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A SPECTRAL SEQUENCE FOR THE TANGENT SHEAF COHOMOLOGY OF A SUPERMANIFOLD

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ABSTRACT. One constructs a spectral sequence permitting to compute the tangent sheaf cohomology of a complex analytic supermanifold in terms of the tangent sheaf cohomology of the associated split supermanifold. The first coboundary operators of this sequence are determined and the example of the superquadric in the projective superplane is studied.

1. PRELIMINARIES

We consider here the complex analytic supermanifolds in the sense of Berezin–Leites (see [1, 5, 7]). The supermanifold, having the complex manifold M as its reduction and the sheaf $\mathcal{O}$ on M as its structure sheaf, will be denoted by $(M, \mathcal{O})$. As in [5, 9], we say that a supermanifold is *split* if it is isomorphic to $(M, \bigwedge_{\mathcal{F}} \mathcal{E})$, where $M = (M, \mathcal{F})$ is a usual complex analytic manifold and $\mathcal{E}$ is a locally free analytic sheaf on M . The structure sheaf $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{E}$ admits in this case the $\mathbb{Z}$ -grading by the subsheaves $\mathcal{O}^p = \bigwedge_{\mathcal{F}}^p \mathcal{E}$, $p \geq 0$. Note that the $\mathbb{Z}_2$ -grading of $\mathcal{O}$ is compatible with this $\mathbb{Z}$ -grading, in the sense that

$$(1) \quad \mathcal{O}_{\bar{0}} = \bigoplus_{p \geq 0} \mathcal{O}_{2p}, \quad \mathcal{O}_{\bar{1}} = \bigoplus_{p \geq 0} \mathcal{O}_{2p+1}.$$

It is well known that with an arbitrary supermanifold $(M, \mathcal{O})$ a split supermanifold $(M, \mathcal{O}_{\text{gr}})$, having the same reduction M and the same dimension as $(M, \mathcal{O})$, can be associated. It is defined in the following way. Let $\mathcal{J}$ denote the subsheaf of ideals in $\mathcal{O}$, generated by the odd elements. Consider the filtration

$$(2) \quad \mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots$$

of $\mathcal{O}$ by the powers $\mathcal{J}^p$ of $\mathcal{J}$, and denote by $\text{gr } \mathcal{O}$ the associated $\mathbb{Z}$ -graded sheaf of superalgebras. Thus,

$$\text{gr } \mathcal{O} = \bigoplus_{p \geq 0} \text{gr}_p \mathcal{O},$$

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where

$$\mathrm{gr}_p \mathcal{O} = \mathcal{J}^p / \mathcal{J}^{p+1}.$$

One checks that $\mathrm{gr} \mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{F} = \mathrm{gr}_0 \mathcal{O}$, $\mathcal{E} = \mathrm{gr}_1 \mathcal{O}$.

A sheaf $\mathcal{S}$ on a supermanifold $(M, \mathcal{O})$ will be called *analytic* if $\mathcal{S}$ is a sheaf of $\mathcal{O}$ -modules. An important example is the *tangent sheaf* $\mathcal{T} = \mathrm{Der} \mathcal{O}$ of $(M, \mathcal{O})$, i.e. the sheaf of derivations of $\mathcal{O}$ over $\mathbb{C}$ in the $\mathbb{Z}_2$ -graded sense. This is a locally free analytic sheaf of rank $n|m = \dim(M, \mathcal{O})$. It has a natural structure of the sheaf of complex Lie superalgebras, the bracket being defined by

$$(3) \quad [u, v] = vu + (-1)^{p(u)p(v)+1}uv.$$

We consider the following filtration of $\mathcal{T}$:

$$(4) \quad \mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \cdots \supset \mathcal{T}_{(m)} \supset \mathcal{T}_{(m+1)} = 0,$$

where

$$\mathcal{T}_{(p)} = \{u \in \mathcal{T} \mid u(\mathcal{O}) \subset \mathcal{J}^p, u(\mathcal{J}) \subset \mathcal{J}^{p+1}\}, \quad p \geq 0.$$

One sees easily that $\mathcal{T}_{(p)}$ are analytic subsheaves of $\mathcal{T}$ such that

$$(5) \quad \mathcal{J}\mathcal{T}_{(p)} \subset \mathcal{T}_{(p+1)}, \quad p \geq -1.$$

On the other hand,

$$[\mathcal{T}_{(p)}, \mathcal{T}_{(q)}] \subset \mathcal{T}_{(p+q)},$$

and so we have a filtered sheaf of Lie superalgebras.

Now we make some remarks concerning the case of a split supermanifold. If $(M, \mathcal{O})$ is split, then $\mathcal{T}$ is a $\mathbb{Z}$ -graded analytic sheaf, the grading being given by

$$(6) \quad \mathcal{T} = \bigoplus_{p \geq -1} \mathcal{T}_p,$$

where

$$\mathcal{T}_p = \mathrm{Der}_p \mathcal{O} = \{u \in \mathcal{T} \mid u(\mathcal{O}_q) \subset \mathcal{O}_{q+p} \text{ for all } q \in \mathbb{Z}\}.$$

As well as for the structure sheaf, the $\mathbb{Z}_2$ -grading is compatible with this $\mathbb{Z}$ -grading, i.e.

$$\mathcal{T}_{\bar{0}} = \bigoplus_{p \geq 0} \mathcal{T}_{2p}, \quad \mathcal{T}_{\bar{1}} = \bigoplus_{p \geq 0} \mathcal{T}_{2p-1}.$$

Under the bracket (3) and the grading (6), the sheaf $\mathcal{T}$ becomes a $\mathbb{Z}$ -graded sheaf of Lie superalgebras.

One also verifies that the filtration (4) of $\mathcal{T}$ coincides with that associated with the grading (6), so that

$$(7) \quad \mathcal{T}_{(p)} = \bigoplus_{r \geq p} \mathcal{T}_r.$$

Since in the split case the inclusion $\mathcal{E} \subset \mathcal{O}$ is valid, any analytic sheaf on $(M, \mathcal{O})$ is an analytic sheaf on the complex manifold M as well. In particular, $\mathcal{T}$ turns out to be a locally free $\mathbb{Z}$ -graded analytic sheaf of a finite rank on M .

Now we return to the general case. The filtration (4) of the tangent sheaf $\mathcal{T}$ of a supermanifold $(M, \mathcal{O})$ gives rise to the associated graded analytic sheaf

$$\mathrm{gr} \mathcal{T} = \bigoplus_{p \geq -1} \mathrm{gr}_p \mathcal{T},$$

where

$$\mathrm{gr}_p \mathcal{T} = \mathcal{T}_{(p)} / \mathcal{T}_{(p+1)}.$$

This sheaf is actually a graded analytic sheaf on $(M, \mathrm{gr} \mathcal{O})$. In fact, due to (5) we have

$$\mathcal{J}^p \mathcal{T}_{(q)} \subset \mathcal{T}_{(p+q)}, \quad \mathcal{J}^{p+1} \mathcal{T}_{(q+1)} \subset \mathcal{T}_{(p+q+2)} \subset \mathcal{T}_{(p+q+1)},$$

which gives rise to a multiplication $\mathrm{gr}_p \mathcal{O} \times \mathrm{gr}_q \mathcal{T} \rightarrow \mathrm{gr}_{p+q} \mathcal{T}$. We want to fix the following simple fact.

