The Evolution of Correlation Functions and Power Spectra in Gravitational Clustering

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ABSTRACT

Hamilton et al. (1991) proposed a simple formula relating the nonlinear autocorrelation function of the mass distribution to the primordial spectrum of density fluctuations for gravitational clustering in an $\Omega = 1$ universe. High resolution N-body simulations show this formula to work well for scale-free spectra $P(k) \propto k^n$ when the spectral index $n \approx 0$, but not when $n \lesssim -1$. We show that a modified version of the formula can work well provided its form depends on $n$. This dependence can be derived from a simple physical model for collapse from Gaussian initial conditions. Our modified formula is easy to apply and is an excellent fit to N-body simulations with $0 \leq n \leq -2$. It can also be applied to non-power law initial spectra such as that of the standard Cold Dark Matter model by using the local spectral index at the current nonlinear scale as the effective value of $n$ at any given redshift. We give analytic expressions both for the nonlinear correlation function and for the nonlinear power spectrum.

Key words: galaxies: clustering-cosmology: theory-dark matter

1 INTRODUCTION

According to the theory of gravitational instability, the clustering of matter in the universe is determined by the power spectrum of primordial density fluctuations. Since present cosmic structures are highly nonlinear on small scales, the relation between primordial and present spectra results from the interplay of complex processes. It is nevertheless highly desirable to find a simple formula which can relate the two at least approximately. Such a formula not only gives the two-point clustering properties of the evolved density field for any given initial model, but may also enable us to reconstruct the primordial density spectrum from the observed correlations of galaxies.

It is generally assumed that in an Einstein-de Sitter universe, structure will grow self-similarly in time if the primordial power spectrum has a power law form:

\[ P(k) = A k^n, \]

where $A$ is a constant and $4 > n > -3$. For a given spectral index $n$, the correlation functions at different epochs are then related by a simple scaling relation (see Efstathiou et al. [88, hereafter EFWD]). Realistic power spectra are not pure power laws, but we will show that their evolution between the linear and highly nonlinear regimes is similar to that of a power law spectrum with an effective index $n_{eB}$ where we define $n_{eB}$ by

\[ n_{eB} \equiv \left[ \frac{d \ln P(k_0)}{d \ln k_0} \right]_{k_0 = 1/r_0}, \]

where $r_0$ is the radius of the top-hat window in which the rms mass fluctuation is unity.

Hamilton et al. (1991, HKLM) made an ansatz for the relation between the linear, and the evolved (or nonlinear) average two-point correlation functions $\xi_{eB}$ and $\xi_{B}$:

\[ \xi_{eB}(R) = F[\xi_{0}(R_0)], \quad R_0 = [1 + \xi_{B}(R)]^{1/3} R. \] (3a)

Here $\xi_{0}(R) = (3/\pi^2) \int_0^R y^2 \xi(y) dy$, and $F$ is a universal function assumed to be independent of the initial spectrum. Nityananda and Padmanabhan (1994) explained this result by suggesting that the relative peculiar velocities of particle pairs separated by distance $R$ are determined solely by the average mass correlation function $\bar{\xi}$ at this separation. HKLM showed that their ansatz is in good agreement with the N-body simulations of EFWD, and used these data to determine the functional form of $F$. Motivated by this work, Peacock & Dodds (1994, PD) studied a similar
power spectrum:

$$\Delta_{\text{E}}(k) = \Phi[\Delta_{s} (k_0)], \quad k_0 = [1 + \Delta_{\text{E}}(k)]^{-1/3} k,$$

where $$\Delta_{\text{E}}(k) = 4\pi k^3 P_{\text{E}}(k)$$ (with $$P_{\text{E}}$$ being the evolved power spectrum), $$\Delta_{s} (k_0) = 4\pi k^3 P(k_0)$$, and $$\Phi$$ is a universal function. The relations in equations (3a) and (3b) both work remarkably well for power-law spectra with $$n = 0$$, but that they are significantly in error for $$n \lesssim -1$$. This deficiency can be rectified by using the functional forms of $$F$$ and $$\Phi$$ depend on $$n$$. We find that this dependence can be derived quite accurately from a simple physical model for collapse from Gaussian initial conditions. We provide improved formulae to replace the old ones and we show how they can be applied to a non-power-law model such as the CDM universe. In Section 2 results from several N-body simulations are presented and compared with the formulae of HKLM and PD. In Section 3 we describe our model and present our new formulae. Section 4 gives our conclusions.

