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SUPERCONDUCTORS
WITH JOSEPHSON COUPLING

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**A SIMPLE MODEL FOR THE dc FLUX TRANSFORMER
IN LAYERED SUPERCONDUCTORS WITH JOSEPHSON COUPLING**

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ABSTRACT

We present a model for the dc flux transformer configuration in layered superconductors with Josephson coupling. It consists of a simple extension of a model introduced by Clem for the traditional dc flux transformer, but including Josephson coupling and dissipation between planes. We calculate the non-linear current-voltage characteristics for different cases. We find two kinds of behavior. For weak pinning or strong interplane coupling (both due to magnetic forces and Josephson effect) there is an onset of dissipation with the same voltages in both the top and the bottom faces of the sample, at a given critical current. At a higher current there is a decoupling of the two faces where they have different voltages. This corresponds to a cutting of the vortices that occurs between the first (top) and the second planes. The decoupling current is independent of pinning and vanishes linearly with temperature close to T_c . For strong pinning or weak interplane coupling, each face behaves independently with corresponding different critical currents for the onset of dissipation and different voltages.

MIRAMARE – TRIESTE

September 1994

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I. INTRODUCTION

The dynamics of motion of magnetic flux structures is important for various applications of superconductivity. In a pioneering work, Giaever¹ obtained the first experimental evidence of flux flow in the so called dc flux transformer. He used two superconducting Type II thin films ($\sim 1000\text{\AA}$ thick) which sandwich an insulating layer thick enough ($\sim 200\text{\AA}$) to avoid electron tunneling between them. When a magnetic field is applied perpendicular to the films, an Abrikosov lattice of vortices is formed, which are magnetically coupled through the insulating layer. The experiment consists in passing a dc current through one of the films (the primary), and the voltage drop is measured across both films. The current in the primary sets in motion the vortex lattice and then a voltage drop is induced because of flux flow. Due to the magnetic coupling, the vortices in the secondary film move dragged by the motion of the ones in the primary, and also induce a voltage drop. In fact, Giaever found that the magnitude of the voltage drops in both films were the same. Because a voltage appeared in the secondary without a driving current, this was a proof that dissipation in the flux flow regime is due to the movement of vortices. Thus, we have a dc flux transformer, where a dc voltage in the primary film induces a dc voltage in an electrically insulated secondary. A simple model by Cladis, Parks and Daniels² was able to describe qualitatively the main features of the transformer. In particular, it shows how for very large currents slippage between the primary and secondary vortex lattices could occur. Later on, Clem^{3,4} included pinning in the model and provided a theory for the maximum magnetic coupling force.

Recently, experiments with a flux transformer contact configuration have been done in High T_c superconductors⁵⁻¹⁰. These experiments are done in single crystals, with (at least) six electrical contacts. On the top face of the sample, there are two contacts for current injection and two contacts for voltage measurement. On the bottom face, there are two additional voltage contacts. The first experiments⁵⁻⁷ were done on $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_x$ crystals and showed that the voltage at the top of the sample V_t was much larger than the voltage

at the bottom V_b . This implies that the vortices are effectively two dimensional in these samples, because of the absence of coupling of their motion on the two faces. More recently, the same kind of experiments have been done in $\text{YBa}_2\text{Cu}_3\text{O}_7$ single crystals⁸⁻¹⁰. Below a certain temperature (which depends on sample thickness and magnetic field) both voltages are equal, $V_t = V_b$, showing that the vortices become effectively three dimensional. In the linear regime, the experimental results can be interpreted by a non-local conductivity¹¹ due to the vortices behaving as a line liquid in the temperature range of the experiments.

The High T_c superconductors are made of two dimensional CuO layers, where superconductivity presumably resides, with a Josephson coupling between the layers¹²⁻¹⁶. In this paper we propose a simple extension of Clem's model^{3,4} for the dc flux transformer, that includes Josephson coupling and dissipation between the planes, and a variable number of planes. Here we try to keep the model as simple as possible, providing only a qualitative description of the non linearities in the I-V characteristics. We focus in an analysis of the characteristic currents and voltages for slippage between the different planes, leading to a classification of the different types of possible IV curves. We assume the existence of a rigid Abrikosov vortex lattice, and therefore the model is valid for low temperatures (well below the melting line) and weak disorder. We also neglect the spatial dependence of the currents along the direction of current injection. This is done for simplicity, but many of our qualitative results are valid beyond that approximation. The paper is organized as follows: in Sec. II we present our model for the dc flux transformer of two Josephson coupled superconducting planes, in Sec. III we extend the model for an arbitrary number of planes, and in Sec. IV we give our conclusions and discuss possible extensions of the model.

II. DC FLUX TRANSFORMER IN TWO SUPERCONDUCTING LAYERS WITH JOSEPHSON COUPLING

Let us consider first the case of two superconducting layers, which are close enough to have Josephson tunneling between them. In the dc flux transformer configuration, a

current is applied in the top plane, and the voltage drop is measured both in the top and bottom planes (or primary and secondary planes). There is a magnetic field applied in the perpendicular direction to the planes. At low temperatures and disorder, this field induces a triangular vortex lattice in both planes. This is the problem studied by Clem in 1974³, and we shall follow the same approach in this section. The difference is that now we allow for the possibility of current flow between planes, due to either Josephson effect or ohmic dissipation.

We assume that far away from the boundaries where the current is applied, the current flow is homogeneous in both planes in a macroscopic scale (i.e larger than the intervortex spacing). Let us call J_t the current density flowing in the bulk of the top plane, and J_b the current density flowing in the bulk of the bottom plane. Following Clem^{3,4}, in each plane we consider a balance of forces acting on each vortex,

$$F_L - F_m - F_p - F_\eta = 0 , \quad (1)$$

where $F_L = J_{t,b}\Phi_0 d/c$ is the Lorentz force acting on the vortices due to the current J_t or J_b in each respective plane, and d is the width of each plane; $F_p = J_c\Phi_0 d/c$ is the pinning force due to the presence of defects, with J_c the critical current of each individual plane; and $F_\eta = \eta \dot{u}_{t,b}d$ is the friction force originated by vortex motion, with $\dot{u}_{t,b}$ the vortex velocity in each corresponding plane, and the dissipation parameter can be taken as $\eta = \Phi_0 H_{c2}/\rho_n c^2$, with ρ_n the resistivity in the normal state. If the vortex lattices are displaced one with respect to the other, there will a magnetic force F_m acting between them. In the one reciprocal lattice vector approximation³, it is given by

$$F_m[u_t - u_b] = F_m \sin[2\pi(u_t - u_b)/a] , \quad (2)$$

for the vortices in the top plane, and $-F_m$ for the vortices in the bottom plane. The coordinates u_t, u_b can be thought either as the coordinates of one individual vortex, or as the center of mass of the vortex lattices. The vortex lattice spacing is $a = (\frac{2\Phi_0}{\sqrt{3}B})^{1/2}$ for a triangular lattice.

