ELECTRON SCATTERING, FORM FACTORS, VECTOR MESONS

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REFERENCES
1. INTRODUCTION

The subject has been divided into three sections:

(1) the classification of electromagnetic scattering, generally referred to as Coulomb scattering, treated in the approximation of one-photon exchange:

(2) the definition, measurement, and interpretation of nucleon, pion, and electron form factors, as they are measured by Coulomb scattering:

(3) the formation of strongly interacting states, such as vector mesons, by annihilation of $e^+e^-$ pairs, and the relation to the crossed process, namely the electron scattering:

Because of the educational purpose of the lectures, the main emphasis has been put on the basic ideas and formalisms and not on completeness. With respect to the derivation of cross-sections, we show the ansatz and the result, omitting the calculation.

References for Sections 1 and 2 can be found in text books, as for example:

Bjorken and Drell (BJORKEN, 1964)
Gasiorowicz (GASIOROWICZ, 1966)
Källén (KÄELLEN, 1964)
Muirhead (MUIRHEAD, 1965)
Nishijima (NISHIJIMA, 1964)

and in the proceedings of the CERN Easter School 1965:

Beckmann (BECKMANN, 1965)
Meyer Berkhout (MEYER-B., 1965).

The second half of Section 2 and the whole of Section 3 are taken from recent publications and conference proceedings.
CLASSIFICATION OF COULOMB SCATTERING

- Rutherford
  - Dirac electron
  - recoil
- Mott
  - recoil
- $^\text{cNS}$
  - Dirac electron
  - c.m. system
  - equal particle
  - antiparticle
- Dirac proton
- proton form factor
- Rosenbluth
- Möller $e^+e^-$
- Bhabha $e^+e^-$
1.1 Kinematics (HOFSTADER, 1957)

In this section we treat the kinematics of the scattering of an electron on a proton, or, more generally, the scattering of a particle with a mass \( m \) on another particle with mass \( M \), where \( M \) is at rest in the laboratory before the scattering. The properties of the particles are described by the four-vectors \( p_0, p_f, P_1, P_f \), where \( P_1 = (M,0) \). Further

\[
\begin{align*}
p_0^2 &= E_0^2 - \vec{p}_0^2 = m^2 \\
p_f^2 &= E^2 - \vec{p}^2 = m^2 \\
P_1 &= M^2 \\
P_f &= E'^2 - \vec{p}'^2 = M^2
\end{align*}
\]

Energy and momentum conservation yields, with \( m \ll E_0, E \), and \( m \ll M \)

\[
E_0 + M = E + E' \\
\vec{p}_0 = \vec{p} + \vec{p}'
\]

\begin{align*}
\text{electron:} & \quad E = \frac{E_0}{1 + \left(2E_0/M\right) \sin^2 \left(\theta/2\right)} \quad (1) \\
\text{proton:} & \quad E' = M + \frac{\left(2E_0^2/M\right) \sin^2 \left(\theta/2\right)}{1 + \left(2E_0/M\right) \sin^2 \left(\theta/2\right)} \\
& \quad = M \frac{(E_0 + M)^2 + E_0^2 \cos^2 \phi}{(E_0 + M)^2 - E_0^2 \cos^2 \phi} \quad (2a)
\end{align*}

It should be noted that the condition \( m \ll M, E_0, E \) is met well by electrons, but may be violated in muon scattering, which, in principle, follows the same laws.

The square of the four-momentum transfer from the incident particle to the target particle (which is equal to the mass\(^2\) of the virtual photon), is
\[-q^2 = t = (p_k - p_f)^2\]
\[= (E_0 - E)^2 - (\vec{p}_0 - \vec{p})^2\]
\[= 2m^2 - 2[E_0E - p_0p \cos \theta]\]
\[m \ll E_0, E, \ p_0 \approx E_0, \ p \approx E:\]

\[q^2 = 2E_0E(1 - \cos \theta) \quad (3a)\]
\[= 4E_0E \sin^2(\theta/2) \quad (3b)\]
\[= \frac{4E_0^2 \sin^2(\theta/2)}{1 + (2E_0/M) \sin^2(\theta/2)} \quad (3c)\]

From here we obtain for the proton

\[E' = \frac{q^2}{2M} + M.\]

The four-momentum transfer is measured in units of (GeV/c)^2 or fermi^{-2}:

\[
\begin{align*}
1 \ (\text{GeV/c})^2 &= 25.68 \ \text{fermi}^{-2} \\
1 \ \text{GeV/c} &= 5.07 \ \text{fermi}^{-1} \\
1 \ \text{fermi}^{-1} &= 197.32 \ \text{MeV/c} \\
1 \ \text{fermi} &= 10^{-13} \ \text{cm}
\end{align*}
\]
1.2 Rutherford scattering (RUTHERFORD, 1930)

Rutherford scattering is the scattering of a particle with mass \( m \) and a point-like charge \( e \) on an infinitely heavy point charge \( Ze \) (potential)

\[
\begin{align*}
\text{Equation (1) gives for } M = \infty : E = E_0. \text{ Using the Feynman prescription (Appendix I) we derive the Rutherford cross-section for } Z = 1:\n
d\sigma &= \frac{1}{v_{rel}} \frac{2\pi}{4E_0^2} |M_{fi}|^2 D \\
v_{rel} &= \frac{p_0}{E_0} \approx 1 \\
t &= 4E_0^2 \sin^2 (\theta/2) \\
M_{fi} &= (p_i + p_f) \frac{4\pi e^2}{t} \\
D &= \frac{1}{(2\pi)^3} pE \, d\Omega \\
d\sigma &= \frac{e^2}{4E_0^2 \sin^2 (\theta/2)} \quad (4a) \\
&= \frac{\alpha^2}{4E_0^2 \sin^2 (\theta/2)} \quad (4b) \\
&= \frac{r_e^2 \, m^2}{4E_0^2 \sin^2 (\theta/2)} \quad (4c)
\end{align*}
\]

where \( \alpha = e^2 \), and \( r_e = \frac{\alpha}{m} = 2.8178 \) fermi (classical electron radius)
1.3 The spin of the electron \textit{(HAGEDORN, 1962)}

The electron is a fermion with spin $\frac{1}{2}$ and a magnetic moment

$$ u = \frac{g}{2} \mu_B,$$

where $\mu_B = e/2m$ is the Bohr magneton and $m$ the electron mass. The $g$-factor of the electron is known to be \textit{(PANOFSKY, 1968)}

\[
\frac{g - 2}{2} \begin{array}{cc}
\text{th} & = (115.964.1 \pm 0.3) \times 10^{-8} \\
\text{exp} & = (115.955.7 \pm 3.0) \times 10^{-8}
\end{array}
\]

A beam of electrons can be decomposed into two states of different alignment with respect to its direction of motion: $m = \pm \frac{1}{2}$. These states are called helicity states. We will show that helicity is a conserved quantum number in high-energy electron interactions. For this purpose we will go through the following semi-classical considerations:

We consider a charged particle (mass $m$, charge $e$, magnetic moment $g \mu_B$, velocity $\vec{v}$, spin $\vec{s}$) moving in the electric field $\vec{E}$ of another charge and a superimposed magnetic field $\vec{H}$. We choose a special case where $\vec{H}$ is perpendicular on the deflection plane: $\vec{H} \parallel \vec{s} \times \vec{E}$. We then obtain a Larmor precession of the spin in the particle's rest frame, with the corresponding proper time $\tau$

\[
\frac{d\vec{s}}{d\tau} = g u \left[ \vec{s} \times \vec{H} + \vec{s} \times (\vec{E} \times \vec{s}) \right],
\]

where the second term comes from the convection current of $Ze$ in the particle's system. In our special frame this is, since $\vec{s} \cdot \vec{E} = 0$,

\[
\frac{d\vec{s}}{d\tau} = g u \left( \vec{s} \times \vec{H} + \vec{s} \vec{E} \right)
\]

\textit{(5a)}
(quantities $\hat{s}$, $\hat{\beta}$, $\hat{h}$ are unit vectors). The particle's deflection in the fields is

$$\frac{d\hat{s}}{d\tau} = \frac{e}{m} \left( \hat{\beta} \times \hat{h} + \hat{E} \right)$$

(6a)

$$\frac{d\hat{\beta}}{d\tau} = \frac{e}{m} \left( \hat{\beta} \times \hat{h} + \frac{\hat{E}}{\beta} \right).$$

(6b)