2 NONLINEAR EVOLUTION AND THE UNIVERSAL SCALING ANSÄTZ

In this paper, we use simulation results for six different spectra. One is the standard CDM spectrum. The other five are power-law spectra with $$n = 0$$, $$-0.5$$, $$-1$$, $$-1.5$$ and $$-2$$. All simulations assume an Einstein-de Sitter universe.

These simulations were performed using high resolution particle-particle/particle-mesh (P³M) codes. For models with $$0 \geq n \geq -1.5$$, the code is the same as in EFWD, but was run with more particles ($$10^6$$, compared to $$32^3$$) and higher force resolution ($$L/2500$$, compared to $$L/200$$, where $$L$$ is the side of the computational box). The power spectrum is normalized as described in EFWD.

The $$n = -2$$ run was performed by E. Bertschinger using an adaptive P³M code. It followed $$128^3$$ particles with a force resolution of $$L/2560$$. The CDM simulation, performed by Gelsb & Bertschinger (1994), is also a P³M simulation. It has $$144^3$$ particles, $$Q = 1$$, $$H_0 = 50$$ km/s/Mpc, $$L = 100$$ Mpc and a force resolution of 65 kpc. It is normalized so that $$\sigma_8$$ (the linear rms mass fluctuation in a sphere of radius 16 Mpc) is unity when the expansion factor $$a = 1$$.

The dotted curves in Figure 1 show $$\xi_{\text{E}}(R)$$ as a function of $$\xi_{\text{E}}(R_0)$$. In each case we show results for four different expansion factors as detailed in the figure caption. For each $$n$$, the curves for different $$a$$ are close to each other, thus demonstrating that clustering is self-similar in the power law models. (The flattening of the curves at high amplitudes, which is severer for earlier output times, is due to insufficient resolution on small scales). The simulation results for different spectra show that the log $$\xi_{\text{E}}(R)$$-log $$\xi_{\text{E}}(R_0)$$ relation depend on $$n$$. The long-dashed curve in each panel shows the fitting formula given by HKLM, which is the same for all spectra. HKLM’s fit works very well for $$n \sim 0$$. However, for $$n \lesssim -1$$, it misses the simulation results by a factor of 3 to 10 at $$\xi_{\text{E}} \sim 1$$. A similar failure is also seen in the CDM case where results for different $$a$$ (or $$\sigma_8$$) differ substantially from each other. The effective power indices $$n_{\text{E}}$$ are $$-0.7, -1.3, -1.8$$ and $$-2$$ for $$n = -1, 0.5, 0.3$$ and 0.2 respectively. The dependence on $$n_{\text{E}}$$ is strikingly similar to that for the corresponding power-law spectra.

The dotted curves in Figure 2 show $$\Delta_{\text{E}}(k)$$ as a function of $$\Delta_{\text{E}}(k_0)$$ for the same simulations as in Figure 1. The self-similarity of clustering is again verified for the power law models, though the results for $$n = -2$$ have some scatter. Again the relation between $$\Delta_{\text{E}}(k)$$ and $$\Delta_{\text{E}}(k_0)$$ depends on $$n$$. The long dashed curves in each panel show the fit proposed by PD. As was the case for the HKLM formula in Figure 1, the PD formula only works well for $$n \sim 0$$. For $$n \lesssim -1$$ and for the CDM spectrum, there are substantial deviations from the simulation data. This is not surprising, because as pointed out by PD, their formula corresponds to the HKLM formula very accurately for $$\Omega = 1$$, and therefore reproduces its shortcomings.

