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1. Introduction

It is well known that the cotangent bundle of any manifold has a canonical symplectic structure. If we specialise to the case when the manifold is a compact Lie group G , then this structure is preserved by the actions of G on T^*G induced by left and right translation on G . We refer to these as the left and right actions of G on T^*G .

Guillemin and Sternberg [GS] show that one may choose a diffeomorphism $\phi: T^*G \rightarrow G \times \mathfrak{g}^*$ with the following properties:

- (i) ϕ intertwines the left action on T^*G with the action

$$(g, \zeta) \mapsto (\gamma g, \zeta), \quad \gamma \in G,$$

on $G \times \mathfrak{g}^*$;

- (ii) ϕ intertwines the right action on T^*G with the action

$$(g, \zeta) \mapsto (g\gamma^{-1}, \text{Ad}(\gamma)\zeta), \quad \gamma \in G,$$

on $G \times \mathfrak{g}^*$;

(iii) under the identification of T^*G with $G \times \mathfrak{g}^*$, the moment map for the right action of G on T^*G is just projection of $G \times \mathfrak{g}^*$ onto the $\mathfrak{g}^*$ factor.

Let H be a closed subgroup of G . By taking a Marsden-Weinstein quotient of the product of T^*G with a vector space V admitting a symplectic H action, Guillemin and Sternberg show the existence of a symplectic structure on $G \times_H (\mathfrak{m}^* \times V)$, where $\mathfrak{m}$ denotes the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form. (In fact $\mathfrak{m}^*$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$ so can be defined without using the Killing form.)

In this paper we shall discuss an analogue to the above construction in the realm of hyperkähler geometry by studying a moduli space of solutions to Nahm's equations. We recall that a hyperkähler structure on a manifold is a quadruple $(\mathbf{g}, I, J, K)$, where $\mathbf{g}$ is a Riemannian metric and I, J, K are complex structures satisfying the quaternionic relations $IJ = K = -JI$ etc., and such that $(\mathbf{g}, I)$, $(\mathbf{g}, J)$, $(\mathbf{g}, K)$ are Kähler structures. One can show that such a metric $\mathbf{g}$ is necessarily Ricci-flat, and in fact all the hyperkähler metrics we discuss are also complete.

For further material on hyperkähler geometry, and in particular on the hyperkähler version of the Marsden-Weinstein quotient, we refer to [H] and [HKLR].

2. A moduli space

We shall first describe a result due to Kronheimer [K] which provides a hyperkähler counterpart to the symplectic manifold T^*G .

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. We shall consider $\mathfrak{g}$ as being embedded in $\mathfrak{u}(n)$. Let $\mathcal{A}$ denote the infinite-dimensional affine quaternionic space whose points are of the form

$$\frac{d}{dt} + T_0 + iT_1 + jT_2 + kT_3,$$

where T_i are smooth maps from $[0, 1]$ to $\mathfrak{g}$. We can define a metric on $\mathcal{A}$ by

$$\langle\langle (Y_0, Y_1, Y_2, Y_3), (Z_0, Z_1, Z_2, Z_3) \rangle\rangle = \int_0^1 \sum_{i=0}^3 \langle Y_i, Z_i \rangle dt,$$

where $\langle Y, Z \rangle = -\text{Tr } YZ^*$. Right multiplication by $-i, -j, -k$ defines complex structures I, J, K on $\mathcal{A}$, which together with the metric endow $\mathcal{A}$ with a hyperkähler structure.

The gauge group $\mathcal{G}_0^0$ of smooth maps from $[0, 1]$ to G which are the identity at $t = 0, 1$, acts on $\mathcal{A}$ by

$$T_0 \mapsto fT_0f^{-1} - \frac{df}{dt}f^{-1}, \quad (1)$$

$$T_i \mapsto fT_if^{-1}, \quad i = 1, 2, 3, \quad (2)$$

and this action preserves the hyperkähler structure. The associated hyperkähler moment map is

$$(T_0, T_1, T_2, T_3) \mapsto \begin{pmatrix} \frac{dT_1}{dt} + [T_0, T_1] - [T_2, T_3] \\ \frac{dT_2}{dt} + [T_0, T_2] - [T_3, T_1] \\ \frac{dT_3}{dt} + [T_0, T_3] - [T_1, T_2] \end{pmatrix}.$$

The hyperkähler quotient construction [HKLR] is used in the proof of the following result (cf. [D]).

