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CLASSICAL BOSONS AND FERMIONS IN THE CONTEMPORARY TRANSITION DESCRIPTION

G. Kaniadakis*, P. Quarati

*Dipartimento di Fisica - Politecnico di Torino - Corso Duca degli Abruzzi 24, 10129 Torino, Italy**Istituto Nazionale di Fisica della Materia - Unità del Politecnico di Torino**Istituto Nazionale di Fisica Nucleare - Sezioni di Cagliari e di Torino*

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Abstract

Contemporary transition description, in the nearest neighbors frame, allows a more complete and rigorous treatment of the kinetics of a system of identical particles obeying an exclusion-inclusion principle than the only individual transition approximation we discussed in a previous work [Phys. Rev. E **49**, 5103 (1994)]. Within this description, statistical distributions are derived, as stationary states of a generalized non linear Fokker-Planck equation, and collective quantum macroscopic effects for both bosons and fermions can be evidenced in a more precise form and with a more appropriate meaning than in the individual transition approximation.

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I. INTRODUCTION

It is well known that, if we have a system of classical particles of a certain kind and if the interaction of the particles is weak enough, each particle has its own motion which is independent of all others. A Fokker-Planck equation (FP) is well adapted to describe their classical kinetics. Suppose that the interaction is not too strong in such a way that we can limit ourselves in considering only interactions with the nearest neighbors and caused only by the exclusion-inclusion principle (EIP) [1,2]. Of course, in the case of a system of fermions, the exclusion principle does work like a repulsive potential, while for bosons the interparticle potential is attractive due to the inclusion principle. A non-linear FP equation is still well based to describe the kinetics of these particles submitted to the EIP [2].

In the nearest neighbors framework, when we are studying the particle kinetics of a not too dense gas, we may use, as a useful approach, the IT approximation: in a time interval dt only one transition is allowed and only two adjacent sites are involved. The most complete treatment, in the nearest neighbors framework, is given by the contemporary transition (CT) description: if two particles occupy two different sites $i = \pm 1$, one of them can make a transition to the site i if this site is empty and if the second particle does not move toward the same site (exclusion) or, on the opposite (inclusion), one particle is invited to the site i if this last and the initial site are occupied and if, in addition, also the second particle is moving to the same site (in this case, three sites in $[t, t + dt]$, are involved). Other descriptions, involving more than three particles, produce very complex formulae without a significant changement both in the comprehension and in the quantitative evaluation of the kinetics.

The exclusion and the inclusion effects on the probability transitions can be weighted in such a way that many different intermediate situations can be described and intermediate statistics can be derived [3]. We wish to mention, among the numerous papers on the fractional statistics subject, the work of March et al. [4] on the statistical distribution function for an anyon liquid and the very recent works of Johnson and Canright [5], Murthy and

*E-mail: kaniadakis@polifs.polito.it

Shankar [6] and of Yong-Shi Wu [7] based mostly on an extension for ideal non-interacting particles of the Haldane statistics [8] which generalizes the exclusion principle.

Our approach differs sensibly from the ones used by the above authors because is based on the inclusion of the quantum statistics effects (EIP) into the transition probability from a site to another one, which is the main, delicate quantity entering into the kinetic equations of identical particles (e.g. Fokker-Plank and Boltzmann equations). Therefore these transition probabilities are expressed in terms of the occupation number distributions. We can, by solving numerically these equations, study the time evolution of a given, initial distribution and deduce the steady states of both non-interacting or interacting particle systems.

The particles considered in this work and submitted to the EIP, are uniquely Brownian particles. We hold this limitation because this is the unique case we are able to obtain analytically, at the infinite time, the stationary distributions. A second reason is related to the features of their random walks, allowing a comparison to the quantum mechanics approach of the kinetics of a system of identical particles [9].

In our previous works on the present subject [1,2], we have derived a kinetic equation for the classical distribution function which incorporates the effect of quantum statistics in the IT approximation. We have already given some hints to study, in the CT description, a system of fermions [1]. The kinetics and the dynamics of interacting particles submitted to an external potential is analyzed in the individual transition (IT) approximation in Ref. [10] by means of a generalized Boltzmann equation. The correctness of semiclassical transport models (Boltzmann equation) [11,12] has been recently discussed and criticized in problems of heavy-ion collisions where quantum statistics must be incorporated into simulations. Our approach differs from the ones used by other authors because introduces quantum effects directly into the transition probability.

