QCD sum rules with finite masses

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The concept of QCD sum rules is extended to bound states composed of particles with finite mass such as scalar quarks or strange quarks. It turns out that mass corrections become important in this context. The number of relevant corrections is analyzed in a systematic discussion of the IR- and UV-divergencies, leading in general to a finite number of corrections. The results are demonstrated for a system of two massless quarks and two heavy scalar quarks.

1 Introduction

In the last fifteen years the concept of QCD sum rules invented by Shifman, Vainshtein and Zakharov [1] has been very well developed. The intention was to include non-perturbative effects in QCD calculations, which naturally become important when dealing with hadron physics. The concept of vacuum condensates is best illustrated in the case of chiral symmetry, which is spontaneously broken for hadrons as one can see by comparing the masses of different mesons with identical quantum numbers (i.e., \( \rho \) and \( a_1 \) meson). A direct consequence of spontaneous breakdown of any symmetry is the appearance of non-vanishing vacuum condensates. The idea of QCD sum rules is to parametrize the non-trivial QCD vacuum with vacuum condensates, including the chiral condensate \( \langle \bar{q}q \rangle \), and in this way to realize a parametrization of non-perturbative effects in QCD.

The condensates appear as matrix elements in the operator-product-expansion [2] (OPE) of a physical function (i.e., the polarization function):

\[
\Pi(q^2) \sim \sum_{j=1}^{\infty} <0|B_j(0)|0> \int d^4x \ E_j(x) e^{ix} \tag{1}
\]

This expansion separates perturbative (Wilson coefficients \( E_j(x) \)) and non-perturbative (condensates \( <0|B_j(0)|0> \)) parts, being thus completely calculable with perturbative Feynman rules. It can be related to the hadronic spectrum through a dispersion relation with one subtraction \( C \)

\[
\Pi_{OPE}(q^2) = \frac{q^2}{\pi} \int_0^{\infty} ds \ \frac{I_m \Pi_{Had}(s)}{s(s-q^2)} + C \tag{2}
\]
where the imaginary part of the function \( \Pi_{had} \) is connected to the hadronic spectrum. In that way it is possible to get hadronic quantities like the mass of a bound state as a function of a few vacuum condensates, which are fitted to a large number of hadronic data.

The kind of condensates being relevant in the OPE depends strongly on the flavour composition and the quantum numbers of the bound state considered. In the case of light quarks (up or down) there is strong evidence for the presence of quark-antiquark pairs in the vacuum, so that apart from the gluon-condensates the quark-condensates have to be taken into account. If the particles are heavy quarks (charm or heavier) the value of the corresponding condensates becomes very small and may be neglected, so that the calculation may lead to a reasonable result by including the gluon-condensate only. The strange quark, however, poses a problem because it is neither light nor heavy. Therefore its condensate can not be neglected a priori, and in order to take it into account, it is necessary to treat massive quarks in QCD sum rules — a problem unsolved up to now. We derive several results necessary to approach this goal and elucidate the problems occurring.

This article is organized as follows. We shall illustrate the problem of finite mass sum rules with a specific example (introduced in section 2), a detailed study of which will be published elsewhere. In the third section we give an expression for the matrix elements of normal-ordered, nonlocal field products — which occurs in the expansion of the correlator under consideration —, demonstrating that the finite masses cause corrections to the commonly used expressions. Afterwards some of the Wilson coefficients introduced above will be discussed with special attention to the appearing infrared and ultraviolet divergencies and to the relevance for the sum rule.

2 The system under consideration

For studying the influence of mass terms in QCD sum rules calculations we will consider a four-particle bound state composed of two light fermionic quarks and two heavy scalar quarks. The last ones are hypothetical particles introduced in QCD phenomenologically. Scalar quarks are predicted by the GUT-theory supersymmetry [3] (they are called squarks), such that we are not merely dealing with a toy model. The physical problem motivating this model is whether or not the quark – squark bound state may be energetically more favourable than a squark state, which is predicted to have a very high mass [4]. The lowest lying quark – squark bound state can be calculated — using QCD sum rules — in dependence of different new condensates and of the squark mass. These parameters are unknown, but if one could find a reasonable choice of them, such that the quark – squark
bound state mass becomes small, this would be most interesting for supersymmetry phenomenology. The results of these calculations will be published elsewhere.