3 AN IMPROVED MODEL

Since for given $$n$$ the correlation functions and dimensionless power spectra obey scaling relations of the form (3), we can write both in the more general form $$y = f_{\text{E}}(x)$$. From Figures 1 and 2 we see that although the relation between the evolved and unevolved quantities differs for different spectra, the shapes of the various curves are similar. This suggests that the $$n$$-dependence can be scaled away by a simple shift in the log-log plane. From linear theory and from the stable clustering hypothesis respectively we know that $$f_{\text{E}}$$ has the following asymptotic behaviour: $$f_{\text{E}}(x) \propto x$$ as $$x \to 0$$ and $$f_{\text{E}}(x) \propto x^{-1/2}$$ as $$x \to \infty$$. Thus the shift can be made only in the direction $$y = x$$. This suggests the ansätze:

$$\frac{\xi_{\text{E}}(R)}{B(n)} = F \left[ \frac{\xi_{\text{E}}(R_0)}{B(n)} \right],$$

and

$$\frac{\Delta_{\text{E}}(k)}{B(n)} = \Phi \left[ \frac{\Delta_{\text{E}}(k_0)}{B(n)} \right],$$

where $$B(n)$$ is a constant depending on $$n$$, and $$F$$ and $$\Phi$$ remain independent of $$n$$. Since the dimensionless power spectrum $$\Delta$$ and the correlation function $$\xi_{\text{E}}$$ are proportional to each other, we assume $$B(n)$$ to be the same in the two relations. In the following we derive $$B(n)$$ from a simple analytical model. This turns out to give results which are almost as good as could have been obtained by fitting an arbitrary functional form to the simulation data.

In a recent paper Mo & White (1994, MW) propose an analytic model for gravitational clustering from Gaussian initial conditions. Modelling the collapse of density perturbations as spherically symmetric, they write the average mass correlation function as

$$\xi_{\text{E}}(r) = \frac{1}{r^3} \int_{r_0}^{r} \frac{\delta^3 \rho(\delta_0 | r) d\delta_0}{\int_{r_0}^{r} \rho(\delta_0 | r) d\delta_0} - 1,$$

where $$(\delta_0 | r | d\delta_0)$$ is the probability that a spherical region with Eulerian radius $$r$$ has mean linear overdensity in the range $$\delta_0 = \delta_0 + d\delta_0$$; $$r_0$$ and $$r$$ are the Lagrangian and Eulerian radius of such a region. MW make the following ansatz
for \( p(\delta_0) \):
\[
p(\delta_0) = \frac{1}{\sqrt{2\pi}} e^{-\delta_0^2/2} d\delta_0,
\]
where \( \delta_0 = \sigma_0 / \sigma(\eta_0) \) and \( \sigma(\eta_0) \) is the rms linear mass fluctuation in a top-hat window with radius \( \eta_0 \) depends on \( \delta_0 \) both directly and through the relation \( \eta_0 = \eta_0(\delta_0, r) \). For uncollapsed regions (i.e., \( \delta_0 < \delta_c \equiv 1.68 \)), MW assumed \( \eta_0 \) to be related to \( r \) and \( \delta_0 \) through a spherical infall model (Peebles 1980, §19). For density perturbations that have collapsed, we need a model to describe the relation of \( \delta_0 \) to \( \eta_0 \) and \( r \). If we assume that, after reaching maximum expansion at \( r = r_{\text{m}} = (3/5) \eta_0 / \delta_0 \), a mass shell will virialize at a radius \( r = r_{\text{m}} \) where \( \alpha \) is a constant, then
\[
\delta_0 = \frac{3 \alpha \eta_0}{5 \alpha_0}.
\]
We will consider the case where \( r \to 0 \). In this case \( \eta_0 \to 0 \) unless \( \delta_0 \to \infty \). The integrations in equation (5) are then dominated by collapsed regions. Since \( \sigma(\eta_0) \to \infty \) as \( r_0 \to 0 \) for \( n > -3 \) (which is true for most physical models), we have, changing the integration variable from \( \delta_0 \) to \( \eta_0 \),
\[
\tilde{\xi}_E(r) = \frac{1}{r^3} \int_0^{\infty} \frac{\sigma(\eta)}{\eta} e^{-\eta^2/4} d\eta_0,
\]
when \( r \to 0 \) (or \( \tilde{\xi}_E \to \infty \)). For the power-law spectrum (1), \( \sigma(\eta_0) \propto \eta_0^{(n+1)/2} \). Using equation (7) in equation (8) we obtain
\[
\tilde{\xi}_E(R) = 2^{3/2} \left( \frac{5}{3 \alpha_0} \right)^{3/2} \left[ \frac{\Gamma\left(\frac{7+n}{3+n}\right)}{\Gamma\left(\frac{1+n}{3+n}\right)} \right]^{(5+n)/2} R^2,
\]
where \( R_0 = [1 + \tilde{\xi}_E(R)]^{1/3} R \).

