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The Generalized Hartree-Fock Method in The Gross-Neveu Model

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Abstract

In the generalized Hartree-Fock scheme one can incorporate non-perturbative effects from the beginning of a diagrammatic expansion. In order to make detailed analysis on the method we apply it to the Gross-Neveu model as an example and calculate the chiral condensate at large N by using a direct diagrammatic expansion. Unperturbed Gaussian kernel is adjusted to be massive by adding fermion mass terms and the mass terms are subtracted from the interaction part, thus the full Lagrangian being not changed. The chiral condensate $\langle g\sigma \rangle$ is expanded in powers of the renormalized coupling constant and then improved by the method of effective charges. We find that our diagrammatic expansion is compatible with renormalization. By the use of the principle of minimal sensitivity we evaluate $\langle g\sigma \rangle$ and show that the exact broken and unbroken solutions for the discrete γ_5 symmetry are obtained at each order of the expansion.

1 INTRODUCTION

Since the pioneering work by Nambu and Jona-Lasinio [1], the generalized Hartree-Fock method and its similars have been investigated by several authors [1-11] as an alternative to the conventional perturbative approach. As stated by Neveu [8] the underlying idea is to approximate a given non-linear system by an appropriate linear system. For the purpose one modifies the ordinary Gaussian kernel by, for instance, adding appropriate terms and subtracting the same terms from the interaction part. The added terms may be considered as representing the effect of original non-linear interactions while the subtracted terms may play the role to moderate the interaction. In changing the Gaussian kernel, one can incorporate terms expected to be induced by interactions even though they are absent in the Lagrangian. The perturbation expansion is performed with respect to a new interaction Lagrangian around a new un-perturbed state. While the conventional perturbation expansion preserves the symmetry of the Lagrangian, if one incorporates symmetry breaking piece in this manner, un-conventional expansion of this kind does not, in general, respect the symmetry at finite orders of expansion. Then by optimizing a calculated quantity by fixing the incorporated terms in some manner one can test whether the symmetry breaking result survives or not. Thus, this approach, which we call GHF approach, may supply a suitable framework for the study of dynamical symmetry breaking.

The whole aspect of GHF approach is not clarified yet. Therefore, to get further insights and develop the method, it is desirable to examine the method in great details by applying it to simple but interesting models. As a first step we used GHF method in the Gross-Neveu model [12] and calculated the effective potential to obtain the chiral condensate [10] (See also ref.[9]),

$$\langle g\sigma \rangle = \int D\bar{\psi} D\psi D\sigma \exp(iS) \cdot g\sigma / \int D\bar{\psi} D\psi D\sigma \exp(iS). \quad (1)$$

The result was satisfactory both at zero and finite temperatures. However, in order to compute composite condensates in 4-dimensional theories such as quark and gluon condensates, effective potential is not available due to the renormalization problem¹. Therefore we like to explore a different approach in GHF scheme in which the

¹However, there is an alternative approach in which one deals with the effective potential for

renormalizability is not lost in 4-dimensional field theories.

One of the popular ways in describing physical processes is perturbation expansion in terms of Feynman diagrams. It is powerful in the sense that the expansion can be applied to any field theory in any space-time dimension and the renormalization procedure is well-established in it. Therefore, in the present paper, we try to calculate the chiral condensate in the Gross-Neveu model at large N by the use of direct diagrammatic expansion of (1). Since the presence of non-vanishing condensate is closely related with the dynamical fermion mass, we add and subtract fermion mass terms, $m \sum_{i=1}^N \bar{\psi}_i \psi_i = m \bar{\psi} \psi$ to the given massless Lagrangian. The incorporated mass terms do not affect the γ_5 symmetry, the invariance under $\psi \rightarrow \gamma_5 \psi$, in the whole Lagrangian although the symmetry is broken in the free part. In the ordinary diagrammatic expansion within dimensional regularization, the perturbative contributions automatically vanish order by order because the one-loop tad pole vanishes. In contrast, in our approach, the condensate can be expanded in powers of the renormalized coupling constant due to the incorporation of mass terms and shows a non-trivial structure. Based on this power series in g^2 we will obtain the broken solution which depends on g^2 non-analytically as $\exp(-\pi/g^2)$. Although the Gross-Neveu model was already solved at large N [12], our main purpose is to make detailed analysis on GHF expansion extending to all orders by taking the model as an example. By this examination a deeper understanding of the method will be obtained and our analysis would help one to perform similar calculus in general field theories.

In the next section we review the Feynman diagrams in GHF scheme which appear in the expansions of (1) in the leading order of large N . In section 3 we define the perturbation scheme to be used in the present paper. It is shown that Feynman diagrams can be calculated un-ambiguously and the diagrammatic expansion of $\langle g\sigma \rangle$ can be made compatible with renormalization. Note that in the conventional massless expansion the diagrams are plagued with both ultra and infrared divergences which prevent us from extracting reliable information from the diagrams. This difficulty is resolved in GHF scheme and $\langle g\sigma \rangle$ is expanded in g^2 . We point out that $\langle g\sigma \rangle$ is non-local quantities. See, for example, refs. [13, 14, 15]. Recently Fukuda proposed inversion method in which the effective potential is not used [16].

renormalization group invariant, as it should be from the standard renormalization group analysis, up to the maximal order of g at each order of expansion. In section 4 we discuss the application of the method of effective charges [17] to the perturbative chiral condensate. The application induces an improvement on the series which is interpreted by the self consistent view point. Section 5 is devoted to the evaluation of the condensate under the principle of minimal sensitivity [18]. We show that the exact values of $\langle g\sigma \rangle$ is obtained to all orders. In section 6 we discuss several issues including the large order aspects of GHF approach, and the possibility of another expansion scheme, e.t.c.. Through out this paper we work in the framework of the dimensional regularization and $\overline{MS}$ renormalization scheme [19]. We confine ourselves to the leading order of large N expansion and omit N for the sake of simplicity.

2 FEYNMAN DIAGRAMS IN GHF SCHEME

The Lagrangian to start with is

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{free} + \mathcal{L}_{int}, \\ \mathcal{L}_{free} &= \bar{\psi} i \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi - \frac{1}{2} \sigma^2, \\ \mathcal{L}_{int} &= m \bar{\psi} \psi - g \sigma \bar{\psi} \psi + \mathcal{L}_{ct}, \\ \mathcal{L}_{ct} &= \frac{1}{2} \sigma^2 \cdot \frac{-g^2}{2\pi} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) \right),\end{aligned}\tag{2}$$

where $\epsilon = 2 - D$. The free part, $\mathcal{L}_{free}$, defines the perturbative vacuum around which we perform perturbation expansion. The free fermion propagator is given by $i/(\not{p} - m)$ and there arises no infra-red singularity. The interaction part, $\mathcal{L}_{int}$, defines two-kinds of vertices, $m \bar{\psi} \psi$ (Fig.1) and $g \sigma \bar{\psi} \psi$. In the large N limit any Feynman diagram is factorized into the products of 1-P-I diagrams of σ each of which is just a fermion loop with possible mass vertices. The 1-P-I diagrams are classified according to the numbers of σ legs and $m \bar{\psi} \psi$ vertex. For example the diagrams shown in Fig.2 have three σ legs and one mass vertex. We denote the sum of diagrams with r number of σ legs and s number of $m \bar{\psi} \psi$ vertex as $\Gamma_s^{(r)}$. In this classification, the sum, $\sum_{s=0}^{\infty} \Gamma_s^{(r)}$, corresponds to the single diagram $\Gamma^{(r)}(m=0)$ in the ordinary perturbation

expansion.

