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A FOUR-FERMION LATTICE MODEL

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S. Randjbar-Daemi

and

J. Strathdee

MIRAMARE-TRIESTE

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

A FOUR-FERMION LATTICE MODEL

S. Randjbar-Daemi and J. Strathdee
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The weak coupling limit of an Ising-type model on the F_4 lattice is examined. It is shown that by imposing some constraints on the Ising couplings, the weak coupling limit can be expressed as a multidimensional Berezin integral with local symmetries. We explore the possibility of deriving a fermion propagator by a self-consistent, Nambu, Jona-Lasinio type of calculation. We argue that at low energies this model describes Dirac fermions coupled to Yang-Mills fields.

MIRAMARE - TRIESTE

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1 Introduction

It is now recognized that local quantum field theories are inevitable in low energy physics: they reconcile quantum mechanics with special relativity. The renormalizable theories are probably the only consistent ones because they are largely insensitive to short distance structure. If we adopt the view that relativistic invariance is not fundamental but merely an approximate feature of the low energy sector, then it follows that little can be deduced about fundamental physics – short distance structure – by studying local quantum field theory. Presumably, to each QFT there must correspond a large class of models whose low energy sectors are identical. For example, any regularized version of the QFT in question would be a member of the class, though perhaps not a very attractive one if it violates general principles such as unitarity or causality. Our view is that, to explore this class it may be instructive to begin with a relatively primitive but well defined model and work backwards, i.e. try to extract its low energy sector in the form of a renormalizable local quantum field theory.

In this paper we examine the weak coupling limit of an Ising-type model defined on a 4-dimensional integral lattice [1]. The dynamical variable is a 2-valued “spin” at the lattice sites and the model is parametrized by an array of coupling constants associated with pairs of sites. A number of restrictions are imposed on the couplings so that the model acquires both a crystal structure and a local symmetry. The weak coupling limit is conveniently expressed in terms of Grassmann variables where it takes the form of a 4-fermion interacting system with a crystal structure and a continuous local symmetry. The model is described in Section 2.

Our main assumption is that the fermions form a condensate whose invariance group involves an embedding of the crystal point group into the local symmetry. The excitations of the condensate will then simulate particles with spin [2]. In the long wavelength limit they will be identified as Dirac fermions and Yang-Mills bosons. Observable features such as fermion masses and gauge couplings are, at least in

principle, computable functions of ratios of the Ising model coupling parameters. The number of such parameters is not limited by any deep principle and the model therefore has no predictive power. Its interest lies, we believe, in the qualitative description it provides for quantum numbers such as spin and colour. Spin, in particular, is viewed as a dynamical effect associated with the formation of a condensate (Section 4).

We have no exact way to compute the condensate amplitude (fermion Green's function). This notorious problem is treated at best as a self-consistent calculation of the Hartree-Fock type. In an attempt to make this more systematic we consider the construction of a generalized effective action in the form of a loop expansion. The fermion Green's function is counted among the independent variables in this functional, and the extremal conditions provide a set of "gap equations" for its determination. If the loop expansion is truncated at the 2-loop level then the resulting gap equations are obtained in the Hartree-Fock approximation. Higher loop corrections to the Hartree-Fock equations are easily obtained from the generalized action but, of course, we have no idea about non-perturbative contributions to this functional. However, it is termwise gauge invariant so that any truncation will at least be compatible with the local symmetry. The generalized effective action is defined in Section 3.

In the effective action formulation the fermion Green's function is a dynamical variable in its own right [3]. Fluctuations of this Green's function above its condensate value therefore represent bosonic excitations of the condensate. Among these we expect to find Yang-Mills particles associated with any unbroken local symmetries [4]. It is relatively easy to identify these by choosing a covariant parametrization of the long wavelength fluctuations in terms of gauge vectors defined on the continuum. This will be illustrated in Section 5 where the standard Yang-Mills Lagrangian is obtained in the Hartree-Fock approximation.

Our approach should be viewed as a generalization of the well known Nambu, Jona-Lasinio mechanism for the dynamical generation of fermion mass through the

spontaneous breaking of chiral symmetry [5]. In our scheme not only mass but also kinetic energy, the entire Green's function, is supposed to be generated. Based as it is, on a lattice model, our fermions are, of course, not chiral [6].

We also wish to offer a speculation about gravity. It is tempting to believe that the coupling parameters of the Ising-type models are not arbitrary parameters but, rather, the ground state values of a dynamical variable, a bilocal field defined over pairs of lattice sites. In the present work this field is "external", i.e. fixed. However, if one allows it to transform, along with the Grassmann variables, as a scalar, then the model acquires a very large symmetry. The action is invariant with respect to permutations of the lattice sites, a discrete analogue of spacetime diffeomorphisms. The long wavelength sector ought therefore to exhibit general covariance if the coupling parameters and Green's functions could be suitably parametrized by an external gravitational field. Our attempts in this direction have been only partially successful because of the need to introduce a set of frame vectors, rather than a metric tensor, to represent the long wavelength gravitational fluctuations. Unfortunately, the local frame rotations are restricted to a discrete subgroup of $O(4)$ and this feature seems to persist in the continuum limit. General relativity is not fully recovered [7].

We conclude that the Ising-type lattice model is an example of a system whose long wavelength behavior is semi-realistic in the sense that it corresponds to a renormalizable local quantum field theory of Dirac fermions coupled to Yang-Mills vectors. Conceptually, this model could be regarded as more fundamental in the sense that the properties of the QFT are derived from it. It fails to be entirely realistic, however, in that it yields Dirac fermions rather than chiral fermions. It also fails, at least up to now, to provide a satisfactory description of the gravitational interaction. The latter problem is, we believe, a technical one that ought to be overcome. The former is probably fundamental and indicates that renormalizable chiral theories are not derivable from lattice models [6].