The main results obtained in this work (a rigorous expression of the quantum distribution of a system of identical particles submitted to the EIP, as steady state of a non-linear FP equation) are related to the effects on the time evolution of the statistical distributions due to additional terms introduced by the CT description respect to the IT approximation and

proportional to the factor containing the distribution function itself: $\kappa n(t, v)[1 + \kappa n(t, v)]$ (where κ is a parameter which has been related to the degree of symmetrization or antisymmetrization of the many-body wave functions).

From the IT approximation we obtain as steady states the FD or the BE distributions, which are the exact quantum statistical distributions for non interacting particles. These distributions are a consequence of a complete symmetrization or antisymmetrization of the many body wave function.

The symmetry properties originates a potential (attractive or repulsive) depending on the temperature, which cannot be regarded as a true interparticle potential. The CT description originates a temperature dependent potential which has a slight influence at $T \neq 0$ on the energy of the fermions, around the Fermi energy, and of the bosons. This is the motivation why the CT steady states differ slightly from the FD and BE distributions.

The consequences of these effects are mainly represented by collective quantum macroscopic behavior of a quantum system of identical many bodies, which does not appear in the IT approximation. In fact, let us consider the two extreme cases of fermions ($\kappa = -1$) and bosons ($\kappa = 1$). In the first case we have that the CT description introduces in the fermion energy levels additional terms respect to the IT approximation, proportional to T , which can be formally related to the quantum correction to the partition function and to the distribution of an ideal ensemble in the theory of metals [13,14]. In the IT approximation we do not obtain such detailed and precise description. In the case of bosons we can foreseen a condensation of particles at the lowest energy level, different from zero and below a critical temperature T_b . This is a quantum mechanics effect in a macroscopic scale that in the IT approximation can be provided only at zero energy and zero temperature. Both these quantum macroscopic effects are better described within the CT description rather than within the IT approximation, because the first one considers more properly, within the nearest neighbors frame, the interaction of the particles, submitted to the EIP.

In Sect.II, we recall the main steps of the IT approximation and report more convenient definitions than in Ref.s [1,2] of the different quantities to be used also in the following

sections. In Sect.III, we discuss the CT description and deduce the stationary state distributions, analyse the results and compare them to those of the IT approximation. Finally, in Sect. IV, we summarize the main results obtained in this paper. Systems of particles satisfying a fractional statistics are not considered in this paper and will be studied elsewhere.

II. INDIVIDUAL TRANSITION APPROXIMATION

The distribution function $n(t, v)$ and the probability current $j(t, v)$ are related through the continuity equation:

$$\frac{\partial n(t, v)}{\partial t} + \frac{\partial j(t, v)}{\partial v} = 0 . \quad (1)$$

The current $j(t, v)$ is defined as:

$$j(t, v)dv = a[\pi(t, v - dv/2 \rightarrow v + dv/2) - \pi(t, v + dv/2 \rightarrow v - dv/2)] , \quad (2)$$

where a is a convenient dimensional constant and the function π is the transition probability.

In the simplest approach, π is a linear (L) function of n

$$\pi_{\pm}(t, v \rightarrow v \pm dv) = r^{\pm}(t, v)n(t, v) , \quad (3)$$

where the adimensional rates $r^{\pm}(t, v)$ are defined as

$$r^{\pm}(t, v) = \frac{1}{a} \left[D(t, v) \mp \frac{1}{2} J(t, v)dv \right] , \quad (4)$$

The two functions $D(t, v)$ and $J(t, v)$ are the diffusion and the drift coefficients, respectively.

