t_B} ; \quad D_1(T_{B2}) \equiv \frac{f_B - f_2}{t_B - t_2} ,$$

with

$$T_{A3} = \frac{t_A - t_3}{2} + t_3 ,$$
$$T_{3B} = t_B + \frac{t_3 - t_B}{2} ,$$
$$T_{B2} = t_2 + \frac{t_B - t_2}{2} ,$$

and

$$D_1(t_j) = \frac{D_1(T_{A3}) - D_1(T_{3B})}{T_{A3} - T_{3B}} ,$$
$$D_2(t_B) = \frac{D_1(T_{3B}) - D_1(T_{B2})}{T_{3B} - T_{B2}} .$$

It can be easily recognized that

$$T_{A3} - T_{3B} = \frac{t_A - t_B}{2} ,$$
$$T_{3B} - T_{B2} = \frac{t_3 - t_2}{2} .$$

The quantity to be minimized is then

$$F(t_B) = D_1(t_j)^2 + D_2(t_B)^2$$

which gives

$$F(t_B) = \frac{4}{(t_A - t_B)^2} \left( \frac{f_A - f_3}{t_A - t_3} - \frac{f_3 - f_B}{t_3 - t_B} \right)^2 + \frac{4}{(t_3 - t_2)^2} \left( \frac{f_3 - f_B}{t_3 - t_2} - \frac{f_B - f_2}{t_B - t_2} \right)^2 .$$

The minimization has been done numerically since the equation $dF(t_B) dt_B = 0$ is of the sixth degree.

Coming back to the previous example, where $f/Miz = (23 + 7 \sin wt)$, the minimum is found at

$$t_B = - \frac{7 \times 10^{-4}}{w} ,$$

and the third derivative given by the polynomial is

$$\dddot{F} = 0.9948 \times 7 \times w^3 ,$$

with an error $E_2 \approx 5\%$. 
It might be surprising that this second method, though intrinsically less accurate, gives better results than the first one. This depends on the fact that in reality the frequency band involved in the first method is roughly twice as broad as that of the second.

If we compare the two systems for equal bandwidth, the first method gives better results. In fact, recalculating with \( f_0 = 0.5 \text{ MHz} \), an error of 2.5% is found.