2 The model

We start with a class of Ising-type or lattice gas [8] models defined on a 4-dimensional integral lattice [1]. To each lattice site, A , there is assigned a 2-valued variable, $S_A = 0, 1$. The partition function is

$$Z(\beta J) = \sum_{\{S\}} \exp \left(-\frac{\beta}{2} \sum_{A,B} S_A J_{AB} S_B + i\pi \sum_A S_A \right) \quad (2.1)$$

where the couplings, J_{AB} , are to be specified. The sign factor is included in order to facilitate the computation of the weak coupling (or high temperature) limit, $\beta J \rightarrow 0$. A simple way to represent this limit as a multiple integral of the Berezin type is obtained as follows. Introduce a set of auxiliary real variables, ϕ_A , to write

$$\begin{aligned} Z(\beta J) &= \sum_{\{S\}} (\det 2\pi\beta J)^{-1/2} \prod_A \int_{-\infty}^{\infty} d\phi_A \cdot \\ &\quad \cdot \exp \left(-\frac{1}{2\beta} \sum \phi_A J_{AB}^{-1} \phi_B + i\sum (\phi_A + \pi) S_A \right) \\ &= (\det 2\pi\beta J)^{-1/2} \prod_A \int_{-\infty}^{\infty} d\phi_A (1 - e^{i\phi_A}) \exp \left(-\frac{1}{2\beta} \sum \phi_A J_{AB}^{-1} \phi_B \right) \\ &= \prod_A (1 - e^{-\partial_A}) \exp \left(-\frac{\beta}{2} \sum \xi_A J_{AB} \xi_B \right) \Big|_{\xi=0} \end{aligned}$$

where $\partial_A = \partial/\partial\xi_A$. Each differentiation yields a factor, βJ . Hence, the leading term is

$$\left(\prod_A \partial_A \right) \exp \left(-\frac{\beta}{2} \sum \xi_A J_{AB} \xi_B \right) \Big|_{\xi=0}$$

which is non-vanishing if the number of sites, $2n$, is even. It takes the form of a sum over all possible pairings of sites and it can be represented by an integral over Grassmann variables. Take the limit $\beta \rightarrow 0$ to define

$$\begin{aligned} Z_0(J) &= \lim_{\beta \rightarrow 0} (\beta^{-n} Z(\beta J)) \\ &= \int (d\psi d\bar{\psi}) \exp \left(-\frac{1}{2} \sum \bar{\psi}_A \psi_A J_{AB} \bar{\psi}_B \psi_B \right) \end{aligned} \quad (2.2)$$

We shall assume henceforth that the number of sites is infinite.

The weak coupling limit of the Ising type models is now described by fermionic

systems with the Euclidean action

$$S(\psi, \bar{\psi}) = \frac{1}{2} \sum_{A,B} \bar{\psi}_A \psi_A J_{AB} \bar{\psi}_B \psi_B \quad (2.3)$$

Our main objective is to learn something about the ground state of such a system and, in particular, whether there can be a non-vanishing condensate,

$$C_{AB} = \langle \psi_A \bar{\psi}_B \rangle$$

The first step is to impose some structure on the system by restricting the coupling parameters. We suppose that our lattice is composed of 4-dimensional cells labelled by the integers, n , and that within each cell there are N sites. We denote the variables ψ_A by $\psi_i(n)$ where $i = 1, 2, \dots, N$ and $n \in \mathbb{Z}^4$. We suppose further that the couplings are translation invariant and that they do not distinguish among the sites within a cell, i.e.

$$J_{AB} = J_{ij}(n, m) = J(n - m) \quad (2.4)$$

The action (2.3) becomes

$$S(\psi, \bar{\psi}) = \frac{1}{2} \sum_{n,m} \bar{\psi}(n) \psi(n) J(n - m) \bar{\psi}(m) \psi(m) \quad (2.5)$$

where $\bar{\psi}(n) \psi(n) = \sum_i \bar{\psi}_i(n) \psi_i(n)$. By virtue of the restrictions (2.4) the model acquires both translation invariance and a local symmetry, $GL(N, \mathcal{F})$. This local symmetry is in part an artifact of the Grassmann representation. In terms of the original Ising variables, $S_A = S_i(n)$, we have

$$Z(\beta J) = \sum_{\{S\}} \exp \left(-\frac{\beta}{2} \sum_{n,m} \left(\sum_i S_i(n) \right) J(n - m) \left(\sum_j S_j(m) \right) + i\pi \sum_{i,n} S_i(n) \right)$$

which exhibits only a local permutation symmetry, S_N .

In order to further limit the number of independent couplings we shall suppose that they are invariant with respect to some space group that acts on the lattice. That is, in addition to the translations there is a point group, H , such that

$$J(nh) = J(n), \quad h \in H \quad (2.6)$$

where H is a subgroup of $GL(4, \mathbb{Z})$, i.e. h is an integer-valued 4×4 matrix satisfying $\det h = \pm 1$. In Section 4 we shall argue that in order to obtain realistic fermions it will be necessary to assume that H is one of the 4-dimensional crystallographic point groups.

3 Gap equations

To show that the system described by the Euclidean action (2.5) actually includes propagating fermions it would be necessary to solve some kind of generalized gap equation to find the propagator. Our aim here is to set up a system of such equations in a semi-systematic way and discuss some of their features. Although we are not able to solve them in even the simplest (Hartree-Fock) approximation, their overall structure is not difficult to describe and is, we believe, of some general interest. (A toy version is solved in the appendix.)

Our approach to the problem of self-consistent calculation relies on the old idea of “generalized” effective action [3]. In this approach the equations for 2-point Green’s functions are obtained as extremal conditions on an action functional that depends on the Green’s functions as well as the fields. This functional is represented by a loop expansion of 2-particle irreducible vacuum graphs in which the lines are associated with the unknown Green’s functions.