Equation (9) shows that this model correctly predicts the scaling relation \( \tilde{\xi}_E(R) \propto [\tilde{\xi}_E(R)]^{3/2} \) for stable clustering. The amplitude of this relation is also determined for given \( n \) once \( \alpha \) is given. In Figure 1 the upper dotted lines show the predictions of equation (9) for different \( n \), with \( \alpha = 1 \). This agrees reasonably well with the simulation results and is expected, since equation (9) depends strongly on \( \alpha \) and we do not understand why \( \alpha \) should be close to \( 1 \) rather than to \( 0.5 \). The latter choice seems more reasonable, because it implies that a mass shell settles at half its maximum expansion radius (e.g., Gott & Rees 1975). In our discussion, we need only the relative amplitude for different \( n \). The actual value of \( \alpha \) is unimportant provided it does not vary with \( n \).

Comparing equations (4) and (9) we find
\[
B(n) \propto \left[ \frac{\Gamma\left(\frac{7+n}{3+n}\right)}{\Gamma\left(\frac{1+n}{3+n}\right)} \right]^{(5+n)/2}.
\]
If we choose \( n = 0 \) as the fiducial case, the shift that has to be made to bring the result for a spectrum with index \( n \) into coincidence with that for \( n = 0 \) is
\[
\frac{B(n)}{B(0)} = 0.795 \times \left[ \frac{\Gamma\left(\frac{7+n}{3+n}\right)}{\Gamma\left(\frac{1+n}{3+n}\right)} \right]^{(5+n)/2}.
\]
For simplicity we choose \( B(0) = 1 \).

In the first panel of Figure 1, the solid curve shows our best fit to the simulation results for \( n = 0 \). Since HKLM’s original formula fits these results quite well, we have chosen a similar functional form for \( F(x) \) to theirs. Our result is
\[
F(x) = x + 0.451 x^3 + 0.0420 x^5.
\]
The inverse of this can be approximated to better than 2% over the range of interest by
\[
\tilde{F}(y) = y \left( \left( 1 + 0.0199 y^2 + 0.000162 y^4 \right)^{1/2} / \left( 1 + 1.166 y^2 - 0.105 y^4 + 0.0247 y^6 \right)^{1/2} \right)^{1/2}.
\]

The solid curves in the other panels of Figure 1 are obtained from equation (4a), with \( F \) given by (12a) and \( B(n) \) given by (11). For the CDM case, we have used the effective power indices given above. It is remarkable that this simple model works accurately in all cases. For spectra with \( n \lesssim -1 \) and the CDM spectrum, it makes a substantial improvement over the formula given by HKLM.