Multiplication from the left by $\hat{h}$, $\hat{h} \cdot \hat{\beta} = 0$ gives for the helicity change [Eqs. (5b) - (6b)]:

$$\frac{d\Theta_K}{d\tau} = \frac{e}{m} \left[ \frac{g - \frac{2}{2}}{2} \hat{\beta} (\hat{h} \times \hat{h}) - \left( \frac{g\hat{h}}{2 \beta} - \frac{1}{\beta} \right) \hat{h} \cdot \hat{E} \right]$$

(7a)

$$= \frac{e}{m} \left[ \frac{g - \frac{2}{2}}{2} \hat{\beta} \cdot (\hat{h} \times \hat{h}) + \frac{g - \frac{2 - (g/\gamma^2)}{2\beta}} \hat{h} \cdot \hat{E} \right].$$

(7b)

We now transform the result from the particle's rest frame to the laboratory:

$$\frac{d\Theta_B}{dt} = \gamma \frac{d\Theta_K}{d\tau} = \frac{d\Theta_K}{d\tau}$$

(8)

$$\frac{d\Theta_B}{dt} = \frac{e}{m} \left[ \frac{g - \frac{2}{2}}{2} \hat{\beta} \cdot (\hat{h} \times \hat{h}) + \frac{g - \frac{2 - (g/\gamma^2)}{2\beta}} \hat{h} \cdot \hat{E} \right]$$

(9)

This result can be obtained under completely general geometrical conditions for $\hat{E}$ and $\hat{h}$ (BARGMANN, 1959). We can conclude that helicity is conserved in electromagnetic interactions, in the limit $g - 2 = 0$ (one-photon exchange) and $g \ll \gamma^2$.

Relation (9) holds for elementary processes irasmuch as fields are homogeneous over the particle extension. This means that relation (9) is strictly valid in the case of electrons and muons with point-like structure.
1.4 Mott scattering (MOTT, 1929)

Mott scattering is the scattering of a Dirac electron on a point charge with infinite mass and charge Ze.

As we know from the previous section, helicity is conserved in this process in the relativistic limit. This allows us to predict that the differential cross-section at 180° will approach zero at that limit:

\[
\lim_{\beta \to 1} \frac{d\sigma}{d\Omega}(180°) \bigg|_{\text{Mott}} = 0.
\]

Since helicity is conserved, the 180° scattering has to be accompanied by a spin-flip of the electron. The resulting unit of angular momentum cannot be transferred to orbital angular momentum, which has to be perpendicular on the scattering plane. So the transition is forbidden by angular momentum conservation:

We obtain the differential cross-section for Mott scattering by means of the Feynman prescription for \( Z = 1 \):

\[
\frac{d\sigma}{d\Omega} = \frac{1}{v_{\text{rel}}} \frac{2\pi}{4E_0^2} |M_{fi}|^2 D
\]

\[
v_{\text{rel}} = \frac{p_e}{E_0} \approx 1
\]

\[
-t = 4E_0^2 \sin^2 \left(\frac{\theta}{2}\right)
\]

\[
M_{fi} = \left[ \bar{u}(p_T) \gamma_u u(p_I) \right] \frac{4\pi e^2}{t}
\]

\[
D = \frac{1}{(2\pi)^2} pE d\Omega
\]

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_0^2 \sin^2 \left(\frac{\theta}{2}\right)} \left(1 - \beta^2 \sin^2 \frac{\theta}{2}\right)
\]

\approx \frac{\alpha^2}{4E_0^2 \sin^2 \left(\frac{\theta}{2}\right)} \cos^2 \frac{\theta}{2} \quad (10a)

\[
\frac{d\sigma}{d\Omega} \approx \frac{\alpha^2}{4E_0^2 \sin^2 \left(\frac{\theta}{2}\right)} \cos^2 \frac{\theta}{2} \quad (10b)
\]
1.5 The target recoil

In this section we investigate the scattering of an electron of mass $m$ on a target particle of mass $M$ with a point-like charge $e$ and without spin:

\[
\begin{align*}
\mathbf{p}_f & \quad \mathbf{p}_f \\
\mathbf{p}_i & \quad \mathbf{p}_i
\end{align*}
\]

The energy of the scattered electrons is given by Eq. (1), its four-momentum transfer by the Eqs. (3). The cross-section is obtained from the Feynman prescription:

\[
\frac{d\sigma}{d\widehat{\Omega}} = \frac{\alpha^2}{4E_0^2 \sin^2(\theta/2)} \frac{\cos^2 \theta}{2} \frac{1}{1 + (2E_0/M) \sin^2(\theta/2)}
\]

(11)
1.6 Electron scattering on a Dirac proton

Now we identify the target particle with a Dirac proton with spin, i.e. we set $M = 0.938$ GeV, but we neglect the anomalous magnetic moment and the spatial extension.

The cross-section is obtained in the conventional way:

\[
\frac{d\sigma}{d\Omega} = \frac{2\pi}{v_{rel}} \left( \frac{2\pi}{16 E_0 E_M} \right)^2 |M_{fi}|^2 \ D
\]

\[
v_{rel} = \frac{E_0}{E_0} \approx 1
\]

\[-t = 4E_0 E \sin^2 \frac{\theta}{2}
\]

\[
M_{fi} = \left[ \bar{u}(p_f) \gamma_\mu u(p_i) \right] \frac{4\pi e^2}{t} \left[ \bar{u}(p_f) \gamma_\mu u(p_i) \right]
\]

\[
D = \frac{1}{(2\pi)^3} \frac{E'E'}{M} \frac{d\Omega}{1 + (2E_0/M) \sin^2 (\theta/2)}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_0^2 \sin^4 (\theta/2)} \frac{\cos^2 (\theta/2)}{1 + (2E_0/M) \sin^2 (\theta/2)} \left[ 1 + \frac{q^2}{2M} \frac{t}{t} \frac{\theta}{2} \right]
\]

(12)
1.7 Möller scattering (MÖLLER, 1932)

Möller scattering is the scattering of two electrons on each other. For historical and practical reasons (storage rings), the process is treated, in this and the next paragraphs, in the c.m. system. Therefore initial and final energy $E$ are the same.

Because there are two identical particles in the initial and final state, the process is represented by two amplitudes, resulting from the indistinguishability of the particles. Since the two particles are fermions, the total amplitude must be antisymmetric with respect to the exchange of the two initial (final) electrons, resulting in a minus sign between the two terms:

The cross-section in the centre-of-mass system calculates as follows:

$$
\frac{d\sigma}{d\Omega} = \frac{1}{v_{\text{rel}}} \frac{2\pi}{16E^2} |M_{fi}|^2 \, D
$$

$$
v_{\text{rel}} = 2 \frac{E}{P} \approx 2
$$

$$
-t_1 = 4E^2 \sin^2 \frac{\theta}{2}
$$

$$
-t_2 = 4E^2 \cos^2 \frac{\theta}{2}
$$

$$
M_{fi} = \frac{4\alpha^2}{(2\pi)^2} \frac{E}{\sin \frac{\theta}{2}} \left\{ \frac{\langle \bar{u}_{1/2} \gamma_\nu u_1 \rangle \langle \bar{u}_{3/2} \gamma_\nu u_3 \rangle}{t_1} - \frac{\langle \bar{u}_{1/2} \gamma_\nu u_3 \rangle \langle \bar{u}_{3/2} \gamma_\nu u_1 \rangle}{t_2} \right\}
$$

$$
D = \frac{1}{2\pi^2} \frac{Ep}{\sin \frac{\theta}{2}} \, d\Omega
$$

$$
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2} \left[ \frac{1 + \cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} + \frac{1 + \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} + \frac{2}{\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \right]
$$

(13)

Note the presence of a forward-backward symmetry, as there should be for identical particles.
1.8 Bhabha scattering (BHABHA, 1932)

Bhabha scattering is the scattering of an electron on a positron, described in the c.m. The process is again given by two amplitudes, the first being a t-channel exchange of a photon, the second an s-channel annihilation and subsequent pair production. The total amplitude now has to be antisymmetric with respect to the exchange of the incoming electron and the outgoing positron (ingoing electron), and vice versa:

\[
\begin{align*}
\begin{array}{c}
p_3 \quad e^- \quad e^+ \quad p_4 \\
p_1 - p_3 \\
p_1 \quad e^- \quad e^+ \\
p_2 
\end{array}
- \begin{array}{c}
p_3 \quad e^- \quad e^+ \\
p_1 + p_2 \\
p_1 \quad e^- \quad e^+ \\
p_2 
\end{array}
\end{align*}
\]