Therefore, the equations of motion for the vortices can be written as,

$$\begin{aligned}\eta\dot{u}_t d &= \mathcal{F}\left\{J_t \frac{\Phi_0}{c} d - F_m \sin(2\pi(u_t - u_b)/a), J_c \frac{\Phi_0}{c} d\right\} \\ \eta\dot{u}_b d &= \mathcal{F}\left\{J_b \frac{\Phi_0}{c} d + F_m \sin(2\pi(u_t - u_b)/a), J_c \frac{\Phi_0}{c} d\right\}.\end{aligned}\quad (3)$$

The “pinning” function $\mathcal{F}(x, c)$ is given by,

$$\mathcal{F}(x, c) = \begin{cases} x - c & \text{if } x > c \\ 0 & \text{if } -c < x < c \\ x + c & \text{if } x < -c \end{cases} \quad (4)$$

The vortex motion will generate an electric field $E_y^{t,b} = \dot{u}_{t,b} B/c$, and thus a voltage drop $V_{t,b} = L_y E_y^{t,b}$, with L_y the length of the sample along the y direction. After replacing in (3), the current-voltage characteristics are given by,

$$\begin{aligned}V_t &= R_f \mathcal{F}\{I_t - f_m \sin(2\pi(u_t - u_b)/a), I_c\} \\ V_b &= R_f \mathcal{F}\{I_b + f_m \sin(2\pi(u_t - u_b)/a), I_c\}.\end{aligned}\quad (5)$$

where the flux flow resistance is $R_f = \frac{L_y}{L_x d} \rho_f$ with flux flow resistivity $\rho_f = \frac{\Phi_0 B}{\eta c^2} = \frac{B}{H_{c2}} \rho_n$, the total currents are $I_{t,b,c} = L_x d J_{t,b,c}$; and $f_m = \frac{c L_x}{\Phi_0} F_m$.

This is the model proposed by Clem^{3,4} for the flux transformer. It assumes that there is a rigid vortex lattice, and therefore it can only be valid at low temperatures, well below the melting line. Thermal fluctuations and the effects of vortex flux creep are completely neglected in this model. Since their main effect is for currents smaller or of the order of J_c , and we are interested in the decoupling mechanisms that occur at currents $J > J_c$, this is not a bad approximation. Moreover, the effect of pinning is modeled as simply giving a critical current for the voltage onset. However, it is known that pinning always destroys the long range order of the vortex lattice¹⁷. The translational crystalline order is only maintained within a correlated volume of characteristic length R_c , and the pinning force results from the incomplete averaging of the random potential over this finite correlated volume¹⁷. We can say, therefore, that the Eq. (2) and Eq. (3) are only valid within this correlated volume.

In general, the average magnetic force can be thought as an average over all the different correlated volumes,

$$F_m[u_t - u_b] \approx F_m \langle \sin[2\pi(u_t - u_b)/a + \alpha_i] \rangle , \quad (6)$$

with α_i random phases contributions of each correlated volume. In any case, within our simplified approach for the qualitative behavior of the IV curves, the relevant contribution is the fact that there is a *maximum* magnetic interaction force, and not the periodicity of the sine in (2).

Also a not very weak pinning induces a plastic or inhomogeneous vortex flow close to the critical current¹⁸. This gives rise to a non-linear grow of the voltage close to the critical current. We can choose a more realistic function $\mathcal{F}(x, c)$, for example with a non-linear rise close to c to account for this behavior in a very phenomenological way. But again, for simplicity, we will only use (4) in this paper.

A. Current flow between planes

If we apply an external current I only on the boundary of the top plane, part of it will flow in the bulk of the top plane, and the other part will be diverted to the bottom plane (see Fig. 1). Current conservation requires

$$I = I_t + I_b , \quad (7)$$

where I_t and I_b are the total currents flowing in the bulk of the top and bottom planes, respectively.

Now we have to provide a model for the current I_b flowing to the bottom plane. We assume that it is given by the sum of a Josephson current I_J due to the displacement of the respective vortex lattices, plus a dissipative current I_{diss} due to the interplane resistance,

$$I_b = I_J + I_{diss} . \quad (8)$$

In general, the Josephson current is,

$$I_J(\vec{r}) = I_0 \sin(\phi_b(\vec{r}) - \phi_t(\vec{r})), \quad (9)$$

with $\phi_{t,b}(\vec{r})$ the phase of the superconducting order parameter in each respective plane. The phase in each plane can be written as the sum of the phase contribution of each vortex,

$$\phi_{t,b}(\vec{r}) = \sum_{\vec{l}} \phi(\vec{r} - \vec{r}_l^{t,b}), \quad (10)$$

where $\vec{l}$ is the lattice vector of the triangular vortex lattice, $\vec{l} = m\vec{a}_1 + n\vec{a}_2$, ($\vec{a}_1 = a\hat{x}$, $\vec{a}_2 = a/2(\hat{x} + \sqrt{3}\hat{y})$), and the position of each vortex is given by $\vec{r}_l^{t,b} = \vec{l} + \vec{u}_{t,b}$. The phase originated by each vortex can be modeled, for example, with the “arctan” approximation, $\phi(\vec{r}) = \tan^{-1} \frac{r_x}{r_y}$. In the reciprocal lattice, Eq. (10) is transformed to,

$$\phi_{t,b}(\vec{r}) = \sum_{\vec{g}} P(\vec{g}) e^{i\vec{g} \cdot (\vec{r} - \vec{u}_{t,b})}, \quad (11)$$

where the reciprocal lattice vector is $\vec{g} = 2\pi(m\vec{b}_1 + n\vec{b}_2)$, ($\vec{b}_1 = 1/a(\hat{x} - \hat{y}/\sqrt{3})$, $\vec{b}_2 = \frac{2}{a\sqrt{3}}\hat{y}$), and $P(\vec{g}) = \int d\vec{\rho} e^{i\vec{g} \cdot \vec{\rho}} \phi(\vec{\rho})$. Therefore the Josephson current can be written as

$$I_J(\vec{r}) = I_0 \sin \left[\sum_{\vec{g}} P(\vec{g}) e^{i\vec{g} \cdot \vec{r}} e^{-i\vec{g} \cdot (\vec{u}_t + \vec{u}_b)/2} 2 \sin(\vec{g} \cdot (\vec{u}_b - \vec{u}_t)/2) \right]. \quad (12)$$

For small displacements $|\vec{u}_t - \vec{u}_b| \ll a$ it can be written as,

$$I_J(\vec{r}) = I_0 \sin \left[\left(- \sum_{\vec{g}} i P(\vec{g}) e^{i\vec{g} \cdot \vec{r}} \vec{g} \right) \cdot (\vec{u}_b - \vec{u}_t) \right]. \quad (13)$$

The average of $I_J(\vec{r})$ over a unit cell is zero. Therefore, there is no net Josephson current in the bulk of the sample. The only contributions are in the surface between the planes. As it is schematically shown in Fig. 1, we will assume that the only relevant current flow between planes occurs at their boundaries. Within this approach, the effective Josephson current, in the one reciprocal lattice vector approximation, is given by

$$I_J \approx I_0 \sin[2\pi(u_t - u_b)/a]. \quad (14)$$

On the other hand, the dissipative contribution to the interplane current is simply given by the Ohm's law,

$$I_{diss} = \frac{V_i}{R_i}, \quad (15)$$

with R_i the interplane resistance. The interplane voltage drop V_i , is given by $V_i = (V_t - V_b)/2$.