In the following calculation the quark masses are neglected and the scalar quarks are treated as massive particles. In order to calculate the mass of the lowest lying bound state of this system, the OPE of the polarization-function corresponding to the diagram in Figure 1 has to be considered (see eq. (1)), which is given by the two-point-function

$$\Pi(k^2) = -\frac{ig^2}{3} \int d^4(x-y) \epsilon^{ik(x-y)} g_{\mu\nu} \langle 0| T\{J^\mu(x),J^\nu(y)\}|0\rangle$$

where $T$ denotes the time ordered product. The current $J^\mu(x)$ contains an incoming and outgoing fermionic and scalar quark:

$$J^\mu(x) = g \overline{\psi}(x)\gamma^\mu\psi(x) \overline{\phi}(x)\phi(x).$$

The present sum rule is calculated in lowest order of QCD perturbation theory. Thus the exchange of gluons and the coupling to the gluon condensate are neglected.

Usually QCD sum rule calculations are carried out in configuration space. In the case of massless particles this choice simplifies the calculations: the propagators are as simple as in momentum space, but the number of integrations is reduced to one, while in momentum space the number of integrations is equal to the number of loops in the corresponding Feynman diagram. However, in the case of particles with non-vanishing masses the propagators become very cumbersome, so that in most cases calculations in momentum space will be more convenient.

The next step is to find the explicit form of the OPE for the polarization function (3). The time-ordered product has to be expanded into normal-ordered products with attention to the non-vanishing vacuum condensates. In this way one gets sixteen terms, each of them corresponding to one diagram: first there is the perturbative diagram (see figure 1), which corresponds to the fully contracted term without any condensates

$$- < 0 | Tr\{S(x-y)\gamma^\mu S(y-x)\gamma^\nu \} Tr\{\Delta(y-x)\Delta(x-y)\} | 0 >$$

where $S(x-y)$ is the Feynman-propagator of the quarks and $\Delta(x-y)$ is the scalar quark propagator. The traces are taken over color- and Dirac-indices in the case of fermionic propagators and in color space only in the case of scalar propagators.

There are eight terms containing one non-contracted pair of fermionic field operators of the form

$$+ i < 0 | (\gamma^\nu)_{ik} S_{kl}(y-x) (\gamma^\mu)_{ij} : \overline{\psi}_a(y)\psi_b(x) : Tr\{\Delta(y-x)\Delta(x-y)\} | 0 >$$
which all vanish independently of the scalar part of the term: the condensate of massless fermions is proportional to $\delta_{ij}$, such that a trace over an odd number of $\gamma$-matrices appears in the fermionic part of the expression.

The term without any contraction is vanishing as well, because it contains no propagator, so that there is no momentum flow through the diagram.

Six diagrams are left with one or two non-contracted pairs of scalar field operators, for example, the contribution from figure 2

$$ -i <0 | Tr \{ S(x-y)\gamma^\mu S(y-x)\gamma^\nu \} \Delta(x-y) :\bar{\phi}(x)\phi(y): |0> .$$

When expanding expressions like that given in eq. (3) one quite generally encounters nonlocal normal ordered product of scalar or fermionic field operators. In these products, the field operators are taken at different points in Minkowski space, while the vacuum condensates of scalar or fermionic fields are defined at the same points. Nevertheless, the nonlocal normal ordered products commonly are identified with the vacuum condensates. We claim that this procedure is correct only in zeroth order in the mass of the particles under consideration.

\section{Mass corrections to the condensates}

\subsection{Condensates of scalars}

In the previous section we have shown, that vacuum expectation values of normal ordered products of scalar fields at different space-time points of the type

$$<0 | :\bar{\phi}_\alpha(x)\phi_\beta(y): |0>$$

\begin{equation}
\end{equation}
occur in the operator product expansion. Here the greek letters denote the color space indices important in the following. With the object of establishing a relation between the expression (8) and the vacuum condensates defined by

\[ <\bar{\phi}\phi> := \sum_{\alpha\beta} <0| \bar{\phi}_{\alpha}(x)\phi_{\beta}(x) : |0> \]  

we will have to expand one of the fields in eq. (8) with respect to the space-time point \( y \) belonging to the second field. The corresponding Taylor series is

\[ \bar{\phi}_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(x-y)^{\mu_1} \cdots (x-y)^{\mu_n} \left( \bar{\phi} \bar{D}_{\mu_1} \cdots \bar{D}_{\mu_n} \right)_{\alpha}(y) \]  

To preserve gauge invariance the ordinary derivatives must be replaced by the covariant ones.

\[ \bar{\phi}_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(x-y)^{\mu_1} \cdots (x-y)^{\mu_n} \left( \bar{\phi} \bar{D}_{\mu_1} \cdots \bar{D}_{\mu_n} \right)_{\alpha}(y) \]  

The replacement is allowed if there exists a gauge for which the above two expressions are equivalent. The coordinate gauge \( x^\mu A^\mu = 0 \) has exactly this property (see for example [5]).