The cross-section calculates as usual:

\[
d\sigma = \frac{1}{v_{rel}} \frac{2\pi}{16 E^4} |M_{fi}|^2 D
\]

\[v_{rel} = 2 \frac{p}{E} \approx 2\]

\[-t_1 = 4E^2 \sin^2 \frac{\Theta}{2}\]

\[t_2 = 4E^2\]

\[M_{fi} = 4\pi e^2 \left\{ \frac{\langle \bar{u}_3 \gamma_\mu u_1 \rangle \langle \bar{u}_2 \gamma_\nu u_4 \rangle}{t_1} - \frac{\langle \bar{u}_1 \gamma_\mu u_2 \rangle \langle \bar{u}_3 \gamma_\nu u_4 \rangle}{t_2} \right\}\]

\[D = \frac{1}{(2\pi)^2} \frac{E p}{2} d\Omega\]

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2} \left[ \frac{1 + \cos^4 (\Theta/2)}{\sin^4 (\Theta/2)} + \frac{1 + \cos^2 \Theta}{2} - \frac{2 \cos^4 (\Theta/2)}{\sin^2 (\Theta/2)} \right]
\]

(14)

Note that the middle term represents the s- and d-wave "decay" of the intermediate photon in the second graph (see next section).
1.9 One-photon exchange (GRIFFY, 1967)

In this section we will show that there is a direct connection between the expression for the cross-section for ep scattering (the Rosenbluth formula), and the assumption of one-photon exchange. This is in as much important, as we will later parametrize the properties of the proton by two form factors. This parametrization is only valid, if the assumption of one-photon exchange is justified.

We will compare electron-proton scattering (s-channel) and its crossed process, $e^+e^-$ annihilation into a $\bar{p}p$ pair. For this purpose we describe the reactions by invariant quantities $s$ and $t$, and we postulate that the transition matrix element $M_{fi}$ obeys crossing symmetry:

\[ s = (p_1 + p_2)^2 \]
\[ t = (p_1 - p_2)^2 \]

\[ e^- p \rightarrow e^- p \]
\[ e^+ E_0 \rightarrow \bar{p}p \]

**laboratory system**

\[ s = M^2 + 2M E_0 \]
\[ t = -\frac{4E_0^2 \sin^2 (\Theta/2)}{1 + (2E_0/M) \sin^2 (\Theta/2)} \]

\[ \sin^2 \frac{\Theta}{2} = -\frac{M^2 t}{(s - M^2)(s - M^2 + t)} \]

\[ \cos^2 \phi = 1 + \frac{\ctg^2 (\Theta/2)}{1 - (t/4M^2)} \]

**c.m. system**

\[ t = 4E_0^2 = 4(M^2 + p^2) \]
\[ s = M^2 - 2E_0^2 + 2E_0 p \cos \phi \]

\[ \cos \phi = \frac{t + (4/t)[s - M^2](s - M^2 + t)}{t - 4M^2} \]

(15)
crossed channel:

\[ |N_{fi}|^2 = a''(t) + b''(t) \cotg \frac{\theta}{2} \]

\[ = \frac{\cotg \frac{\theta}{2}}{t} \left[ t a''(t) \cotg \frac{\theta}{2} + b''(t) \right] \]

\[ = \frac{\cos^2 \frac{\theta}{2}}{4E_0E \sin^2 \frac{\theta}{2}} \left[ a(t) + b(t) \cotg \frac{\theta}{2} \right] \]

(17)

The fermion pair with \( J^{PC} = 1^{--} \) is in a \( ^3S_1 \) or \( ^3D_1 \) state. s- and d-waves give an angular distribution

\[ |M_{fi}|^2 = a'(t) + b'(t) \cos^2 \phi \]  \hspace{1cm} (16)

This latter equation represents in essence the Rosenbluth formula for ep scattering.
1.10 Two-photon exchange (GRIFFY, 1967)

The contribution of a possible two-photon exchange amplitude, as seen in the graph:

\[
\Lambda = \alpha A_1 + \alpha^2 A_2
\]

\[
\frac{d\sigma}{d\Omega} \propto \alpha^2 A_1^2 + 2\alpha^3 \text{Re}(A_1A_2) + \alpha^4 |A_2|^2
\]

where \(A_1\) and \(A_2\) represent the one- and two-photon amplitude, respectively.

In the presence of form factors, the amplitudes can no longer be predicted; we may say, however, that for the proton as many as 12 form factors can be defined. The scattering amplitude, which is real for the one-photon exchange amplitude (first Born approximation), will be complex for the two-photon exchange (second Born approximation).

1.10.1 Modification of the Rosenbluth formula

In order to derive a more precise expression for the cross-section including a two-photon contribution, we turn again to the crossed reaction: \(e^+e^- \rightarrow p\bar{p}\).

The two photons can couple only to a state with charge conjugation \(C = +1\). Helicity conservation at the electron vertex forbids \(J = 0\), so that we choose to study \(J = 1\) as the lowest state. Then the following combinations for the baryon pair are possible:
\[ \begin{array}{|c|c|c|}
\hline
\text{2S+1L}_{ij} & p = (-1)^{L} & c = (-1)^{L+S} \\
\hline
^3S_1 & - & - \quad \text{one photon} \\
^3P_1 & + & + \quad \text{two photon} \\
^1P_1 & + & - \quad \text{forbidden} \\
^3D_1 & - & - \quad \text{one photon} \\
\hline
\end{array} \]

So the only allowed two-photon state is the \(^3P_1\) or \(1^{++}\) state.

Now we turn to the angular distribution in the \(\vec{p}\) rest frame. Whereas as a state with unique parity the one-photon process requires a symmetrical angular distribution from the s- and d-waves:

\[ |M_{rl}|_{\text{lan}}^2 = a'(t) + b'(t) \cos^2 \phi , \quad (16) \]

the two-photon contribution causes an interference between opposite parities and permits odd powers in \(\cos \phi\):

\[ |M_{rl}|_{\text{lan}}^2 = a'(t) + b'(t) \cos^2 \phi + c'(t) \cos \phi . \quad (20) \]

Returning to the scattering channel and replacing \(\cos \phi\) by use of Eq. (15), we obtain

\[ |M_{rl}|_{\text{sc}}^2 = a''(t) + b''(t) \cotg^2 \frac{\theta}{2} + c''(t) \left[ 1 + \frac{\cotg^2 \left( \frac{\theta}{2} \right)}{1 - (t/4M^2)} \right] \]

which modifies the expression for the Rosenbluth formula (17) to

\[ |M_{rl}|_{\text{sc}}^2 = \frac{\cos^2 \left( \frac{\theta}{2} \right)}{4E_E \sin^2 \left( \frac{\theta}{2} \right)} \left\{ a(t) + b(t) \cotg \frac{\theta}{2} + c(t) \cotg \frac{\theta}{2} \left( \cotg \frac{\theta}{2} + \frac{1}{1 - (t/4M^2)} \right) \right\} \quad (21) \]

**1.10.2 Difference between \(e^+p\) and \(e^-p\) scattering**

The sign of the one-photon term in the scattering amplitude is opposite for \(e^+p\) scattering:

\[ \frac{d\sigma}{d\Omega} = \left| \pm A_1 + A_2^* \right|^2 \]

\[ = \alpha^2 A_1^2 \pm 2 \alpha A_1 A_2 \Re A_2 + \alpha^4 \left| A_2^* \right|^2 \quad . \quad (22) \]

So the measurement of a difference in \(e^+p\) scattering at fixed \(\theta\) and \(q^2\)

\[ \Delta \left( \frac{d\sigma}{d\Omega} \right) = 4\alpha A_1 \Re A_2 \quad (23) \]

or the ratio

\[ \frac{\sigma_+}{\sigma_-} \approx 1 + \frac{4\alpha \Re A_2}{A_1} \quad (24) \]

is a test for two-photon exchange.
1.10.3 Polarization of the recoil proton

The polarization of the recoil nucleon in a scattering process is

\[ p = \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} = \frac{2 \text{Im} f g^* \hat{n}}{|f|^2 + |g|^2}, \]  

where \( \sigma_\pm \) are the yields of outgoing protons with spin parallel and antiparallel to the normal \( \hat{n} \), and \( f \) and \( g \) are non-spin-flip and spin-flip amplitudes.