According to Neveu [8] the sum of diagrams having $m\bar{\psi}\psi$ vertices can be easily calculated from the diagram having no $m\bar{\psi}\psi$ vertex: We first change m in $\Gamma_0^{(r)}(m)$ to $m(1-\delta)$ and then expand $\Gamma_0^{(r)}(m(1-\delta))$ in powers of δ . By noting that the power of δ represents the number of the mass vertex we find

$$\Gamma_0^{(r)}(m(1-\delta)) = \sum_{s=0}^{\infty} \Gamma_s^{(r)}(m) \delta^s. \quad (3)$$

By truncating the series at δ^n and then setting $\delta = 1$ we obtain $\sum_{s=0}^n \Gamma_s^{(r)}(m)$ which plays the role of basic building blocks in constructing perturbation expansion.

For the later argument it is convenient to make here the calculation of $\sum_{s=0}^n \Gamma_s^{(r)}(m)$. For $\langle g\sigma \rangle$ in the leading order of $1/N$ expansion, it is sufficient to deal with diagrams in the limit of zero external momenta. Let us start with the formulae

$$\begin{aligned} \text{tr} \int \frac{d^D p}{(2\pi)^D} \frac{1}{\not{p} - m} &= \frac{-im \hat{2}}{2\pi \epsilon} \\ \text{tr} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(\not{p} - m)^2} &= \frac{-i \hat{2}}{2\pi \epsilon} + \frac{i}{\pi} \\ \text{tr} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(\not{p} - m)^r} &= \frac{i(-1)^{r-1}}{(r-1)(r-2)\pi m^{r-2}} \quad (r \geq 3) \end{aligned} \quad (4)$$

where

$$\frac{\hat{2}}{\epsilon} = \frac{2}{\epsilon} - \gamma - \log \frac{m^2}{4\pi\mu^2}. \quad (5)$$

Since $\Gamma_0^{(r)}$ is given by the integral,

$$\Gamma_0^{(r)} = -(r-1)! g^r \text{tr} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(\not{p} - m)^r}, \quad (6)$$

we find

$$\begin{aligned} \Gamma_0^{(1)} &= \frac{img \hat{2}}{2\pi \epsilon} \\ \Gamma_0^{(2)} &= \frac{ig^2}{2\pi} \left(\frac{\hat{2}}{\epsilon} - 2 \right) \\ \Gamma_0^{(r)} &= \frac{i(-g)^r (r-3)!}{\pi m^{r-2}} \quad (r \geq 3). \end{aligned} \quad (7)$$

In GHF expansion, there are ultraviolet divergences which can not be subtracted in the usual manner. In the Gross-Neveu model in the leading order of $1/N$ expansion, those divergences come only from the diagrams shown in Figs.3 and 4, and the pole

in Fig.3 is shown in the first line of (7). However we note that Fig.4 cancels the divergence of Fig.3 [10] and thus the conventional renormalization procedure works as long as the two diagrams are kept in pairs. By making the shift $m \rightarrow m(1-\delta)$ and expanding $\Gamma_0^{(r)}(m(1-\delta))$ in δ up to δ^n , we find

$$\begin{aligned} \sum_{s=0}^n \Gamma_s^{(1)} &= \frac{img}{\pi} \frac{1}{n} \\ \sum_{s=0}^n \Gamma_s^{(2)} &= \frac{ig^2}{2\pi} \left(-\frac{1}{2} \log \frac{m^2}{\mu^2} - 1 + \sum_{s=1}^n \frac{1}{s} \right) \\ \sum_{s=0}^n \Gamma_s^{(r)} &= (-1)^r \frac{ig^r}{(r-2)\pi} \cdot \frac{(n+1)(n+2)\cdots(n+r-2)}{m^{r-2}} \quad (r \geq 3) \end{aligned} \quad (8)$$

where the ordinary renormalization with $\overline{MS}$ prescription has been made.

One can easily read the asymptotic behavior of $\sum_{s=0}^n \Gamma_s^{(r)}$ for large n from (8);

$$\begin{aligned} \sum_{s=0}^n \Gamma_s^{(2)} &\sim \frac{ig^2}{4\pi} \log \left(\frac{n^2 \mu^2}{m^2} \right), \\ \sum_{s=0}^n \Gamma_s^{(r)} &\sim (-1)^r \frac{ig^r}{(r-2)\pi} \left(\frac{n}{m} \right)^{r-2} \quad (r \neq 2), \end{aligned} \quad (9)$$

where we used

$$\sum_{s=1}^n \frac{1}{s} \rightarrow \log(n) + \gamma + O(1/n). \quad (10)$$

For $r \geq 3$ the above asymptotic behavior has a close relation with the infra-red behavior of the corresponding diagram in the ordinary massless expansion. Namely, the degree of divergence in the $n \rightarrow \infty$ limit represents the degree of infra-red divergence in the massless diagram. In other words, n in (8) and (9) plays the role of infra-red cut-off. It is worth noting that the infra-red divergence is separated unambiguously from the ultraviolet divergences only which show up as poles in ϵ . It is then reasonable that m enters into $\sum_{s=0}^n \Gamma_s^{(r)}$ through the combination, m/n , because the degree of the infra-red divergence is determined also by its mass dimension which is represented by the power of m .

3 PERTURBATION EXPANSION

As noted in the previous section we must construct a perturbation scheme in which the diagrams shown in Figs.3 and 4 are kept in pairs for preserving the renormalizability. The requirement is fulfilled in the following perturbation scheme.

At the leading order of $1/N$ expansion any Feynman diagram contributing to $\langle g\sigma \rangle$ is constructed by fermion loops with possible mass insertions and σ propagators at the tree level which connect the fermion loops. Therefore we first define the order of sub-diagrams, $\Gamma_s^{(r)}$, as $r + s = k$. Next, given a Feynman diagram with l loops, let the orders of l sub-diagrams as $k_1, k_2, \dots, k_l$. Then we define the order of the diagram by $\max(l, k_1, k_2, \dots, k_l)$. In this definition, included diagrams to n -th order of expansion have loops less than or equal to n . And each loop, $\Gamma_s^{(r)}$ ($r \leq n$), is allowed to have the mass vertex up to $n - r$. Hence the set of diagrams with the same fermion loop structure can be summed up into one "corrected" diagram where $\sum_{s=0}^{n-r} \Gamma_s^{(r)}$ plays the role of a basic building block. In simple words, we can obtain the expansion by first considering the ordinary massless loop expansion and then replacing each $\Gamma_s^{(r)}(m=0)$ by $\sum_{s=0}^{n-r} \Gamma_s^{(r)}(m)$.