The solid curve in the first panel of Figure 2 is given by the fitting formula
\[
\Phi(x) = x \left( 1 + 2x^2 - 0.6x^3 - 4.8x^{5/2} + 1.15x^4 \right)^{1/2} / \left( 1 + 0.037x^3 \right)^{1/2}.
\]
The inverse of this function is fitted to an accuracy of better than 2% over the range of interest by
\[
\tilde{\Phi}(y) = y \left( 1 + 0.235 y^{1.875} + 0.000353 y^3 \right)^{1/3} / \left( 1 + 1.58 y^{1.258} - 0.00303 y^3 + 0.073 y^6 \right)^{1/3}.
\]

The solid curves in other panels are obtained from equation (4b), with \( \tilde{\Phi} \) given by (13a) and \( B(n) \) given by (11). For the CDM spectrum, we have again used the effective power index \( n_{\text{eff}} \). The figure shows that our model also works for the dimensionless power spectrum. There are small differences in the shapes of the N-body curves for different spectra, thus suggesting that one can do marginally better by transforming the relation for the correlation function to obtain the power spectrum. However our fit is adequate given the scatter in the N-body data. For \( n \lesssim -1 \) and CDM spectra, our model makes a substantial improvement over PD’s original formula.

4 DISCUSSION

For a given linear power spectrum \( P(k_0) \), the effective power index \( n_{\text{eff}} \) can be obtained from equation (2). The linear dimensionless power spectrum \( \Delta_0(k_0) \) and the linear average correlation function \( \tilde{\xi}_E(R_0) \) can be calculated directly from \( P(k_0) \). The evolved dimensionless spectrum \( \Delta_E(k) \) and the evolved average correlation function \( \tilde{\xi}_E(R) \) can then be calculated from equations (4a) and (4b), respectively, with \( F \) given by equation (12a), \( \tilde{\Phi} \) given by equation (13a), and \( B(n) \) given by equation (11). To obtain the linear spectrum from the evolved spectrum (or correlation function), one can use equations (12b) and (13b). In this case, an iterative procedure may have to be used, because the effective power index is unknown a priori. As noted by PD the effects of redshift space distortions and biasing also need to be considered in such reconstructions. For other applications, such as using the nonlinear power spectrum to compute the statistics of photon trajectories, our results can be used directly. It would clearly be useful to extend these results to cosmological models with \( \Omega < 1 \) or which involve a hot dark...
matter component (see PD).

As a cautionary note, we point out that the improvements presented in this paper were made possible by the availability of higher resolution N-body simulations. The fitting formulae we propose are only as accurate as these simulations and should be tested further as better simulations become available. We also note that the scaling proposed in equations (4) and (11) is not the only possibility. Other modifications of equations (3) can do equally well, for example one can introduce a simple $n$-dependent function in the relation between linear and nonlinear scales. The actual form clearly does not matter, the use of these relations lies in the case with which they can be used to calculate many properties of the nonlinear mass distribution for a wide range of cosmologies.

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REFERENCES


FIGURE CAPTIONS

Figure 1. The evolved average mass correlation function $\xi_R(R)$ as a function of the linear average mass correlation function $\bar{\xi}_L(R_0)$. Note that these two functions are calculated at two different scales, as discussed in the text. The dotted curves show the results derived from N-body simulations of different spectra. Results are shown for four different expansion factors $a$; a curve that flattens earlier corresponds to a lower value of $a$. The expansion factors are $a = (6.10, 14.91, 36.86, 90.88)$ for $n = 0$, $(4.52, 9.53, 20.22, 42.32)$ for $n = -0.5$, $(3.34, 6.08, 11.08, 20.20)$ for $n = -1$, $(2.47, 3.86, 6.07, 9.52)$ for $n = -1.5$, $(0.96, 1.62, 2.72, 3.85)$ for $n = -2$, and $(0.2, 0.3, 0.5, 1.0)$ for CDM. Long-dashed curves show the fitting formula of HKLM. Solid curves show the results of our improved formula (see text). The two light dotted lines show the expected behaviour of the relation in the linear and stable clustering (highly nonlinear) regimes.

Figure 2. The evolved dimensionless power spectrum $\Delta_{E}(k)$ as a function of the linear dimensionless power spectrum $\Delta_{L}(k_0)$. Note that these two functions are calculated at two different scales, as discussed in the text. The dotted curves show the results derived from the same N-body simulations as described in Figure 1. Long-dashed curves show the fitting formula of PD. Solid curves show the results of our improved formula (see text). The two light dotted lines show the expected behaviour of the relation in the linear and stable clustering (highly nonlinear) regimes.

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