In the one-photon exchange the amplitudes \( f \) and \( g \) are real, and therefore \( P = 0 \). In a two-photon exchange approximation we decompose

\[ f = \alpha f_1 + \alpha' f_2, \]
\[ g = \alpha g_1 + \alpha' g_2, \]

\[ P \approx \frac{2 \hat{n}}{|f|^2 + |g|^2} \text{Im} \left[ \alpha^2 f_1 g_1^* + \alpha' \left( f_1 g_2^* + f_2 g_1^* + \alpha'' f_2 g_2^* \right) \right] \]
\[ \approx \frac{2 \hat{n}}{|f|^2 + |g|^2} \alpha^3 \left( f_1 \text{Im} g_1^* + g_1^* \text{Im} f_2 \right). \]

The polarization measurement is a test for a complex amplitude. As such, it is also a test for a two-photon amplitude. Experimental results are given in Fig. 1.

There is no experimental evidence for the presence of a two-photon exchange contribution (BERGER, 1968; MAR, 1968; BIZOT, 1963).

---

**Fig. 1** Ratio of cross-sections for \( e^+ p \) and \( e^- p \) scattering. [MAR et al., Phys. Rev. Letters 21, 482 (1968).]
2. FORM FACTORS

2.1 Definition of the form factor

The form factor was introduced historically to describe the scattering on an extended charge distribution. To do this, we replace the point charge e by a charge distribution \( \rho(x) \):

\[
\rho(x) = e f(x)
\]  

(28)

with the normalization \( \int f(x) \, d^3x = 1 \). The distribution function \( f(x) \) can be interpreted as a classical spatial extension of the particle, or as a statistical probability distribution of more elementary, point-like constituents. From the charge distribution we obtain the Coulomb potential and the Hamiltonian (Appendix I)

\[
H = - \int j_\nu(x) V(x) \, d^3x.
\]  

(29)

Expressing the current \( j(x) \) by plane waves we obtain

\[
H = j_\nu(p) \int e^{i[p_\nu - p\nu]x} V(x) \, d^3x.
\]  

(30)

We now introduce the following purely mathematical transformations:

a) double differentiation

\[
\Box^2 e^{i q x} = -q^2 e^{i q x}
\]  

(31a)

b) Green's formula

\[
\int \left( u \Box^2 v - v \Box^2 u \right) \, d^3x = 0
\]  

(31b)

c) Poisson equation

\[
\Box^2 V = -4\pi \rho(x)
\]  

(31c)

\[
\int e^{i q x} V(x) \, d^3x = -\frac{1}{q^2} \int e^{i q x} \, V(x) \, d^3x
\]  

(32a)

\[
= -\frac{1}{q^2} \int e^{i q x} \Box^2 V(x) \, d^3x
\]  

(32b)

\[
= \frac{4\pi}{q^2} \int e^{i q x} \rho(x) \, d^3x
\]  

(32c)

\[
= \frac{4\pi e}{q^2} \int e^{i q x} f(x) \, d^3x.
\]  

(32d)

Now we define the form factor

\[
P(q) = \int e^{i q x} f(x) \, d^3x
\]  

(33)
as the Fourier transform of the spatial distribution function. Its normalization is

\[ F(0) = 1. \] (34)

Examples of form factors of different spatial distributions are given in the table below (WILSON, 1969).

<table>
<thead>
<tr>
<th>Charge distribution</th>
<th>Form factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>point ( f(r) = \delta(r - r_0) )</td>
<td>( F(q^2) = 1 ) unity</td>
</tr>
<tr>
<td>exponential ( f(r) = \frac{a^3}{8\pi} e^{-ar} )</td>
<td>( F(q^2) = \left[ \frac{1}{1 + q^2/a^2} \right]^2 ) dipole</td>
</tr>
<tr>
<td>Yukawa ( f(r) = \frac{a^2}{4\pi r} e^{-ar} )</td>
<td>( F(q^2) = \frac{1}{1 + q^2/a^2} ) pole</td>
</tr>
<tr>
<td>Gaussian ( f(r) = \left( \frac{a^2}{2\pi} \right)^{\frac{3}{2}} e^{-a^2 r^2/2} )</td>
<td>( F(q^2) = e^{-q^2/2a^2} ) Gaussian</td>
</tr>
</tbody>
</table>

We finally note that the spatial interpretation of the form factor is unsatisfactory in the scattering of high-energy particles, since the probe particle (electron) does not see a static charge distribution, but an accelerated one. We will therefore introduce the form factor in a more formal way in the following section.
2.2 The Rosenbluth formula (GRIFFY, 1967)

We turn to the construction of the proton current. For a Dirac proton without structure it is given by

\[
J_{\mu}(p) = ei \bar{U}(p_f) \gamma_{\mu} U(p_i).
\]  
(35)

Including a spatial extension of the Dirac proton, (35) modifies to

\[
J_{\mu}(p) = ei \bar{U}(p_f) \gamma_{\mu} F(q) U(p_i).
\]  
(36)

We see that the Dirac proton has only one form factor. This form factor depends only on the variable

\[
q^2 = -(p_i - p_f)^2 = -(p_f - p_i)^2
\]

\[
= -2m^2 + 2p_ip_f
\]

\[
= -2M^2 + 2p_ip_f
\]

since all other scalars formed of the four vectors involved are either trivial or equivalent:

\[
p_i \cdot p_i = p_f \cdot p_f = m^2
\]

\[
p_i \cdot p_f = p_f \cdot p_i = M^2
\]

\[
(p_i + p_f)^2 = 2m^2 + 2p_ip_f.
\]

Hermiticity requires the reality of the form factor. This can be seen from

\[
T_{fi} = T_{if}^X
\]

\[
\bar{U}(p_f) \gamma_\mu F(q^2) U(p_i) = \left[ \bar{U}(p_f) \gamma_\mu F(q^2) U(p_i) \right]^X
\]

\[
F(q^2) \left[ \bar{U}(p_f) \gamma_\mu U(p_i) \right] = F^*(q^2) \left[ \bar{U}(p_f) \gamma_\mu U(p_i) \right]^X
\]

Hermitian Hermitian

\[
F(q^2) = F^*(q^2),
\]

i.e. \(F(q^2)\) is real.

We have already seen in Section 1.9 that the matrix element for the annihilation process \(e^+e^- \rightarrow pp\) is described by two parameters, which can be interpreted in terms of amplitudes of S- and D-waves:

\[
\frac{d\sigma}{d\Omega}_{\text{annih.}} \propto a'(t) + b'(t) \cos^2 \phi. 
\]  
(37)

\[
S \rightarrow D\text{-wave} \quad D\text{-wave}
\]

Turning to the crossed channel, these two parameters appear again:

\[
\frac{d\sigma}{d\Omega}_{\text{scatt.}} \propto a(t) + b(t) \tan^2 \frac{\theta}{2}.
\]  
(38)
Thus we conclude that the real proton can have two form factors.

In order to obtain the current for the real proton we consider the energy density of a proton in an electromagnetic potential, including the Pauli term arising from the anomalous magnetic moment of the proton:

\[
\hat{H}(x) = j_0(x)A_\mu(x) + \mu_\text{an} \left( \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} \right),
\]

(39)

where \( \mu_\text{an} = \kappa_\text{nuc} = \kappa(e/2M) \) is the anomalous magnetic moment. This energy density in configuration space is translated in the usual way by a Fourier transformation into momentum space, and we obtain a transition matrix element

\[
M_{fi} = j_0(p) \frac{e}{q^2} \bar{\tilde{U}}(p_f) \left( \gamma_\mu + i \frac{\kappa}{2M} \sigma_\nu \gamma^\nu \right) \tilde{U}(p_i) \]

(40)

\[
\sigma_{\nu \nu} = \frac{i}{2} \left( \gamma_\nu \gamma_\nu - \gamma^\nu \gamma^\nu \right)
\]

with a proton current

\[
J_\mu = e i \bar{\tilde{U}}(p_f) \left( \gamma_\mu + i \frac{\kappa}{2M} \sigma_\nu \gamma^\nu \right) \tilde{U}(p_i).
\]

(41a)

We now introduce the structure of the proton by the Dirac and Pauli form factors:

\[
J_\mu = e i \bar{\tilde{U}}(p_f) \left( \gamma_\mu F_1(q^2) + i \frac{1}{2M} \sigma_\nu \gamma^\nu F_2(q^2) \right) \tilde{U}(p_i).
\]

(41b)