Under thus defined perturbation scheme, one finds no un-renormalizable divergence at and above the second order because the sub-diagram shown in Fig.3 is always accompanied by the sub-diagram shown in Fig.4. Since the infra-red divergence is "regularized" by the incorporated fermion mass terms and the ultraviolet structure is thus not changed at every order, $\langle g\sigma \rangle$ can be calculated without ambiguity by expanding it in terms of Feynman diagrams.

In this perturbation expansion, second, third and fourth order calculations give

$$\begin{aligned} \langle g\sigma \rangle &= \frac{m}{1} \alpha (1 - t\alpha), \quad t = \frac{1}{2} \log\left(\frac{m^2}{\mu^2}\right) + 1, \\ \langle g\sigma \rangle &= \frac{m}{2} \alpha \left(1 - t\alpha + \left(t^2 - \frac{1}{4}\right)\alpha^2\right), \quad t = \frac{1}{2} \log\left(\frac{m^2}{\mu^2}\right), \\ \langle g\sigma \rangle &= \frac{m}{3} \alpha \left(1 - t\alpha + \left(t^2 - \frac{1}{3}\right)\alpha^2 + \left(-t^3 + t + \frac{1}{54}\right)\alpha^3\right), \quad t = \frac{1}{2} \log\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}, \end{aligned} \quad (11)$$

where $\alpha = g^2(\mu)/\pi = 1/\log(\mu/\Lambda)$ ($\Lambda = \Lambda_{\overline{MS}}$) in the leading order of $1/N$ expansion. The above result already shows us a basic feature in GHF expansion; that is, the condensate is represented by analytic function of the renormalized coupling constant. On the other hand, from the renormalization group analysis, the true dependence of the condensate on $g^2(\mu)$ is such that $\langle g\sigma \rangle \sim \mu \exp(-\pi/g^2(\mu))$. The difference comes from the mass parameter associated with the perturbative vacuum and we find that the shift of perturbative vacuum changes the structure of functional dependence of

$\langle g\sigma \rangle$. We will later eliminate the mass parameter, m , and recover the true dependence on g^2 .

Another point we note is that our expansion scheme realizes the μ independence maximally; that is, to the n -th order $\langle g\sigma \rangle$ is μ independent up to the order α^n . We emphasize that this is a characteristic feature in our expansion and not found in the expansion where the mass and the $\sigma\bar{\psi}\psi$ interactions are regarded as being of the same order. Let us modify, however, the perturbation expansion into exactly μ independent at every order. For the purpose, we adopt the rule to replace the free σ propagator by the dressed σ propagator which involves all the chain diagrams with no tad pole (see Fig.5). Thus corrected σ propagator multiplied by the coupling constant on both end points yields at the zero momentum

$$\frac{\alpha}{1 + \alpha\left(\frac{1}{2} \log \frac{m^2}{\mu^2} + \text{const.}\right)} = \left(\frac{1}{2} \log \frac{m^2}{\Lambda^2} + \text{const.}\right)^{-1}. \quad (12)$$

Since the logarithm, $\log(m/\mu)$, comes only from the self-energy of σ (see (8)), we have $\langle g\sigma \rangle$ as m times a function of $\log(m/\Lambda)$, and thus the μ independence is rigorously realized. We apply the rule of the order of sub-diagrams to the self-energy diagram of σ : To the n -th order, we use $\sum_{s=0}^{n-2} \Gamma_s^{(2)}$ for the fermion loop. By this we complete the definition of the modified perturbation scheme to be used hereafter. To the second order a simple calculation yields

$$\langle g\sigma \rangle = \frac{m}{1} x, \quad 1/x = \frac{1}{2} \log\left(\frac{m^2}{\Lambda^2}\right) + 1, \quad (13)$$

and in Fig.6 $\langle g\sigma \rangle$ is shown as a function of m/Λ . This result agrees with the result obtained by summing up the leading logs in the un-modified expansion. Therefore the modification is not restricted to the model under consideration. At higher orders we obtain $\langle g\sigma \rangle$ as a polynomial of x (see (17)). Since x can be expanded as a power series of α , the analytic structure of $\langle g\sigma \rangle$ is still different from the true one. When the condensate is evaluated, however, we can recover the true g dependence as we can see below.

Since μ independence is rigorously satisfied in (13) the evaluation is easier than in the case (11). Our evaluation method is based on the fact that the theory is independent of the incorporated parameter, m . The rigorous m independence, however,

is met only when the theory is completely solved out. At a finite order as we shown in (13), the m dependence remains and in this sense the result do not respect the rigorous masslessness. Still, we should note that the transformation property under the renormalization group agrees with the massless case in which $\langle g\sigma \rangle$ is renormalization group invariant. Now, in the evaluation of the condensate (13), we follow refs.[5,8]. The idea is that the most insensitive value of $\langle g\sigma \rangle$ with respect to m represents the best approximation and this is provided by PMS condition [18],

$$\frac{d\langle g\sigma \rangle}{dm} = x - x^2 = 0. \quad (14)$$

The solutions are

$$x = 0, 1. \quad (15)$$

Since $x = 0^+$ leads to $m = +\infty$ which gives $\langle g\sigma \rangle = mx = +\infty$, the first solution should be understood as $x = 0^-$ which corresponds to $m = 0$. The second solution, $x = 1$, leads to $m = \pm\Lambda$. Thus we obtain from (13) the exact broken and unbroken solutions,

$$\langle g\sigma \rangle = \pm\Lambda, 0 \quad (16)$$

for $m = \pm\Lambda$ and 0, respectively. It would now be clear how the true g dependence is recovered: The use of PMS condition optimizes m such that $m = \text{const.} \cdot \Lambda$ because of apparent μ independence of m . At higher orders, however, one finds the problem of the existence of the several solutions some of which have the large numerical error. The status of GHF method is therefore not satisfactory yet. To resolve the problems we improve the series of $\langle g\sigma \rangle$ by summing up infinite subsets of Feynman diagrams in the next section.