Proposition 1 [K] *Let M be the quotient by $\mathcal{G}_0^0$ of the set of quadruples (T_0, T_1, T_2, T_3) of smooth maps from $[0, 1]$ to $\mathfrak{g}$ satisfying the Nahm equations*

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k],$$

where (i, j, k) ranges over cyclic permutations of $(1, 2, 3)$.

Then M is a complete hyperkähler manifold. $\square$

We recall the following facts from [K]. Consider the space $\mathcal{B}$ of pairs (α, β) where α, β are smooth maps from $[0, 1]$ to $\mathfrak{g}_{\mathbb{C}}$ satisfying the *complex Nahm equation*

$$\frac{d\beta}{dt} + [\alpha, \beta] = 0. \quad (3)$$

The complexification of the gauge group $\mathcal{G}_0^0$ acts on $\mathcal{B}$ as in (1), (2), where f now takes values in $G_{\mathbb{C}}$. We refer to the quotient as N .

The map which assigns to (T_0, T_1, T_2, T_3) the pair (α, β) , where

$$\alpha = T_0 - iT_1, \quad \beta = T_2 + iT_3, \quad (4)$$

induces a bijection between M and N .

Given a point in N represented by a pair (α, β) , we let $f: [0, 1] \rightarrow G_{\mathbb{C}}$ be the solution to the linear equation

$$\frac{df}{dt} = f\alpha$$

subject to the condition $f(1) = \text{Id}$. The map ψ defined by

$$\psi: (\alpha, \beta) \mapsto (f(0)^{-1}, \beta(1)) \quad (5)$$

is well-defined on N and in fact is a diffeomorphism between N and $G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \cong T^*G_{\mathbb{C}}$.

Finally, we need to discuss the group actions on M . Let $\mathcal{G}_0$ and $\mathcal{G}^0$ denote the groups of smooth maps from $[0, 1]$ to G which are the identity at $t = 0$ (respectively $t = 1$). Both these groups contain the gauge group $\mathcal{G}_0^0$ as a normal subgroup, and evaluation at $t = 1$ (resp. $t = 0$) identifies the quotients, which we denote by G_R and G_L , with G .

Both G_R and G_L act on M as in (1), (2). These actions commute and preserve the hyperkähler structure on M .

There is also an $SO(3)$ action on M defined by

$$T_0 \mapsto T_0, \quad T_i \mapsto \sum_{j=1}^3 a_{ij} T_j, \quad i = 1, 2, 3.$$

This action preserves the metric and permutes the Kähler structures. It commutes with the actions of G_R and G_L .

3. Moment maps

We shall use the moduli space M as a hyperkähler analogue of the symplectic manifold T^*G . We have seen that M admits commuting hyperkähler actions of the groups G_R and G_L , both of which are isomorphic to G . We now wish to calculate the moment

maps for these actions. The calculation will use the following observation concerning the tangent space to M .

The horizontal lift to $\mathcal{A}$ of a tangent vector to M at a point (T_0, T_1, T_2, T_3) is a quadruple (Y_0, Y_1, Y_2, Y_3) of smooth maps from $[0, 1]$ to $\mathfrak{g}$ satisfying

$$\frac{dY_1}{dt} + [T_0, Y_1] + [Y_0, T_1] = [T_2, Y_3] + [Y_2, T_3], \quad (6)$$

$$\frac{dY_2}{dt} + [T_0, Y_2] + [Y_0, T_2] = [T_3, Y_1] + [Y_3, T_1], \quad (7)$$

$$\frac{dY_3}{dt} + [T_0, Y_3] + [Y_0, T_3] = [T_1, Y_2] + [Y_1, T_2] \quad (8)$$

and

$$\frac{dY_0}{dt} + \sum_{i=0}^3 [T_i, Y_i] = 0. \quad (9)$$

The first three equations are just the linearisation of the Nahm equations at (T_0, T_1, T_2, T_3) , while (9) expresses the condition of orthogonality to the orbits of $\mathcal{G}_0^0$.