Therefore, the total current flowing to the bottom plane is,

$$I_b = I_0 \sin[2\pi(u_t - u_b)/a] + \frac{V_t - V_b}{2R_i}. \quad (16)$$

Again, the same warnings given in the discussion after Eq. (5) also apply in this case. Eq. (16) should be substituted by an average over the different correlated volumes. But within our qualitative analysis, the periodicity of the sine in (16) is not relevant. The only relevant fact from (16) is the existence of a *maximum* supercurrent capacity between planes, above which there is ohmic dissipation.

B. Equations of motion

Now, we replace in Eq. (5) I_b from (16), and $I_t = I - I_b$ from (7), with I the applied current, and we get

$$\begin{aligned} V_t &= R_f \mathcal{F}\{I - (I_0 + f_m) \sin(2\pi(u_t - u_b)/a) - \frac{V_t - V_b}{2R_i}, I_c\} \\ V_b &= R_f \mathcal{F}\{(I_0 + f_m) \sin(2\pi(u_t - u_b)/a) + \frac{V_t - V_b}{2R_i}, I_c\}. \end{aligned} \quad (17)$$

We make these equations of motion adimensional by defining $\theta_{t,b} = \frac{2\pi}{a}u_{t,b}$, with the normalized time $\tau = \frac{2\pi c R_f (I_0 + f_m)}{L_y a B} t = \frac{2\pi c \rho_n (I_0 + f_m)}{L_z d H_{c2} a} t$, and normalized currents $i = I/(I_0 + f_m)$,

$$\begin{aligned} \dot{\theta}_t &= \mathcal{F}\{i - \sin(\theta_t - \theta_b) - \sigma(\dot{\theta}_t - \dot{\theta}_b), i_c\} \\ \dot{\theta}_b &= \mathcal{F}\{\sin(\theta_t - \theta_b) + \sigma(\dot{\theta}_t - \dot{\theta}_b), i_c\}, \end{aligned} \quad (18)$$

with $i_c = I_c/(I_0 + f_m)$ and $\sigma = R_f/2R_i$. This simple system of equations can be studied analytically as a function of i_c and σ , obtaining different current-voltage characteristics.

To carry out the analysis of Eq. (18), we have to consider different possible solutions for $\dot{\theta}_{t,b}$, according to the different possible values of $\mathcal{F}$ in (4).

(i) $\dot{\theta}_t = \dot{\theta}_b = 0$, i.e., absence of dissipation in both planes. This can be achieved in (18) whenever

$$\begin{aligned} -i_c < i - \sin(\theta_t - \theta_b) < i_c \\ -i_c < \sin(\theta_t - \theta_b) < i_c . \end{aligned} \quad (19)$$

If $i_c < 1$, the above conditions imply $-2i_c < i < 2i_c$. If $i_c > 1$, the second condition in (19) has to be replaced by

$$-1 < \sin(\theta_t - \theta_b) < 1 \quad (20)$$

and therefore the absence of dissipation happens for $-(1 + i_c) < i < 1 + i_c$. In conclusion, there is a critical current below which there is no dissipation in both planes, given by

$$i_{crit} = \begin{cases} 2i_c & \text{if } i_c < 1 \\ i_c + 1 & \text{if } i_c > 1 \end{cases} . \quad (21)$$

(ii) $\dot{\theta}_t \neq 0$, $\dot{\theta}_b = 0$, i.e. there is dissipation in the top plane, but not in the bottom plane.

In this case, from Eq. (18), we have

$$\begin{aligned} \dot{\theta}_t &= i - \sin(\theta_t - \theta_b) - \sigma \dot{\theta}_t - i_c \\ i_c &> |\sin(\theta_t - \theta_b) + \sigma \dot{\theta}_t| , \end{aligned} \quad (22)$$

The second condition in (22) is satisfied if $|\dot{\theta}_t| < (i_c - 1)/\sigma$, which is only possible when $i_c > 1$. Applying this condition into the first equation in (22), we get

$$i - i_c - \sin(\theta_t - \theta_b) < (i_c - 1)(1 + \sigma)/\sigma . \quad (23)$$

This requires that $i < 2i_c + \frac{i_c - 1}{\sigma}$. Therefore, there is a second critical current for the onset of voltage in the bottom plane given by

$$i_b = 2i_c + \frac{i_c - 1}{\sigma} . \quad (24)$$

(iii) $\dot{\theta}_t \neq 0$, $\dot{\theta}_b \neq 0$, both planes dissipate. In this case the equations of motion are, from (18),

$$\begin{aligned}
\dot{\theta}_t &= i - \sin(\theta_t - \theta_b) - \sigma(\dot{\theta}_t - \dot{\theta}_b) - i_c \\
\dot{\theta}_b &= \sin(\theta_t - \theta_b) + \sigma(\dot{\theta}_t - \dot{\theta}_b) - i_c,
\end{aligned} \tag{25}$$

It is better to rewrite these equations for the sum and the difference of the phases,

$$\begin{aligned}
\dot{\theta}_t + \dot{\theta}_b &= i - 2i_c \\
\dot{\theta}_t - \dot{\theta}_b &= \frac{i}{1 + 2\sigma} - \frac{2}{1 + 2\sigma} \sin(\theta_t - \theta_b).
\end{aligned} \tag{26}$$

The second equation in (26) is equivalent to the RSJ model of a single Josephson junction¹⁹. The normalized voltages will be given by the time averages $v_{t,b} = \langle \dot{\theta}_{t,b} \rangle$. From the solution of the second equation¹⁹, we have that $v_t = v_b$ for $i < 2$, and $v_t \neq v_b$ for $i > 2$. Therefore, the critical current for separation of the top and bottom voltages is

$$i_d = 2. \tag{27}$$

In general, we can solve Eq. (26) obtaining for the voltages¹⁹,

$$\begin{aligned}
v_t &= \begin{cases} \frac{i - 2i_c}{2} & \text{if } i < 2 \\ \frac{i - 2i_c}{2} + \frac{1}{2 + 4\sigma} \sqrt{i^2 - 4} & \text{if } i > 2 \end{cases} \\
v_b &= \begin{cases} \frac{i - 2i_c}{2} & \text{if } i < 2 \\ \frac{i - 2i_c}{2} - \frac{1}{2 + 4\sigma} \sqrt{i^2 - 4} & \text{if } i > 2 \end{cases}
\end{aligned} \tag{28}$$

However, this result is only valid if $i_{crit} < i_d$, in other words, only for $i_c < 1$. For $i_c > 1$, we will have that after $i > i_b$ both planes will dissipate. But for currents close to i_b , we can have alternatively $\dot{\theta}_b = 0$ and $\dot{\theta}_b \neq 0$ as a function of time, like in a stick-slip motion, which is not described by Eq. (26). Only for currents $i \gg i_b$, the Eq. (28) will be valid in this case.