Using this expansion we will derive an expression of the full series describing the nonlocal field product in eq. (8):

\[ <0| :\bar{\phi}_{\alpha}(x)\phi_{\beta}(y) : |0> = <0| :\bar{\phi}_{\alpha}(y)\phi_{\beta}(y) : |0> + \frac{1}{2} <0| :\bar{\phi} \bar{D}_{\mu_1} \bar{D}_{\mu_2}(y)\phi_{\beta}(y) : |0> \xi^{\mu_1}\xi^{\mu_2} + \cdots + \frac{1}{m!} <0| :\bar{\phi} \bar{D}_{\mu_1} \cdots \bar{D}_{\mu_m}(y)\phi_{\beta}(y) : |0> \xi^{\mu_1} \cdots \xi^{\mu_m} \]

where \( \xi := x - y \).

The terms in eq. (12) can contribute to the series only if the corresponding operators do not have any uncontracted Lorentz index, for the vacuum state is a scalar state. Consequently only the terms with an even number of covariant derivatives are of interest (the remaining terms will be renumbered using \( n := m/2 \)). All color and Lorentz indices must be contracted in this terms, so that the structure
of the vacuum expectation value in color and Lorentz space is fixed a priori:

\[
<0| \phi_\alpha(y) \phi_\beta(y) : |0> = C_0 \delta_{\alpha\beta}
\]

\[
<0| \left( \phi \bar{D}_{\mu_1} \bar{D}_{\mu_2} \right)_\alpha(y) \phi_\beta(y) : |0> = C_1 g_{\mu_1\mu_2} \delta_{\alpha\beta}
\]

\[
<0| \left( \phi \bar{D}_{\mu_1} \cdots \bar{D}_{\mu_2} \right)_\alpha(y) \phi_\beta(y) : |0> = C_2 (g_{\mu_1\mu_2} g_{\rho_3\rho_4} + g_{\mu_1\rho_3} g_{\mu_2\rho_4} + g_{\mu_1\rho_4} g_{\mu_2\rho_3}) \delta_{\alpha\beta}
\]

\[
<0| \left( \phi \bar{D}_{\mu_1} \cdots \bar{D}_{\mu_2n} \right)_\alpha(y) \phi_\beta(y) : |0> = C_n \left( g_{\mu_1\mu_2} g_{\rho_3\rho_4} \cdots g_{\mu_{2n-1}\mu_{2n}} \text{ + perm.} \right) \delta_{\alpha\beta}
\]

\[(2n-1)!! \text{ terms} \] (13)

The coefficients \( C_n \) are to be calculated by contracting the equations (13) with the metric tensor \( g^{\mu_1\mu_2} \), using the Klein-Gordon field equation \((+m^2)\phi(y) = 0\) with \( :g^{\mu_\nu} D_\mu D_\nu : \), and summing over all remaining indices. By this procedure one gets the condensates (9) multiplied with the mass in the corresponding dimension on the left hand side and the coefficient of interest with a numerical factor on the right hand side. For instance:

\[
<\bar{\phi} \phi> = \sum_{\alpha\beta} <0| \phi_\alpha(y) \phi_\beta(y) : |0>
\]

\[
= C_0 \sum_{\alpha\beta} \delta_{\alpha\beta} = N_c C_0
\]

\[
\Rightarrow \quad C_0 = \frac{1}{N_c} <\bar{\phi} \phi>
\] (14)

\[
-m^2 <\bar{\phi} \phi> = \sum_{\alpha\beta} <0| \left( \phi g^{\mu_1\mu_2} \bar{D}_{\mu_1} \bar{D}_{\mu_2} \right)_\alpha(y) \phi_\beta(y) : |0>
\]

\[
= C_1 \sum_{\alpha\beta} \delta_{\alpha\beta} g^{\mu_1\mu_2} g_{\mu_1\mu_2} = 4N_c C_1
\]

\[
\Rightarrow \quad C_1 = -\frac{m^2}{4N_c} <\bar{\phi} \phi>
\] (15)

\[
m^4 <\bar{\phi} \phi> = \sum_{\alpha\beta} <0| \left( \phi g^{\mu_1\mu_2} g^{\rho_3\rho_4} \bar{D}_{\mu_1} \bar{D}_{\mu_2} \bar{D}_{\rho_3} \bar{D}_{\rho_4} \right)_\alpha(y) \phi_\beta(y) : |0>
\]