It is convenient to replace the purely fermionic action (2.5) by an equivalent one involving a real scalar, $\sigma(n)$, given by

$$S(\sigma, \psi, \bar{\psi}) = -\frac{1}{2} \sum_{n,m} \sigma(n) J^{-1}(n-m) \sigma(m) + \sum_n \sigma(n) \bar{\psi}(n) \psi(n), \quad (3.1)$$

where J^{-1} is the matrix inverse to J ,

$$\sum_m J^{-1}(n-m) J(m-\ell) = \delta_{n\ell}.$$

It is clear that (3.1) reduces to (2.5) when $\sigma(n)$ is eliminated by solving the variational equation, $\delta S / \delta \sigma(n) = 0$.

The generalized effective action $\Gamma(\sigma, \Delta, G)$ is defined by [3]

$$\begin{aligned} \Gamma = & -\frac{1}{2} \sum_{n,m} J^{-1}(n-m) (\sigma(n) \sigma(m) + \Delta(n, m)) - \frac{1}{2} \ell n \det \Delta \\ & + \ell n \det G - \sum_n \sigma(n) \text{tr} G(n, n) + \Gamma_2(\Delta, G) \end{aligned} \quad (3.2)$$

where Γ_2 is represented by a series of 2-particle irreducible vacuum graphs with two or more loops. The Feynman rules are

$$\begin{aligned} \bullet \text{---} \bullet &= \Delta(n, m) \\ \bullet \text{---} \bullet &= G_{ij}(n, m) \\ \bullet \text{---} \bullet &= \delta_{\ell n} \delta_{\ell m} \delta_{ij} \end{aligned} \quad (3.3)$$

Associate the usual numerical factors for vacuum graphs and the sign factor $(-1)^{L+1}$ where L is the number of fermion loops. There is only one 2-loop graph

$$\text{Diagram: a circle with two vertices and two internal lines (one solid, one dashed)} = \frac{1}{2} \sum_{n,m} \text{tr}(G(n, m) G(m, n)) \Delta(n, m) \quad (3.4)$$

and one 3-loop graph

$$\text{Diagram: a circle with three vertices and three internal lines (one solid, one dashed, one solid)} = \frac{1}{8} \sum_{n_1, \dots, n_4} \text{tr}(G(n_1, n_2) G(n_2, n_3) G(n_3, n_4) G(n_4, n_1)) \Delta(n_1, n_3) \Delta(n_2, n_4)$$

The graphs begin to proliferate in higher orders and the expansion for Γ_2 therefore has a limited usefulness. In effect we have an approximate expression for Γ , valid only where one or other, or both, of the Green’s functions Δ and G is small.

The more familiar effective action, $\Gamma_1(\sigma)$, the generating functional for 1-particle irreducible vertices, is obtained from (3.2) by using the variational equations, $\delta \Gamma / \delta \Delta = 0$ and $\delta \Gamma / \delta G = 0$, to eliminate the Green’s functions. Imposing $\delta \Gamma / \delta \sigma = 0$ as well would yield the ground state value $\langle \sigma \rangle$ and the corresponding Greens functions,

$$\Delta(n, m) = \langle \sigma(n) \sigma(m) \rangle \quad \text{and} \quad G_{ij}(n, m) = \langle \psi_i(n) \bar{\psi}_j(m) \rangle$$

To determine the condensate and its properties one must therefore solve the simultaneous equations,

$$\sigma(n) = - \sum_m J(n-m) \text{tr} G(m, m)$$

$$\begin{aligned}
\Delta^{-1}(n, m) &= -J^{-1}(n - m) + \frac{\delta\Gamma_2(\Delta, G)}{\delta\Delta(n, m)} \\
G_{ij}^{-1}(n, m) &= \sigma(n) \delta_{nm} \delta_{ij} - \frac{\delta\Gamma_2(\Delta, G)}{\delta G_{ji}(m, n)}
\end{aligned} \tag{3.5}$$

A systematic approach to this problem would be to replace J by J/N , assume $G_{ij} = \delta_{ij}G$, and examine the large N behaviour. It is easy to see that Δ vanishes like $1/N$ while G and σ remain finite. The effective action is dominated by 1-loop terms while Γ_2 contributes small corrections. In this case the loop expansion is justified and the solution is probably valid. This regime is not very interesting, however, since it gives $G(n, m) \sim \delta_{n, m}$ in each order of the expansion, so that the fermions fail to propagate.

Another approach, which is interesting but not very systematic, would be to truncate the loop expansion for Γ_2 . The Hartree-Fock approximation, for example, is obtained by including only the 2-loop contribution (3.4). In this approximation, assuming translation invariance, the equations (3.5) reduce to

$$\begin{aligned}
\sigma &= -\sum_s J(s) \text{tr} G(0) \\
\Delta^{-1}(s) &= -J^{-1}(s) + \text{tr}(G(s) G(-s)) \\
G^{-1}(s) &= \sigma \delta_{s,0} - \Delta(s) G(s)
\end{aligned} \tag{3.6}$$

where $s = n - m$, $G(n, m) = G(s)$, etc.

In a previous report we proposed, on the basis of a semi-classical and even less systematic approximation, the relatively simple gap equations,

$$G^{-1}(s) = J(s) G(s) \tag{3.7}$$

These equations do not agree with (3.6) unless we impose the constraint, $\Sigma J(s) = 0$ and discard $\text{tr} GG$ from the equation for Δ^{-1} . Probably neither of the equations (3.6) (3.7) should be taken very seriously since they both involve uncontrolled approximations. Furthermore, in a complete treatment the effects of gauge fixing should be included. Nevertheless, we believe it may be worthwhile to study such equations as a qualitative guide. For example, it would be encouraging to learn that

there is a choice of couplings, $J(s)$, for which the equations (3.6) or (3.7) yield a physically reasonable Green's function. In the following we shall simply assume that this is true. A simple example is discussed in the appendix.