We are now able to derive the Rosenbluth formula for the scattering of an electron on a real proton (ROSENBURG, 1950):

\[
\frac{d\sigma}{d\Omega} = \frac{1}{\nu_{rel}} \frac{2\pi}{16E_0E E M E} \left| M_{fi} \right|^2 D
\]

\[
\nu_{rel} = \frac{p_0}{E_0} \approx 1
\]

\[-t = 4E_0 E \sin^2 (\theta/2)\]

\[
M_{fi} = (\bar{\tilde{U}}(p_f) U_i) \frac{4\pi e^2}{\nu} \left[ \bar{\tilde{U}}(p_f) \gamma_\mu F_1 + i \sigma_\nu \gamma^\nu F_2 \right] U_i
\]

\[
D = \frac{1}{(2\pi)^3} \frac{E^2 E'}{M} \int \frac{d\Omega}{1 + (2E_0/M) \sin^2 (\theta/2) + 2 \left( F_1 + 2M F_2 \right)^2 \tan^2 \frac{\theta}{2}}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_0^2 \sin^2 (\theta/2)} \cos^2 \frac{\theta}{2} \frac{1}{1 + (2E_0/M) \sin^2 (\theta/2)}
\]

\[
\text{Rutherford} \quad \text{Mott} \quad \text{recoil}
\]

(42)
2.3 Nucleon form factors

2.3.1 Definition

In order to describe the two states of the nucleon, the neutron and the proton, one obviously needs four form factors. We will introduce three definitions of them:

<table>
<thead>
<tr>
<th>Dirac and Pauli form factors:</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1^\text{proton}(q^2)$</td>
<td>normalization:</td>
<td>$F_1^\text{proton}(0) = 1$</td>
</tr>
<tr>
<td>$F_2^\text{proton}(q^2)$</td>
<td></td>
<td>$F_2^\text{proton}(0) = 1.79$</td>
</tr>
<tr>
<td>$F_1^\text{neutron}(q^2)$</td>
<td></td>
<td>$F_1^\text{neutron}(0) = 0$</td>
</tr>
<tr>
<td>$F_2^\text{neutron}(q^2)$</td>
<td></td>
<td>$F_2^\text{neutron}(0) = -1.91$</td>
</tr>
</tbody>
</table>

It is convenient to perform a linear transformation as follows

$$G_E(q^2) = F_E(q^2) - \frac{q^2}{4M^2} F_2(q^2)$$  \hspace{1cm} (43a)

$$G_M(q^2) = F_M(q^2) + F_2(q^2)$$  \hspace{1cm} (43b)

These transformed "G" form factors are called "electric" and "magnetic" form factors because they normalize to the total charge and total magnetic moment, respectively:

<table>
<thead>
<tr>
<th>Electric and magnetic form factors:</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_E^\text{proton}(q^2)$</td>
<td>normalization:</td>
<td>$G_E^\text{proton}(0) = 1$</td>
</tr>
<tr>
<td>$G_M^\text{proton}(q^2)$</td>
<td></td>
<td>$G_M^\text{proton}(0) = 2.79$</td>
</tr>
<tr>
<td>$G_E^\text{neutron}(q^2)$</td>
<td></td>
<td>$G_E^\text{neutron}(0) = 0$</td>
</tr>
<tr>
<td>$G_M^\text{neutron}(q^2)$</td>
<td></td>
<td>$G_M^\text{neutron}(0) = -1.91$</td>
</tr>
</tbody>
</table>

The introduction of the "G" form factors in the Rosenbluth formula gives an expression without the interference between two form factors:

$$\frac{d\sigma}{d\Omega} = \sigma_{NS} \left[ \frac{G_E^2 + \left(q^2/4M^2\right) G_M^2}{1 + \left(q^2/4M^2\right)} + \frac{q^2}{2M^2} G_M^2 \tan^2 \frac{\theta}{2} \right].$$  \hspace{1cm} (44)
Finally, we introduce generalized isotopic spin form factors by the following transformation:

\begin{align}
G_{\text{scalar}}^l &= \frac{1}{2} \left( G_{\text{proton}}^l + G_{\text{neutron}}^l \right) \\
G_{\text{vector}}^l &= \frac{1}{2} \left( G_{\text{proton}}^l - G_{\text{neutron}}^l \right)
\end{align}

(45a) (45b)

Isotopic spin form factors:

\begin{align*}
G_E^S(q^2) &\quad \text{normalization:} \quad G_E^S(0) = 0.5 \\
G_E^V(q^2) &\quad G_E^V(0) = 0.5 \\
G_M^S(q^2) &\quad G_M^S(0) = -0.06 \\
G_M^V(q^2) &\quad G_M^V(0) = 1.85
\end{align*}

2.3.2 Measurement

The nucleon form factors are, in general, obtained from a measurement of the differential cross-sections of electron-nucleon scattering at a fixed value of the four-momentum transfer \( q^2 \) and at different angles \( \theta \). The energy of the scattered electron is given by Eq. 1. Only the electron is detected. The experimental result, after applying a "radiative correction" (Appendix II), is displayed in a "Rosenbluth plot" by plotting

\[
\frac{d\sigma}{d\Omega} \bigg|_{\text{corr}} / \sigma_{\text{NS}} \quad \text{versus} \quad \tan^2 \frac{\theta}{2}
\]

\[
\frac{d\sigma}{d\Omega} / \sigma_{\text{NS}} \quad \text{best fit}
\]

\[
t = \frac{q^2}{4M^2}
\]

The form factors are then extracted from

\[
\text{slope} = 2\tau G_M^2 \\
\text{intersect} = \frac{G_E^2 + \tau G_M^2}{1 + \tau}
\]
Another version of the Rosenbluth plot is to plot (see Fig. 2):

\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{corr}} / \sigma_{\text{NS}} \cot^2 \frac{\theta}{2} \] versus \( \cot^2 \frac{\theta}{2} \).

We see that the square of the magnetic form factor is always multiplied by \( \tau = q^2/4M^2 \); we will also find that \( G_M^2 \) scales approximately with \( \mu^2 G_E^2 \). This is the reason why \( G_M^2 \) is always better measured than \( G_E^2 \), and in experiments at large values of \( q^2 \) the contribution of \( G_E^2 \) is neglected.

The following experimental data are available:

a) Electron-proton scattering on proton targets, giving \( G_E \) and \( G_M \). Experiments were done at Stanford, Orsay, Cornell, Harvard, DESY, SLAC and Bonn.

b) Electron-neutron scattering by quasi-elastic scattering on deuteron targets, invoking the validity of the impulse approximation:

The en cross-section is obtained by a subtraction procedure:

\[ \sigma(e\text{n}) = \sigma(e\text{d}) - \sigma(e\text{p}) + \text{nuclear physics.} \]

Therefore the results are less precise than those for the proton.
c) Elastic electron-deuteron scattering. Because of the isospin of the deuteron, this measurement yields directly the isoscalar form factor:

\[
\sigma_{\text{ed}} = \frac{1}{2} \left[ G_{q^2}^{\text{proton}} + G_{q^2}^{\text{neutron}} \right]^2 F_D(q^2)
\]

Here \( F_D(q^2) \) is the deuteron form factor, which decreases rapidly with \( q^2 \), so that the experiment is only possible at small values of \( q^2 < 1 \text{ (GeV/c)}^2 \).

d) Neutron-electron scattering. Thermal neutrons from a reactor are scattered on atomic electrons of noble gases. The interference of isotropic nuclear scattering with Coulomb scattering is observed:

\[
\frac{d\sigma}{d\Omega} = \left| a_{\text{nuclear}} + b_{\text{Coulomb}} F(q^2) \right|^2.
\]

From measurements at two angles one obtains (Knox, 1966):

\[
R = \frac{(d\sigma/d\Omega) \, (45^\circ)}{(d\sigma/d\Omega) \, (135^\circ)} \approx 1 + \frac{8\pi b \, \Delta F \, \text{Re} \, a}{|a|^2} + \frac{8\pi b_{\text{Coulomb}} \, \Delta F(q^2) \, \text{Re} \, a_{\text{nuclear}}}{|a|^2}
\]

where \( \Delta F = F(45^\circ) - F(135^\circ) \).

result:

\[
\frac{d\sigma_{\text{neutron}}}{dq^2} \bigg|_{q^2=0} = (0.0193 \pm 0.0004) \text{ fermi}^2.
\]

2.3.3 Results

The world values of nucleon form factor determinations are given in Figs. 3-6. The results can be parametrized by the empirical expressions, the "scaling law" and the "dipole formula" (Fig. 7):

**Scaling Law**

\[
G(q^2) = G_{\text{N}}^{P}(q^2) = \frac{G_{\text{N}}^{P}(q^2)}{m_N} = G_{\text{N}}^{M}(q^2)
\]

\[ G_{\text{N}}^{N}(q^2) = 0 \]  \hspace{1cm} (46)

**Dipole Formula**

\[
G(q^2) = \left( \frac{1}{1 + (q^2/0.71)} \right)^2 \left[ q^2 \text{ in (GeV/c)}^2 \right]
\]

\[
G(q^2) = \left( \frac{1}{1 + (q^2/18.1)} \right)^2 \left[ q^2 \text{ in fermi}^{-2} \right]
\]  \hspace{1cm} (47a)

The scaling law and the dipole formula are purely empirical, but nevertheless of surprising simplicity. Their validity may, however, be limited, as seen from Fig. 8.
Fig. 3 The magnetic form factor $G_{MP}$ divided by the magnetic moment of the proton $\mu_p$ as a function of the momentum transfer. [WEBER, Proc. Symposium on Electron and Photon Interactions, Stanford (1967), p. 59.]