\[
= C_2 \sum_{\alpha\beta} \delta_{\alpha\beta} g^{\mu_1\mu_2} g^{\rho_3\rho_4} (g_{\mu_1\mu_2} g_{\rho_3\rho_4} + g_{\mu_1\rho_3} g_{\mu_2\rho_4} + g_{\mu_1\rho_4} g_{\mu_2\rho_3})
\]

\[
= N_c C_2 \left( 16 + g^{\rho_2\rho_4} g^{\mu_3\mu_4} + g^{\mu_2\rho_4} g^{\mu_3\rho_2} \right) = 24N_c C_2
\]

\[
\Rightarrow \quad C_2 = \frac{m^4}{24N_c} <\bar{\phi} \phi>
\] (16)

where \( N_c \) is the number of colors.
The general form of $C_{n>2}$ is

$$C_n = (-1)^n \frac{m^{2n}}{N_c G_n} \langle \bar{\phi} \phi \rangle,$$

with

$$G_n := g_{\mu_1 \nu_2} \cdots g_{\mu_{2n-1} \nu_{2n}} \left( g_{\mu_1 \nu_2} \cdots g_{\mu_{2n-1} \nu_{2n}} + \text{permutations} \right).$$

For each contraction of two covariant derivatives one gets a factor $-m^2$ coming from the Klein-Gordon field equation, all terms contain the same factor $N_c$ from the summation in color space, and the factor $G_n$ is appearing by construction. One can easily verify that the relation $G_n = 2^n (n+1)!$ holds for all $n$.

In this way the coefficients $C_n$ of eq. (17) are known explicitly and thus we have found an expression for all the vacuum expectation values of eq. (13). Inserting this into the expansion (12) we get:

$$\langle 0| : \bar{\phi}_\alpha(x) \phi_\beta(y) : |0 \rangle = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \frac{m^{2n}}{n!(n+1)!} \langle \bar{\phi} \phi \rangle \delta_{\alpha\beta} \cdot$$

$$\times \left( g_{\mu_1 \nu_2} g_{\mu_3 \mu_4} \cdots g_{\mu_{2n-1} \nu_{2n}} + \text{permutations} \right) \xi^{\mu_1} \cdots \xi^{\mu_{2n}}$$

$$= J_n^2 \delta_{\alpha\beta} \langle \bar{\phi} \phi \rangle,$$

The $(2n-1)!!$ terms contract the $2n$ dimensional tensor $\xi^{\mu_1} \cdots \xi^{\mu_{2n}}$ yielding $(2n-1)!!(\xi^2)^n$, so that we are left with the full expression of the nonlocal field product in eq. (8) to all orders in the mass:

$$\langle 0| : \bar{\phi}_\alpha(x) \phi_\beta(y) : |0 \rangle = \frac{1}{N_c} \delta_{\alpha\beta} \langle \bar{\phi} \phi \rangle \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+1)!} \frac{m^{2n} \xi^2}{4^n}$$

The sum can be performed giving:

$$\langle 0| : \bar{\phi}_\alpha(x) \phi_\beta(y) : |0 \rangle = \frac{1}{N_c} \delta_{\alpha\beta} \langle \bar{\phi} \phi \rangle \frac{2}{m|\xi|} J_1 (m|\xi|).$$

This result is not so surprising, because there is a connection between the scalar propagator and the Bessel functions. The left hand side of eq. (21) has the structure of a scalar propagator as well. In this sense it may be meaningful to call the vacuum expectation value of a nonlocal normal ordered product of two fields a non-perturbative propagator.

In the case of vanishing mass the series reduces to the commonly used relation

$$\langle 0| : \bar{\phi}_\alpha(x) \phi_\beta(y) : |0 \rangle = \frac{1}{N_c} \delta_{\alpha\beta} \langle \bar{\phi} \phi \rangle.$$
For nonvanishing mass the higher-order terms of the series have to be taken into account. The question arises whether the series is convergent and which order still is important in a specific calculation? The answer depends strongly on the value $m^2 \xi^2$ where $\xi^2$ is the distance between the two vertices at $x$ and $y$, which is inversely proportional to the mass $M^2$ of the incoming state to be evaluated by QCD sum rules. So the ratio $m/M$ of the composite particle mass and the bound state mass determine the importance of the higher-order terms. For instance $m/M = 1$ leads to a 10% correction in first order, while $m/M = 2$ blows up the first order correction to 50% and the second order term makes another 10% contribution.

To conclude: the higher-order corrections may be neglected for systems where a heavy bound state is constructed from light particles. However, the higher-order corrections become dominant if the bound state is lighter than its constituent particles. For instance this could happen in the supersymmetric system introduced in the second section. In the last case the exact form (20) or (21) has to be used.