Proposition 1. *For any supermanifold $(M, \mathcal{O})$, the sheaf $\mathrm{gr} \mathcal{T}$ is isomorphic (as a $\mathbb{Z}$ -graded analytic sheaf or a $\mathbb{Z}$ -graded sheaf of complex Lie superalgebras on $(M, \mathrm{gr} \mathcal{O})$) to the tangent sheaf $\mathrm{Der}(\mathrm{gr} \mathcal{O})$.*

Proof. For any $u \in (\mathcal{T}_{(p)})_x$, $x \in M$, we have $u(\mathcal{J}_x^q) \subset \mathcal{J}_x^{q+p}$, $u(\mathcal{J}_x^{q+1}) \subset \mathcal{J}_x^{q+p+1}$, and so u defines a $\mathbb{C}$ -linear mapping $\hat{u} : (\mathrm{gr} \mathcal{O})_x \rightarrow (\mathrm{gr} \mathcal{O})_x$ which clearly is a derivation of degree p . Now, $\hat{u} = 0$ if and only if $u(\mathcal{J}_x^q) \subset \mathcal{J}_x^{q+p+1}$ for all q , i.e. if $u \in (\mathcal{T}_{(p+1)})_x$. One verifies easily that the correspondence $u + (\mathcal{T}_{(p+1)})_x \mapsto \hat{u}$ gives the desired isomorphism.

In what follows, we will identify the sheaves $\mathrm{gr} \mathcal{T}$ and $\mathrm{Der}(\mathrm{gr} \mathcal{O})$ by means of the isomorphism of Proposition 1.

In the study of a supermanifold $(M, \mathcal{O})$, the cohomology groups $H^q(M, \mathcal{T})$ of its tangent sheaf are of great importance. For example, $H^0(M, \mathcal{T})$ is the Lie superalgebra of all holomorphic vector fields on $(M, \mathcal{O})$, $H^1(M, \mathcal{T})$ is the set of all infinitesimal deformations of $(M, \mathcal{O})$, $H^2(M, \mathcal{T})$ contains the obstructions for existing of analytic deformations (see [3, 10]). In the case when $(M, \mathcal{O})$ is split, these cohomology groups can sometimes be calculated, using the well elaborated methods of complex analytic geometry, since $\mathcal{T}$ is a locally free analytic sheaf of a finite rank on the complex manifold M . In the non-split case, these methods cannot be applied directly. In what follows, we will construct a spectral sequence which helps to reduce the calculation of the tangent sheaf cohomology to the split case.

To describe the cohomology $H^q(M, \mathcal{A})$, where $\mathcal{A}$ is a sheaf of groups on M , we use Čech cochains with values in $\mathcal{A}$. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of M , and let $C^q(\mathcal{U}, \mathcal{A})$ denote the group of q -cochains of $\mathcal{U}$ with values in $\mathcal{A}$ (without any conditions of skew-simmetricity) and $d : C^{q-1}(\mathcal{U}, \mathcal{A}) \rightarrow C^q(\mathcal{U}, \mathcal{A})$ the coboundary operator. As usually, we define $Z^q(\mathcal{U}, \mathcal{A}) = \mathrm{Ker} d$ and $H^q(\mathcal{U}, \mathcal{A}) = Z^q(\mathcal{U}, \mathcal{A}) / dC^{q-1}(\mathcal{U}, \mathcal{A})$, so that $H^*(\mathcal{U}, \mathcal{A}) = \bigoplus_{q \geq 0} H^q(\mathcal{U}, \mathcal{A})$ is the cohomology of the cochain complex $C^*(\mathcal{U}, \mathcal{A}) = \bigoplus_{q \geq 0} C^q(\mathcal{U}, \mathcal{A})$. Suppose that the cover $\mathcal{U}$ is Stein, i.e. that all the open subsets U_i (and hence all their finite intersections) are Stein. If $\mathcal{A}$ is a coherent analytic sheaf on M or on $(M, \mathcal{O})$, then any Stein cover is acyclic relative to $\mathcal{A}$ (see [3, 10]), and, by the well known theorem

of Leray (see [4]), the natural mapping $H^*(\mathcal{U}, \mathcal{A}) \rightarrow H^*(M, \mathcal{A}) = \bigoplus_{q \geq 0} H^q(M, \mathcal{A})$ is an isomorphism of graded groups.

Now we make some remarks concerning the tangent sheaf cohomology $H^*(M, \mathcal{T})$.

First, the $\mathbb{Z}_2$ -grading of $\mathcal{T}$ gives rise to the $\mathbb{Z}_2$ -gradings in $C^*(\mathcal{U}, \mathcal{T})$ and $H^*(M, \mathcal{T})$ given by

$$\begin{aligned}
 C_0(\mathcal{U}, \mathcal{T}) &= \bigoplus_{q \geq 0} C^{2q}(\mathcal{U}, \mathcal{T}_0) \oplus \bigoplus_{q \geq 0} C^{2q+1}(\mathcal{U}, \mathcal{T}_1), \\
 C_1(\mathcal{U}, \mathcal{T}) &= \bigoplus_{q \geq 0} C^{2q}(\mathcal{U}, \mathcal{T}_1) \oplus \bigoplus_{q \geq 0} C^{2q+1}(\mathcal{U}, \mathcal{T}_0). \\
 H_0(M, \mathcal{T}) &= \bigoplus_{q \geq 0} H^{2q}(M, \mathcal{T}_0) \oplus \bigoplus_{q \geq 0} H^{2q+1}(M, \mathcal{T}_1), \\
 H_1(M, \mathcal{T}) &= \bigoplus_{q \geq 0} H^{2q}(M, \mathcal{T}_1) \oplus \bigoplus_{q \geq 0} H^{2q+1}(M, \mathcal{T}_0).
 \end{aligned}
 \tag{8}$$

If $(M, \mathcal{O})$ is split, then we actually have the bigradings

$$\begin{aligned}
 C^*(\mathcal{U}, \mathcal{T}) &= \bigoplus_{p, q} C^p(\mathcal{U}, \mathcal{T}_q), \\
 H^*(M, \mathcal{T}) &= \bigoplus_{p, q} H^p(M, \mathcal{T}_q),
 \end{aligned}$$

and the $\mathbb{Z}_2$ -gradings (8) are compatible with the complete degree of these two bigradings.

Secondly, the bracket (3) on $\mathcal{T}$ gives rise to bilinear operations in $C^*(\mathcal{U}, \mathcal{T})$ and $H^*(M, \mathcal{T})$ which we denote by $[,]$ as well (see [4]). The operation in $C^*(\mathcal{U}, \mathcal{T})$ is given by

$$[c, c']_{i_0, \dots, i_{p+q}} = [c_{i_0, \dots, i_p}, c'_{i_p, \dots, i_{p+q}}],
 \tag{9}$$

where $c \in Z^p(\mathcal{U}, \mathcal{T})$, $c' \in Z^q(\mathcal{U}, \mathcal{T})$, and the operation in $H^*(M, \mathcal{T})$ is inherited from that in $C^*(\mathcal{U}, \mathcal{T})$. Clearly, $C^*(\mathcal{U}, \mathcal{T})$ and $H^*(M, \mathcal{T})$ are $(\mathbb{Z} \times \mathbb{Z}_2)$ -graded algebras, and they are bigraded algebras whenever $(M, \mathcal{O})$ is split. They also are superalgebras with respect to the $\mathbb{Z}_2$ -gradings (8) ($H^*(M, \mathcal{T})$ actually is a Lie superalgebra).