The constant a has the dimensions of the diffusion coefficient D . We can define a linear probability current:

$$j_L(t, v) = - \left\{ \left[J(t, v) + \frac{\partial D(t, v)}{\partial v} \right] n(t, v) + D(t, v) \frac{\partial n(t, v)}{\partial v} \right\} . \quad (5)$$

Let us now define a transition probability different from the one defined in Eq.(3). We call this function the individual transition probability; it depends on the distribution of the initial site and on the distribution of the arrival site. This quantity is correct for all kind

of particles, depending on the value of κ ($\kappa = 1$ for bosons, $\kappa = 0$ for classical particles, $\kappa = -1$ for fermions, otherwise for anyons) :

$$\pi_{IT}(t, v \rightarrow v \pm dv) = r^{\pm}(t, v)n(t, v)[1 + \kappa n(t, v \pm dv)] . \quad (6)$$

The probability current, related to the above transition probability, is

$$j_{IT}(t, v) = - \left\{ \left[J(t, v) + \frac{\partial D(t, v)}{\partial v} \right] n(t, v)[1 + \kappa n(t, v)] + D(t, v) \frac{\partial n(t, v)}{\partial v} \right\} . \quad (7)$$

We pose :

$$j_{IT}(t, v) = j_L(t, v) + \kappa j_J(t, v) , \quad (8)$$

where:

$$j_J(t, v) = - \left[J(t, v) + \frac{\partial D(t, v)}{\partial v} \right] [n(t, v)]^2 . \quad (9)$$

The current $\kappa j_J(t, v)$ is the term introduced by the EIP. This current is absent if we consider a linear kinetics ($\kappa=0$), in which the particles are free to move without the limits imposed by the EIP. By means of Eq.(7), the continuity equation in a three dimensional space becomes:

$$\frac{\partial n(t, \mathbf{v})}{\partial t} = \nabla \cdot \{ [J(t, \mathbf{v}) + \nabla D(t, \mathbf{v})] n(t, \mathbf{v}) [1 + \kappa n(t, \mathbf{v})] + D(t, \mathbf{v}) \nabla n(t, \mathbf{v}) \} , \quad (10)$$

or in a more compact form:

$$\frac{\partial n(t, \mathbf{v})}{\partial t} = \nabla \cdot \left(\frac{1}{\beta} D(t, \mathbf{v}) \{ [\nabla U_{IT}(t, \mathbf{v}, n)] n(t, \mathbf{v}) + \beta \nabla n(t, \mathbf{v}) \} \right) , \quad (11)$$

where $\beta = 1/kT$ and

$$\nabla U_{IT}(t, \mathbf{v}, n) = \frac{\beta}{D(t, \mathbf{v})} [J(t, \mathbf{v}) + \nabla D(t, \mathbf{v})] [1 + \kappa n(t, \mathbf{v})] . \quad (12)$$

We can decompose the potential $U_{IT}(t, \mathbf{v}, n)$ in two terms:

$$U_{IT}(t, \mathbf{v}, n) = U_L(t, \mathbf{v}) + U_I(t, \mathbf{v}, n) . \quad (13)$$

The first term $U_L(t, \mathbf{v})$ represents the linear potential due to the anisotropy and heterogeneity of the space and is given by:

$$\nabla U_L(t, \mathbf{v}) = \frac{\beta}{D(t, \mathbf{v})} [\mathbf{J}(t, \mathbf{v}) + \nabla D(t, \mathbf{v})] . \quad (14)$$

The second term is due to the EIP and has the expression:

$$\nabla U_I(t, \mathbf{v}, n) = \kappa n(t, \mathbf{v}) \nabla U_L(t, \mathbf{v}) . \quad (15)$$

Let us now consider Brownian particles . Then the diffusion and the drift coefficients are:

$$\mathbf{J}(t, \mathbf{v}) = c\mathbf{v} \quad , \quad D(t, \mathbf{v}) = c \frac{kT}{m} , \quad (16)$$

where c is an inverse time dimensional constant which does not affect the steady states in the IT approximation. We can calculate at any time from Eq.(14) the potential $U_L(t, \mathbf{v})$

$$U_L(t, \mathbf{v}) = \frac{1}{2} m v^2 , \quad (17)$$

If we pose $E = mv^2/2$ the stationary distributions of Eq.(10) are given by the expression

$$n(\mathbf{v}) = [\exp \beta(E - \mu) - \kappa]^{-1} . \quad (18)$$

and differ each other depending on the value of the parameter κ . The parameter κ is related to the parameter α of March et al. [4] by: $\kappa = -\cos \alpha$ and to the parameter α of Wu [7] by: $\kappa = 1 - 2\alpha$. It is very difficult to determine the analytical form of the potential $U_I(t, \mathbf{v}, n)$ at any time. The steady form, at infinite time, is given by:

$$U_I(\infty, \mathbf{v}) = \beta \log \{1 - \kappa \exp[-\beta(E - \mu)]\} . \quad (19)$$