### 3.2 Condensates of fermions

A treatment analogous to eq. (8) is possible also for fermion fields instead of the scalar fields used in the previous section. This is especially important if the interest lies in the treatment of finite masses in QCD sum rules emerging in mass calculations of strange baryons [6].

The following nonlocal normal-ordered product of fermion field operators is to be expanded

$$<0 \mid \overline{\psi}_{i\alpha} (x) \psi_{j\beta} (y) \mid 0 > \quad .$$

Here the fields carry a Dirac index (Latin letter) in addition to the color index (Greek letter). The fermion condensate is defined by

$$< \overline{\psi} \psi > := \sum_{i,j,\alpha,\beta} <0 \mid \overline{\psi}_{i\alpha} (x) \psi_{j\beta} (x) \mid 0 >$$

and the corresponding expansion of one fermion field preserving gauge invariance reads (see eq. (11))

$$\psi_{i\alpha} (y) = \sum_{n=0}^{\infty} \frac{1}{n!} (y - x)^{\mu_1} \cdots (y - x)^{\mu_n} \left( D_{\mu_1} \cdots D_{\mu_n} \psi \right)_{i\alpha} (x)$$

This series can be evaluated in close analogy to the case of the scalar field treated in sect. 3.1. There is one important difference: In the scalar case only the terms in the expansion of the field with an even number of derivatives give a contribution.
to the series (see discussion after eq. (12)), while in the case of fermion fields the terms with an odd number of derivatives are non-vanishing, for the Dirac equation \((i \not \! \! D - m)\psi = 0\) is of first order in the derivatives.

The Lorentz structure of the even terms is the one shown in eq. (13) with an additional factor \(\delta_{ij}\). Each odd term has the same structure as the corresponding even term with one more derivative, except for the replacement of one \(g_{\mu \nu}\) by \(\gamma_{\mu}\). It turns out that the coefficients \(C_n\) \((n \in 0, 1, 2, \ldots)\) corresponding to eq. (17) are

\[
C_n = \frac{(-im)^n}{4N_c G_n} \psi \psi
\]

where the even coefficients \(G_{2n}\) coincide with those for scalar fields while the odd coefficients are given by \(G_{2n-1} = G_{2n}\). This means that the coefficient of a term with an odd number of derivatives is equal to the coefficient of the following term with an even number of derivatives (except a factor \(im\), which are determined by \(G_{2n} = 2^n(n + 1)!\) like in the case of scalar fields. After some algebra the whole series corresponding to eq. (20) reads:

\[
< 0| : \bar{\psi}_i(x) \psi_j(y) : |0 > = \frac{\delta_{\alpha \beta} \delta_{ij}}{4N_c} \psi \psi \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+1)!} \left( \frac{m^2 \xi^2}{4} \right)^n
\]

\[
+ \frac{im}{2} \frac{\delta_{\alpha \beta} \delta_{ij}}{4N_c} \psi \psi \sum_{n=1}^{\infty} (-1)^n \frac{1}{(n-1)!n!(n+1)!} \left( \frac{m^2 \xi^2}{4} \right)^{n-1}
\]

This result can be verified by acting with \((+i \not \! \! D + m)\) on the series (20) for scalar fields

\[
< 0| : \bar{\psi}_i(x) \psi_j(y) : |0 > \rightarrow (i \not \! \! D + m) < 0| : \bar{\psi}_i(x) \psi_j(y) : |0 >
\]

with the replacement

\[
m < \bar{\phi} \phi > \rightarrow < \bar{\psi} \psi > \frac{\delta_{ij}}{4}\]

necessary on dimensional grounds. So that the above series can be rewritten:

\[
< 0| : \bar{\psi}_i(x) \psi_j(y) : |0 > = \frac{\delta_{\alpha \beta} \delta_{ij}}{4N_c} \psi \psi \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+1)!} \left( \frac{m^2 \xi^2}{4} \right)^n
\]

\[
= \frac{\delta_{\alpha \beta} \delta_{ij}}{2N_c} \psi \psi \sum_{n=1}^{\infty} (-1)^n \frac{1}{(n-1)!n!(n+1)!} \left( \frac{m^2 \xi^2}{4} \right)^{n-1}
\]

\[
\times \frac{1}{m^2 |\xi|} J_1 \left( \frac{m |\xi|}{\sqrt{2}} \right)
\]

Again the commonly used formula can be recovered as lowest order term in the mass expansion looking at eq. (27)

\[
< 0| : \bar{\psi}_i(x) \psi_j(y) : |0 > = \frac{\delta_{\alpha \beta} \delta_{ij}}{4N_c} \psi \psi
\]
which remains correct for vanishing mass. In the case of non-vanishing mass the higher-order contributions of eq. (30) should be taken into account. Belyaev and Ioffe tried to explain the mass splitting of strange and nonstrange baryons using QCD sum rules [6]. It would be interesting to calculate the importance of these mass corrections to their results.