Note that all the results obtained in this section, with the exception of Eq. (28), hold if we replace $\sin(x)$ for any function $f(x)$ (periodic or non-periodic) such that $|f(x)| < 1$.

C. Numerical simulations

We can better illustrate our previous analysis with a numerical solution of the equations of motion for the different cases. We solve Eq. (18) with a fourth order Runge-Kutta

integration method, after solving for $\{\dot{\theta}_t, \dot{\theta}_b\}$ in (18) with an iterative Newton-Rapson-Fourier algorithm²⁰.

In Fig. 2 we show our results for the I-V characteristics. Consistently with our previous analysis, we find that there are two different kinds of behavior in the non-linear IVs:

(1) For $i_c < 1$, after $i > i_{crit} = 2i_c$ both planes start to dissipate with the same voltage drop, and for higher currents, $i > i_d = 2$, the voltages become different. This is shown in Fig. 2(a) for $i_c = 0.5$, $\sigma = 0.7$. In this case the numerical IVs correspond exactly to Eq. (28).

(2) For $i_c > 1$, after $i > i_{crit} = i_c + 1$ only the top plane starts to dissipate, and only at higher currents, $i > i_b$, the bottom plane starts to show a voltage drop. This is shown in Fig. 2(b) for $i_c = 2.$, $\sigma = 0.7$. In this case the numerical IVs can be fitted with Eq. (28) only for $i \gg i_b$.

All the characteristic currents i_{crit} , i_d and i_b obtained numerically for different parameter values are correctly described by the Eqs. (21), (27) and (24) respectively.

These two kinds of behavior depend on $i_c < 1$ or $i_c > 1$ ($I_c < I_0 + f_m$ or $I_c > I_0 + f_m$). In other words, the IV characteristics are qualitatively different depending on the in-plane pinning (weighted by I_c) being weaker or stronger than the interplane coupling (either due to Josephson effect or magnetic forces).

The model discussed by Clem in the past^{3,4} corresponds to $\sigma = 0$ in our equations. In the case (1) it gives a voltage in the bottom plane that always decreases, and tends to zero for large currents. In our case, v_b always increases for large currents. This can be seen from Eq. (28) in the limit of large currents, $i \gg 2$, where $v_t = \frac{1+\sigma}{1+2\sigma}i$ and $v_b = \frac{\sigma}{1+2\sigma}i$. In fact, changing different values of σ only affects the linear slopes at large currents. Physically, what is happening is that above the decoupling current i_d there is now flow of current to the bottom plane that always induces vortex motion, whereas that was impossible in the Clem model. In the case (2) the difference with the Clem model is more clear, since for $\sigma = 0$, $i_b \rightarrow \infty$ and there is never dissipation in the bottom plane.

III. DC TRANSFORMER FOR LAYERED SUPERCONDUCTORS

A. Equations of motion

It is straightforward to generalize the equations of motion of Sec.II for the case of N superconducting planes with Josephson coupling. Now in a given k th plane ($k = 1, \dots, N$) the balance of forces will be,

$$F_L^k - F_m^{k,k+1} + F_m^{k-1,k} - F_p^k - F_\eta^k = 0 \quad (29)$$

where $F_L^k = J_k \Phi_0 d / c$, with J_k the current density in plane k , $F_p^k = J_c \Phi_0 d / c$ (we assume the same pinning in all the planes), $F_\eta^k = \eta \dot{u}_k d$, and $F_m^{k,k+1}$ is the magnetic force on the vortices in plane k due to the displacement of the vortices in the plane $k + 1$,

$$F_m^{k,k+1} = F_m \sin [2\pi(u_k - u_{k+1})/a] \quad (30)$$

for $1 < k < N$, and the boundary condition $F_m^{0,1} = F_m^{N,N+1} = 0$. Now, we allow for the possibility of flow of a current $I_{k,k+1}$ between the planes k and $k + 1$. Again this current is given by the sum of a Josephson current and a dissipative current,

$$I_{k,k+1} = I_0 \sin [2\pi(u_k - u_{k+1})/a] + \frac{V_k - V_{k+1}}{2R_i} \quad (31)$$

for $1 < k < N$, and the boundary condition $I_{0,1} = I$ (the external current applied in the top plane, $k = 1$) and $I_{N,N+1} = 0$. We assume that these currents flow in the boundaries between planes as it is schematically represented in Fig. 3. Therefore, current conservation requires

$$I_k = I_{k-1,k} - I_{k,k+1} . \quad (32)$$

Proceeding as in the previous section, with the same normalizations, we obtain the equations of motion

$$\dot{\theta}_k = \mathcal{F}\{\sin(\theta_{k-1} - \theta_k) - \sin(\theta_k - \theta_{k+1}) + \sigma(\dot{\theta}_{k+1} - 2\dot{\theta}_k + \dot{\theta}_{k-1}), i_c\} \quad (33)$$

for $1 < k < N$ and the boundary condition

$$\begin{aligned}\dot{\theta}_1 &= \mathcal{F}\{i - \sin(\theta_1 - \theta_2) - \sigma(\dot{\theta}_1 - \dot{\theta}_2), i_c\} \\ \dot{\theta}_N &= \mathcal{F}\{\sin(\theta_{N-1} - \theta_N) + \sigma(\dot{\theta}_{N-1} - \dot{\theta}_N), i_c\} .\end{aligned}\quad (34)$$

We can again analyze the different possible solutions of Eqs. (33) and (34), before proceeding to a direct numerical simulation of them.