4 Ground state symmetry

The generalized effective action (3.2) is invariant with respect to a space group that acts on the lattice,

$$n \rightarrow n' = nh + b, \tag{4.1}$$

where h belongs to a point group $H \subset GL(4, \mathbb{Z})$ and $b \in \mathbb{Z}^4$. The independent variables σ, Δ and G transform as scalars,

$$\begin{aligned}
\sigma'(n') &= \sigma(n), \\
\Delta'(n', m') &= \Delta(n, m), \\
G'(n', m') &= G(n, m),
\end{aligned} \tag{4.2}$$

and the couplings are invariant,

$$J(n' - m') = J(n - m) \tag{4.3}$$

In addition there is a local invariance, $GL(N, \mathcal{T})$, with respect to which σ and Δ are neutral while G transforms according to

$$G'(n, m) = e^{i\theta(n)} G(n, m) e^{-i\theta(m)}, \tag{4.4}$$

where $\theta(n)$ is an $N \times N$ matrix.

The equations that determine the ground state, the gap equations (3.5), are of course covariant under the transformations (4.2) and (4.4). We shall suppose that some of them are spontaneously broken, i.e. that the ground state, or condensate, is invariant under a more restricted group where the transformations (4.2) and (4.4) act jointly. We assume that the solution of the gap equations determines a homomorphism, $H \rightarrow GL(N, \mathcal{T})$, such that

$$\Delta(sh) = \Delta(s),$$

$$G(sh) = a(h)^{-1} G(s) a(h), \quad (4.5)$$

where $a(h) \in GL(N, \mathbb{T})$. If such a solution could be found it would imply not only that the fermions propagate, but that they belong to non-trivial representations of the ground state symmetry group, they carry spin.

In order that the fermion spectrum be reasonable, i.e. comprise particles with spin 1/2, it is necessary that the invariance group of the ground state should include, as a factor, some discrete subgroup of $\text{spin}(4)$, the covering group of $\text{SO}(4)$. We therefore require that the point group, H , itself be contained in $\text{O}(4)$. This is a strong requirement. It implies that our lattice must be a 4-dimensional crystal. There are 227 crystal groups in four dimensions [9] but, to be specific, we choose the largest one. It has 1152 elements and coincides with the Weyl group of the root lattice of the exceptional Lie algebra, F_4 . (The corresponding lattice is often referred to as the “ F_4 lattice” [10].)

The invariance conditions (4.3) and (4.5) give significant restrictions. Note that the action of the point group serves to partition the lattice into “orbits”. Any point on an orbit can be mapped to any other by the action of some element of H . The number of points on an orbit can be as large as $|H|$, the number of elements in H , but it can be smaller if there is a non-trivial subgroup of H that leaves a point fixed. The couplings $J(s)$ are equal for all points on the same orbit and the Green’s function $G(s)$ is fully specified if it is known at one point on each orbit. (If the couplings are assumed to vanish except for a limited number of orbits, nearest neighbor, next nearest neighbor, etc., then the simplest version (3.7) of the gap equations will reduce to a finite set of equations determining $G(s)$ in terms of these parameters.)

The F_4 lattice comprises the points, $n^\mu \underline{b}_\mu$, with integer coefficients where the basis vectors, $\underline{b}_\mu$, are expressible in the orthonormal basis, $\underline{e}_\mu$, by

$$\underline{b}_1 = \underline{e}_1 - \underline{e}_2, \quad \underline{b}_2 = \underline{e}_2 - \underline{e}_3, \quad \underline{b}_3 = \underline{e}_3 - \underline{e}_4, \quad \underline{b}_4 = 2\underline{e}_4 \quad (4.6)$$

Equivalently, this lattice comprises the vectors $x^\mu \underline{e}_\mu$ with integer coefficients subject

to the constraint, $\Sigma x_\mu = \text{even integer}$. The squared length $\Sigma x^\mu x_\mu$ is necessarily an even integer. Some nearby orbits are

1) Nearest neighbors: 24 points with $\underline{s}^2 = 2$

$$\underline{s} = \pm \underline{e}_\mu \pm \underline{e}_\nu, \quad \mu \neq \nu$$

2) Next nearest neighbors: 24 points with $\underline{s}^2 = 4$

$$\underline{s} = \pm 2\underline{e}_\mu, \quad \pm \underline{e}_1 \pm \underline{e}_2 \pm \underline{e}_3 \pm \underline{e}_4$$

3) Next next nearest: 96 points with $\underline{s}^2 = 6$

$$\underline{s} = \pm 2\underline{e}_\mu \pm \underline{e}_\nu \pm \underline{e}_\lambda, \quad \mu \neq \nu \neq \lambda$$

The subgroup H_0 of transformations with $\det h = +1$ contains 576 elements. Its covering group, H'_0 , has two inequivalent 2-component spinor representations. These can be combined into a Dirac spinor that transforms according to

$$\psi \rightarrow a\psi, \quad \bar{\psi} \rightarrow \bar{\psi}a^{-1}$$

where

$$a = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \quad (4.7)$$

with u_1 and u_2 belonging to the spinor representation of the 3-dimensional octahedral group, O' . They can be expressed as products of two generators,

$$\alpha = \frac{1}{\sqrt{2}}(1 + i\sigma_3) \quad \text{and} \quad \beta = \frac{1}{2}(1 + i\sigma_1 + i\sigma_2 + i\sigma_3) \quad (4.8)$$

which satisfy the relations

$$\alpha^4 = \beta^3 = (\alpha\beta)^2 = -1 \quad (4.9)$$

It can be shown with a little effort that the 1152 elements of H'_0 are expressible as products of four generators,

$$a_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad a_2 = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}, \quad a_3 = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \quad a_4 = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix} \quad (4.10)$$

while the improper elements are obtained by multiplication with the generator

$$a_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.11)$$

The vector representation is obtained in the usual way,

$$a \bar{\gamma}^\mu a^{-1} = \bar{\gamma}^\nu \bar{R}_\nu{}^\mu(a) \quad (4.12)$$

where the Dirac matrices, $\bar{\gamma}_\mu$, are defined in the orthonormal basis by

$$\bar{\gamma}^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, j = 1, 2, 3 \quad \text{and} \quad \bar{\gamma}^4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad (4.13)$$

In this basis the matrices $\bar{R}_\mu{}^\nu(a)$ are orthogonal and it is straightforward to identify the generators (4.10) with various 4-dimensional rotations: (a_1) and (a_2) are 90° rotations in the $(\underline{e}_1, \underline{e}_2)$ and $(\underline{e}_3, \underline{e}_4)$ planes, respectively; (a_3) and (a_4) are 120° rotations in the $(\underline{e}_1 - \underline{e}_2, \underline{e}_1 - \underline{e}_3)$ and $(\underline{e}_1 + \underline{e}_2 + \underline{e}_3, \underline{e}_4)$ planes, respectively. Finally, (a_5) is a reflection in the direction, $\underline{e}_4$.