Fig. 4 The charge form factor of the proton $G_{EP}$ as a function of $q^2$. [WEBER, Proc. Symposium on Electron and Photon Interactions, Stanford (1967), p. 59.]
Fig. 5 The magnetic form factor of the neutron $G_{MN}$, divided by the neutron magnetic moment $\mu_N$. [WEBER, Proc. Symposium on Electron and Photon Interactions, Stanford (1967), p. 59.]

Fig. 6 The charge form factor of the neutron as a function of $q^2$. The dotted line represents the thermal neutron slope. [WEBER, Proc. Symposium on Electron and Photon Interactions, Stanford (1967), p. 59.]
Fig. 7 The SLAC values of $G_M/\mu$ are plotted against the square of the four momentum transfer $q^2$. The solid line is the dipole model. [SLAC group (unpublished).]

Fig. 8 Compilation of electron-proton elastic scattering cross-sections for $q^2$ greater than 1 (GeV/c)$^2$. The cross-sections are normalized to the Rosenbluth formula and the dipole relation. [SLAC group (unpublished).]
2.4 Interpretation of form factors

2.4.1 Spatial interpretation (Muirhead, 1965)

In Section 2.1 we introduced the electromagnetic form factor as the Fourier transform of a spatial charge distribution. In reverse, we can retransform the experimental findings by the inverse operation.

For the dipole form factor

\[ G(q^2) = \left( \frac{1}{1 + \left[ q^2 F(q^2)/18.1 \right]} \right)^2 \tag{48} \]

we find an exponential charge distribution

\[ f(r) = \int G(q^2) \ e^{-iqr} \ d^3q \tag{49} \]

\[ f(r) \bigg|_{\text{dipole}} = 3.06 \ \text{exp} \ (-4.25 \ r) \tag{50} \]

For the approximation of small momentum transfer we can define an electromagnetic radius of charged particles:

\[ F(q^2) = \int f(r) \ e^{iqr} \ d^3r \tag{51} \]

\[ e^{iqr} = 1 + i(q \cdot r) - \frac{(q \cdot r)^2}{2} + \ldots \]

\[ F(q^2) \approx F(0) \left( 1 - \frac{q^2}{2} \int r^2 \cos^2 \Theta \ d^3r \right) \tag{52a} \]

\[ \approx F(0) \left( 1 - \frac{1}{6} q^2 \langle r^2 \rangle \right) \tag{52b} \]

\[ \langle r^2 \rangle = \frac{\int r^2 \rho(r) \ d^3r}{\int \rho(r) \ d^3r} . \tag{53} \]

The linear term in \( (q \cdot r) \) vanishes because of central symmetry.

As an inversion of Eq. (52b) we can derive the radius for a given form factor parametrization by

\[ \langle r^2 \rangle = -\frac{6}{F(0)} \frac{d}{dq^2} F(q^2) \bigg|_{q^2=0} . \tag{54} \]

Inserting the dipole expression of the proton form factor we obtain for the r.m.s. proton radius

\[ \langle r^2 \rangle_{\text{proton}}^{1/2} = 0.81 \ \text{fermi} . \]

2.4.2 \( ep \) and \( pp \) scattering

Wu and Yang (Wu, 1965) suggested that the charge distribution in the proton may be the same as that of strong matter. Therefore \( ep \) scattering and \( pp \) scattering should be related. For comparison, the electron is considered as a point probe, the proton as a
cloud, so that pp scattering is a point-cloud interaction, pp scattering a cloud-cloud process. The following conventions are used (VAN HOVE, 1966):

\[
\frac{d\sigma}{dt}\bigg|_{pp} = \frac{d\sigma}{dt}\bigg|_{t=0} G^2(t)_{pp} \tag{55}
\]

\[
G^2(t)_{pp} = G^2(q^2)_{pp} \tag{56}
\]

\[t = -q^2.\]

The latest comparison of this kind is shown in Fig. 9 (ABARBANEL, 1966). It turns out that \(G^2(t)\) is really a function of \(s\) and \(t\): \(G^2(s,t)\). The authors postulate that the above prediction (55) is valid asymptotically at \(s \to \infty\).

![Fig. 9 Plot of \(X(s,t) = \left[\frac{d\sigma(t)/dt}{d\sigma(0)/dt}\right]\) for pp scattering and of \(G^2(t)/G^2(0)\). [ABARBANEL et al., Phys. Rev. Letters 20, 280 (1968).]

2.4.3 Dispersion relations (GRIFFY, 1987)

From causality arguments it is expected that form factors follow dispersion relations (Appendix III). Then the real form factor \(G(t)\) with \(t = -q^2 < 0\) can be expressed by a dispersion integral over the imaginary (absorptive) part of \(G(t)\) for \(t \geq t_0 > 0\):

\[
G(t) = G(0) + \frac{1}{i\pi} \int_{t_0}^{\infty} \frac{\text{Im} G(z)}{z - t} \, dz.
\]

Here the core term describes a possible point charge in the proton core. Measurements give \(G(0) \leq 10^{-3}\). The pole terms represent all absorptive states of the photon, coupling
to the nucleon-antinucleon state. The threshold $t_0$ is the square of the lowest possible strongly interacting state with the compulsory quantum numbers of the photon $J^{PC} = 1^{--}$, generally identified with the two-pion threshold, i.e. $t_0 = (2m_\pi)^2$.

Even before the discovery of meson resonances, it was postulated (Frazer, 1959) that the intermediate states were represented by particles with $J^{PC} = 1^{--}$. These were later identified with the vector mesons $\rho^0$, $\omega$, and $\phi$. In the hypothesis of "pole dominance", the form factors are represented by these three poles:

$$
\begin{align*}
\rho & \quad + \quad \omega & \quad + \quad \phi \\
\pi & \quad \pi & \quad K & \quad K
\end{align*}
$$

Limiting the vector mesons to $\delta$-functions at their mass, the integral reduces to the "Clemental-Villi" formula for isoscalar and isovector form factors:

$$
G_E^S(t) = a_s + \frac{a_\omega}{m_\omega^2 - t} + \frac{a_\phi}{m_\phi^2 - t} \quad (57)
$$

$$
G_M^V(t) = a_V + \frac{a_\rho}{m_\rho^2 - t} \quad (58)
$$

and a corresponding set for the $G_M$'s. The full set contains, adding the normalization at $t = 0$, a total of six free parameters.

As stated in Section 2.3.3, the experimental form factors are described quite accurately by the dipole formula:

$$
G(t) = \left(\frac{1}{1 - (t/0.71)}\right)^2 . \quad (59)
$$

The dipole expression is equivalent to the sum of two pole terms:

$$
G(t) = \frac{b_1}{m_1^2 - t} + \frac{b_2}{m_2^2 - t} \quad (60)
$$

which reduces to a dipole expression for $b_1 = -b_2$ and $m_1 \approx m_2$. Since there are two isoscalar vector mesons, it may be possible to describe the isoscalar form factor, but pole dominance is certainly ruled out for the isovector form factor, which should be represented by the $\rho$ pole alone.