(i) $\dot{\theta}_1 = \dot{\theta}_2 = \dots = \dot{\theta}_N = 0$, i.e. absence of dissipation in all the planes. In this case the Eqs. (33) lead to the conditions,

$$\begin{aligned}-i_c &< i - \sin(\theta_1 - \theta_2) < i_c \\ -i_c &< \sin(\theta_1 - \theta_2) - \sin(\theta_2 - \theta_3) < i_c \\ &\dots\dots\dots \\ -i_c &< \sin(\theta_{k-1} - \theta_k) - \sin(\theta_k - \theta_{k+1}) < i_c \\ &\dots\dots\dots \\ -i_c &< \sin(\theta_{N-2} - \theta_{N-1}) - \sin(\theta_{N-1} - \theta_N) < i_c \\ -i_c &< \sin(\theta_{N-1} - \theta_N) < i_c .\end{aligned}\quad (35)$$

Adding up all the rows except the first one, we have

$$\begin{aligned}-i_c &< i - \sin(\theta_1 - \theta_2) < i_c \\ -(N-1)i_c &< \sin(\theta_1 - \theta_2) < (N-1)i_c \quad \text{if } i_c < \frac{1}{N-1} \\ &\dots\dots\dots \\ -(N-k+1)i_c &< \sin(\theta_{k-1} - \theta_k) < (N-k+1)i_c \quad \text{if } i_c < \frac{1}{N-k+1} \\ &\dots\dots\dots \\ -2i_c &< \sin(\theta_{N-2} - \theta_{N-1}) < 2i_c \quad \text{if } i_c < \frac{1}{2} \\ -i_c &< \sin(\theta_{N-1} - \theta_N) < i_c \quad \text{if } i_c < 1 .\end{aligned}\quad (36)$$

If $i_c < \frac{1}{N-1}$, we add the first and the second row, and we get the condition $|i| < Ni_c$. If $i_c > \frac{1}{N-1}$, we have to consider for the first two rows in (35)

$$\begin{aligned}-i_c &< i - \sin(\theta_1 - \theta_2) < i_c \\ -1 &< \sin(\theta_1 - \theta_2) < 1\end{aligned}\quad (37)$$

Then we have the condition $|i| < i_c + 1$ for this case. Therefore, there is a critical current i_{crit} below which there is no dissipation in any plane, given by

$$i_{crit} = \begin{cases} Ni_c & \text{if } i_c < \frac{1}{N-1} \\ i_c + 1 & \text{if } i_c > \frac{1}{N-1} \end{cases} . \quad (38)$$

(ii) $\dot{\theta}_1 \neq 0$, $\dot{\theta}_2 = \dots = \dot{\theta}_N = 0$, there is dissipation only in the top plane. Proceeding as before for Eq. (36), we obtain now

$$\begin{aligned} \dot{\theta}_1 &= i - \sin(\theta_1 - \theta_2) - \sigma\dot{\theta}_1 - i_c \\ -(N-1)i_c &< \sin(\theta_1 - \theta_2) + \sigma\dot{\theta}_1 < (N-1)i_c . \end{aligned} \quad (39)$$

The second equation requires that $|\sigma\dot{\theta}_1| < (N-1)i_c - 1$, therefore this behavior is possible only when $i_c > \frac{1}{N-1}$. From the first equation in (39) we get,

$$\dot{\theta}_1 = \frac{1 - i_c}{1 + \sigma} - \frac{1}{1 + \sigma} \sin(\theta_1 - \theta_2) , \quad (40)$$

for $\dot{\theta}_1 > 0$, and together with the condition $\sigma\dot{\theta}_1 < (N-1)i_c - 1$, we get

$$\frac{1 - i_c}{1 + \sigma} - \frac{1}{1 + \sigma} \sin(\theta_1 - \theta_2) < \frac{(N-1)i_c - 1}{\sigma} . \quad (41)$$

We obtain the condition $|i| < Ni_c + \frac{(N-1)i_c - 1}{\sigma}$. Therefore, above a critical current

$$i_b = Ni_c + \frac{(N-1)i_c - 1}{\sigma} , \quad (42)$$

the second plane starts to dissipate (when $i_c > \frac{1}{N-1}$).

(iii) $\dot{\theta}_1 \neq 0$, $\dot{\theta}_2 \neq 0$, $\dots$, $\dot{\theta}_{k-1} \neq 0$, $\dot{\theta}_k = \dot{\theta}_{k+1} = \dots = \dot{\theta}_N = 0$; there is dissipation only in the first $k-1$ planes. In this case the equations of motion (33) can be reduced to,

$$\begin{aligned} \dot{\theta}_1 &= i - \sin(\theta_1 - \theta_2) - \sigma(\dot{\theta}_1 - \dot{\theta}_2) - i_c \\ \dot{\theta}_2 &= \sin(\theta_1 - \theta_2) - \sin(\theta_2 - \theta_3) + \sigma(\dot{\theta}_1 - 2\dot{\theta}_2 + \dot{\theta}_3) - i_c \\ &\dots\dots\dots \\ \dot{\theta}_{k-1} &= \sin(\theta_{k-2} - \theta_{k-1}) - \sin(\theta_{k-1} - \theta_k) + \sigma(\dot{\theta}_{k-2} - 2\dot{\theta}_{k-1}) - i_c \\ -(N-k+1)i_c &< \sin(\theta_{k-1} - \theta_k) + \sigma\dot{\theta}_{k-1} < (N-k+1)i_c . \end{aligned} \quad (43)$$

The last condition implies that $|\sigma\dot{\theta}_{k-1}| < (N - k + 1)i_c - 1$. Therefore, this kind of solution is only possible when $i_c > \frac{1}{N-k+1}$. We are not able to determine the critical current for the next plane to start to dissipate for the general case $k > 2$ as we did in case (ii). When $k = N$, all the planes dissipate except the last one at the bottom, which is only possible for $i_c > 1$.

(iv) $\dot{\theta}_k \neq 0$ for all k ; all the planes dissipate. In this case the equations of motion Eqs. (33) are,

$$\begin{aligned}
\dot{\theta}_1 &= i - \sin(\theta_1 - \theta_2) - \sigma(\dot{\theta}_1 - \dot{\theta}_2) - i_c \\
\dot{\theta}_2 &= \sin(\theta_1 - \theta_2) - \sin(\theta_2 - \theta_3) + \sigma(\dot{\theta}_1 - 2\dot{\theta}_2 + \dot{\theta}_3) - i_c \\
&\dots\dots\dots \\
\dot{\theta}_k &= \sin(\theta_{k-1} - \theta_k) - \sin(\theta_k - \theta_{k+1}) + \sigma(\dot{\theta}_{k-1} - 2\dot{\theta}_k + \dot{\theta}_{k+1}) - i_c \\
&\dots\dots\dots \\
\dot{\theta}_{N-1} &= \sin(\theta_{N-2} - \theta_{N-1}) - \sin(\theta_{N-1} - \theta_N) + \sigma(\dot{\theta}_{N-2} - 2\dot{\theta}_{N-1} + \dot{\theta}_N) - i_c \\
\dot{\theta}_N &= \sin(\theta_{N-1} - \theta_N) + \sigma(\dot{\theta}_{N-1} - \dot{\theta}_N) - i_c .
\end{aligned} \tag{44}$$