Corresponding to the lattice basis defined by (4.6), we define the Dirac matrices γ^μ such that

$$\gamma^\mu \underline{b}_\mu = \bar{\gamma}^\mu \underline{e}_\mu \quad (4.14)$$

The transformations (4.12) then take the form

$$a \gamma^\mu a^{-1} = \gamma^\nu R_\nu{}^\mu(a) \quad (4.15)$$

where the representatives, $R(a)$, belong to $GL(4, \mathbb{Z})$.

Suppose now that the solution of the gap equations assigns the fermions to the Dirac representation, i.e. that the ground state invariance condition (4.5) takes the form

$$a^{-1} G(s) a = G(s R(a)), \quad (4.16)$$

where a ranges over the group. For the Fourier transform,

$$\tilde{G}(k) = \sum_s G(s) e^{-ik \cdot s},$$

the invariance condition becomes

$$a^{-1} \tilde{G}(k) a = \tilde{G}(R(a^{-1})k) \quad (4.17)$$

This implies that, near $k = 0$,

$$\tilde{G}(k)^{-1} = Z_2^{-1} (M + i\gamma^\mu k_\mu + \dots) \quad (4.18)$$

where Z_2 and M are numbers to be determined by solving the gap equations. For such a solution to be possible it is of course necessary to choose the number of fermi components, N , to be a multiple of 4. The full ground state symmetry could then be the direct product of the space group with $U(N/4)$ or one of its subgroups. Unfortunately, the gap equations, even in the simplified version (3.7), are very complicated. We have not been able to show that such a solution exists.

The leading terms (4.18) are clearly invariant with respect to the continuous rotations, $O(4)$. Higher order terms, of course, would break this symmetry. In order for the model to be interesting it is necessary that at least some of the eigenvalues of M should be small. While it should always be possible to produce light fermions by tuning the coupling parameters, one may hope that this could be achieved in some more natural way. Another possibility would be to choose an odd value for $N/2$ and look for solutions that violate parity. In such cases there would necessarily be some massless fermions.

5 Gauge sector

In order to obtain a local action functional describing the low energy or long wavelength features of the system, it is necessary to introduce continuum fields that interpolate the discrete variables. For example,

$$\psi(x) = \int \left(\frac{dk}{2\pi} \right)^4 \tilde{\psi}(k) e^{ikx} \quad (5.1)$$

where $\tilde{\psi}(k)$ is defined by the lattice sum

$$\tilde{\psi}(k) = \sum_n \psi(n) e^{-ikn} \quad (5.2)$$

If $\psi(n)$ is slowly varying then $\tilde{\psi}(k)$ is concentrated around $k = 0$ and the integral (5.1) makes sense. Fermion propagation at long wavelengths is governed by the term,

$$\begin{aligned} \sum_{n,m} \bar{\psi}(n) \langle G^{-1}(n-m) \rangle \psi(m) &= \int \left(\frac{dk}{2\pi} \right)^4 \tilde{\bar{\psi}}(-k) \tilde{G}(k)^{-1} \tilde{\psi}(k) \\ &\simeq \int \left(\frac{dk}{2\pi} \right)^4 \tilde{\bar{\psi}}(-k) Z_2^{-1}(M + i\not{k} + \dots) \tilde{\psi}(k) \\ &= \int d^4x \bar{\psi}(x) Z_2^{-1}(M + \not{\partial} + \dots) \psi(x), \end{aligned} \quad (5.3)$$

where $\langle G^{-1}(s) \rangle$ is a solution of the gap equations and we have substituted the approximate form (4.18).

On the other hand, the matrix $G(n, m)$ that appears in the generalized effective action (3.2) is itself a dynamical variable. (In (5.3) we have used only its ground state expectation value.) In the long wavelength regime the dynamical aspect of $G(n, m)$ should be expressible in terms of local fields, scalars, vectors, tensors etc. Among the light degrees of freedom we should expect to find, because of the local symmetry of (3.2), a Yang-Mills sector. Guided by this symmetry, under which $G(n, m)$ transforms according to

$$G(n, m) \rightarrow e^{i\theta(n)} G(n, m) e^{-i\theta(m)},$$

we adopt the ansatz,

$$G(n, m) = \langle G(n-m) \rangle T \left(\exp i \int_m^n A \right), \quad (5.4)$$

where $A = dx^\mu A_\mu(x)$ and the path ordered integral is along the straight line joining the lattice sites n and m . Under local transformations the vector A_μ behaves in the standard way,

$$A_\mu(x) \rightarrow e^{i\theta(x)} (A_\mu(x) + i\partial_\mu) e^{-i\theta(x)} \quad (5.5)$$

It is clear that the ansatz (5.4) reproduces the correct transformation behavior if $\theta(x)$ is restricted to the unbroken symmetries, i.e. if $\theta(x)$ commutes with $\langle G(s) \rangle$. The unbroken local symmetries must be associated with a set of Yang-Mills vectors and

these fields must contribute a factor like (5.4) in the long wavelength approximation. Of course, the functions $G(n, m)$ must represent many other degrees of freedom but, for simplicity, we shall assume that they are all heavy and therefore irrelevant.