Lemma 2 *Let H be a subgroup of G . The moment map for the action of H on M is given by*

(i) *evaluation of the $\mathfrak{h}$ -component of $(-T_1, -T_2, -T_3)$ at $t = 1$, if H acts as a subgroup of G_R ,*

(ii) *evaluation of the $\mathfrak{h}$ -component of (T_1, T_2, T_3) at $t = 0$, if H acts as a subgroup of G_L .*

Proof. We shall only prove (i) in detail. Let ξ be an element of the Lie algebra $\mathfrak{g}$. Identifying G with G_R , and viewing ξ as an element of the Lie algebra of the latter group, we see that ξ generates the one-parameter subgroup whose elements are represented by maps f_θ defined by

$$f_\theta: t \mapsto \exp(t\theta\xi).$$

Differentiating the action of this subgroup at $\theta = 0$, we see that a lift to $\mathcal{A}$ of the tangent vector at (T_0, T_1, T_2, T_3) defined by ξ is

$$([t\xi, T_0] - \xi, [t\xi, T_1], [t\xi, T_2], [t\xi, T_3]).$$

Let $Y = (Y_0, Y_1, Y_2, Y_3)$ be a lift to $\mathcal{A}$ of an arbitrary tangent vector to M . We shall calculate $\omega_1(X, Y)$, where ω_1 is the Kähler form on $\mathcal{A}$ associated to the complex structure defined by multiplication by $-i$. We have

$$\begin{aligned}\omega_1(X, Y) &= \int_0^1 \langle [t\xi, T_0] - \xi, Y_1 \rangle - \langle [t\xi, T_1], Y_0 \rangle + \langle [t\xi, T_3], Y_2 \rangle - \langle [t\xi, T_2], Y_3 \rangle dt \\ &= \int_0^1 -\langle \xi, Y_1 \rangle + \langle [T_0, Y_1] + [Y_0, T_1] - [T_2, Y_3] - [Y_2, T_3], t\xi \rangle dt \\ &= \int_0^1 -\langle \xi, Y_1 \rangle - \left\langle t\xi, \frac{dY_1}{dt} \right\rangle dt \\ &= -\langle \xi, Y_1(1) \rangle.\end{aligned}$$

Similar calculations for the other Kähler forms show that the Hamiltonian function for ξ is given by

$$(T_0, T_1, T_2, T_3) \mapsto (-\langle \xi, T_1(1) \rangle, -\langle \xi, T_2(1) \rangle, -\langle \xi, T_3(1) \rangle),$$

which implies our claim. Note that this function is invariant under the action of $\mathcal{G}_0^0$ so is well-defined on M .

The proof of (ii) is almost identical except that in this case we take the one-parameter subgroup whose elements are represented by maps $f_\theta: t \mapsto \exp((1-t)\theta\xi)$. $\square$

The $SO(3)$ action on M is not hyperkähler. However each $U(1)$ subgroup of $SO(3)$ preserves one of the Kähler forms of M and defines a Kähler moment map. If we let I, J and K denote the complex structures on M induced by multiplication by $-i, -j, -k$ on $\mathcal{A}$, then the moment map for the $U(1)$ subgroup preserving K is

$$\mu_K: (T_0, T_1, T_2, T_3) \mapsto \frac{1}{2} \int_0^1 \langle T_1, T_1 \rangle + \langle T_2, T_2 \rangle dt$$

This $U(1)$ subgroup fixes the Nahm matrix T_3 and rotates T_1, T_2 .