It is not possible to give an analytical solution of this full system of equations, but some results can be obtained. First, by adding up all the equations, we obtain the property,

$$\sum_{k=1}^N \dot{\theta}_k = i - Ni_c . \tag{45}$$

One case that can be analyzed is when all the $\dot{\theta}_k$ are equal, and we want to know the current i_d at which $\dot{\theta}_1 \neq \dot{\theta}_2$ and $\dot{\theta}_2 = \dots = \dot{\theta}_k = \dots = \dot{\theta}_N$. In that case we consider the sum,

$$\sum_{k=2}^N \dot{\theta}_k = -\sin(\theta_2 - \theta_1) - \sigma(\dot{\theta}_2 - \dot{\theta}_1) - (N-1)i_c = (N-1)\dot{\theta}_2 , \tag{46}$$

the last step because we assume that all $\dot{\theta}_k$ are equal for $k > 1$. After using the equation for $\dot{\theta}_1$ in Eq. (44), we get

$$\dot{\theta}_1 - \dot{\theta}_2 = i - \frac{N}{N-1} \sin(\theta_1 - \theta_2) - \frac{N}{N-1} \sigma(\dot{\theta}_1 - \dot{\theta}_2) . \tag{47}$$

Therefore, from the solution of this equation¹⁹ we will have $v_1 = v_2$ ($v_{1,2} = \langle \dot{\theta}_{1,2} \rangle$) for $i < \frac{N}{N-1}$, and $v_1 \neq v_2$ for $i > \frac{N}{N-1}$. This defines a critical current for the separation of the top and bottom voltages (since still $v_2 = v_N$),

$$i_d = \frac{N}{N-1} . \quad (48)$$

We can solve Eq. (47) and (45) obtaining¹⁹

$$v_1 = \begin{cases} \frac{i - N i_c}{N} & \text{if } i < \frac{N}{N-1} \\ \frac{i - N i_c}{N} + \frac{1}{N + \frac{N^2}{N-1} \sigma} \sqrt{i^2 - \left(\frac{N}{N-1}\right)^2} & \text{if } i > \frac{N}{N-1} \end{cases} \quad (49)$$

$$v_2 = \begin{cases} \frac{i - N i_c}{N} & \text{if } i < \frac{N}{N-1} \\ \frac{i - N i_c}{N} - \frac{1}{N + \frac{N^2}{N-1} \sigma} \sqrt{i^2 - \left(\frac{N}{N-1}\right)^2} & \text{if } i > \frac{N}{N-1} \end{cases} .$$

This case is only possible when $i_d > i_{crit}$, i.e. for $i_c < \frac{1}{N-1}$. Of course, for much larger bias currents $i \gg \frac{N}{N-1}$, Eq. (49) will not be valid, since the other voltages will start to differ ($v_2 \neq v_3$, etc.). We can not calculate exactly the currents for the subsequent voltage separations, but an estimation (valid for very large N) will be given below.

We can also discuss the fully resistive case, when $v_k = \langle \dot{\theta}_k \rangle \gg 1$ for all k . In this case we can assume $\langle \sin(\theta_{k-1} - \theta_k) \rangle \approx 0$, and the Eqs. (33) reduce to,

$$\begin{aligned} v_1 &= i - \sigma(v_1 - v_2) - i_c \\ v_2 &= \sigma(v_1 - 2v_2 + v_3) - i_c \\ &\dots\dots\dots \\ v_k &= \sigma(v_{k-1} - 2v_k + v_{k+1}) - i_c \\ &\dots\dots\dots \\ v_{N-1} &= \sigma(v_{N-2} - 2v_{N-1} + v_N) - i_c \\ v_N &= \sigma(v_{N-1} - v_N) - i_c . \end{aligned} \quad (50)$$

This is a linear system of equations with solution

$$v_k = \frac{1}{\sigma} \frac{\cosh[\alpha(N + \frac{1}{2} - k)]}{2 \sinh(\alpha N) \sinh(\alpha/2)} i - i_c, \quad (51)$$

with $\cosh \alpha = 1 + \frac{1}{2\sigma}$. This means that for large N the voltages decrease exponentially,

$$v_k \approx \frac{e^{-\alpha k}}{\sigma} i - i_c. \quad (52)$$

We can also estimate that the critical currents $i_d^{(k)}$ for the decoupling of the k th plane, for $k \gg 1$ and $N \gg 1$, will happen when $v_k \approx 1$, obtaining

$$i_d^{(k)} \approx (1 + i_c) \sigma e^{\alpha k}. \quad (53)$$

Note that most of the results obtained in this section, with the exception of Eq. (49), hold if we replace $\sin(x)$ for any function $f(x)$ (periodic or non-periodic) such that $|f(x)| < 1$. The only specific effects of the approximations considered, are in the particular shape of the IV curves. Close to i_{crit} the voltage raises linearly due to the form of $\mathcal{F}(x, c)$, Eq. (4). Close to $i_d^{(k)}$ the voltages separate with a square root dependence (as given in (49)) due to the $\sin(x)$ in the equations of motion.

B. Numerical results

We have done numerical simulations of Eqs. (33), (34) using the same numerical procedure as in Sec. II.C. Consistently with our previous analysis, we can classify the IV curves in N different kinds of behavior, depending on the value of i_c :

(1) For $i_c < \frac{1}{N-1}$. Above a critical current $i > i_{crit} = N i_c$ all the planes start to dissipate with the same voltages. After $i > i_d^{(1)} = \frac{N}{N-1}$ there is a decoupling between the first and the second plane, where $v_1 \neq v_2$ and $v_2 = v_3 = \dots = v_N$. At a much higher current $i_d^{(2)}$ the second and third plane decouple, and then after $i_d^{(3)}$ the third and the fourth plane decouple, etc. The decoupling currents $i_d^{(k)}$ grow exponentially with k , and are approximately given by Eq. (53) for large N . Also, Eq. (49) reproduces the IV curve for $i < i_d^{(2)}$. In Fig.4(a) we show an example of this case for $i_c = 0.05$, $\sigma = 0.7$, $N = 8$.

(2) For $\frac{1}{N-1} < i_c < \frac{1}{N-2}$. Above a critical current $i > i_{crit} = i_c + 1$ only the first plane starts to dissipate. After a higher current $i > i_b$ (given by Eq. (42)) all the other $N - 1$ planes start to dissipate with the same voltage drop. Then after a current $i_d^{(2)}$ there is a decoupling between the second and third plane, after a current $i_d^{(3)}$ the third and fourth plane decouple, etc. In Fig. 4(b) we show an example of this case for $i_c = 0.15$, $\sigma = 0.7$, $N = 8$.

(k) For $\frac{1}{N-k+1} < i_c < \frac{1}{N-k}$. Above a critical current $i > i_{crit} = i_c + 1$ only the first plane starts to dissipate. After a higher current $i_b^{(1)}$ the second plane starts to dissipate, then when $i > i_b^{(2)}$ the third plane starts to dissipate, ..., and when $i > i_b^{(k-2)}$ the $k - 1$ th plane starts to dissipate. Then after a critical current $i_b^{(k-1)}$ all the remaining $N - k + 1$ planes start to dissipate with the same voltage. At a higher current $i_d^{(k)}$ the k th and the $k + 1$ th plane decouple, etc. In Fig. 4(c) we show an example of this case for $i_c = 0.23$, $\sigma = 0.7$, $N = 8$ ($k = 4$).