4 THE METHOD OF EFFECTIVE CHARGES

To n -th order in our modified perturbation expansion, we find

$$\langle g\sigma \rangle = \frac{m}{n-1} x \left(1 + c_2(n)x^2 + c_3(n)x^3 + \dots + c_{n-1}(n)x^{n-1} \right) \quad (17)$$

where x is given by

$$x = \left(\frac{1}{2} \log(m^2/\Lambda^2) + 1 - \sum_{r=1}^{n-2} \frac{1}{r} \right)^{-1}. \quad (18)$$

The first few coefficients $c_r(n)$ are found to be

$$c_2 = \frac{-(n-2)}{2(n-1)}, \quad c_3 = \frac{(n-2)(n-3)}{12(n-1)^2}, \quad c_4 = \frac{(n-2)^2}{2(n-1)^2} - \frac{(n-2)(n-3)(n-4)}{72(n-1)^3}. \quad (19)$$

The factor $1/(n-1)$ contained in the R.H.S. of (17) comes from a tad pole, $\sum_{s=0}^{n-1} \Gamma_s^{(1)}$, which is any one of tad-poles involved in a Feynman diagram. The absence of x^2 in (17) is a consequence of our modification that in the expansion the dressed σ propagator is used in the place of free one. Unlike the ordinary perturbation expansion, c_r varies with the order n . Note that, in the $n \rightarrow \infty$ limit, $c_r(n)$ ($r = 2, 3, 4$) converge to finite values, $-1/2, 1/12$ and $35/72$, respectively. By the power counting with respect to n or dimensional arguments, one can check that the $n \rightarrow \infty$ limit of $c_r(n)$ exists for all r . More precisely we can say that, in a given diagram, the infra-red divergences involved in sub-graphs are cancelled by tad-poles and, in accordance with mass dimension of $\langle g\sigma \rangle$, the effect of just one tad-pole remains giving $\sim m/(n-1)$. We thus obtain $\langle g\sigma \rangle$ as $m/(n-1)$ times a series of logs. Then, since x behaves in this limit as

$$x \sim \frac{1}{-\frac{1}{2} \log(n^2/m^2)} \rightarrow 0^- \quad (n \rightarrow \infty), \quad (20)$$

we find

$$\langle g\sigma \rangle \sim \frac{m}{n} x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 + O(x^3) \right) \sim \frac{m}{n - \frac{1}{2} \log(n^2/m^2)} \rightarrow 0^- \quad (n \rightarrow \infty). \quad (21)$$

Thus, as the order becomes large, it seems that the condensate tends to zero for any finite m . In addition to the problem mentioned at the end of the previous section, this seems preventing us from obtaining the broken solution itself. However the things are not so simple as we can see below.

We note that $\langle g\sigma \rangle$ goes to zero from the negative side while m is fixed positive. Since $\langle g\sigma \rangle$ agrees with the dynamical mass itself in the leading order of $1/N$ expansion, the discrepancy of the sign is curious and requires a closer look. Indeed when $m/n\Lambda$ decreases from large enough region we meet discontinuous change of $\langle g\sigma \rangle$ from positive to negative region as follows. For m large enough ($m/n\Lambda \gg 1$) the expansion (17) makes sense since x is positive and small. When m decreases, x increases and eventually at the value, $m_c = \Lambda \exp C(n)$, x diverges to $+\infty$ and turns

to $-\infty$ when m passes across m_c (see (18)). Then x tends to zero from the negative side as m further decreases to zero. Thus, for m close to m_c , the expansion (17) loses the meaning and hence the series must be summed up, in some manner, when we let m smaller. The resulting behavior of $\langle g\sigma \rangle$ for large or negative small x may differ from that represented by (17) and indicates another branch different from the perturbative one given by (17) in the naive $n \rightarrow \infty$ limit with fixed m . Non-zero condensate would be found in this non-trivial branch and therefore we turn to discuss the summation procedure.

In order to sum up the perturbative series, we use the method of effective charges [19]. Following Grunberg, we define $\bar{\alpha}$ by

$$\langle g\sigma \rangle = \frac{m}{n-1} \bar{\alpha}, \quad (22)$$

which leads to

$$\bar{\alpha} = x(1 + c_2(n)x^2 + c_3(n)x^3 + \cdots + c_{n-1}(n)x^{n-1}), \quad (23)$$

or in terms of α ,

$$\bar{\alpha} = \alpha \left(1 - \alpha \left(\frac{1}{2} \log \frac{m^2}{\mu^2} + 1 - \sum_{r=1}^{n-2} \frac{1}{r} \right) + \mathcal{O}(\alpha^2) \right). \quad (24)$$

From its renormalization scheme independence property represented by (22) and from (24), $\bar{\alpha}$ can be regarded as the effective charge [17] associated with the chiral condensate². The behavior of $\bar{\alpha}$ is governed by the associated beta function, $\bar{\beta}(\bar{\alpha}) = m \partial \bar{\alpha} / \partial m$, which is calculated directly from (23) or (24). By expanding $\bar{\beta}(\bar{\alpha})$ in powers of $\bar{\alpha}$, we obtain

$$\bar{\beta}(\bar{\alpha}) = b_0 \bar{\alpha}^2 + b_1 \bar{\alpha}^3 + b_2 \bar{\alpha}^4 + b_3 \bar{\alpha}^5 + b_4 \bar{\alpha}^6 + \cdots, \quad (25)$$

where

$$b_0 = -1, \quad b_1 = 0, \quad b_2 = -c_2, \quad b_3 = -2c_3, \quad b_4 = -3c_4 + 5c_2^2, \cdots \quad (26)$$

Note that b_i ($i = 1, 2, \cdots, n-1$) is uniquely determined by $c_2, c_3, \cdots, c_{n-1}$. The minus sign of b_0 shows the asymptotic freedom in the Gross-Neveu model and the

²A prototypical quantity of effective charges due to Grunberg is seen in the earlier work of Susskind [20]

first two terms in $\bar{\beta}(\bar{\alpha})$ agree exactly with those of $\beta(\alpha) = \mu \partial \alpha / \partial \mu$. Although (17) is a polynomial of x , $\bar{\beta}$ does not terminate at a finite power of $\bar{\alpha}$. Here we apply the truncation rule such that

$$\begin{aligned} \frac{1}{\bar{\beta}(\bar{\alpha})} &= \frac{-1}{\bar{\alpha}^2} (1 + \bar{b}_2(n)\bar{\alpha}^2 + \cdots + \bar{b}_{n-1}(n)\bar{\alpha}^{n-1}) \\ &\equiv \theta(\bar{\alpha}) \\ &\equiv \frac{-1}{\bar{\alpha}^2} + \theta^{reg}(\bar{\alpha}), \end{aligned} \quad (27)$$

where $\theta^{reg}(\bar{\alpha})$ represents the regular part of $\theta(\bar{\alpha})$. The coefficients $\bar{b}_i$ ($i = 2, 3, \cdots, n-1$) are given by b_i ($i = 2, 3, \cdots, n-1$) and hence by c_i ($i = 2, 3, \cdots, n-1$) in (17).