III. CONTEMPORARY TRANSITION DESCRIPTION

Contemporary transitions are strongly influenced by the space dimension; in fact, the number of the nearest neighbors increases when the space dimension increases. This complicates the particle kinetics. In this section we consider the simplest case of one dimensional space, where the number of the nearest neighbors are two. In the IT description two fermions occupying, at a given time, the two sites $i \pm 1$, can, at the same time, initiate a transition

toward the site i . This transition, if allowed, violates the EP. For this reason we must define the transition probability in such a way the EP be rigorously considered. In the nearest neighbors framework, we assume that the transition probability $\pi_{cT}(t, v \rightarrow v \pm dv)$ in the CT description, be given by the expression:

$$\pi_{cT}(t, v \rightarrow v \pm dv) = \pi_{IT}(t, v \rightarrow v \pm dv) [1 + \kappa \pi_{IT}(t, v \pm 2dv \rightarrow v \pm dv)] . \quad (20)$$

When considering fermions, the transitions $v \rightarrow v \pm dv$ are allowed only if the transitions $v \pm 2dv \rightarrow v \pm dv$ are forbidden. In the case of bosons the transitions $v \rightarrow v \pm dv$ are stimulated by the transitions $v \pm 2dv \rightarrow v \pm dv$. The expression of $\pi_{cT}(t, v \rightarrow v \pm dv)$ given by Eq.(20) is the simplest expression we may write using $\pi_{IT}(t, v \rightarrow v \pm dv)$ and imposing that contemporaneous transitions toward the same site do not violate the EIP.

The particle current related to the transition probability, given by Eq.(20), assumes the form:

$$j_{cT}(t, v) = - \left(\left[J(t, v) + \frac{\partial D(t, v)}{\partial v} \right] n(t, v) [1 + \kappa n(t, v)] + D(t, v) \frac{\partial n(t, v)}{\partial v} - \frac{\kappa}{a} \frac{\partial}{\partial v} \{ D(t, v) n(t, v) [1 + \kappa n(t, v)] \}^2 \right) . \quad (21)$$

This quantity can be expressed as the sum of three terms

$$j_{cT}(t, v) = j_L(t, v) + \kappa j_I(t, v) + \kappa j_C(t, v) . \quad (22)$$

The first two quantities j_L and j_I have been defined in Sect. II, the third is

$$j_C(t, v) = \frac{1}{a} \frac{\partial}{\partial v} \{ D(t, v) n(t, v) [1 + \kappa n(t, v)] \}^2 . \quad (23)$$

The new current $j_C(t, v)$ does not enter into the expression of $j_{IT}(t, v)$ and assures that the EIP is not violated during contemporaneous transitions toward the same site. With the current defined by Eq.(21), the continuity equation becomes

$$\frac{\partial n(t, v)}{\partial t} = \frac{\partial}{\partial v} \left(\left[J(t, v) + \frac{\partial D(t, v)}{\partial v} \right] n(t, v) [1 + \kappa n(t, v)] + D(t, v) \frac{\partial n(t, v)}{\partial v} - \frac{\kappa}{a} \frac{\partial}{\partial v} \{ D(t, v) n(t, v) [1 + \kappa n(t, v)] \}^2 \right) . \quad (24)$$

Let us introduce the following variables:

$$\tau = ct \quad , \quad u = \sqrt{\frac{m}{kT}} v \quad . \quad (25)$$

If we define now the two quantities:

$$E_o = \frac{2ma}{c} = kT_o \quad , \quad \xi = \frac{kT}{E_o} \quad , \quad (26)$$

Eq.(24), in the case of Brownian particles, can be written in the form

$$\frac{\partial n(\tau, u)}{\partial \tau} = \frac{\partial}{\partial u} \{un(\tau, u)[1 + \kappa n(\tau, u)] + \frac{\partial n(\tau, u)}{\partial u} - 2\kappa \xi \frac{\partial}{\partial u} \{n(\tau, u)[1 + \kappa n(\tau, u)]\}^2\} \quad . \quad (27)$$