Now we remind some known facts concerning the classification of the non-split supermanifolds having the given associated split one.

Let $(M, \mathcal{O})$ be a supermanifold. Consider the sheaf $\mathcal{A}ut \mathcal{O}$ of automorphisms of the structure sheaf $\mathcal{O}$ (as usually, any automorphism is even and maps each stalk $\mathcal{O}_x$, $x \in M$, onto itself). This is a sheaf of groups. Its sections are the automorphisms of $(M, \mathcal{O})$ that are identical on M . Clearly, any $a \in \mathcal{A}ut \mathcal{O}_x$, $x \in M$, maps $\mathcal{J}_x$ onto itself, and hence preserves the filtration (2) and induces a germ of an automorphism of $\mathcal{O}_{gr}$. By definition, a induces the identity mapping on $\mathcal{F}_x = \mathcal{O}_x / \mathcal{J}_x$. Consider the filtration

$$\mathcal{A}ut \mathcal{O} = \mathcal{A}ut_{(1)} \mathcal{O} \supset \dots \supset \mathcal{A}ut_{(p)} \mathcal{O} \supset \dots,
 \tag{10}$$

where

$$\mathcal{A}ut_{(p)} \mathcal{O}_x = \{a \in \mathcal{A}ut \mathcal{O}_x \mid a(f) - f \in \mathcal{J}_x^p \text{ for all } f \in \mathcal{O}_x\}, \quad x \in M.$$

Clearly, any $\mathcal{A}ut_{(p)}\mathcal{O}$ is a subsheaf of invariant subgroups of $\mathcal{A}ut\mathcal{O}$. It is the subsheaf of trivial subgroups whenever $p \geq m+1$, where, as above, m denotes the odd dimension of $(M, \mathcal{O})$. We also notice that the group $\text{Aut}(M, \mathcal{O})$ of automorphisms of the supermanifold $(M, \mathcal{O})$ acts on $\mathcal{A}ut\mathcal{O}$ by automorphisms of this sheaf, leaving any $\mathcal{A}ut_{(p)}\mathcal{O}$ invariant.

We apply the sheaves of automorphisms to our classification problem. Let $\mathbf{E}$ be a holomorphic vector bundle over a complex manifold M , and $\mathcal{E}$ the sheaf of holomorphic sections of $\mathbf{E}$. Consider the split supermanifold $(M, \mathcal{O}_{\text{gr}})$, where $\mathcal{O}_{\text{gr}} = \bigwedge \mathcal{E}$. We want to describe the family of all supermanifolds $(M, \mathcal{O})$ such that $\text{gr}\mathcal{O} = \mathcal{O}_{\text{gr}}$. Let $\text{Aut}\mathbf{E}$ be the group of all holomorphic automorphisms of the vector bundle $\mathbf{E}$. Since $\text{Aut}\mathbf{E} \subset \text{Aut}(M, \mathcal{O}_{\text{gr}})$, this group acts on the sheaf $\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$, and hence on its 1-cohomology set. The following assertion was proved in [5].

Proposition 2. *To any supermanifold $(M, \mathcal{O})$ satisfying $\text{gr}\mathcal{O} = \mathcal{O}_{\text{gr}}$ there corresponds an element of the set $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$. This correspondence gives rise to a bijection between the isomorphy classes of supermanifolds, satisfying the above condition, and the orbits of the group $\text{Aut}\mathbf{E}$ on $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$.*

Let us describe the correspondence mentioned in Proposition 2. Let $(M, \mathcal{O})$ be a supermanifold such that $\text{gr}\mathcal{O} = \mathcal{O}_{\text{gr}}$. We can choose an open cover $\mathcal{U} = (U_i)_{i \in I}$ of M such that there exist isomorphisms of sheaves $\sigma_i : \mathcal{O}|_{U_i} \rightarrow \mathcal{O}_{\text{gr}}|_{U_i}$, $i \in I$, inducing the identity isomorphisms of $\text{gr}\mathcal{O}|_{U_i}$ onto $\mathcal{O}_{\text{gr}}|_{U_i}$. Setting $g_{ij} = \sigma_i \circ \sigma_j^{-1}$, we get a 1-cocycle $g = (g_{ij}) \in Z^1(\mathcal{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$. Its cohomology class $\gamma \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ does not depend of the choice of σ_i .

Clearly, the given split supermanifold $(M, \mathcal{O}_{\text{gr}})$ corresponds to the distinguished point $e \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ which is fixed under the action of $\text{Aut}\mathbf{E}$.

Now we use the even items $\mathcal{A}ut_{(2p)}\mathcal{O}_{\text{gr}}$ of the filtration (10) of $\mathcal{O}_{\text{gr}}$, $p = 1, \dots, [\frac{m}{2}] + 1$, where m is the odd dimension of $(M, \mathcal{O}_{\text{gr}})$, which is equal to the rank of $\mathbf{E}$. Denoting by $H_{(2p)}$ the image of the natural mapping $H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}_{\text{gr}}) \rightarrow H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$, we get the $(\text{Aut}\mathbf{E})$ -invariant filtration

$$(11) \quad H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}) = H_{(2)} \supset \dots \supset H_{(2p)} \supset \dots \supset H_{(2([\frac{m}{2}]+1))} = \{e\}.$$

Take $\gamma \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$. Following [9], we define the *order* of γ (denoted $o(\gamma)$) the maximal one of the numbers $2p$ such that $\gamma \in H_{(2p)}$. The order of a supermanifold $(M, \mathcal{O})$ such that $\text{gr}\mathcal{O} = \mathcal{O}_{\text{gr}}$ is, by definition, the order of the corresponding cohomology class.

There exists a natural relationship between automorphisms and derivations of $\mathcal{O}_{\text{gr}}$. Let $\mathcal{T}_{\text{gr}} = \text{Der}\mathcal{O}_{\text{gr}}$ be the tangent sheaf of $(M, \mathcal{O}_{\text{gr}})$. Then we have the filtration (see (5))

$$(\mathcal{T}_{\text{gr}})_{\bar{0}} \supset (\mathcal{T}_{\text{gr}})_{(2)\bar{0}} \supset \dots \supset (\mathcal{T}_{\text{gr}})_{(2p)\bar{0}} \supset \dots,$$

where, by (7),

$$(\mathcal{T}_{\text{gr}})_{(2p)\bar{0}} = \bigoplus_{r \geq p} (\mathcal{T}_{\text{gr}})_{2r}.$$

Using the exponential series, define the mapping $\exp : (\mathcal{T}_{\text{gr}})_{(2)\bar{0}} \rightarrow \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$. Since any derivation from $(\mathcal{T}_{\text{gr}})_{(2)\bar{0}}$ is nilpotent, this series is actually a polynomial. One proves that $\exp$ is an isomorphism of sheaves of sets (but not of sheaves of groups, in general) and maps $(\mathcal{T}_{\text{gr}})_{(2p)\bar{0}}$ onto $\mathcal{A}ut_{(2p)}\mathcal{O}_{\text{gr}}$ [2, 9].