(N) For $1 < i_c$. Above a critical current $i > i_{crit} = i_c + 1$ only the first plane starts to dissipate. After a higher current $i_b^{(1)}$ the second plane starts to dissipate, then when $i > i_b^{(2)}$ the third plane starts to dissipate, ..., and finally when $i > i_b^{(N-1)}$ the N th plane starts to dissipate. In Fig. 4(d) we show an example of this case for $i_c = 2$, $\sigma = 0.7$, $N = 8$.

In all the cases, the voltages in the linear regime of high currents, are given by Eq. (51).

In practice, when we are looking only at the top ($k = 1$) and bottom ($k = N$) planes only two kinds of behavior will be distinguished in the IV curves. For $i_c < \frac{1}{N-1}$ we have case (1) and for $i_c > \frac{1}{N-1}$ we have that both planes will start to dissipate at different critical currents (cases (2) to (N)).

C. Multiplane current injection

Let us consider the case when the external current is injected not only in the top plane, but in the first m planes. For example, we consider the case of $N = 8$ planes, where the same current i is injected in the first $m = 3$ planes. Now the equations of motions are the

same as in (33) but with the boundary condition for the 3 top planes,

$$\begin{aligned}\dot{\theta}_1 &= \mathcal{F}\{i - \sin(\theta_1 - \theta_2) - \sigma(\dot{\theta}_1 - \dot{\theta}_2), i_c\} \\ \dot{\theta}_2 &= \mathcal{F}\{i + \sin(\theta_1 - \theta_2) - \sin(\theta_2 - \theta_3) + \sigma(\dot{\theta}_1 - 2\dot{\theta}_2 + \dot{\theta}_3), i_c\}\end{aligned}\quad (54)$$

$$\dot{\theta}_3 = \mathcal{F}\{i + \sin(\theta_2 - \theta_3) - \sin(\theta_3 - \theta_4) + \sigma(\dot{\theta}_2 - 2\dot{\theta}_3 + \dot{\theta}_4), i_c\} \quad (55)$$

One of the few analytical results we were able to obtain for this problem is that, proceeding as for case (i) in Sec. III.A, the critical current for dissipation at least in the first plane is,

$$i_{crit} = \begin{cases} \frac{N}{m} i_c & \text{if } i_c < \frac{1}{N-m} \\ i_c + \frac{1}{m} & \text{if } i_c > \frac{1}{N-m} \end{cases} \quad (56)$$

In Fig.5 we show our results for the IV characteristics for $N = 8$, $m = 3$ and $\sigma = 0.7$. For $i_c = 0.1 < \frac{1}{5}$ the results are shown in Fig.5(a). After a critical current $i_{crit} = \frac{N}{m} = \frac{8}{3}$ all the planes start to dissipate with the same voltage. Then at a current i_d there is a decoupling between the third and the fourth plane, where the first 3 planes have the same voltage, and the other $N - 3$ planes have a different voltage. Then at higher currents there are successive decouplings of the other planes, either in the layer of the first 3 or in the region of the last $N - 3$. The first decoupling current i_d can be calculated analytically proceeding in a similar way as in case (iv) of Sec.III.A, obtaining

$$i_d = \frac{N}{m(N - m)} \quad (57)$$

In Fig.5(b) we show another case with $i_c = 0.3 > \frac{1}{5}$. Here, after a critical current $i_{crit} = i_c + 1/3$ only the first 3 planes start to dissipate with the same voltage, and they decouple at higher currents. The fourth plane starts to dissipate at a higher current $i_b^{(3)}$, then the remaining $N - 4$ start dissipating at $i_b^{(4)}$ with the same voltage, and after they decouple at much higher driving currents.

Of course, other different cases occur for different values of i_c . But again, if we only look at the top and bottom planes, only two different kinds of behavior can be distinguished in

the IV curves, depending on $i_c < \frac{1}{N-m}$ or $i_c > \frac{1}{N-m}$. One important consequence is that now the decoupling occurs where there is the highest gradient in the applied current, between the m th and the $m + 1$ th plane.

IV. SUMMARY AND CONCLUSIONS

Let us summarize our main results. We found that there are two different kinds of behavior in the IV characteristics for the top and bottom planes. This depends on the relative strength of pinning as compared with the coupling between planes (both magnetic and Josephson).

For weak pinning or strong interplane coupling, $((N-1)I_c < I_0 + f_m$, with I_c the depinning critical current per plane, I_0 the Josephson critical current and f_m the maximum “magnetic current”), both the top and the bottom planes start to dissipate at the same current with the same voltages. This critical current depends on pinning and is given by NI_c . At a higher current the top and the bottom plane decouple, having different voltages. This corresponds actually to a decoupling between the first and the second planes (or where there is the highest gradient in the applied current), where we can say that there is a “cutting” of the vortices. This decoupling or cutting current is independent of pinning, as given by Eq. (48) in normalized units. Taking into account that $I_0 = \frac{L_x c \Phi_0 s^2}{8\pi^2 \lambda_{ab}^2 \lambda_J^2}$ and $f_m = \frac{L_x c \Phi_0 s^2}{8\pi^2 \lambda_{ab}^4}$ (see for example Ref. 15), we can write it as

$$I_d = \frac{N}{N-1} \frac{L_x c \Phi_0 s^2}{8\pi^2 \lambda_{ab}^2} \left(\frac{1}{\lambda_J^2} + \frac{1}{\lambda_{ab}^2} \right), \quad (58)$$

where s is the interplane spacing, $\lambda_{ab}(T, B)$ is the penetration depth in the ab planes, and $\lambda_J = s\gamma = s\lambda_c/\lambda_{ab}$ is the Josephson penetration depth. For large Josephson coupling, $\lambda_{ab} \gg \lambda_J$, I_d decreases linearly close to T_c (since $\lambda_{ab} \sim (T_c - T)^{-1/2}$). This dependence has been observed experimentally in $\text{YBa}_2\text{Cu}_3\text{O}_7$ ^{8,9,21}. The decoupling current was also calculated recently in Ref. 21 obtaining a similar result. It is possible to have further decouplings between the successive planes at much higher currents. However these decoupling currents

$i_d^{(k)}$ increase exponentially with k , as approximately given by Eq. (53), and therefore it should be difficult to observe their effect experimentally.

For strong pinning or weak interplane coupling ($(N - 1)I_c > I_0 + f_m$) the top and the bottom plane start to dissipate at different critical currents. There is no dynamical coupling of vortices in this case. This situation corresponds to the experiments in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_x$ ^{5,6}.