Now we rewrite the chiral condensate in terms of $\bar{\alpha}$. The mass parameter, m , can be expressed with $\bar{\alpha}$ as

$$m = m_0 \exp \int_{\bar{\alpha}_0}^{\bar{\alpha}} dz \theta(z) = m_0 \exp \left(\frac{1}{\bar{\alpha}} - \frac{1}{\bar{\alpha}_0} + \int_{\bar{\alpha}_0}^{\bar{\alpha}} dz \theta^{reg}(z) \right). \quad (28)$$

By taking the $\bar{\alpha}_0 \rightarrow 0$ limit, we obtain from (24)

$$m = \pm \Lambda \exp \left(C(n) + \frac{1}{\bar{\alpha}} + \int_0^{\bar{\alpha}} dz \theta^{reg}(z) \right) \equiv \pm \Lambda \exp \left(C(n) + \int_0^{\bar{\alpha}} dz \theta(z) \right) \quad (29)$$

where

$$C(n) = -1 + \sum_{r=1}^{n-2} \frac{1}{r}. \quad (30)$$

The plus sign in (29) is for positive m and the minus sign for negative m . From here on we confine ourselves to the case of positive m . We thus arrive at a new form,

$$\langle g\sigma \rangle = \frac{m}{n-1} \bar{\alpha} = \frac{\Lambda}{n-1} \bar{\alpha} \exp \left(C(n) + \int_0^{\bar{\alpha}} dz \theta(z) \right). \quad (31)$$

In this form it is easy to see that by optimizing $\bar{\alpha}$ we have the condensate as Λ times a pure number.

The next task is to calculate θ order by order based on our modified perturbative expansion. The coefficient $\bar{b}_i(n)$ can be obtained from the series (17). In the Gross-Neveu model, however, it is more convenient to substitute (27) into (29). From (18) and (29) we then have

$$\bar{\alpha} = x \left(1 - \sum_{r=2}^{n-1} \frac{\bar{b}_r(n)}{r-1} \bar{\alpha}^r \right). \quad (32)$$

Equation (32) is rewritten in terms of $\langle g\sigma \rangle$ by using (22). The result reads

$$\langle g\sigma \rangle = \alpha(\mu) \left(\frac{m}{n-1} - \left(\log \frac{m}{\mu} - \mathcal{C}(n) \right) \langle g\sigma \rangle - \sum_{r=2}^{n-1} \frac{\bar{b}_r(n)}{r-1} \left(\frac{n-1}{m} \right)^{r-1} \langle g\sigma \rangle^r \right). \quad (33)$$

This can be interpreted as an approximated self-consistent equation to be fulfilled by $\langle g\sigma \rangle$. The exact version of the self-consistent equation is represented in Fig.7. Thus we understand how the method of effective charges improves our previous result (17).

Then from diagrammatic consideration $\bar{b}_r(n)$ is extracted from $\sum_{r=0}^{n-1} \Gamma_r^{(r)}$ as

$$\bar{b}_r(n) = \frac{(-1)^r n-1 C_r}{(n-1)^r} \quad (r \geq 2), \quad n-1 C_r = \frac{(n-1)!}{r!(n-1-r)!}. \quad (34)$$

We thus obtain

$$\theta_n(\bar{\alpha}) = \frac{-1}{\bar{\alpha}^2} \left(\bar{\alpha} + \left(1 - \frac{\bar{\alpha}}{n-1} \right)^{n-1} \right), \quad (35)$$

$$m = \Lambda \exp \left(\mathcal{C}(n) + \int^{\bar{\alpha}} dz \theta(z) \right) = \Lambda \exp \left(\mathcal{C}(n) + \frac{1}{\bar{\alpha}} - \sum_{r=2}^{n-1} \frac{(-1)^r n-1 C_r}{(r-1)(n-1)^r} \bar{\alpha}^{r-1} \right), \quad (36)$$

$$\langle g\sigma \rangle = \frac{m}{n-1} \bar{\alpha} = \frac{\Lambda}{n-1} \bar{\alpha} \exp \left(\mathcal{C}(n) + \int^{\bar{\alpha}} dz \theta(z) \right), \quad (37)$$

and

$$\bar{\alpha} = x \left(1 - \sum_{r=2}^{n-1} \frac{(-1)^r n-1 C_r}{(r-1)(n-1)^r} \bar{\alpha}^r \right). \quad (38)$$

At $n = 2, 3$ and 4 , $\langle g\sigma \rangle$ is plotted as a function of $\bar{\alpha}$ in Fig.8.

Let us analyze the structure of the effective charge as a function of x or $\log(m/\Lambda)$. For example consider the case, $n = 3$, at which order the non-trivial structure begins to appear (The second order case just gives $\bar{\alpha} = x$). For $n = 3$ we have from (33), (34) and (38)

$$\bar{\alpha} = x \left(1 - \frac{1}{4} \bar{\alpha}^2 \right), \quad (39)$$

and

$$\langle g\sigma \rangle = \alpha(\mu) \left(\frac{m}{2} - \log \frac{m}{\mu} \langle g\sigma \rangle - \frac{2}{m} \langle g\sigma \rangle^2 \right). \quad (40)$$

From (40) and by recalling the definition of the expansion, we find that $\langle g\sigma \rangle$ at this order is given by the sum of diagrams which are constructed in all possible ways by the building blocks shown in Fig.9. We can solve (39) for $\bar{\alpha}$ to give

$$\bar{\alpha} = \begin{cases} 2(-\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}) = 2(-\log(m/\Lambda) + \sqrt{\log^2(m/\Lambda) + 1}) \equiv \bar{\alpha}_1(x) \\ 2(-\frac{1}{x} - \sqrt{\frac{1}{x^2} + 1}) = 2(-\log(m/\Lambda) - \sqrt{\log^2(m/\Lambda) + 1}) \equiv \bar{\alpha}_2(x) \end{cases} \quad (41)$$

and find that $\bar{\alpha}(x)$ is double-valued. The functions, $\bar{\alpha}_1$ and $\bar{\alpha}_2$, are positive and negative definite, respectively. For small x they are expanded as follows,

$$\bar{\alpha}_1(x) = \begin{cases} x + O(x^3) & (x > 0) \\ -4/x + O(x) & (x < 0), \end{cases} \quad (42)$$

$$\bar{\alpha}_2(x) = \begin{cases} -4/x + O(x) & (x > 0) \\ x + O(x^3) & (x < 0). \end{cases} \quad (43)$$

We note that, for negative small x , $\bar{\alpha}$ behaves in two ways as shown in (42) and (43). If we vary m from large enough region (where x is positive and small) in the branch $\bar{\alpha}_1$ to smaller region (in this course, x first grows to $+\infty$ and then decreases to zero from the negative side from $-\infty$), we reach the non-perturbative behavior as shown in the second line in (42). As expected before, we will see later that at higher orders small x ($x < 0$) behavior of positive $\bar{\alpha}$ is relevant for broken condensate.

The multi-valued structure mentioned above manifests itself in $\beta_3(\bar{\alpha}) = -\bar{\alpha}/(1 + \bar{\alpha}^2/4)$ as one can see below. For the positive part of $\bar{\alpha}$ which is described by $\bar{\alpha}_1(x(m))$, $\bar{\alpha} = 0^+$ is an ultraviolet fixed point and as m decreases $\bar{\alpha}$ goes to $+\infty$. For the negative part described by $\bar{\alpha}_2(x)$, $\bar{\alpha} = 0^-$ is an infra-red fixed point and as m increases $\bar{\alpha}$ goes to $-\infty$. We can thus see the multi-valued nature from $\beta(\bar{\alpha})$ without explicitly solving (39) for $\bar{\alpha}$.