Choose a class $\gamma \in H^1(M, \text{Aut}_{(2)} \mathcal{O}_{\text{gr}})$. Clearly, $o(\gamma) = 2p$ if and only if γ can be represented by a cocycle $g \in Z^1(\mathcal{U}, \text{Aut}_{(2p)} \mathcal{O}_{\text{gr}})$. We have $g = \exp z$, where $z = z_{2p} + z_{2p+2} + \dots$, $z_{2r} \in C^1(\mathcal{U}, (\mathcal{T}_{\text{gr}})_{2r})$. One proves (see [9]) that $z_{2r} \in Z^1(\mathcal{U}, (\mathcal{T}_{\text{gr}})_{2r})$ for all $r < 2p$. In particular, the cocycle z_{2p} gives rise to a cohomology class $\lambda_p(\gamma) \in H^1(M, (\mathcal{T}_{\text{gr}})_{2p})$ which is uniquely determined by γ .

2. THE SPECTRAL SEQUENCE

Now we pass to the construction of a spectral sequence for the cohomology of the tangent sheaf $\mathcal{T}$ of an arbitrary supermanifold $(M, \mathcal{O})$ of dimension $m|n$. We fix an open Stein cover $\mathcal{U} = (U_i)_{i \in I}$ of M and consider the corresponding Čech cochain complex $C^*(\mathcal{U}, \mathcal{T}) = \bigoplus_{p \geq 0} C^p(\mathcal{U}, \mathcal{T})$. The filtration (4) gives rise to the filtration

$$(12) \quad C^*(\mathcal{U}, \mathcal{T}) = C_{(-1)} \supset C_{(0)} \supset \dots \supset C_{(p)} \supset \dots \supset C_{(m+1)} = 0$$

of this complex by the subcomplexes

$$C_{(p)} = C^*(\mathcal{U}, \mathcal{T}_{(p)}).$$

Denoting by $H(M, \mathcal{T})_{(p)}$ the image of the natural mapping $H^*(M, \mathcal{T}_{(p)}) \rightarrow H^*(M, \mathcal{T})$, we get the filtration

$$(13) \quad H^*(M, \mathcal{T}) = H(M, \mathcal{T})_{(-1)} \supset \dots \supset H(M, \mathcal{T})_{(p)} \supset \dots \supset H(M, \mathcal{T})_{(m+1)} = 0.$$

Note that (12) is a filtration of the graded differential algebra $C^*(\mathcal{U}, \mathcal{T})$ (under the bracket (9)) by graded differential subalgebras, and hence (13) is a filtration of the graded algebra $H^*(M, \mathcal{T})$ by graded subalgebras. Denote by $\text{gr } H^*(M, \mathcal{T})$ the bigraded algebra associated with the filtration (13); its bigrading is given by

$$\text{gr } H^*(M, \mathcal{T}) = \bigoplus_{\substack{p \geq -1 \\ q \geq 0}} \text{gr}_p H^q(M, \mathcal{T}).$$

By the general procedure, invented by Leray, the filtration (12) gives rise to a spectral sequence of bigraded algebras E_r converging to $E_\infty \simeq \text{gr } H^*(M, \mathcal{T})$. It is constructed in the following way.

For any $p \geq -1, r \geq 0$, define the vector spaces

$$C_r^p = \{c \in C_{(p)} \mid dc \in C_{(p+r)}\}.$$

Then, for a fixed p , we have

$$C_{(p)} = C_0^p \supset \dots \supset C_r^p \supset C_{r+1}^p \supset \dots$$

The r -th term of the spectral sequence is defined by

$$E_r = \bigoplus_{r=-1}^m E_r^p, \quad r \geq 0,$$

where

$$E_r^p = C_r^p / C_{r-1}^{p+1} + dC_{r-1}^{p-r+1}.$$

The bracket (9) gives rise to a structure of the graded algebra in E_r . Since $d(C_r^p) \subset C_r^{p+r}$, d induces a derivation d_r of E_r of degree r such that $d_r^2 = 0$. Then E_{r+1} is naturally isomorphic to the homology algebra $H(E_r, d_r)$. Denoting $Z_r = \text{Ker } d_r$, we have the natural mapping $\kappa_{r+1}^r : Z_r \rightarrow E_{r+1}^{r, r+1}$. For any $s > r$, denote $\kappa_s^r = \kappa_s^{s-1} \circ \dots \circ \kappa_{r+1}^r$ (this composition is not defined on the entire Z_r).

The $\mathbb{Z}_2$ -grading (8) in $C^*(\mathcal{U}, T)$ gives rise to certain $\mathbb{Z}_2$ -gradings in C_r^p and E_r^p , turning E_r into a superalgebra. Clearly, the coboundary operator d in $C^*(\mathcal{U}, T)$ is odd. It follows that the coboundary d_r is odd for any $r \geq 0$.

The superalgebras E_r are also endowed with a second $\mathbb{Z}$ -grading. Namely, for any $q \in \mathbb{Z}$, set

$$\begin{aligned} C_r^{p,q} &= C_r^p \cap C^{p+q}(\mathcal{U}, T), \\ E_r^{p,q} &= C_r^{p,q} / C_{r-1}^{p+1, q-1} + dC_{r-1}^{p-r+1, q+r-2}. \end{aligned}$$

Then

$$E_r = \bigoplus_{p,q} E_r^{p,q}.$$

Clearly,

$$(14) \quad d_r(E_r^{p,q}) \subset E_r^{p+r, q-r+1}$$

for any r, p, q .

One sees easily that $C_r^{p,q} = 0$ for all p, r if $q \leq -(m+1)$. Therefore, for a fixed q , we have $d(C_r^{p,q}) = 0$ for all $p \geq -1$ and all $r \geq q - m + 2$. This implies that $\kappa_{r+1}^r : E_r^{p,q} \rightarrow E_{r+1}^{p,q}$ is an isomorphism for all p and $r \geq r_0(q) = q - m + 2$. Setting $E_\infty^{p,q} = E_{r_0(q)}^{p,q}$, we get the bigraded superalgebra

$$E_\infty = \bigoplus_{p,q} E_\infty^{p,q}.$$

Now we prove certain properties of the spectral sequence (E_r) . Some of them are well known and are valid in a more general situation.

Proposition 3. *The first three terms of the spectral sequence (E_r) can be identified with the following bigraded algebras:*

$$\begin{aligned} E_0 &= C^*(\mathcal{U}, T_{\text{gr}}), \\ E_1 &= E_2 = H^*(M, T_{\text{gr}}). \end{aligned}$$

Here

$$\begin{aligned} E_0^{p,q} &= C^{p+q}(\mathcal{U}, (T_{\text{gr}})_p), \\ E_1^{p,q} &= E_2^{p,q} = H^{p+q}(M, (T_{\text{gr}})_p). \end{aligned}$$

We have $d_{2k+1} = 0$ and, hence, $E_{2k+1} = E_{2k+2}$ for all $k \geq 0$.