One of the limitations of our model is that in Eq. (9) we have considered only the contribution of the relative displacements of the vortex lattice to the Josephson phase differences. An extension of the model including the contribution of the flowing currents to the superconducting phases is presently being studied by us. However, the main results of this paper remain unchanged. The other main approximation consisted in considering a rigid Abrikosov lattice, valid for very weak pinning and low temperatures below the melting line. Thus, our model can strictly be applied only for the multilayered low temperature superconductors (see for example Ref. 22). However, most of the results obtained here, regarding the characteristics currents for dissipation and decoupling, do not depend on the periodicity of the sines in the equation of motion, but on the existence of maximum interaction forces and currents. Therefore, our main qualitative results should still be of application in the ceramic superconductors below the melting line.

In conclusion, it may be of interest to do experiments in samples of the same compound with different degrees of disorder. For weak disorder, the first kind of IV curves with I_d independent of pinning should be observed, and then for large disorder the second kind of IV curves without coupling should be observed.

ACKNOWLEDGMENTS

We have benefited from discussions with F. de la Cruz and E. Rodríguez about their experimental results. We acknowledge L. N. Bulaevskii for a critical reading of the manuscript.

One of the authors (K.K.U.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, during the 1992–1993 ICTP Diploma Course where this work was started.

REFERENCES

- ¹ I. Giaever, Phys. Rev. Lett. **15**, 825 (1965).
- ² P. E. Cladis, R. D. Parks, J. M. Daniels, Phys. Rev. Lett. **21**, 1521 (1968).
- ³ J. R. Clem, Phys. Rev. B **9**, 898 (1974); J. W. Ekin, B. Serin and J. R. Clem, Phys. Rev. B **9**, 912 (1974).
- ⁴ J. R. Clem, Phys. Rev. B **12**, 1742 (1975). J. W. Ekin and J. R. Clem, Phys. Rev. B **12**, 1753 (1975).
- ⁵ R. Busch, G. Ries, H. Werthner, G. Kreiselmeyer, and G. Saemann-Ischenko, Phys. Rev. Lett. **69**, 522 (1992).
- ⁶ H. Safar, E. Rodríguez, F. de la Cruz, P. L. Gammel, L. F. Schneemeyer, D. J. Bishop, Phys. Rev. B **46**, 14238 (1992).
- ⁷ Y. M. Wan, S. E. Hebboul, D. C. Harris, J. C. Garland, Phys. Rev. Lett. **71**, 157 (1993).
- ⁸ H. Safar, P. L. Gammel, D. A. Huse, S. N. Majumdar, L. F. Schneemeyer, D. J. Bishop, D. López, G. Nieva, F. de la Cruz, Phys. Rev. Lett. **72**, 1272 (1994).
- ⁹ D. López, E. Rodríguez, G. Nieva, F. de la Cruz, S. W. Cheong, *Proceedings of the 20th International Conference on Low Temperature Physics*, Physica B **194-196**, 1977 (1994).
- ¹⁰ F. de la Cruz, D. López, G. Nieva, preprint.
- ¹¹ D. A. Huse and S. N. Majumdar, Phys. Rev. Lett. **71**, 2473 (1993).
- ¹² J. R. Clem, Phys. Rev. B **43**, 7837 (1991).
- ¹³ L. N. Bulaevskii and J. R. Clem, Phys. Rev. B **44**, 10234 (1991).
- ¹⁴ L. N. Bulaevskii, M. Ledvij, V. G. Kogan, Phys. Rev. B **46**, 366 (1992).
- ¹⁵ L. N. Bulaevskii, M. Ledvij, V. G. Kogan, Phys. Rev. B **46**, 11807 (1992).

- ¹⁶ W. E. Lawrence and S. Doniach, in *Proceedings of the Twelfth International Conference on Low Temperature Physics*, edited by E. Kanda (Academic Press of Japan, Kyoto, 1971), p. 361.
- ¹⁷ A. I. Larkin and Yu. N. Ovchinnikov, *J. Low Temp. Phys.* **34**, 409 (1979).
- ¹⁸ A. C. Shi and A. J. Berlinsky, *Phys. Rev. Lett.* **67**, 1926 (1991); S. Bhattacharya and M. J. Higgins, *Phys. Rev. Lett.* **70**, 2617 (1993); D. Domínguez, *Phys. Rev. Lett.* **72**, 3096 (1994).
- ¹⁹ A. Barone and G. Paterno, *Physics and applications of the Josephson effect* (Wiley, New York, 1982).
- ²⁰ W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes* (Cambridge University Press, Cambridge, 1989).
- ²¹ D. López, G. Nieva, F. de la Cruz, H. J. Jensen and D. O’Kane, preprint; D. O’Kane and H. J. Jensen, preprint.
- ²² P. Koorevaar, W. Maj, P. H. Kes, and J. Aarts, *Phys. Rev. B* **47**, 934 (1993).

FIGURES

FIG. 1. Schematic representation of the flow of current in two Josephson coupled superconducting planes in the flux transformer configuration. The dotted arrows indicate the Josephson currents in the bulk due to vortex lattice displacements.

FIG. 2. Current-voltage characteristics of the dc flux transformer for two superconducting planes with Josephson coupling. (a) For $i_c = 0.1$, $\sigma = 0.7$. (b) For $i_c = 2.$, $\sigma = 0.7$. Currents are normalized by $I_0 + f_m$, voltages by $R_f(I_0 + f_m)$. V_t and V_b indicate the voltages in the top and the bottom planes respectively.

FIG. 3. Schematic representation of the flow of current in a layered superconductor with N Josephson coupled planes in the flux transformer configuration.

FIG. 4. Current-voltage characteristics of the dc flux transformer for $N = 8$ superconducting planes with Josephson coupling. (a) For $i_c = 0.05$, $\sigma = 0.7$. (b) For $i_c = 0.15$, $\sigma = 0.7$. (c) For $i_c = 0.23$, $\sigma = 0.7$. (d) For $i_c = 1.2$, $\sigma = 0.7$. Currents are normalized by $I_0 + f_m$, voltages by $R_f(I_0 + f_m)$. The different curves correspond, going from the highest voltage to the lowest, to the planes going from the first to the last (the 8th). The top and bottom voltages are highlighted for clarity. Note that the plots are in log-log scales.

FIG. 5. Current-voltage characteristics of the dc flux transformer for $N = 8$ superconducting planes with Josephson coupling and current injection in the first $m = 3$ planes. (a) For $i_c = 0.1$, $\sigma = 0.7$. (b) For $i_c = 0.3$, $\sigma = 0.7$. Currents are normalized by $I_0 + f_m$, voltages by $R_f(I_0 + f_m)$. The different curves correspond, going from the highest voltage to the lowest, to the planes going from the first to the last (the 8th). The top and bottom voltages are highlighted for clarity.

Fig.1

Fig.2

Fig.3

Fig.4

Fig.5