If the validity of dispersion relations should be maintained, one of the following three possibilities are open.

i) There is an undiscovered vector meson with isospin 1. This possibility cannot be discarded, because the strong decay mode of this particle can be anything with $I = 1$.

ii) Pole dominance is wrong. There are background waves with $J^{PC} = 1^{--}$, which behave in such a way that they give a dipole expression.
iii) The coupling of the vector meson to the nucleon contains another form factor. The total form factor is the product of two components, coming from the $\rho$ propagator and from the $\rho NN$ vertex:

$$G^V(t) = \frac{a}{m_\rho^2 - t} \cdot \frac{1}{1 - (t/\Lambda^2)}.$$ (61)

The experiment gives $\Lambda^2 \approx 1 \text{ GeV}^2$ (MASSAM, 1966). This is no conventional pole dominance, because $\text{Im } G(t)$ is not completely represented by the poles.
2.5 The pion form factor

Three methods have been used to measure the pion form factor for \( t < 0 \).

i) Electroproduction of pions.

\[
\begin{align*}
\left( r_\pi^+ \right)^\frac{1}{2} & = (0.80 \pm 0.10) \text{ fermi (AKERLOF, 1967)} \\
\left( r_\pi^- \right)^\frac{1}{2} & = (0.86 \pm 0.14) \text{ fermi (MISTRETTA, 1968)}.
\end{align*}
\]

Kinematical conditions are chosen such that the second graph is enhanced, but the amplitude of the first graph has to be known. So the result is model dependent. Results:

\[
\begin{align*}
\left( r_\pi^+ \right)^\frac{1}{2} & \leq 0.9 \text{ fermi (BLOCK, 1968)} \\
\left( r_\pi^- \right)^\frac{1}{2} & = (2.96 \pm 0.43) \text{ fermi (CROWE, 1968)}.
\end{align*}
\]

ii) Scattering of positive and negative pions on helium, which gives opposite sign for the interference between nuclear and Coulomb amplitude. Model dependent:

\[
\sigma = \left| A_N \pm A_C(q^2) \right|^2 \\
= \left| A_N \right|^2 + A_C^2 \pm 2A_C \text{ Re } A_N.
\]

Results:

\[
\begin{align*}
\left( r_\pi^+ \right)^\frac{1}{2} & \leq 0.9 \text{ fermi (BLOCK, 1968)} \\
\left( r_\pi^- \right)^\frac{1}{2} & = (2.96 \pm 0.43) \text{ fermi (CROWE, 1968)}.
\end{align*}
\]

iii) Pion-electron scattering. This is the only experiment which may give a model-independent result. Because of the small mass of the electron, only low values of \( q^2 \) can be obtained at present pion energies.

In the frame of dispersion relations, the pion form factor should be determined by the \( \rho \)-meson alone, since it is the only vector meson decaying into two pions. The "\( \rho \) dominance" predicts the pion form factor to be

\[
F_\pi(t) = \frac{m_\rho^2}{m_\rho^2 - t}.
\]

Formula (54) gives the prescription for the pion radius:

\[
\left( r_\pi^2 \right)^\frac{1}{2} = \left\{ \frac{6}{F(0)} \left[ \frac{d}{dt} F(t) \right]_{t=0} \right\}^\frac{1}{2} = \frac{1}{m_\rho} \sqrt{\frac{6}{25.68}}
\]

\[
\left( r_\pi^2 \right)^\frac{2}{3}_\rho = 0.63 \text{ fermi}.
\]
Experimental results from method (i) are not incompatible with this prediction (Fig. 10).

Fig. 10  Estimates of the pion form factor based on dispersion theory. [MISTRETTA et al., Phys. Rev. Letters 20, 1525 (1968).]
2.6 The electron form factor

The investigation of hadron form factors by electron scattering is based on the assumption that the electron has no structure, i.e. that

\[ \rho_e(r) = e \delta(x - x_0) \]

\[ F_e(q^2) = 1. \]

This hypothesis was verified by e⁻e⁻ scattering at the Stanford Storage Ring. To account for an electron structure, we write the electron current

\[ j_{\mu}(p) = \text{i} e [ \bar{u}(p_f) \gamma_{\mu} F_e(q^2) u(p_i) ] . \]

The two graphs with exchange symmetry in Section 1.7 give two momentum transfers

\[ q_1^2 = 4 E^2 \sin^2 \frac{\theta}{2} \]

\[ q_2^2 = 4 E^2 \cos^2 \frac{\theta}{2} . \]

The Möller cross-section is then

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2} \left[ \frac{1 + \cos^4 \left( \frac{\theta}{2} \right)}{\sin^4 \left( \frac{\theta}{2} \right)} F_e(q_1^2) + \frac{1 + \sin^4 \left( \frac{\theta}{2} \right)}{\cos^4 \left( \frac{\theta}{2} \right)} F_e(q_2^2) + \frac{2}{\sin^2 \left( \frac{\theta}{2} \right) \cos^2 \left( \frac{\theta}{2} \right)} F_2(q_1^2) F_2(q_2^2) \right] . \]

The parametrization of \( F_e(q^2) \) is

\[ F_e(q^2) = \frac{1}{1 + \left( q^2/\Lambda^2 \right)} , \]

where \( \Lambda^2 \) is a fitted parameter. The experiments at (single) energies of 300 MeV and 550 MeV yielded

\[ \Lambda^{-2} \leq 0.03 \text{ GeV}^{-2} \]

corresponding to

\[ \langle r_e^2 \rangle \leq 0.085 \text{ fermi} \]

(95% confidence level, GITTELSPAN, 1969)

which is compatible with a point charge.
3. VECTOR MESONS

3.1 The vector dominance model (J005, 1967)

The vector dominance model (VDM) postulates that the interaction between photons and hadrons is mediated by vector mesons, i.e. by strongly interacting particles with the quantum numbers $J^{PC} = 1^{--}$:

$$\gamma \leftrightarrow V^0 \leftrightarrow \text{hadrons}$$

The relation between the electromagnetic current $j_{\mu}(x)$ and the vector meson current $V^0_{\mu}(x)$ is given by

$$j_{\mu}(x) = -\left[\frac{m_\gamma^2}{2\gamma_0} \rho_\mu(x) + \frac{m_\omega^2}{2\gamma_\omega} \omega_\mu(x) + \frac{m_\phi^2}{2\gamma_\phi} \phi_\mu(x)\right], \quad (66)$$

where $\rho, \omega, \phi$ are the known vector mesons, $m_\gamma$ their masses, and the $\gamma_i$'s are constants characteristic of the coupling. In the Feynman language, the $\gamma V^0$ vertex is described by a coupling constant

$$g_{\gamma V} = -\frac{e}{2\gamma_0} m_\gamma^2 \epsilon_{\gamma V} \quad (67)$$

where the $\epsilon_{\gamma V}$ is the polarization vector of vector particles. The VDM is evidently valid on the mass shell of the vector mesons, since they are known to have a well-defined decay rate

$$V^0 \rightarrow \gamma \rightarrow e^+ e^-.$$

The non-trivial statement of the model is, however, that the same coupling constant is valid at all masses of the photon:

a) $t = m_\gamma^2$ e.m. decay ("time-like")

$$\begin{align*}
(a) & \quad V^0 \quad \text{leptonic decay} \\
& \quad t = m_\gamma^2
\end{align*}$$

b) $t = 0$ photoproduction

$$\begin{align*}
(b) & \quad \gamma \rightarrow V^0 \\
& \quad \text{photoproduction of pion} \\
& \quad t = 0 \\
& \quad \text{V}^0 \text{- production by pion}
\end{align*}$$
c) \( t < 0 \) Coulomb scattering ("space-like")

\[
\begin{array}{c}
(c) \quad e \quad Y \quad V^0 \\
\quad e
\end{array}
\]

form factor

\( t < 0 \)
3.2 Leptonic decays of vector mesons (TING, 1968)

The quantity determined experimentally is the branching ratio

\[
\text{BR} = \frac{N(V^0 \rightarrow \ell^+ \ell^-)}{N(V^0 \rightarrow \text{all})} \approx \frac{N(V^0 \rightarrow \ell^+ \ell^-)}{N(V^0 \rightarrow \text{hadrons})}.
\]

Thus leptonic and hadronic decay have to be measured. A difficulty of the experiment is, at least for the $\rho^0$ and $\omega^0$, that their decays interfere in the leptonic final state.