The gauge invariant fermion bilinear term now takes the form

$$\begin{aligned} \sum_{n,m} \bar{\psi}(n) G^{-1}(n, m) \psi(m) &= \sum_{n,m} \bar{\psi}(n) \langle G^{-1}(n-m) \rangle T \left(\exp i \int_m^n A \right) \psi(m) \\ &\simeq \int d^4x \bar{\psi}(x) Z_2^{-1}(M + \not{\partial} - i\not{A} + \dots) \psi(x) \end{aligned} \quad (5.6)$$

where we have expanded in powers of A_μ , treating this interpolating field as a small quantity comparable to $\partial_\mu \psi / \psi$.

On substituting the ansatz (5.4) into the generalized effective action (3.2), the variables σ and Δ could be eliminated from the functional $\Gamma(\sigma, \Delta, G(A))$ by solving the equations $\delta\Gamma/\delta\sigma = 0$ and $\delta\Gamma/\delta\Delta = 0$, i.e.

$$\begin{aligned} \sigma(n) &= - \sum_m J(n-m) \text{tr} G(m, m) \\ \Delta^{-1}(n, m) &= -J^{-1}(n-m) + \frac{\delta\Gamma_2(\Delta, G)}{\delta\Delta(n, m)} \end{aligned}$$

It is clear that $\sigma(n)$ cannot depend on A but $\Delta(n, m)$ would become a gauge invariant functional of A . The functional $\mathbb{I}(A)$ obtained by eliminating σ and Δ must be invariant with respect to the local transformations (5.5). On expanding in powers of A_μ this functional must take the form,

$$\mathbb{I}(A) = \mathbb{I}(0) + \frac{1}{4e^2} \int d^4x \text{tr}(F_{\mu\nu} F^{\mu\nu}) + \dots \quad (5.7)$$

where $F_{\mu\nu}$ is the usual field strength tensor and e^2 is a computable coupling constant. It is not obvious that the continuum action for A_μ should be $O(4)$ invariant. This is true only of the leading terms in the expansion (5.7). It is a consequence of the invariance of $\langle G(s) \rangle$ with respect to the point group (4.16) [11].

It is not difficult to obtain an expression for the polarization tensor in terms of the fermion propagator. Since σ is independent of A and Δ cannot contain a term linear in A , it follows that at the point $A = 0$,

$$\frac{\delta^2 \mathbb{I}(A)}{\delta A(1) \delta A(2)} = \Sigma \left\langle \frac{\delta^2 \Gamma}{\delta G(n, m) \delta G(n', m')} \right\rangle \frac{\delta G(n, m)}{\delta A(1)} \frac{\delta G(n', m')}{\delta A(2)} \quad (5.8)$$

where the second derivatives of $\Gamma(\sigma, \Delta, G)$ are computed for fixed σ and Δ at the extremum, i.e. at a solution of the gap equations (3.5). Expanding in powers of $G^2\Delta$,

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta G(n, m) \delta G(n', m')} &= -G^{-1}(m, n') G^{-1}(m', n) + \frac{\delta^2 \Gamma_2}{\delta G(n, m) \delta G(n', m')} \\ &= -G^{-1}(m, n') G^{-1}(m', n) + \Delta(n, m) \delta_{mn'} \delta_{m'n} + \dots \end{aligned} \quad (5.9)$$

where the Hartree-Fock approximation is indicated. The derivatives $\delta G/\delta A$ are obtained from (5.4). At $A_\mu = 0$,

$$\frac{\delta G(n, m)}{\delta \tilde{A}_\mu^\alpha(k)} = \langle G(n - m) \rangle T_\alpha i(n - m)^\mu \int_0^1 du e^{ik(un + (1-u)m)} \quad (5.10)$$

where the matrices T_α commutes with $\langle G(n - m) \rangle$.

In the momentum basis defined by $\langle n|p \rangle = e^{ipn}$ the fermion Greens's function is diagonal at $A = 0$,

$$\langle p|G|q \rangle = (2\pi)^4 \delta_{2\pi}(p - q) \tilde{G}(q)$$

Its derivative at $A = 0$, from (5.10)

$$\langle p| \frac{\delta G}{\delta \tilde{A}_\mu^\alpha(k)} |q \rangle = -(2\pi)^4 \delta_{2\pi}(p - q - k) \int_0^1 du \tilde{G}(p - uk)^\mu T_\alpha$$

where $\tilde{G}^\mu = \partial \tilde{G} / \partial p_\mu$. In this basis the polarization tensor (5.8) is written,

$$\begin{aligned} \Pi_{\alpha\beta}^{\mu\nu}(k, k') &= \frac{\delta^2 \Pi}{\delta \tilde{A}_\mu^\alpha(k) \delta \tilde{A}_\nu^\beta(k')} \\ &= \int \left(\frac{dp}{2\pi} \right)^4 \left(\frac{dq}{2\pi} \right)^4 \text{tr} \left[- \langle p| \frac{\delta G}{\delta \tilde{A}_\mu^\alpha(k)} |q \rangle \tilde{G}(q)^{-1} \langle q| \frac{\delta G}{\delta \tilde{A}_\nu^\beta(k')} |p \rangle \tilde{G}(p)^{-1} + \right. \\ &\quad \left. + \int \left(\frac{d\ell}{2\pi} \right)^4 \tilde{\Delta}(\ell) \langle p| \frac{\delta G}{\delta \tilde{A}_\mu^\alpha(k)} |q \rangle \langle q + \ell| \frac{\delta G}{\delta \tilde{A}_\nu^\beta(k')} |p + \ell \rangle + \dots \right] \\ &= (2\pi)^4 \delta_{2\pi}(k + k') \delta_{\alpha\beta} \int \left(\frac{dp}{2\pi} \right)^4 \int_0^1 du_1 du_2 \cdot \\ &\quad \cdot \text{tr} \left[-\tilde{G}(p - u_1 k)^\mu \tilde{G}(p - k)^{-1} \tilde{G}(p + (1 - u_2)k')^\nu \tilde{G}(p)^{-1} + \right. \\ &\quad \left. + \int \left(\frac{d\ell}{2\pi} \right)^4 \tilde{\Delta}(\ell) \tilde{G}(p - u_1 k)^\mu \tilde{G}(p + (1 - u_2)k')^\nu + \dots \right] \end{aligned} \quad (5.11)$$