Proof. By definition, we have

$$E_0^p = C_{(p)}/C_{(p+1)}, \quad p \geq -1,$$

where the coboundary operator d_0 of degree 0 is induced by $d : C_{(p)} \rightarrow C_{(p)}$. On the other hand, the exact sequence

$$(15) \quad 0 \rightarrow \mathcal{T}_{(p+1)} \rightarrow \mathcal{T}_{(p)} \rightarrow (\mathcal{T}_{\text{gr}})_p \rightarrow 0$$

and Theorem B for Stein supermanifolds (see [3, 10]) imply the exact sequence

$$0 \rightarrow \mathcal{T}_{(p+1)}(U) \rightarrow \mathcal{T}_{(p)}(U) \rightarrow (\mathcal{T}_{\text{gr}})_p(U) \rightarrow 0$$

for any Stein open subset $U \subset M$. Therefore

$$C^*(\mathcal{U}, (\mathcal{T}_{\text{gr}})_p) \simeq C_{(p)}/C_{(p+1)} = E_0^p, \quad p \geq -1.$$

One sees easily that this is an isomorphism of complexes and that the resulting isomorphism $C^*(\mathcal{U}, \mathcal{T}_{\text{gr}}) \simeq E_0$ is an isomorphism of bigraded algebras. It follows that

$$E_1 \simeq H(E_0, d_0) \simeq H^*(\mathcal{U}, \mathcal{T}_{\text{gr}}) \simeq H^*(M, \mathcal{T}_{\text{gr}}).$$

We see that

$$(E_0)_{\bar{0}} = \bigoplus_{p,q} E_0^{p,2q}, \quad (E_0)_{\bar{1}} = \bigoplus_{p,q} E_0^{p,2q+1}.$$

Hence, the parity of elements lying in $E_0^{p,q}$ is the same as the parity of q . Passing to the homology, we see that this is true for any E_r , $r \geq 0$. This implies, by (14), that the coboundary operator d_{2k+1} is even. Since it is odd, we have $d_{2k+1} = 0$ for any $k \geq 0$.

Proposition 4. *There is the following identification of bigraded algebras:*

$$E_\infty = \text{gr } H^*(M, \mathcal{T}),$$

where

$$E_\infty^{p,q} = \text{gr}_p H^{p+q}(M, \mathcal{T}).$$

Proof. Clearly, for $r \geq r_0(q)$ we have $C_r^{p,q} = Z^{p+q}(\mathcal{U}, \mathcal{T}_{(p)})$. It follows that

$$\begin{aligned} E_\infty^{p,q} &= Z^{p+q}(\mathcal{U}, \mathcal{T}_{(p)})/Z^{p+q}(\mathcal{U}, \mathcal{T}_{(p+1)}) + dC^{p+q-1}(\mathcal{U}, \mathcal{T}) \cap Z^{p+q}(\mathcal{U}, \mathcal{T}_{(p)}) \\ &= H^{p+q}(M, \mathcal{T})_{(p)}/(Z^{p+q}(\mathcal{U}, \mathcal{T}_{(p+1)})/dC^{p+q-1}(\mathcal{U}, \mathcal{T}) \cap Z^{p+q}(\mathcal{U}, \mathcal{T}_{(p+1)})) \\ &= H^{p+q}(M, \mathcal{T})_{(p)}/H^{p+q}(M, \mathcal{T})_{(p+1)} = \text{gr}_p H^{p+q}(M, \mathcal{T}). \end{aligned}$$

Corollary. *If M is compact, then*

$$\dim H^k(M, \mathcal{T}) = \sum_{p+q=k} \dim E_{\infty}^{p,q}.$$

Proof. In fact, if M is compact, then all cohomology groups with values in a coherent analytic sheaf on $(M, \mathcal{O})$ or M are of finite dimension.

Consider the cohomology exact sequence

$$H^{p+q}(M, \mathcal{T}_{(p+1)}) \rightarrow H^{p+q}(M, \mathcal{T}_{(p)}) \xrightarrow{\sigma} H^{p+q}(M, (\mathcal{T}_{\text{gr}})_p) = E_2^{p,q}$$

associated with (15). We would like to interpret the subspace $\text{Im } \sigma \subset H^{p+q}(M, (\mathcal{T}_{\text{gr}})_p)$ by means of our spectral sequence. An element $a \in E_2^{p,q}$ shall be called a *permanent cocycle* if $d_2 a = 0$, $d_4(\kappa_4^2 a) = 0$, $d_6(\kappa_6^2 a) = 0$, etc. Let us denote by $Z_{\infty}^{p,q}$ the subspace of permanent cocycles. Any $a \in Z_{\infty}^{p,q}$ determines an element $a^* \in E_{\infty}^{p,q}$, so we have a linear mapping $\kappa_{\infty}^2 : Z_{\infty}^{p,q} \rightarrow H_{\infty}^{p,q}$.

We note that $\text{Im } \sigma \subset Z_{\infty}^{p,q}$. In fact, if $a = \sigma(b)$, where $b \in H^{p+q}(M, \mathcal{T}_{(p)})$, then b is represented by a cocycle $z \in Z^{p+q}(\mathcal{U}, \mathcal{T}_{(p)})$, and a is the coset of the same z regarded as an element of $E_2^{p,q}$. Since $dz = 0$, a is a permanent cocycle of the spectral sequence.

Proposition 5. *We have the commutative diagram*

$$\begin{array}{ccc} H^{p+q}(M, \mathcal{T}_{(p)}) & \xrightarrow{\sigma} & Z_{\infty}^{p,q} \subset E_2^{p,q} = H^{p+q}(M, (\mathcal{T}_{\text{gr}})_p) \\ \downarrow & & \downarrow \kappa_{\infty}^2 \\ H^{p+q}(M, \mathcal{T})_{(p)} & \rightarrow & E_{\infty}^{p,q} = \text{gr}_p H^{p+q}(M, \mathcal{T}) \rightarrow 0, \end{array}$$

where the bottom sequence is exact, and the vertical mappings are natural surjections.

In the case when $p + q = 0$, we have $\sigma(H^0(M, \mathcal{T}_{(p)})) = Z_{\infty}^{p,-p}$, and $\kappa_{\infty}^2 : Z_{\infty}^{p,-p} \rightarrow E_{\infty}^{p,-p} = \text{gr}_p H^0(M, \mathcal{T})$ is an isomorphism.

Proof. The commutativity of the diagram follows easily from the definitions. Clearly, $H^0(M, \mathcal{T})_{(p)} = H^0(M, \mathcal{T}_{(p)})$, which implies that κ_{∞}^2 is bijective for $q = -p$. Since the bottom line is exact, σ is surjective.

Now we prove our main result concerning the first non-zero coboundary operators among $d_2, d_4, \dots$. As usually, we denote by $\text{ad } x$ the adjoint operator determined by an element x of an algebra L with the product denoted by $[\cdot, \cdot]$, i.e. $(\text{ad } x)y = [x, y]$, $y \in L$. We may suppose that for each $i \in I$ there exists an isomorphism of sheaves $\sigma_i : \mathcal{O}|U_i \rightarrow \mathcal{O}_{\text{gr}}|U_i$, inducing the identity isomorphism $(\text{gr } \mathcal{O})|U_i \rightarrow \mathcal{O}_{\text{gr}}|U_i$. By Proposition 2, $(M, \mathcal{O})$ corresponds to the cohomology class γ of the 1-cocycle $g = (g_{ij}) \in Z^1(\mathcal{U}, \text{Aut}_{(2)} \mathcal{O}_{\text{gr}})$, where $g_{ij} = \sigma_i \circ \sigma_j^{-1}$. If $\sigma(\gamma) = 2k$, then we may choose σ_i , $i \in I$, in such a way that $g \in Z^1(\mathcal{U}, \text{Aut}_{(2k)} \mathcal{O}_{\text{gr}})$. We can write $g = \exp z$, where $z \in C^1(\mathcal{U}, (\mathcal{T}_{\text{gr}})_{(2)\bar{0}})$.