The complex structure on M induced by the identification of M with $T^*G_{\mathbb{C}}$ discussed in section 2 is I . The results of section 3(E) of [HKLR] show that $2\mu_K$ is a globally defined Kähler potential for I , that is, we have

$$2i\partial_I\bar{\partial}_I\mu_K = \omega_I.$$

We may also ask if there is a hyperkähler potential on M , that is, a function λ such that

$$i\partial_C\bar{\partial}_C\lambda = \omega_C$$

for each complex structure C compatible with the hyperkähler structure. It is shown in [S] that the existence of a hyperkähler potential implies that there is a local action of $Sp(1)$ permuting the complex structures. Moreover, if λ_C denotes the Kähler moment map for the action of the circle subgroup of $Sp(1)$ fixing ω_C , then for each C_1, C_2 we have that $\lambda_{C_1} - \lambda_{C_2}$ is constant. It is clear from our formula for μ_K and the analogous expressions for μ_I and μ_J that this fails to hold in our situation. We deduce that there is no hyperkähler potential on M associated to the $SO(3)$ action defined in §2.

We shall now use the hyperkähler moment map for the G_R action to construct a map ϕ from M into $G \times \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$ with certain equivariance properties. We define ϕ by

$$\phi: (T_0, T_1, T_2, T_3) \mapsto (f(0)^{-1}, T_1(1), T_2(1), T_3(1)), \quad (10)$$

where $f: [0, 1] \rightarrow G$ is the solution of the equation

$$\frac{df}{dt} = fT_0,$$

with $f(1) = \text{Id}$.

Note that ϕ is invariant under the action of $\mathcal{G}_0^0$ so is well-defined as a function on M . It is useful to observe that f is the unique gauge transformation with $f(1) = \text{Id}$ which gauges T_0 to zero. The image under gaugeing by f of (T_0, T_1, T_2, T_3) is $(0, P_1, P_2, P_3)$ where $P_i = fT_i f^{-1}$ satisfy

$$\frac{dP_1}{dt} = [P_2, P_3] \quad \text{and cyclically.} \quad (11)$$

We refer to (11) as the *reduced Nahm equations*.

Theorem 3 *The map ϕ is a diffeomorphism of M onto an open set W in $G \times \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$ containing $G \times (0, 0, 0)$.*

Proof. Suppose that $T = (T_0, T_1, T_2, T_3)$ and $S = (S_0, S_1, S_2, S_3)$ are both mapped to $(A, \zeta_1, \zeta_2, \zeta_3)$ by ϕ .

It follows that there exist gauge transformations e, f with $e(0) = f(0) = A^{-1}$ and $e(1) = f(1) = \text{Id}$ such that $e \cdot T = P$, $f \cdot S = Q$ and both P and Q satisfy the reduced Nahm equations (11). (Here we use $e \cdot T$ to denote the image of T under the gauge transformation e). Now

$$P_i(1) = eT_i e^{-1}(1) = T_i(1) = \zeta_i = S_i(1) = Q_i(1), \quad i = 1, 2, 3,$$

so in fact $P = Q$.

Therefore T and S are equivalent via a gauge transformation which is the identity at $t = 0, 1$, so represent the same point of the moduli space M . We have shown that ϕ is injective.

If $(\zeta_1, \zeta_2, \zeta_3)$ is close to the origin in $\mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$ then we can find smooth $\mathfrak{g}$ -valued maps P_1, P_2, P_3 on $[0, 1]$ satisfying the reduced Nahm equations subject to the conditions $P_i(1) = \zeta_i$. Applying a gauge transformation f with $f(0) = A, f(1) = \text{Id}$ yields a point in M mapping to $(A, \zeta_1, \zeta_2, \zeta_3)$ under ϕ . Of course, different choices of f will give the same point in the moduli space M .

We have shown that the image W of ϕ contains a neighbourhood of $G \times (0, 0, 0)$. The condition that the reduced Nahm equations have a smooth solution on $[0, 1]$ with values $\zeta_1, \zeta_2, \zeta_3$ at $t = 1$ is an open one in ζ_i , so we see that W is open. The discussion of the preceding paragraph shows that ϕ^{-1} is smooth on W . It is also clear that ϕ is smooth, so we are done. $\square$

The next result summarises the relevant properties of ϕ .

Proposition 4 (i) *The action of G_L on M and the action*

$$(g, \zeta_1, \zeta_2, \zeta_3) \mapsto (\gamma g, \zeta_1, \zeta_2, \zeta_3), \quad \gamma \in G,$$

on W are intertwined by ϕ .