where $\tilde{\Delta}(\ell)$ is the Fourier transform of the scalar Green's function $\Delta(n - m)$. All momentum integrations range over a fundamental cell of the dual lattice (Brillouin

zone). The Green's functions $\tilde{G}$ and $\tilde{\Delta}$ are periodic as are the delta functions,

$$\delta_{2\pi}(p - q) = \sum_{n \in \mathbb{Z}^4} \delta(p - q + 2\pi n)$$

The expression (5.11) can be simplified, at least in the Hartree-Fock approximation, by using the gap equation

$$\tilde{G}(p)^{-1} = \sigma - \int \left(\frac{d\ell}{2\pi} \right)^4 \tilde{\Delta}(\ell) \tilde{G}(p - \ell) + \dots \quad (5.12)$$

which serves to eliminate $\tilde{\Delta}$. One obtains,

$$\begin{aligned} \Pi_{\alpha\beta}^{\mu\nu}(k, k') &= (2\pi)^4 \delta_{2\pi}(k + k') \delta_{\alpha\beta} \int \left(\frac{dp}{2\pi} \right)^4 \int_0^1 du_1 du_2 \cdot \\ &\quad \cdot \text{tr} \left[-\tilde{G}(p - u_1 k)^\mu \tilde{G}(p - k)^{-1} \tilde{G}(p + (1 - u_2)k')^\nu \tilde{G}(p)^{-1} - \right. \\ &\quad \left. - (\tilde{G}(p - u_1 k)^{-1})^\mu \tilde{G}(p + (1 - u_2)k')^\nu + \dots \right] \end{aligned} \quad (5.13)$$

(It is easy to check the Ward identity, $k_\mu \Pi^{\mu\nu} = 0$, by writing

$$k_\mu G(p - u_1 k)^\mu = -\frac{\partial}{\partial u_1} G(p - u_1 k)$$

and integrating u_1 .) In the vicinity of $k = k' = 0$ write

$$\Pi_{\alpha\beta}^{\mu\nu}(k, k') = (2\pi)^4 \delta(k + k') \delta_{\alpha\beta} \Pi^{\mu\nu}(k)$$

where $\Pi^{\mu\nu}(k)$ is expanded in powers of k . The coefficients in this expansion must be invariant tensors of the point group, H , discussed in Section 4. The leading term is given by

$$\Pi^{\mu\nu}(k) = \frac{1}{e^2} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma}) k_\rho k_\sigma \quad (5.14)$$

where the metric tensor is defined in the lattice basis (4.6), $g_{\mu\nu} = \underline{b}_\mu \cdot \underline{b}_\nu$. The coupling constant, e^2 , is thereby expressed, in the Hartree-Fock approximation, as an integral over a Brillouin zone involving the Green's function $\tilde{G}(p)$ and its derivatives.

6 Outlook

In this paper a Nambu, Jona-Lasinio mechanism is examined in order to extract information about the low energy, long wavelength excitations of a rather primitive

lattice model. The main supposition is that the fermionic variables, in terms of which the model is expressed, will form a condensate. Propagation of fermions in the condensate is governed by a Green's function to be obtained by solving effective equations of motion. In the low energy regime it takes the form of a Dirac propagator. This approximately Lorentz invariant structure results from the four dimensional crystal symmetry of the underlying lattice model.

The fundamental quantity in our formulation of this problem is a so-called generalized effective action, $\Gamma(\sigma, \Delta, G)$, in which the independent variables are a scalar field, $\sigma(n)$, its Green's function, $\Delta(n, m)$, and the fermion Green's function, $G(n, m)$. The functional is invariant with respect to the space group of the lattice and a group of local gauge transformations that act on $G(n, m)$. We conjecture that the effective equations of motion will have solutions that break some of these symmetries.

The generalized action functional is given in the form of a loop expansion and one can only hope that finite truncations of this series give meaningful information about the condensate. In the appendix we have examined a simplified case in the 2-loop (Hartree-Fock) approximation. There it is shown that, by a suitable choice of the coupling parameters, reasonable solutions can be found. The main simplification in the toy model of the appendix consists in replacing the N fermionic variables of the actual model by a single scalar. Our intention here is merely to show that with such simplifications it is indeed possible to generate a reasonable Green's function dynamically. An extension to the fermionic case with $N > 1$ would be more interesting, of course, but also much more difficult.

Another issue, not dealt with in this paper, is the fixing of a gauge. Perhaps a suitable framework in which to discuss this problem is the alternative formulation of our model given in Ref.[1]. In this formulation the starting point, instead of (3.1), is the Euclidean action

$$S(\chi, \psi, \bar{\psi}) = \sum_{n,m} \left(\frac{\text{tr}(\chi(n,m)\chi(m,n))}{2J(n-m)} + \bar{\psi}(n) \chi(n,m) \psi(m) \right) \quad (6.1)$$

where the $N \times N$ matrix $\chi(n, m)$ is an auxiliary bilocal field. Elimination of χ

from (6.1) leads back to the original 4-fermion action (2.5). On the other hand, elimination of ψ and $\bar{\psi}$ leads to a gauge invariant semi-classical action for χ ,

$$s(\chi) = \sum_{n,m} \frac{\text{tr}(\chi(n,m)\chi(m,n))}{2J(n-m)} - \ell n \det \chi \quad (6.2)$$

The approximate gap equations (3.7) coincide with the semi-classical equations of motion deriving from (6.2) with the identification,

$$G^{-1}(n, m) = \chi(n, m) \quad (6.3)$$

Quantum corrections and gauge fixing could be adjoined to (6.2)

An unsatisfactory aspect of the model considered in this paper is the lack of control over the coupling parameters, $J(n, m)$. Our hope is that the model could be extended so as to make $J(n, m)$ into a dynamical variable. If this could be achieved then the full model would be invariant under arbitrary invertible lattice maps. It should then follow that in the continuum limit a generally covariant theory involving gravity would be recovered. Some preliminary calculations in this direction will be published elsewhere [7].