We will identify the differential algebras (E_0, d_0) and $(C^*(\mathcal{U}, \mathcal{T}_{\text{gr}}), d)$ via the isomorphism of Proposition 3. On the other hand, the isomorphism σ_i , $i \in I$ gives rise to the injective homomorphism of sheaves $\psi_i : (\mathcal{T}_{\text{gr}})_p|U_i \rightarrow \mathcal{T}_{(p)}|U_i$ given by $\psi_i(v) = \sigma_i \circ v \circ \sigma_i^{-1}$. Clearly, $\psi_i : (\mathcal{T}_{\text{gr}})_{(p)}|U_i = \sum_{r \geq p} (\mathcal{T}_{\text{gr}})_r|U_i \rightarrow \mathcal{T}_{(p)}|U_i$ is an isomorphism of sheaves for any

$i \in I$, $p \geq -1$. These local sheaf isomorphisms permit us to define an isomorphism of graded cochain groups

$$\psi : C^*(\mathcal{U}, T_{\text{gr}}) \rightarrow C^*(\mathcal{U}, T)$$

such that

$$\psi : C^*(\mathcal{U}, (T_{\text{gr}})_{(p)}) \rightarrow C^*(\mathcal{U}, T_{(p)}), \quad p \geq -1.$$

We give it by

$$\psi(c)_{i_0 \dots i_q} = \sigma_{i_0} \circ c_{i_0 \dots i_q} \circ \sigma_{i_0}^{-1}$$

for any $(i_0, \dots, i_q)$ such that $U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset$. In general, ψ is not an isomorphism of complexes. Nevertheless, we can express explicitly the coboundary d of the complex $C^*(\mathcal{U}, T)$ by means of d_0 and g .

Proposition 6. For any $c \in C^q(\mathcal{U}, T_{\text{gr}}) = E_0^q$, we have

$$(\psi^{-1}(d\psi(c)))_{i_0 \dots i_{q+1}} = (d_0 c)_{i_0 \dots i_{q+1}} + \sum_{l \geq 1} \frac{1}{l!} (\text{ad } z_{i_0 i_1})^l c_{i_1 \dots i_{q+1}}.$$

Proof. We can write

$$\begin{aligned} (d\psi(c))_{i_0 \dots i_{q+1}} &= \sum_{\alpha=0}^{q+1} (-1)^\alpha \psi(c)_{i_0 \dots \hat{i}_\alpha \dots i_{q+1}} \\ &= \sum_{\alpha=1}^{q+1} (-1)^\alpha \psi(c)_{i_0 \dots \hat{i}_\alpha \dots i_{q+1}} + \psi(c)_{i_1 \dots i_{q+1}} \\ &= \sigma_{i_0} \circ \left(\sum_{\alpha=1}^{q+1} (-1)^\alpha c_{i_0 \dots \hat{i}_\alpha \dots i_{q+1}} \right) \circ \sigma_{i_0}^{-1} + \sigma_{i_1} \circ c_{i_1 \dots i_{q+1}} \circ \sigma_{i_1}^{-1} \\ &= \sigma_{i_0} \circ ((d_0 c)_{i_0 \dots i_{q+1}} - c_{i_1 \dots i_{q+1}}) \circ \sigma_{i_0}^{-1} + \sigma_{i_1} \circ c_{i_1 \dots i_{q+1}} \circ \sigma_{i_1}^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} (\psi^{-1}(d\psi(c)))_{i_0 \dots i_{q+1}} &= \sigma_{i_0}^{-1} \circ (d\psi(c))_{i_0 \dots i_{q+1}} \sigma_{i_0} \\ &= (d_0 c)_{i_0 \dots i_{q+1}} - c_{i_1 \dots i_{q+1}} + g_{i_0 i_1} \circ c_{i_1 \dots i_{q+1}} \circ g_{i_0 i_1}^{-1} \\ &= (d_0 c)_{i_0 \dots i_{q+1}} - c_{i_1 \dots i_{q+1}} + (\exp \text{ad } z_{i_0 i_1})(c_{i_1 \dots i_{q+1}}) \\ &= (d_0 c)_{i_0 \dots i_{q+1}} + \sum_{l \geq 1} \frac{1}{l!} (\text{ad } z_{i_0 i_1})^l c_{i_1 \dots i_{q+1}}. \end{aligned}$$

This implies our assertion.

This proposition makes it possible, to calculate the spectral sequence (E_r) whenever d_0 and the cochain z are known. Now we find the explicit form of certain coboundary operators d_r , $r \geq 2$.

Theorem 1. Suppose that the supermanifold $(M, \mathcal{O})$ has order $2k$ and denote by $\gamma \in H_{(2k)}$ the cohomology class corresponding to $(M, \mathcal{O})$ by Proposition 2. Then $d_r = 0$ for $r = 1, \dots, 2k - 1$, and $d_{2k} = \text{ad } \lambda_k(\gamma)$.

Proof. Take a cocycle $c \in E_0^{p, q-p}$, $d_0 c = 0$, and denote by c^* its cohomology class in $E_1^{p, q-p}$. Clearly, c and c^* are represented by the cochain $\psi(c) \in C_0^p$. By Proposition 6,

$$(\psi^{-1}(d\psi(c)))_{i_0 \dots i_{q+1}} = \sum_{l \geq 1} \frac{1}{l!} (\text{ad } z_{i_0 i_1})^l c_{i_1 \dots i_{q+1}}.$$

Writing $z = z^{(2k)} + z^{(2k+2)} + \dots$, where $z^{(2l)} \in C^1(\mathcal{U}, (\mathcal{T}_{\text{gr}})_{2l})$, we see that

$$(\psi^{-1}(d\psi(c)))_{i_0 \dots i_{q+1}} = [z_{i_0 i_1}^{(2k)}, c_{i_1 \dots i_{q+1}}] + u_{i_0 \dots i_{q+1}},$$

where $u \in C_{(2k+2)}$. This means that

$$\psi^{-1}(d\psi(c)) = [z^{(2k)}, c] + u,$$

whence $d_1 = d_2 = \dots = d_{(2k-1)} = 0$. Identifying $E_{(2k)}$ with E_1 , we also see that $d_{2k} c^*$ is represented by the cochain $\psi([z^{(2k)}, c])$. It follows that

$$d_{2k} c^* = [\lambda_k(\gamma), c^*],$$

since $z^{(2k)}$ is a cocycle representing $\lambda_k(\gamma)$.

Theorem 1 and Proposition 5 imply the following assertion, proved in [7], for $k = 1$ and $p = -1, 0$, in another way.

Corollary. If $\alpha(\gamma) = k$ and a vector field $v \in H^0(M, \mathcal{T}_p)$ lies in $\text{Im } \sigma$, then $[\lambda_k(\gamma), v] = 0$. In the case when $m \leq 2k + 1$, the converse is also true.