5 EVALUATION OF THE CHIRAL CONDENSATE

Now, we turn to the evaluation of $\langle g\sigma \rangle$ by using PMS condition. In terms of $\bar{\alpha}$ PMS condition reads

$$\frac{d\langle g\sigma \rangle}{d\bar{\alpha}} = \frac{m(\bar{\alpha})}{n-1} (1 + \bar{\alpha} \theta_n(\bar{\alpha})) = 0. \quad (44)$$

From $m(\bar{\alpha}) = 0$, we find $\bar{\alpha} = 0^-$ and from $1 + \bar{\alpha} \theta_n(\bar{\alpha}) = 0$,

$$\bar{\alpha} = n-1. \quad (45)$$

The solution $\bar{\alpha} = 0^-$ clearly leads the un-broken solution, $\langle g\sigma \rangle = 0$. The rest solution, (45), gives the broken solution as follows: From (36) the corresponding m is calculated as

$$m = \Lambda \exp \left(\mathcal{C}(n) + \frac{1}{n-1} - \frac{1}{n-1} \sum_{r=2}^{n-1} \frac{(-1)^r n-1 C_r}{r-1} \right) = \Lambda, \quad (46)$$

where we used

$$\sum_{r=2}^{n-1} (-1)^r \frac{n-1}{r-1} C_r = (n-1) \sum_{r=1}^{n-2} \frac{1}{r} - (n-2). \quad (47)$$

And from (45) and (47), we obtain the exact answer for the broken solution,

$$\langle g\sigma \rangle = m \frac{\bar{\alpha}}{n-1} = \Lambda. \quad (48)$$

Note that the factor, $1/(n-1)$, is cancelled by optimal $\bar{\alpha}$ given by (45).

The location of the broken solution is found as follows: Since the optimal m is equal to Λ at every order, the optimal x , denoted as x^* , is

$$x^* = \frac{1}{-C(n)} = \frac{1}{1 - \sum_{r=1}^{n-2} 1/r}. \quad (49)$$

Except for the cases of $n = 2$ and $n = 3$ where $x^* = \pm\infty$, all of x^* are negative valued. And for large n , x^* behaves as $-1/\log(n)$ and goes to zero taking negative values in the $n \rightarrow \infty$ limit. This shows that x^* sits on the non-perturbative part in the branch of positive $\bar{\alpha}$, which part shows up only after the series sum.

6 DISCUSSION AND SUMMARY

In order to look into the structure of improved GHF expansion in more details, we discuss the following issues: Some large order aspects of our expansion, Supplementary condition of optimization and another possible expansion scheme.

6-1 Some large order aspects of GHF expansion

In this sub-section we discuss the higher order derivatives of $\langle g\sigma \rangle$, the correspondence between $\bar{\alpha}$ and m at large orders and the independence of the condensate on m in the $n \rightarrow \infty$ limit.

We recall that the effect of adding and subtracting the mass terms should disappear at large orders. This tendency would manifest such that the higher order derivatives at the stationary point decreases as the order increases. In order to confirm this point we discuss the behavior of $\langle g\sigma \rangle$ around the optimal $\bar{\alpha}$ at n -th order, denoted as $\bar{\alpha}^*(n-1)$. The higher order derivatives of the logarithm of $\langle g\sigma \rangle$,

$$\log \frac{\langle g\sigma \rangle}{\Lambda} = \log(n-1) + C(n) - \log \bar{\alpha} + \int_0^{\bar{\alpha}} dz \theta_n(z), \quad (50)$$

is given by

$$\frac{d^r}{d\bar{\alpha}^r} \log \frac{\langle g\sigma \rangle}{\Lambda} = \frac{(-1)^r (r-1)!}{\bar{\alpha}^r} + \frac{d^{r-1}}{d\bar{\alpha}^{r-1}} \theta_n. \quad (51)$$

At $\bar{\alpha}^*$, we find from (35)

$$\frac{d^r}{d\bar{\alpha}^r} \log \frac{\langle g\sigma \rangle}{\Lambda} \Big|_{\bar{\alpha}=\bar{\alpha}^*} = 0, \quad (r = 1, 2, \dots, n-1), \quad (52)$$

and hence

$$\frac{d^r \langle g\sigma \rangle}{d\bar{\alpha}^r} \Big|_{\bar{\alpha}=\bar{\alpha}^*} = 0 \quad (r = 1, 2, \dots, n-1). \quad (53)$$

Thus we find that, as the order n grows larger, the almost flat region for $\langle g\sigma \rangle$ around $\bar{\alpha}^*$ becomes broader. This tendency is a signal that we have chosen good perturbation expansion.

Although the growth of the almost flat region is thus shown it may be desirable to study the same issue directly in terms of m and how the condensate depends on m around the optimal value. As we found in the previous section, optimal value of m , $m^* = \Lambda$, corresponds to $\bar{\alpha}^*$ which increases to infinity. This indicates that the correspondence between the two variables becomes singular in the $n \rightarrow \infty$ limit. Therefore we first consider the relation between m and $\bar{\alpha}$ at large n .

Since for large n

$$C(n) = \log(n-1) - 1 + \gamma + O\left(\frac{1}{n}\right), \quad (54)$$

we find

$$m \sim (n-1)\Lambda \exp\left[-1 + \gamma + \frac{1}{\bar{\alpha}} + \int_0^{\bar{\alpha}} dz \frac{1}{z^2} \left(1 - z - \left(1 - \frac{z}{n-1}\right)^{n-1}\right)\right]. \quad (55)$$

Since $\lim_{n \rightarrow \infty} \left(1 - \frac{z}{n-1}\right)^{n-1} = \exp(-z)$ for fixed z and the integral converges, we find that any finite $\bar{\alpha}$ corresponds to large $m \sim n\Lambda \cdot \exp[O(1)]$. In other words, only $\bar{\alpha}$ which makes the exponential factor small as $\sim 1/n$ gives finite m . The neighborhoods of infra-red fixed points of $\bar{\beta}$ are therefore relevant for finite m . Since

$$\lim_{n \rightarrow \infty} \bar{\beta}_n = \frac{-\bar{\alpha}^2}{\bar{\alpha} + e^{-\bar{\alpha}}} \equiv \bar{\beta}_\infty \equiv \frac{1}{\theta_\infty}, \quad (56)$$

we find that, in the $n \rightarrow \infty$ limit, infra-red fixed points are given by $\bar{\alpha} = 0^-, +\infty$. Then we show that, in the $n \rightarrow \infty$ limit, the exact broken solution comes out for all

m by examining the $\bar{\alpha} \rightarrow \infty$ limit as follows: In the $n \rightarrow \infty$ limit the condensate becomes

$$\langle g\sigma \rangle = \Lambda \exp(-1 + \gamma) \bar{\alpha} \exp\left(\frac{1}{\bar{\alpha}} + \sum_{r=1}^{\infty} \frac{(-1)^r \bar{\alpha}^r}{r(r+1)!}\right) \quad (57)$$

$$= \begin{cases} \Lambda \bar{\alpha} \exp(-\log \bar{\alpha} + e^{-\bar{\alpha}}/\bar{\alpha} - \int_{\bar{\alpha}}^{\infty} dt e^{-t}/t) & (\bar{\alpha} > 0) \\ \Lambda \bar{\alpha} \exp(\gamma + e^{-\bar{\alpha}}/\bar{\alpha} + \int_0^{-\bar{\alpha}} dt (e^t - 1)/t) & (\bar{\alpha} < 0). \end{cases} \quad (58)$$