4 IR-divergencies and higher-order mass terms

The relevant equation in QCD sum rules calculations is a dispersion relation with one subtraction (see eq. (2)), which gives a relation between the polarization function on the quark level — expanded with the use of the OPE (eq. (1)) — and the hadronic spectrum. The sum rule is obtained by taking the Borel transformation of eq. (2), which has the property of isolating the lowest state of the hadronic spectrum (the calculation of moments is an alternative method of isolating the lowest state [7]). The further the second state in the spectrum is separated from the lowest state, the better the isolation works. A typical formula giving an explicit value for the mass of the lowest state is

\[
\frac{\int_{m_{\text{OPE}}}^s ds \, s \, \text{Im} \Pi_{\text{OPE}}(s)}{\int_{m_{\text{OPE}}}^s ds \, s \, \text{Im} \Pi_{\text{OPE}}(s) \, e^{-st}} \approx M^2
\]

(32)

where \(t\) is the Borel transformation parameter.

The main point in this formula is the sole appearance of the imaginary part of \(\Pi_{\text{OPE}}\), so that one needs only to calculate the terms being relevant for the imaginary part. The typical structure of a dimensional-regularized Feynman graph \(G(q^2)\) \((\varepsilon = 4 - d)\) is:

\[
G(q^2) \sim \left\{ \frac{2}{\varepsilon} + \text{const.} + \ln \left( -\frac{4\pi\mu^2}{q^2} \right) \right\} P(q)
\]

(33)

where \(P(q)\) is a purely polynomial expression and \(\mu\) is the renormalization parameter. Remembering the analytic continuation to \(q^2 > 0\) the whole imaginary part of such a graph is hidden in the logarithm:

\[
\ln \left( -\frac{4\pi\mu^2}{q^2} \right) = \ln \left| \frac{4\pi\mu^2}{q^2} \right| + i\pi \Theta(q^2)
\]

(34)

In the previous section a complete expression for nonlocal field products to all orders in the mass was derived and the notion of a nonperturbative propagator was introduced. In the following we show that in general it is justified to restrict the calculation of a Feynman diagram containing a condensate to a finite number of
higher-order corrections in the mass. To illustrate the argument we treat the case of one condensate of scalar fields, which represents a large class of graphs of the same structure. The corresponding graph is shown in figure 2. The broken line represents

Figure 2: Feynman diagram with a scalar propagator, a condensate of scalar fields and an arbitrary additional part

the condensate, the full line is a scalar propagator and the hatched area represents an arbitrary additional structure. This could be a massless fermion loop, for example, which would occur in the sum rule of the 2 quark – 2 squark system introduced in the second section and which corresponds to eq. (7). The corresponding algebraic expression $G(k^2)$ including the mass corrections to all orders reads

$$
G(k^2) \sim \langle \overline{\phi} \phi \rangle \int \frac{d^4 q}{(2\pi)^4} f \left( (k - q)^2 \right) \int \frac{d^4 p}{(2\pi)^4} \Delta(p - q) \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left( \frac{m_p^2}{4} \right)^n \delta^4(p) 
$$

(35)

where the Fourier transformation of eq. (20) has been used and the symbol $p = \left( \frac{\partial}{\partial p_0}, -\nabla p \right) \cdot \left( \frac{\partial}{\partial p_0}, -\nabla p \right)$ was introduced. The function $f$ represents the hatched area in figure 2 and $\Delta(p - q)$ is the scalar propagator.

To evaluate the integral $I_p$, it is useful to reflect once more on the significance of the notion of the nonperturbative propagator $\Delta_{NP}$ given by the series (20). The nonperturbative propagator satisfies the free Klein-Gordon equation so that the relation

$$
\int d^4 p \ g(p^2, p, \ldots) \Delta_{NP}(p) = \int d^4 p \ g(m^2, p, \ldots) \Delta_{NP}(p) 
$$

(36)

holds for an arbitrary function $g(p^2, p, \ldots)$ in lowest order of perturbation theory. Note that the calculations already before were restricted to lowest order of perturbation theory, so that this is no additional restriction. By applying the replacement rule (36) to the scalar propagator in eq.(35), it can be reduced to
\[ \Delta(p - q) = [q^2 - 2pq + i\epsilon]^{-1}. \]
The \(n\)-th order differentiation of the distribution \(\delta(p)\) is carried out by \(n\) differentiations of the simplified propagator and after some algebra, eq. (35) reads:

\[
G(k^2) \sim \langle \bar{\phi} \phi \rangle \int \frac{d^4q}{(2\pi)^4} f ((k - q)^2) \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} \frac{m^{2n}}{(q^2 + i\epsilon)^{n+1}}
\]

A similar argument can be repeated for any other function \(g\) instead of \(g(p) = \Delta(p - q)\). In particular a fermion propagator would lead to a similar result.