3. AN EXAMPLE

As an application of our spectral sequence, we are going to calculate here the tangent sheaf cohomology of a series of homogeneous supermanifolds considered in [2]. The reduction of these supermanifolds is the projective line $M = \mathbb{CP}^1$. Namely, let $(M, \mathcal{O}) = Q_{(q)}^{1|m}$ be the superquadric in $\mathbb{CP}^{2|m}$, determined by a non-degenerate skew-symmetric bilinear form ω of rank $q \leq \frac{m}{2}$ in $\mathbb{C}^m$. In homogeneous coordinates $z_0, z_1, z_2, \zeta_1, \dots, \zeta_m$ it is given by the equation

$$(16) \quad z_0^2 - z_1 z_2 + \omega(\zeta) = 0,$$

where we may suppose that

$$\omega(\zeta) = \sum_{j=1}^q \zeta_j \zeta_{r+j}.$$

Introducing the local coordinates

$$\begin{aligned} x &= \frac{z_0}{z_1}, \quad \xi_j = \frac{\zeta_j}{z_1} \text{ in } U = \{z_1 \neq 0\}; \\ y &= \frac{z_0}{z_2}, \quad \eta_j = \frac{\zeta_j}{z_2} \text{ in } V = \{z_2 \neq 0\}, \end{aligned}$$

we get the following transition functions in $U \cap V$:

$$\begin{aligned} y &= x(x^2 + \omega(\xi))^{-1}, \\ \eta_j &= (x^2 + \omega(\xi))^{-1} \xi_j, \quad j = 1, \dots, m. \end{aligned}$$

It follows that the corresponding split supermanifold is given by the transition functions

$$\begin{aligned} y &= x^{-1}, \\ \eta_j &= x^{-2} \xi_j, \quad j = 1, \dots, m, \end{aligned}$$

and so corresponds to the vector bundle $\mathbf{E} = m\mathbf{L}_{-2}$ over M , where we denote by $\mathbf{L}_k$ the holomorphic line bundle over $\mathbf{CP}^1$, having the degree $k \in \mathbf{Z}$. The cohomology class $\gamma \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ (see Proposition 2) is given by the cocycle $g \in \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}(U \cap V)$ such that

$$\begin{aligned} g(x^{-1}) &= x(x^2 + \omega(\xi))^{-1}, \\ g(x^{-2})g(\xi_j) &= (x^2 + \omega(\xi))^{-1} \xi_j, \quad j = 1, \dots, m. \end{aligned}$$

It follows that

$$\begin{aligned} g(x) &= x + x^{-1}\omega(\xi), \\ g(\xi_j) &= (1 + x^{-2}\omega(\xi))\xi_j, \quad j = 1, \dots, m. \end{aligned}$$

Therefore the 2-component $z^{(2)}$ of the cochain $z = \log g$ has the form

$$(17) \quad z^{(2)} = x^{-1}\omega \frac{\partial}{\partial x} + x^{-2}\omega(\xi)\varepsilon,$$

where

$$\varepsilon = \sum_{j=1}^m \xi_j \frac{\partial}{\partial \xi_j}.$$

This cocycle determines a cohomology class $\zeta \in H^1(M, (\mathcal{T}_{\text{gr}})_2)$. It is non-zero (see [2]), and hence $\alpha(\gamma) = 2$, and $\zeta = \lambda_1(\gamma)$.

Now we have to compute the groups $E_2^{p,q-p} = H^q(M, (\mathcal{T}_{\text{gr}})_p)$. To do this, we use the theorem of Bott on the cohomology of homogeneous vector bundles combined with the calculation of the supertangent bundle, made in [2]. We regard $\mathbf{CP}^1$ as the homogeneous space of the Lie group $G = \text{SL}_m(\mathbf{C}) \times \text{SL}_2(\mathbf{C})$ with the stabilizer $H = \text{SL}_m(\mathbf{C}) \times B_-$ of the point $x_0 = \{x = 0\} \in U$, where B_- is the Borel subgroup corresponding to the (unique) negative root $-\lambda$ of $\text{SL}_2(\mathbf{C})$. Let us denote by $\mu_1, \dots, \mu_m$ the weights of the standard representation ρ of $\text{SL}_m(\mathbf{C})$, by Ad the adjoint representation of this group, and by σ the character of B_- satisfying $d\sigma = \lambda$. Let φ_s , $s \geq 0$, be the irreducible representation of $\text{SL}_2(\mathbf{C})$ (of dimension $s+1$) with the highest weight $s\lambda$.

It is well known that there is an exact sequence

$$(18) \quad 0 \rightarrow \mathcal{A}_{p+1} \rightarrow (\mathcal{T}_{\text{gr}})_p \rightarrow \mathcal{B}_p \rightarrow 0, \quad p \geq 1,$$

where

$$\mathcal{A}_p = \mathcal{E}^* \otimes \bigwedge^p \mathcal{E}, \mathcal{B}_p = \Theta \otimes \bigwedge^p \mathcal{E},$$

Θ being the tangent sheaf of M .

Clearly, $\mathbf{E} = m\mathbf{L}_{-2}$ can be regarded as the G -homogeneous vector bundle over $\mathbf{CP}^1$ determined by the representation $\rho \otimes \sigma^{-2}$ of H . Then $\mathcal{A}_p$ and $\mathcal{B}_p$ are determined by the representations $\rho^* \bigwedge^p \rho \otimes \sigma^{2(1-p)}$ and $\bigwedge^p \rho \otimes \sigma^{2(1-p)}$ of H , respectively, since the isotropy representation for $\mathbf{CP}^1$ is $1 \otimes \sigma^2$. Using the theorem of Bott, we easily get the following result.

Proposition 7. *In the following table the non-zero groups $H^q(M, \mathcal{A}_p)$, $H^q(M, \mathcal{B}_p)$ with the induced representations of G are indicated:*

$$\begin{array}{ccc} H^0(M, \mathcal{A}_0) & H^0(M, \mathcal{A}_1) & H^1(M, \mathcal{A}_p), 2 \leq p \leq m \\ \rho^* \otimes \varphi_2 & 1 + \text{Ad} \otimes \varphi_0 & \rho^* \bigwedge^p \rho \otimes \varphi_{2p-4} \\ \\ H^0(M, \mathcal{B}_0) & H^0(M, \mathcal{B}_1) & H^1(M, \mathcal{B}_p), 2 \leq p \leq m \\ 1 \otimes \varphi_2 & \rho \otimes \varphi_0 & \bigwedge^p \rho \otimes \varphi_{2p-4}. \end{array}$$

Note that for $p \leq m-1$ we have

$$(19) \quad \rho^* \bigwedge^p \rho \simeq \bigwedge^{p-1} \rho + \psi_p,$$

where ψ_p is the irreducible representation of $\text{SL}_m(\mathbb{C})$ with the highest weight $\mu_1 + \dots + \mu_p - \mu_m$.

Using (18) and some results of [2], we can deduce from this the following description of the cohomology of $\mathcal{T}_{\text{gr}}$.