Then noting that, for positive $\bar{\alpha}$ all finite m map to $+\infty$ of $\bar{\alpha}$, we find the corresponding value of the condensate without using PMS condition,

$$\lim_{\bar{\alpha} \rightarrow \infty} \langle g\sigma \rangle = \Lambda. \quad (59)$$

This indeed tells us the m independence of the condensate.

Now we express $\langle g\sigma \rangle$ directly in terms of m around the optimal value m^* . We will find that $\langle g\sigma \rangle$ admits a simple expansion in terms of $\log(m/\Lambda)$. We begin by rewriting (29) as

$$\begin{aligned} m &= \Lambda \exp \int_{n-1}^{\bar{\alpha}} dz \theta_n(z) \\ &= \Lambda \exp\left(-\log \xi - \frac{1}{n-1} \int_1^{\xi} dw \frac{(1-w)^{n-1}}{w^2}\right). \end{aligned} \quad (60)$$

where

$$\xi = \frac{\bar{\alpha}}{n-1}. \quad (61)$$

Note that, by shifting the reference point of the integral to $n-1$, $C(n)$ is exactly absorbed into the integral. The chiral condensate is then written as

$$\langle g\sigma \rangle = \Lambda \exp\left(-\frac{1}{n-1} \int_1^{\xi} dw \frac{(1-w)^{n-1}}{w^2}\right). \quad (62)$$

The integral in the exponent exists in the $n \rightarrow \infty$ limit only for

$$\xi < 2. \quad (63)$$

This gives the lower bound for m as follows: By substituting $\xi = 2$ into (60) we find $m(\xi = 2) \rightarrow (1/2)\Lambda$ ($n \rightarrow \infty$) and therefore on convergence grounds of $\langle g\sigma \rangle$ we obtain

$$m > \frac{1}{2} \Lambda. \quad (64)$$

We do not know whether this restriction is removed or not if some other expansion schemes are applied. Now we change the variable, ξ , to η as,

$$\xi = 1 - \eta, \quad (65)$$

which leads

$$\begin{aligned} m &= \frac{\Lambda}{1-\eta} \exp\left(\frac{1}{n-1} \int_0^{\eta} dv \frac{v^{n-1}}{(1-v)^2}\right) \\ &= \frac{\Lambda}{1-\eta} \left(1 + \frac{\eta^n}{(n-1)n} + O(\eta^{n+1})\right) \\ &= \Lambda(1 + \eta + O(\eta^2)). \end{aligned} \quad (66)$$

The chiral condensate is then given as

$$\langle g\sigma \rangle = m\xi = \Lambda \left(1 + \frac{\eta^n}{(n-1)n} + O(\eta^{n+1})\right). \quad (67)$$

Since we obtain by taking the logarithm of (66) and inverting the series,

$$\eta = \log \frac{m}{\Lambda} + \frac{1}{2} \log^2 \frac{m}{\Lambda} + \dots, \quad (68)$$

we arrive from (67) at

$$\langle g\sigma \rangle = \Lambda \left(1 + \frac{1}{(n-1)n} \log^n \left(\frac{m}{\Lambda}\right) + O(\log^{n+1} \frac{m}{\Lambda})\right). \quad (69)$$

This shows the behavior of $\langle g\sigma \rangle$ around $m = \Lambda$. In particular it is quite transparent that the higher order derivatives at the stationary point tends to vanish successively as n increases.

6-2 Supplementary condition in the optimization

As seen in the previous sections, our expansion scheme with PMS condition produces good results to all orders. Our approach, however, does not allow straightforward error estimation for a few lowest order ansatz due to the absence of the small expansion parameter. For general field theories where the answers are not known, we consider this problem in the Gross-Neveu model as an example and suggest a supplementary condition which would help us to measure the reliability. In the Gross-Neveu model, the condensate $\langle g\sigma \rangle$ has connection with the dynamical fermion mass m_D such as

$$m_D = m + (-m + \langle g\sigma \rangle) \equiv m + \Delta m \quad (70)$$

where the first and second terms represent the free-level contribution and the radiative correction, respectively (see Fig.10). Note the distinction of the zero-th order m from $-m$ contained in the first order contribution. The radiative correction is given by the perturbation expansion as

$$\Delta m = d_{1+2} + d_3 + d_4 + \dots, \quad (71)$$

where d_{1+2} and d_n ($n = 3, 4, \dots$) denote respectively the first plus the second order corrections and n -th order correction³. In addition to the inequality, $m > d_{1+2} > d_3 > \dots$ at the optimal m , which should hold for reliable results, we like to add the following condition.

Let the dynamical mass be $M_D^* = M_D(m^*)$ at m^* . Then if M_D^* approximates well the true mass, the radiative correction Δm evaluated at $m = M_D^*$, should be small compared with M_D^* . If not we interpret that our expansion applied is unstable and the optimized ansatz is hard to be relied upon. In other words, we require that the optimal dynamical mass M_D^* under PMS condition should be close to the solution in the gap condition⁴,

$$M_D = m \quad (72)$$

or

$$\Delta m = 0. \quad (73)$$

Hence, for the approximate fulfillment of (73), we put the condition,

$$\left. \frac{\Delta m}{m} \right|_{m=M_D^*} < 1. \quad (74)$$

This condition is in accord with GHF spirit mentioned in the introduction that the added terms represents the effect of a given non-linear interaction and the subtracted terms in turn reduce the effect.

The optimal solution in section 4 belongs to idealistic case because the gap condition also gives $\bar{\alpha} = n - 1$ and hence the results of the evaluation due to PMS

³The first order correction is given by the sum of the $(-m)$, which corresponds to the second diagram in Fig.10, and the diagram in Fig.2 in which the ultra-violet pole exists. We therefore keep the first and second order corrections combined.

⁴If the expansion can be carried out to several higher orders in general field theories, the smallness of the higher order derivatives at the stationary point would be used as a signature of good expansion scheme. If the higher order derivatives are small enough, the dynamical masses at m^* and at m_{gap} in the gap condition are close with each other as long as the two values of m are not far apart.

and gap conditions exactly agree with each other. This fact shows the particular appropriateness of our choice of the expansion scheme.