The infrared and ultraviolet behaviour of eq. (37) is determined by the lowest and highest power \((m_i\) and \(m_h)\) of \(q\) in the function \(f ((k - q)^2)\). So \(G(k^2)\) is UV-divergent if \(n \leq \frac{m_i + 2}{2}\) is fulfilled, while it is IR-divergent if \(n \geq \frac{m_h + 2}{2}\) is fulfilled. One important conclusion can be drawn: only a finite number of terms containing UV-divergencies exists. There is a problem region \(\frac{m_i + 1}{2} \leq n \leq \frac{m_h + 1}{2}\) with both UV- and IR-divergencies occurring, and finally an infinite number of terms with possible IR-divergencies exists.

To illustrate the above rules, the 2 quark - 2 squark system introduced in the second section is treated explicitly. In this case the function \(f ((k - q)^2)\) in eq. (37) becomes the regularized massless fermion loop diagram, so that eq. (37) now reads:

\[
G(k^2) = -g^2 \frac{8N_c \pi^2}{3(2\pi)^4} \langle \bar{\phi} \phi \rangle \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} \frac{(m^2)^n}{(q^2 + i\epsilon)^{n+1}}
\]

\[
\times i \int \frac{d^4q}{(2\pi)^4} \ln \left( \frac{(k - q)^2}{-4\pi \mu^2} \right) \frac{(k - q)^2}{(q^2 + i\epsilon)^{n+1}}
\]

where all proportionality factors have been reintroduced. The constant term in eq. (33) is omitted, because it does not contribute to the imaginary part of \(G(k^2)\).

The above rule to analyze the divergencies occurring in this expression leads to the following expectation: \(f ((k - q)^2) \sim (k - q)^2\) has the lowest and highest power \(m_i = 0\) and \(m_h = 2\), so that UV-divergencies occur for \(n \leq 2\), IR-divergencies occur for \(n \geq 1\), and double divergencies occur for \(n \in \{1, 2\}\). This is exactly the result of the straightforward regularization of the last integral in the expression (38), which can be verified by counting the number of divergent \(\Gamma\)-functions in the limit \(\varepsilon \to 0\):

\[
G(k^2) = g^2 \frac{8N_c \pi^4}{3(2\pi)^8} \langle \bar{\phi} \phi \rangle \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} \frac{(m^2)^n (k^2)^{2-n}}{(4\pi \mu^2)^{\frac{5}{k^2}}} \frac{\Gamma \left( 1 - n - \frac{\varepsilon}{2} \right) \Gamma \left( 3 - \frac{\varepsilon}{2} \right) \Gamma \left( -2 + \frac{\varepsilon}{2} + n \right)}{\Gamma(n+1) \Gamma(4 - n - \varepsilon)}
\]

The double divergent terms for \(n = 1, 2\) lead to a divergent imaginary part, for the integral \(I\) in eq. (38) is IR- and UV-divergent at the same time, which leads to
meaningless results for the sum rule. A more detailed examination of this integral is unavoidable. Decomposing $I$ in three parts, in which the integrand is proportional to $k^2$, $k \cdot q$ and $q^2$ respectively, one can easily verify that the only part having IR- and UV-divergencies is the $k^2$-term. It turns out that the IR-divergency in this part can be regularized by introducing a small IR-cutoff parameter $a^2$ into the scalar propagator (or equivalently using a lower integration limit), reading:

$$I_{k^2} = i \int \frac{d^4 q}{(2\pi)^4} \ln \left(\frac{(k - q)^2}{-4\pi \mu^2}\right) \frac{k^2}{(q^2 - a^2 + i\epsilon)^{n+1}}$$

(40)