Proposition 8. *In the following table the non-zero groups $H^q(M, (\mathcal{T}_{\text{gr}})_p)$ with the induced representations of G are indicated:*

$$\begin{array}{ccc} H^0(M, (\mathcal{T}_{\text{gr}})_{-1}) & & H^0(M, (\mathcal{T}_{\text{gr}})_0) \\ \rho^* \otimes \varphi_2 & & 1 + \text{Ad} \otimes \varphi_0 + 1 \otimes \varphi_2 \\ \\ H^1(M, (\mathcal{T}_{\text{gr}})_1) \text{ for } m \geq 3 & & H^1(M, (\mathcal{T}_{\text{gr}})_p), 2 \leq p \leq m \\ \psi_2 \otimes \varphi_0 & & \bigwedge^p \rho \otimes \varphi_{2p-4} + \bigwedge^p \rho \otimes \varphi_{2p-2} + \psi_{p+1} \otimes \varphi_{2p-2}. \end{array}$$

Proof. For $p = -1, 0$ or $p = 2$, the assertion follows immediately from Proposition 7. In the case $p = 1$, we deduce from (18) and Proposition 7 the following exact sequence

$$0 \rightarrow H^0(M, (\mathcal{T}_{\text{gr}})_1) \rightarrow H^0(M, \mathcal{B}_1) \xrightarrow{\delta^*} H^1(M, \mathcal{A}_2) \rightarrow H^1(M, (\mathcal{T}_{\text{gr}})_1) \rightarrow 0.$$

By Proposition 18 of [2],

$$(\mathcal{T}_{\text{gr}})_1 \simeq 2\mathcal{F}(-1) \oplus \frac{1}{2}(m-2)(m^2-m+2)\mathcal{F}(-2),$$

where we denote by $\mathcal{F}(k)$ the sheaf of holomorphic sections of the bundle $\mathbf{L}_k$. Hence $H^0(M, (\mathcal{T}_{\text{gr}})_1) = 0$ for all m and $H^1(M, (\mathcal{T}_{\text{gr}})_1) = 0$ for $m = 2$. Thus, ∂^* is injective, and our assertion follows from Proposition 7 and (19).

For simplicity, we consider now the case when $m = 2q$ or $2q + 1$, i.e. when ω has the maximal possible rank. Denote the corresponding superquadric by $\mathbf{Q}^{1|m}$. Let $\mathfrak{osp}_{3|m}(\mathbb{C})$ be the subalgebra of $\mathfrak{gl}_{3|m}(\mathbb{C})$ consisting of linear transformations that annihilate the left-hand side of (16). If $m = 2q$ is even, then this is a simple Lie superalgebra, entering into the orthosymplectic series (see [6]). For odd q , it is not simple. Our main result here is as follows.

Theorem 2. *For the superquadric $(M, \mathcal{O}) = \mathbf{Q}^{1|m}$, $m \geq 2$, we have*

$$H^0(M, \mathcal{T}) \simeq \mathfrak{osp}_{3|m}(\mathbb{C})$$

(as the Lie superalgebras);

$$\begin{aligned} H^1(M, \mathcal{T}_0) &\simeq \bigoplus_{p \geq 1} H^1(M, \mathcal{A}_{2p+1}) \oplus \bigoplus_{p \geq 2} H^1(M, \mathcal{B}_{2p}), \\ H^1(M, \mathcal{T}_1) &\simeq H^1(M, (\mathcal{T}_{\text{gr}})_1) \oplus \bigoplus_{p \geq 2} H^1(M, \mathcal{A}_{2p}) \oplus \bigoplus_{p \geq 1} H^1(M, \mathcal{B}_{2p+1}). \end{aligned}$$

Proof. By Proposition 3, $E_2^{p,q} = H^{p+q}(M, (\mathcal{T}_{\text{gr}})_p)$. By Proposition 8, we have to calculate $d_2 = \text{ad } \zeta$ (see Theorem 1), where ζ is represented by the cocycle (17), on

$$H^0(M, \mathcal{T}_{\text{gr}}) = H^0(M, (\mathcal{T}_{\text{gr}})_0) \oplus H^0(M, (\mathcal{T}_{\text{gr}})_{-1}).$$

Consider the natural projective action of $\mathfrak{gl}_{3|m}(\mathbb{C})$ on $\mathbf{CP}^{2|m}$ (see, e.g., [7]). Clearly, $\mathfrak{osp}_{3|m}(\mathbb{C})$ leaves invariant the ideal sheaf of the submanifold $\mathbf{Q}^{1|m}$ of $\mathbf{CP}^{2|m}$, and so this action induces an action of $\mathfrak{osp}_{3|m}(\mathbb{C})$ on $\mathbf{Q}^{1|m}$, i.e. a homomorphism $\alpha : \mathfrak{osp}_{3|m}(\mathbb{C}) \rightarrow H^0(M, \mathcal{T})$. One verifies easily that α is injective. Since $\dim(\mathfrak{osp}_{3|m}(\mathbb{C}))_{\bar{1}} = 3m = \dim H^0(M, (\mathcal{T}_{\text{gr}})_{-1})$, we see that α is an isomorphism on the odd parts and, by Proposition 5, that $E_2^{-1,1} = H^0(M, (\mathcal{T}_{\text{gr}})_{-1})$ consists of permanent cocycles.

It is also clear (see [2]), that $E_2^{0,0} = H^0(M, (\mathcal{T}_{\text{gr}})_0)$ can be identified with

$$\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_m(\mathbb{C}) \oplus \langle \varepsilon \rangle \oplus \mathfrak{sl}_2(\mathbb{C}).$$

The class $d_2\varepsilon = [\zeta, \varepsilon]$ is represented by the cocycle $[\zeta^{(2)}, \varepsilon] = -2\zeta^{(2)}$. Thus,

$$d_2\varepsilon = -2\zeta.$$

The mapping $d_2 : \mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \rightarrow E_2^{2,-1} = H^1(M, (\mathcal{T}_{\text{gr}})_2)$ is determined by the representation of G on this latter cohomology space described in Proposition 8. It follows that

$$E_2^{0,0} \cap \text{Ker } d_2 = \mathfrak{osp}_{3|m}(\mathbb{C}).$$

Identifying $H^1(M, (\mathcal{T}_{\text{gr}})_2)$ with $H^1(M, \mathcal{A}_3) \oplus H^1(M, \mathcal{B}_2)$, we see that $\text{Im } d_2$ coincides with the tangent space to the orbit $G(\zeta')$ in $H^1(M, \mathcal{B}_2) \simeq \bigwedge^2 \mathbb{C}^m$, where ζ' is determined by

the cocycle $x^{-1} \frac{\partial}{\partial x} \otimes \omega$. Since ω has the maximal rank, this orbit is open, and hence $\text{Im } d_2 = H^1(M, \mathcal{B}_2)$.

Therefore only the following groups $E_3^{p,q}$ are non-zero:

$$\begin{aligned} E_3^{-1,1} &= H^0(M, (\mathcal{T}_{\text{gr}})_{-1}) \simeq (\mathfrak{osp}_{3|m})_{\bar{1}}, \quad E_3^{0,0} = (\mathfrak{osp}_{3|m})_{\bar{0}}, \\ E_3^{1,0} &= H^1(M, (\mathcal{T}_{\text{gr}})_1), \quad E_3^{2,-1} \simeq H^1(M, \mathcal{A}_3), \\ E_3^{p,1-p} &\simeq H^1(M, (\mathcal{T}_{\text{gr}})_p) \simeq H^1(M, \mathcal{A}_{p+1}) \oplus H^1(M, \mathcal{B}_p), \quad 3 \leq p \leq m. \end{aligned}$$

This implies that α is an isomorphism, and that the elements of $E_3^{0,0}$ are permanent cocycles. Therefore $d_3 = d_4 = \dots = 0$, and $E_\infty = E_3$. The assertion concerning $H^1(M, \mathcal{T})$ follows now from Proposition 4.

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