(ii) *The action of G_R on M and the action*

$$(g, \zeta_1, \zeta_2, \zeta_3) \mapsto (g\gamma^{-1}, \text{Ad}(\gamma)\zeta_1, \text{Ad}(\gamma)\zeta_2, \text{Ad}(\gamma)\zeta_3), \quad \gamma \in G,$$

on W are intertwined by ϕ .

(iii) Under the identification of M with W , the hyperkähler moment map for the action of a subgroup H of $G_R \cong G$ on M is given by orthogonal projection onto $\mathfrak{h}^*$ in each $\mathfrak{g}^*$ factor.

(iv) The $SO(3)$ action on M and the action on $W \subset G \times (\mathfrak{g}^* \otimes \mathbb{R}^3)$ induced by the standard action on $\mathbb{R}^3$ are intertwined by ϕ .

Proof. (i) If f gauges T_0 to zero and $f(1) = \text{Id}$, and if u is a gauge transformation with $u(0) = \gamma$, $u(1) = \text{Id}$, then fu^{-1} gauges $uT_0u^{-1} - \frac{du}{dt}u^{-1}$ to zero and $fu^{-1}(1) = \text{Id}$. The equivariance property for the G_L action now follows.

(ii) If, on the other hand, s is a gauge transformation with $s(0) = \text{Id}$, $s(1) = \gamma$, let $\hat{\gamma}$ be the constant gauge transformation identically equal to γ . Then $\hat{\gamma}fs^{-1}$ gauges $sT_0s^{-1} - \frac{ds}{dt}s^{-1}$ to zero and is the identity at $t = 1$. This yields the desired equivariance property for the G_R action.

(iii) and (iv) are immediate. $\square$

Proposition 4 is a generalisation to the hyperkähler situation of the symplectic results of [GS] discussed in the introduction.

4. Hyperkähler quotients

Continuing the analogy with the symplectic case, we can form new hyperkähler manifolds as hyperkähler quotients of the product of M with a vector space.

Let V be a quaternionic vector space admitting a linear hyperkähler action of H , where H is a closed subgroup of G . Viewing H also as a subgroup of $G_R \cong G$, we therefore have a free hyperkähler action of H on the product $M \times V$, or equivalently on $W \times V \subset G \times (\mathfrak{g}^* \otimes \mathbb{R}^3) \times V$. Let Ψ and Φ denote the hyperkähler moment maps for the actions of H on V and $W \times V$ respectively.

Letting $\mathfrak{m}$ denote the $\text{Ad}(H)$ -invariant orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$, we can express $W \times V$ as a subset of

$$G \times (\mathfrak{h}^* \otimes \mathbb{R}^3) \times (\mathfrak{m}^* \otimes \mathbb{R}^3) \times V.$$

including the zero section.

Using part (iii) of Proposition 4, we see that the moment map for the action of H is given by

$$\Phi: (g, h_1, h_2, h_3, m_1, m_2, m_3, v) \mapsto (-h_1 + \Psi_1(v), -h_2 + \Psi_2(v), -h_3 + \Psi_3(v)).$$

It follows that $\Phi^{-1}(0, 0, 0)$ may be identified with a neighbourhood of the zero section in $G \times (\mathfrak{m}^* \otimes \mathbb{R}^3) \times V$ via the H -equivariant diffeomorphism

$$(g, m_1, m_2, m_3, v) \mapsto (g, \Psi_1(v), \Psi_2(v), \Psi_3(v), m_1, m_2, m_3, v).$$

We have thus proved the following result (recall that the metric on W is complete so the hyperkähler quotient of $W \times V$ by H is also complete.)