This integral has no IR-divergent part any more, so that it is possible to regularize the remaining UV-divergency in the dimensional regularization scheme. The result to order $O(\varepsilon)$ (where $\varepsilon = 4 - d$) depends on the cutoff parameter $a^2$:

$$I_{k^2} = -\frac{\pi^2 k^2}{(2\pi)^4} \left\{ \frac{4}{\varepsilon^2} + \frac{2}{\varepsilon} (1 - \gamma) + \text{const.} + \frac{\varepsilon}{2} \gamma^2 \ln \left(\frac{-k^2}{16\pi \mu^2}\right) \right.\\ + \left. \left(\frac{1}{2} + \frac{\varepsilon}{2}\gamma\right) \ln^2 \left(\frac{-k^2}{16\pi \mu^2}\right) - \frac{\varepsilon}{2} \ln^2 \left(\frac{a^2}{16\pi \mu^2}\right) \right\}$$

(41)

The essential point of this result is the exact cancellation of all terms to the order $O(\varepsilon)$ or lower which depend on the cutoff parameter $a^2$. The imaginary part relevant for the sum rule in this way remains cutoff parameter independent. The whole imaginary part of eq. (38) becomes finite:

$$\text{Im}\{G(k^2)\} = \frac{8N_c \pi^5 g^2 m^2 k^2}{3(2\pi)^8} \left\langle \bar{\phi} \phi \right\rangle \left\{ \frac{3}{2} + \ln \left(\frac{32\pi \mu^2}{k^2}\right) \right\} \Theta(k^2)$$

(42)

This confirms the expectation that the IR divergencies are unphysical.

Independently of the choice of the function $f$ in eq. (37) it must be possible to get rid of all IR-divergent terms. This becomes clear when we look back to the very beginning. The OPE is the basic formula of QCD sum rules and its idea is to separate the long- and short-distance effects of nonperturbative QCD. The short-distance effects are completely embodied in the Wilson coefficients just calculated. So any remaining IR-divergency occurring while calculating Wilson coefficients proves an admixture of long-distance effects, which indicates that the OPE did not really separate the two scales of QCD. Nowikov et al. argued [8] that for the consistent use of OPE one has to divide the integration domain (while calculating the Wilson coefficients) into two parts by introducing the normalization point. In this way one can get rid of all IR-divergencies, for only the high momentum part remains relevant for the Wilson coefficients, just as it was demonstrated in the above example. The remaining question is whether or not the resulting coefficients depend on the normalization point. A meaningful OPE or a meaningful sum rule can be guaranteed only if the result is not or weakly dependent on the normalization point.
Supposing that the Wilson coefficients are independent of the normalization point separating the scales, one has the possibility to restrict the calculation of the coefficients to the terms containing an UV-divergent part. And the above rule \( n \leq \frac{m_b^2 + 2}{2} \) shows that this is a finite number of terms. A confrontation with IR-divergent terms is inevitable only if we calculate the double divergent terms \( \frac{m_{s+b}}{2} \leq n \leq \frac{m_s + 1}{2} \) where the UV and IR divergent parts have to be separated carefully (for example by introducing an IR-cutoff, like it was done in the last example).

Summarizing, there is only a finite number of mass corrections in eq. (20) to be included in the calculation of the Wilson coefficients, for only a finite number of terms is UV-divergent and the IR-divergent terms cannot contribute to the Wilson coefficient if — and only if — the OPE is meaningful. But this last question leads us to the main problem of enlarging the concept of QCD sum rules to finite mass particles. A particle of several \( 100 \text{ MeV} \) like the strange quark introduces a new scale in the theory so that the separation of long- and short-distance effects in an OPE becomes much more difficult than in the massless case and remains a delicate task.

5 Conclusions

The application of QCD sum rules to bound states composed of particles with finite mass leads to a correction of the vacuum expectation value of the nonlocal normal-ordered field product, which does not remain simply a vacuum condensate (eq. (22) for scalar fields and eq. (27) for fermions) like it does in the massless case. We derived higher-order corrections and the results are given in eq. (20) for scalars and eq. (26) for fermions to all orders in the mass. The importance of higher-order terms grows in the same measure as the mass ratio of the constituent particles and the bound state mass. These investigations are of great importance for the application of QCD sum rules to mesons and baryons with strangeness.

Analysing the structure of the Wilson coefficients corresponding to the higher order mass corrections, we found that in general a restriction to a finite number of corrections leads not to any neglection of mass effects, for only a finite number of corrections is UV divergent. On the other hand, the number of IR divergent terms is unknown and unlimited. The problem of IR divergencies may be circumvented by introduction of an IR-cutoff provided that a separation of short and long distance effects in the operator product expansion is assured to a good degree of accuracy and provided that the final expressions does not depend strongly on the cutoff.

We wish to thank Dr. Lech Mankiewicz for very helpful discussions. This work was supported by DFG (G. Hess program).
References