The steady state $n(v)$ of this equation is implicitly defined by the algebraic equation

$$n(v) = [\exp\{\beta[E(v, n) - \mu]\} - \kappa]^{-1} \quad , \quad (28)$$

and is related to the stationary distribution of a classical ($\kappa = 0$), or a Fermi ($\kappa = -1$), or a Bose ($\kappa = 1$) system of particles. The energy $E(v, n)$ is given by:

$$E(v, n) = \frac{1}{2}mv^2 + E_p(n) \quad , \quad (29)$$

which is the sum of the kinetic energy $E = mv^2/2$ and of a potential energy dependent on $n(v)$, given by:

$$E_p(n) = E_p^o n(1 + \kappa n) \quad , \quad (30)$$

with

$$E_p^o = -\frac{4\kappa k^2 T^2}{E_o} \quad . \quad (31)$$

We can write Eq.(28) in the form

$$n(v) = \left(\exp \left\{ \beta \left[\frac{1}{2}mv^2 - \kappa \frac{T}{T_o} n(v)[1 + \kappa n(v)] - \mu \right] \right\} - \kappa \right)^{-1} \quad . \quad (32)$$

This expression of $n(v)$ differs from the analogous distribution in the IT approximation given by Eq.(18) when $T \neq 0$. The distribution function $n(v)$ depends on the temperature T of

the system and on the characteristic temperature T_o defined in Eq.(26). When $T_o \rightarrow \infty$, $n(v)$ becomes the FD or the BE statistical distribution. The chemical potential μ can be derived by means of the normalization conditions and depends on the particle density N/V .

We are particularly interested in the $T \rightarrow 0$ region because in this region quantum physics is required and collective quantum macroscopic effects are more evident and important. In the CT description the collective effects are, of course, introduced by the term proportional to $-\kappa T n(v)[1 + \kappa n(v)]$. This term, which does not appear in the IT approximation, vanishes for $T \rightarrow 0$. In the case of fermions when $T \rightarrow 0$, $n(v)$, just as the FD distribution n_{FD} , becomes identical to the step function with value one when $E < E_F$ and zero when $E > E_F$. For $T \rightarrow 0$, the CT and IT distributions coincide because the diffusion coefficient $D = ckT/m$ for Brownian particles is proportional to the temperature, therefore, the capacity of the particles to make transitions from a site to another is greatly reduced. The probability that two particles initiate contemporarily a transition toward the same site is much lowered.

An evaluation of the energy $E_p(n)$ given by Eq.(30), can be obtained using the approximation $n(v) \approx n_{IT}(v)$ which is valid for $T \rightarrow 0$. We have:

$$E_p \approx \frac{4E_p^o}{\cosh^2[\beta(E - \mu)/2]} \quad , \quad \kappa = -1 \quad (33)$$

$$E_p \approx \frac{4E_p^o}{\sinh^2[\beta(E - \mu)/2]} \quad , \quad \kappa = 1 \quad (34)$$

When considering fermions, E_p shows a maximum at $E = \mu$. This means that the correction introduced by the CT description to the FD distribution is more sensible in the energy region $E \approx \mu$. In the case of bosons, being $\mu < 0$, the correction to the BE distribution is more sensible in the energy region $E \rightarrow 0$. In this region we can define the lowest energy level as $E_b = kT_b$. It is straightforward to show that at temperature $T < T_b$, we have that $n_b(T) > n_b(T_b)$ which clearly shows the condensation phenomenon that in the IT approximation appears for $E_b = 0$ at $T_b = 0$.

IV. CONCLUSIONS

In the framework of the CT description, we obtain a generalized FP equation which allows a very complete and precise treatment of the kinetics, in a 1-D space, of particles obeying an EIP. This description represents an improvement respect to the IT approximation, because, in the nearest neighbors transition hypothesis, takes fully into account the restrictions imposed by the EIP.

As steady states of this generalized FP equation we obtain statistical distributions which depend on a parameter κ . When this parameter is equal to ± 1 the particles are bosons or fermions, respectively. For $-1 < \kappa < 1$ we have a boson-fermion intermediate statistics.

Collective quantum macroscopic effects can be appropriately derived in the CT description. This supports the validity and utility of the present semiclassical approach to study the identical particles kinetics.

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