Proposition 5 *Let G be a compact Lie group and H a closed subgroup. Let $\mathfrak{m}$ denote the orthogonal complement (with respect to the Killing form) to $\mathfrak{h}$ in $\mathfrak{g}$ and let V be a quaternionic vector space with a linear hyperkähler action of H .*

Then there is a complete hyperkähler metric defined on a neighbourhood of the zero section in the vector bundle over G/H whose total space is $G \times_H ((\mathfrak{m}^ \otimes \mathbb{R}^3) \times V)$. $\square$*

As mentioned earlier, M can be identified with the moduli space N of solutions to the complex Nahm equation (3) modulo the complexification of $\mathcal{G}_0^0$, and hence, via the map ψ of (5), with $T^*G_{\mathbb{C}}$. If we pick a complex structure I and regard V as a complex vector space V_I , then the hyperkähler quotient $\mathcal{M}$ of Proposition 5 is identified with the complex manifold $G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (\mathfrak{m}_{\mathbb{C}} \times V_I)$. In particular, if we take $V = 0$ we obtain a hyperkähler metric on $T^*(G_{\mathbb{C}}/H_{\mathbb{C}})$.

From now on we shall take $\mathcal{M}$ to be $T^*(G_{\mathbb{C}}/H_{\mathbb{C}})$. We let $\mathcal{C}$ denote the space of solutions (α, β) to the complex Nahm equation (3) such that $\beta(1) \in \mathfrak{m}_{\mathbb{C}}$, and consider the group $\mathcal{H}$ of complex gauge transformations which are the identity at $t = 0$ and lie in the subgroup $H_{\mathbb{C}}$ at $t = 1$. Then (4) identifies $\mathcal{M}$ with $\mathcal{N} = \mathcal{C}/\mathcal{H}$. Note that the action of $\mathcal{H}$ does not preserve the hyperkähler metric on $\mathcal{A}$, but it does preserve the complex symplectic form $\omega_{\mathbb{C}} = \omega_J + i\omega_K$, where ω_J denotes the Kähler form associated to J , etc. Thus (4) identifies $\mathcal{M}$ and $\mathcal{N}$ as complex symplectic manifolds. Note that

Lemma 2 implies that $\mathcal{C}$ is the zero set for the complex moment map associated to the $\mathcal{H}$ -action. Now the map ψ induces an identification of $\mathcal{N}$ with $T^*(G_{\mathbb{C}}/H_{\mathbb{C}})$, but this latter space, being the cotangent bundle of a complex manifold, already has a standard complex symplectic structure.

Proposition 6 (i) *Under the identification between $\mathcal{M}$ and $T^*(G_{\mathbb{C}}/H_{\mathbb{C}})$, the complex symplectic form on $\mathcal{M}$ defined by the hyperkähler structure is the standard symplectic form on $T^*(G_{\mathbb{C}}/H_{\mathbb{C}})$.*

(ii) *At points of the zero section $G/H \times (0, 0, 0) \subset G \times_H (\mathfrak{m}^* \otimes \mathbb{R}^3)$, the hyperkähler metric agrees with the standard metric on $G \times_H (\mathfrak{m}^* \otimes \mathbb{R}^3)$ induced by the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$.*

Proof. (i) The above discussion gives the following formula for the complex symplectic form on $\mathcal{N}$:

$$\omega_{\mathbb{C}}((Y_{\alpha}, Y_{\beta}), (Z_{\alpha}, Z_{\beta})) = \int_0^1 \text{Tr } \tilde{Z}_{\alpha} \tilde{Y}_{\beta} - \text{Tr } \tilde{Y}_{\alpha} \tilde{Z}_{\beta} dt,$$

where $(\tilde{Y}_{\alpha}, \tilde{Y}_{\beta})$ and $(\tilde{Z}_{\alpha}, \tilde{Z}_{\beta})$ are lifts to $\mathcal{C}$ of tangent vectors (Y_{α}, Y_{β}) and (Z_{α}, Z_{β}) to $\mathcal{N}$ at (α, β) . The integral on the right-hand side is degenerate along orbits of $\mathcal{H}$ in $\mathcal{C}$, so is independent of the choice of lifts.

We now calculate the pullback of $\omega_{\mathbb{C}}$ to $G_{\mathbb{C}} \times_{H_{\mathbb{C}}} \mathfrak{m}_{\mathbb{C}}$ by ψ^{-1} . This map is defined by

$$\psi^{-1}: (Q, B) \mapsto \left(-\frac{dh}{dt}h^{-1}, hBh^{-1}\right),$$

where h is a $G_{\mathbb{C}}$ -valued gauge transformation with $h(0) = Q$ and $h(1) \in H_{\mathbb{C}}$. Different choices of h define the same point in $\mathcal{N}$.