Appendix: The gap equation for a scalar Green's function

As an exercise to explore the feasibility of self-consistent computations, consider the simplified model in which the fermion fields, $\psi_i(n)$, are replaced by a single complex scalar, $\phi(n)$. The Green's function, $G(s)$, in this model is a scalar rather than an $N \times N$ matrix, and the gap equations are therefore much easier to solve. Our strategy is to regard $G(s)$ as input, solving the equations for the coupling parameters, $J(s)$. The aim of this somewhat artificial calculation, possible only for $N = 1$, is simply to show that a set of couplings can be found such that the gap equations yield a physically reasonable Green's function.

The Euclidean action for the scalar model is

$$S(\sigma, \phi, \bar{\phi}) = -\frac{1}{2} \sum_{n,m} \sigma(n) J^{-1}(n-m) \sigma(m) + \sum_n \sigma(n) \bar{\phi}(n) \phi(n) \quad (\text{A.1})$$

and the generalized effective action that replaces (3.2) is

$$\Gamma(\sigma, \Delta, G) = -\frac{1}{2} \sum_{n,m} J^{-1}(\sigma(n)\sigma(m) + \Delta(n,m)) - \frac{1}{2} \ell n \det \Delta - \ell n \det G + \sum_n \sigma(n) G(n,n) + \Gamma_2(\Delta, G) \quad (\text{A.2})$$

In the Hartree-Fock approximation,

$$\Gamma_2(\Delta, G) = -\frac{1}{2} \sum_{n,m} G(n,m) G(m,n) \Delta(n,m) \quad (\text{A.3})$$

and the corresponding gap equations are, assuming translation invariance,

$$\sigma = \sum_s J(s) G(0) \quad (\text{A.4})$$

$$\Delta^{-1}(s) = -J^{-1}(s) - G(s) G(-s) \quad (\text{A.5})$$

$$G^{-1}(s) = \sigma \delta_{s,0} - \Delta(s) G(s) \quad (\text{A.6})$$

In order to make the exercise as simple as possible we consider the square lattice in two dimensions and choose the inverse Green's function

$$\begin{aligned} \tilde{G}(k)^{-1} &= \sum_s G^{-1}(s) e^{-iks} \\ &= M^2 + 4 - 2 \cos k_1 - 2 \cos k_2 \end{aligned} \quad (\text{A.7})$$

i.e. we choose $\tilde{G}^{-1}(s) = 0$ everywhere except at the origin and its nearest neighbors,

$$\begin{aligned} G^{-1}(0,0) &= M^2 + 4 \\ G^{-1}(\pm 1,0) &= G^{-1}(0,\pm 1) = -1 \end{aligned} \quad (\text{A.8})$$

If M^2 is small and positive this Green's function describes the propagation of a scalar particle of mass M . More precisely, interpreting s_2 as Euclidean time,

$$\begin{aligned} G(s) &= \int \left(\frac{dk}{2\pi} \right)^2 \frac{e^{iks}}{M^2 + 4 - 2 \cos k_1 - 2 \cos k_2} \\ &= \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} ((M^2 + 4 - 2 \cos k_1)^2 - 4)^{-1/2} e^{ik_1 s_1 - \omega(k_1)|s_2|} \end{aligned} \quad (\text{A.9})$$

where the energy spectrum is given by

$$\omega(k_1) = -\ell n \left[\frac{1}{2}(M^2 + 4 - 2 \cos k_1) - \frac{1}{2} \sqrt{(M^2 + 4 - 2 \cos k_1)^2 - 4} \right] \quad (\text{A.10})$$

If M and k_1 are small then $\omega(k_1) \simeq \sqrt{k_1^2 + M^2}$ but in general $\omega(k_1)$ increases monotonically for $0 \leq |k_1| \leq \pi$. There is no ghost.

To solve the gap equations we begin with the observation that (A.6) implies $\Delta(s) = 0$ everywhere except the origin and its nearest neighbors,

$$\begin{aligned} \Delta(0,0) &= \frac{\sigma - G^{-1}(0,0)}{G(0,0)} \\ \Delta(1,0) &= -\frac{G^{-1}(1,0)}{G(1,0)} \end{aligned} \quad (\text{A.11})$$

where $G(0,0)$ and $G(1,0)$ are given by (A.9). The Fourier transform of $\Delta(s)$ is simple,

$$\tilde{\Delta}(k) = \Delta(0,0) + 2\Delta(1,0)(\cos k_1 + \cos k_2) \quad (\text{A.12})$$

The transform of (A.5) gives

$$\frac{1}{\tilde{J}(k)} = -\frac{1}{\tilde{\Delta}(k)} - \int \left(\frac{dp}{2\pi} \right)^2 \tilde{G}((k-p)) \tilde{G}(p) \quad (\text{A.13})$$

Hence the couplings are obtained in terms of known quantities except for σ . To eliminate σ , write (A.4) in the form

$$\frac{1}{\tilde{J}(0)} = \frac{G(0,0)}{\sigma} \quad (\text{A.14})$$

and compare with (A.13) at $k = 0$,

$$\begin{aligned}\frac{G(0,0)}{\sigma} &= -\frac{1}{\Delta(0)} - \int \left(\frac{dp}{2\pi}\right)^2 \tilde{G}(-p) \tilde{G}(p) \\ &= \frac{-G(0,0)}{\sigma - G^{-1}(0,0) - 4G^{-1}(1,0)G(0,0)/G(1,0)} - \int \left(\frac{dp}{2\pi}\right)^2 \tilde{G}(-p) \tilde{G}(p)\end{aligned}$$

This equation has two real solutions for σ . Hence there are two distinct solutions for $J(s)$.

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