6-3 Another choice of expansion scheme

GHF approach turns, in a rough sense, the ordinary single series into the double series and therefore various choices would be possible in assigning the single order to each Feynman diagram which is labelled by the numbers of $\sigma\bar{\psi}\psi$ and mass vertices. Even if we impose the condition that the preservation of the renormalizability and the maximal μ invariance, GHF framework may allow other expansion schemes. In this section we like to discuss one interesting expansion scheme of them which is worth noting. This expansion is based on the result that $\langle g\sigma \rangle$ formulated with $\bar{\alpha}$ allows the naive $n \rightarrow \infty$ limit (see (31)),

$$\langle g\sigma \rangle = \Lambda \bar{\alpha} \exp\left(-1 + \gamma + \int^{\bar{\alpha}} dz \theta_{\infty}\right). \quad (75)$$

Then the whole task with us is to calculate θ_{∞} and optimize the condensate by PMS condition. Now the expansion we are interested in is such as to approximate $\langle g\sigma \rangle$ by expanding $\theta_{\infty} = -(\bar{\alpha} + \exp(-\bar{\alpha}))/\bar{\alpha}^2$ in the usual manner as $-1/\bar{\alpha}^2, -(1 + \bar{\alpha}^2/2)/\bar{\alpha}^2, \dots$. Let us look this scheme starting with the original perturbation series, (17). Since the naive $n \rightarrow \infty$ limit means that mass insertions are incorporated to all orders in fermion loops, finite $\bar{\alpha}$ leads to infinite m as we found in a previous discussion. Thus, by taking the limit, $m, n \rightarrow \infty$ with $m/(n-1)$ fixed finite, we find $\lim_{n \rightarrow \infty} c_i(n)$ from the expansion (17) and can obtain the coefficients, $\lim_{n \rightarrow \infty} b_i(n)$ of θ . It would be surprising that by staying such an ultraviolet regime for m we can obtain satisfactory results under PMS condition:

$$\begin{aligned} \theta &= \frac{-1}{\bar{\alpha}^2} & \langle g\sigma \rangle &= \Lambda \cdot 1.781 \dots \text{ at } \bar{\alpha} = 1 \\ \theta &= \frac{-1}{\bar{\alpha}^2} \left(1 + \frac{1}{2}\bar{\alpha}^2 - \frac{1}{6}\bar{\alpha}^3\right) & \langle g\sigma \rangle &= \Lambda \cdot 1.089 \dots \text{ at } \bar{\alpha} = 1.596 \dots \\ \theta &= \frac{-1}{\bar{\alpha}^2} \left(1 + \frac{1}{2}\bar{\alpha}^2 - \frac{1}{6}\bar{\alpha}^3 + \frac{1}{24}\bar{\alpha}^4 - \frac{1}{120}\bar{\alpha}^5\right) & \langle g\sigma \rangle &= \Lambda \cdot 1.024 \dots \text{ at } \bar{\alpha} = 2.1806 \dots \end{aligned} \quad (76)$$

As the order increases the optimal $\bar{\alpha}$ increases to ∞ and the corresponding value of $m/(n-1) = \Lambda \exp(-1 + \gamma + \int^{\bar{\alpha}} dz \theta(z))$ becomes smaller as, 1.781 $\dots$, 0.682 $\dots$, 0.469 $\dots$, and tends to zero as the order further increases to infinity, as it should.

The success of this expansion procedure depends on the convergence radius of $\int_0^{\bar{\alpha}} dz \theta_{reg}(z)$. In our case $\int_0^{\bar{\alpha}} dz \theta_{reg}(z)$ has infinite convergence radius and θ shows the rapid approach to the asymptotic form, $\theta \sim -1/\bar{\alpha}$. This is the reason of stable and accurate sequence of the optimized condensate even though we truncated higher terms of θ and the optimal $\bar{\alpha}$ increases to infinity. We do not know why the convergence property of $\langle g\sigma \rangle$ becomes so good when it is formulated in terms of $\bar{\alpha}$. In fact the series (17) reaches in the $n \rightarrow \infty$ limit to

$$\begin{aligned} \langle g\sigma \rangle = & \frac{m}{n-1} x \left(1 - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{35}{72}x^4 - \frac{33}{160}x^5 - \frac{1013}{1800}x^6 + \frac{25531}{60480}x^7 \right. \\ & \left. + \frac{8685107}{12700800}x^8 - \frac{11731999}{14515200}x^9 + O(x^{10}) \right), \end{aligned} \quad (77)$$

and the radius seems $O(1)$ or zero. This point, the drastic improvement of the convergence property, is important and deserves further investigation. For theories where the convergence radius is finite this procedure might give sequence with less stable behavior. We can obtain precise result, though, if $\int_0^{\bar{\alpha}} dz \theta_{reg}(z)$ for a given theory reaches fast enough to its asymptotic behavior within the convergence radius.

To summarize we have succeeded to obtain the broken solution for the chiral condensate based on GHF expansion with improvement by the method of effective charges. By changing the un-perturbed state from massless to massive one, functional dependence of the condensate on the coupling constant have changed and we are able to approach the non-analytic condensate via analytic series with a redundant parameter, m or $\bar{\alpha}$. After the summation by the method of effective charges, the use of PMS condition recovered the true g dependence with accurate numerical result. The recovery of the true non-analytic g dependence is due to that m is independent of μ and therefore it is fixed necessarily of the form, $\mu \exp(-\pi/g^2)$. Thus, GHF scheme would supply a good framework to calculate non-perturbative dimensionfull quantities such as the dynamical mass and various condensates.

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Figure captions

- Fig.1 The vertex due to the incorporated mass term, $m\bar{\psi}\psi$, in the interaction Lagrangian.
- Fig.2 A set of diagrams of three point functions with one mass vertex.
- Fig.3 A tad pole diagram of σ which has the un-renormalizable ultraviolet divergence.
- Fig.4 A tad pole diagram of σ with one mass vertex which has the un-renormalizable ultraviolet divergence.
- Fig.5 Dressed σ propagator which is represented by the thick line. Shaded diagram represents the self-energy of σ with no tad pole contribution.
- Fig.6 The condensate $\langle g\sigma \rangle$ is plotted as a function of m/Λ at the second order.
- Fig.7 Diagrammatic representation of self-consistent equation to be fulfilled by the condensate $\langle g\sigma \rangle$.
- Fig.8 As explicit examples, $\langle g\sigma \rangle$ at $n = 2, 3$ and 4 is shown as functions of $\bar{\alpha}$.
- Fig.9 1-P-I diagrams of σ to the third order of expansion.
- Fig.10 The dynamical fermion mass m_D is represented in terms of Feynman diagrams. The first diagram represents the inverse free fermion propagator.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig.8

Fig.9

$$m_D = \left(\text{---} \cdot \right)^{-1} + \text{---} \bullet \text{---} + \text{---} \overset{\circ}{\underset{\circ}{\circ}}$$

Fig. 10