Now ψ^{-1} maps the point in $G_{\mathbb{C}} \times_{H_{\mathbb{C}}} \mathfrak{m}_{\mathbb{C}}$ represented by (Id, B) to the point in $\mathcal{N}$ with representative $(0, B)$. Using the one-parameter subgroup $s \mapsto (\exp(sq), B + sb)$ and the gauge transformations $h_s: t \mapsto \exp((1-t)sq)$, it is straightforward to calculate that the derivative of ψ^{-1} at (Id, B) is

$$(\psi^{-1})_*: (q, b) \mapsto (q, b + (1-t)[q, B]), \quad q \in \mathfrak{g}_{\mathbb{C}}, \quad b \in \mathfrak{m}_{\mathbb{C}}.$$

(Strictly speaking, the expression on the right-hand side is the lift of a tangent vector to $\mathcal{C}$. However, from above we can use any such lift for calculations involving the symplectic form $\omega_{\mathbb{C}}$).

Another short calculation shows that at (Id, B)

$$(\psi^{-1})^*\omega_{\mathbb{C}}: ((q, b), (r, c)) \mapsto \text{Tr } rb - \text{Tr } qc - \text{Tr } B[q, r].$$

Identifying $\mathfrak{m}_{\mathbb{C}}$ with $\mathfrak{m}_{\mathbb{C}}^*$ via the pairing $(q, c) \mapsto \text{Tr } qc$, we find that at (Id, B) the pullback of $\omega_{\mathbb{C}}$ by ψ^{-1} is the standard symplectic form on $G_{\mathbb{C}} \times_{H_{\mathbb{C}}} \mathfrak{m}_{\mathbb{C}}^*$ obtained by viewing the latter space as $T^*(G_{\mathbb{C}}/H_{\mathbb{C}})$. As both the pullback of $\omega_{\mathbb{C}}$ and the standard symplectic form are $G_{\mathbb{C}}$ -invariant, the proof is completed.

(ii) The proof is essentially the same as for (i). The map ϕ defined in equation (10) restricts to a map from the zero level set of the hyperkähler moment map for the H action, into $G \times (\mathfrak{m}^* \otimes \mathbb{R}^3)$. The equivariance properties of ϕ mean that it induces a map (which we also denote by ϕ) from the hyperkähler quotient $\mathcal{M}$ into $G \times_H (\mathfrak{m}^* \otimes \mathbb{R}^3)$. The inverse of this map is given by

$$\phi^{-1}: (A, \zeta_1, \zeta_2, \zeta_3) \mapsto \left(-\frac{df}{dt}f^{-1}, fP_1f^{-1}, fP_2f^{-1}, fP_3f^{-1}\right),$$

where P_i satisfy (11), $P_i(1) = \zeta_i$, $f(0) = A$ and $f(1) \in H$. Using the gauge transformations $f_s: t \mapsto \exp((1-t)sa)$, we have at $(\text{Id}, \zeta_1, \zeta_2, \zeta_3)$ that a lift to $\mathcal{A}$ of $(\phi^{-1})_*(a, b_1, b_2, b_3)$ is

$$(a, p_1 + (1-t)[a, P_1], p_2 + (1-t)[a, P_2], p_3 + (1-t)[a, P_3]), \quad (12)$$

where $dp_1/dt = [p_2, P_3] + [P_2, p_3]$, etc. and $p_i(1) = b_i$. Here $a \in \mathfrak{m}$ and $b_i \in \mathfrak{m}^*$.

At $(\zeta_1, \zeta_2, \zeta_3) = (0, 0, 0)$, we have $P_i \equiv 0$ and $p_i \equiv b_i$. The expression in (12) is now (a, b_1, b_2, b_3) , which is the *horizontal* lift of a tangent vector to $\mathcal{M}$. Part (ii) now follows. $\square$

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