From the experimental branching fraction $\text{BR}$, the leptonic decay width and further the $\gamma V^0$ coupling constant $\gamma_\nu$ can be obtained:

\[
\Gamma = \frac{2\pi}{8m_\nu E_\ell^2} |M_{F1}|^2 \, D
\]

\[
t = m_\nu^2
\]

\[
M_{F1} = -\frac{e m_\nu^2}{2\gamma_\nu} \frac{4\pi e}{t} (\bar{u}_1 \gamma_\nu u_2)
\]

\[
D = -\frac{1}{(2\pi)^3} \frac{E^2 p}{2E} \, d\Omega
\]

\[
\Gamma = \frac{\alpha^2}{12} \left( \frac{\gamma_\nu^2}{4\pi} \right)^{-1} m_\nu \left( 1 - \frac{4m_\nu^2}{m_\nu^2} \right)^{\frac{3}{2}} \left( 1 + \frac{2m_\nu^2}{m_\nu^2} \right)
\]

(68a)

\[
\approx \frac{\alpha^2}{12} \left( \frac{\gamma_\nu^2}{4\pi} \right)^{-1} m_\nu \quad \text{for electrons.}
\]

(68b)
Fig. 11 Invariant mass spectra in $\rho \to \pi^+\pi^-$ from DESY-MIT and from Harvard. The $\rho \to \pi^+\pi^-$ spectra measured by the DESY-MIT group under the same kinematical conditions are also shown. [TING, Proc. Int. Conf. on High-Energy Physics, Vienna (1968) (CERN, Geneva, 1968), p. 43.]
Fig. 12 \((e^+e^-)\) mass distribution from the reaction \(\pi^- p \rightarrow n e^+ e^-\), indicating the decay of \(\rho\) and \(\omega\) into electron pairs. [BOLLINI, Nuovo Cimento 57 A, 404 (1968).]
Fig. 13 Observation of the rare decay mode of the $\phi$ meson: $\phi \to e^+e^-$. 
[BOLLINI, Nuovo Cimento 56 A, 1173 (1968).]
Fig. 14 Invariant mass spectra of $\phi \rightarrow e^+e^-$ and $\phi \rightarrow K^+K^-$ measured by the DESY-MIT group. [TING, Proc. Int. Conf. on High-Energy Physics, Vienna (1968) (CERN, Geneva, 1968), p. 43.]
3.3 Leptonic formation of vector mesons (TING, 1968)

Here we describe the experiments performed during the last years at the electron-
positron storage rings at Novosibirsk and at Orsay. These experiments have the following
advantages with respect to leptonic decay studies:

i) the energy resolution is about 100 keV, only limited by fluctuations of synchrotron
radiation;

ii) there is no interference between ρ and ω because of different final states.

Cross-sections were normalized with Bhabha scattering (see Section 1.8) which, as an
electromagnetic process, can be calculated. The following reactions were studied:

a) \( e^+e^- \rightarrow ρ + π^+π^- \) (AUSLANDER, 1967; AUGUSTIN, 1969A); two collinear particles from
   the interaction region;

b) \( e^+e^- \rightarrow ω + π^+π^- (π^0) \) (AUGUSTIN, 1969B); two non-collinear particles;

c) \( e^+e^- \rightarrow ϕ \rightarrow π^+π^- (π^0) \) (AUGUSTIN, 1969C)
   \( e^+e^- \rightarrow ϕ \rightarrow π^+π^- (π^0) \) non-collinear;
   the \( ϕ \rightarrow K^+K^- \) decay could not be studied because the charged kaons stop in the vacuum
   chamber walls.

\[
\begin{align*}
\text{Using the phenomenological parametrization of resonance formation, we obtain} \\
\frac{dσ}{dΩ} (2E, θ) = \frac{1}{4} \lambda^2 (2J + 1) \frac{\Gamma_i \Gamma_f}{4 \left(2E - m_\pi\right)^2 + \left(r^2/4\right)} \frac{3}{2} \sin^2 θ . \quad (69)
\end{align*}
\]

Here the factor \( \frac{1}{4} \) accounts for the fact that for \( J = 1 \) only the spin combination

\[
\frac{1}{\sqrt{2}} \left[ \bar{u}(m = \frac{1}{2}), u(-\frac{1}{2}) + \bar{u}(-\frac{1}{2}), u(\frac{1}{2}) \right] \quad (70)
\]

satisfies the Pauli principle (GATTO, 1965). \( E \) is the energy of each electron, \( \lambda = 1/E \),
\( \Gamma_i, \Gamma_f \) are the widths of the initial state and final state, respectively, and \( \Gamma \) is the
total width. \( θ \) is the pion (π), kaon (ϕ) or decay normal (ω) angle with respect to the
collision line of the electrons. Integration over \( θ \) gives

\[
σ(2E) = \frac{π\lambda^2}{4} (2J + 1) \frac{\Gamma_i \Gamma_f}{\left(2E - m_π\right)^2 + \left(r^2/4\right)} \quad (71)
\]
and, at resonance, with $J = 1$, and $2E = m_{\nu}$

$$\sigma(m_{\nu}) = \frac{12\pi}{m_{\nu}^2} \frac{\Gamma_{f}^{f}}{\Gamma^{2}}.$$  \hspace{1cm} (72)

Since $\Gamma_{f}$ is known and $\Gamma$ is measured, we obtain $\gamma_{\nu}$ from

$$\Gamma_{f} (\nu^{0} + e^{+}e^{-}) = \frac{\alpha^2}{12} \left( \frac{\gamma_{\nu}^2}{4\pi} \right)^{-1} m_{\nu}.$$  \hspace{1cm} (68b')

The following tables summarize the results obtained during the last years.

(TING, 1968)

| Decay | $M_{\nu}$ (MeV) | $\Gamma_{\nu}$ (MeV) | BR $\times 10^{6}$ | $\Gamma_{\nu \rightarrow \ell}$ (keV) | Prod. reaction | Lab.
|-------|-----------------|---------------------|-----------------|------------------|------------|------|
| $\rho \rightarrow \nu^{+}\nu^{-}$ | $-97 \pm 20$ | $5.8 \pm 1.2$ | $-\nu p \rightarrow \rho N$ | Harv.
| $e^{+}e^{-}$ | $-6.4 \pm 1.5$ | $-\nu p \rightarrow \rho N$ | Novosib.
| $e^{+}e^{-}$ | $754 \pm 9$ | $105 \pm 20$ | $5.0 \pm 1.0$ | $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}$ | Orsay
| $\omega \rightarrow e^{+}e^{-}$ | $6.54 \pm 0.72$ | $7.13 \pm 0.51$ | $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}$ | CERN
| $\omega \rightarrow e^{+}e^{-}$ | $3.3 \pm 0.7$ | $\gamma_{\nu} \rightarrow \nu \nu \nu$ | Orsay
| $\omega \rightarrow e^{+}e^{-}$ | $9.7 \pm 2.0$ | $-\nu N \rightarrow \nu N$ | CERN
| $\omega \rightarrow e^{+}e^{-}$ | $4.0 \pm 1.5$ | $0.49 \pm 0.19$ | $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}$ | Bol-CERN
| $\omega \rightarrow e^{+}e^{-}$ | $7.9 \pm 1.47$ | $1.11 \pm 0.26$ | Orsay
| $\phi \rightarrow e^{+}e^{-}$ | $3.55 \pm 0.48$ | $1.49 \pm 0.35$ | $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}$ | Bol-CERN
| $\phi \rightarrow e^{+}e^{-}$ | $2.7 \pm 0.8$ | $\nu p \rightarrow \rho N$ | Orsay
| $\phi \rightarrow e^{+}e^{-}$ | $4.2 \pm 0.9$ | $3.9 \pm 0.62$ | Orsay
| $\phi \rightarrow e^{+}e^{-}$ | $2.1 \pm 0.9$ | $1.62 \pm 0.26$ | Orsay

### Average experimental results

<table>
<thead>
<tr>
<th>Decay</th>
<th>BR $\times 10^{-4}$</th>
<th>$\Gamma_{\nu \rightarrow \ell}$ (keV)</th>
<th>$\frac{\gamma_{\nu}^2}{4\pi}$</th>
<th>$\gamma_{\nu}^2$</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho \rightarrow e^{+}e^{-}$</td>
<td>$6.04$</td>
<td>$108.0$</td>
<td>$0.52^{+0.07}_{-0.06}$</td>
<td>$9$</td>
<td>DESY-MIT Novosib. Orsay Harv.</td>
</tr>
<tr>
<td>$\phi \rightarrow e^{+}e^{-}$</td>
<td>$3.55$</td>
<td>$4.2$</td>
<td>$3.04^{+1.07}_{-0.66}$</td>
<td>$1.54^{+0.43}_{-0.42}$</td>
<td>Bol-CERN DESY-MIT Orsay</td>
</tr>
<tr>
<td>$\omega \rightarrow e^{+}e^{-}$</td>
<td>$6.1$</td>
<td>$12.2$</td>
<td>$4.69^{+1.24}_{-0.81}$</td>
<td>$1.00^{+0.21}_{-0.21}$</td>
<td>Bol-CERN Orsay</td>
</tr>
</tbody>
</table>
Fig. 15  Experimental total cross-section for the process $e^+ e^- \rightarrow \pi^+ \pi^- \pi^0$ versus energy. Solid curve is a best fit with a Breit-Wigner expression. [AUGUSTIN et al., Phys. Letters 28 B, 514 (1969).]

Fig. 16  a) Excitation curve of $\phi \rightarrow K_{S}^{0} (K_{S}^{0} \rightarrow \pi^+ \pi^-) K_{S}^{0}$. [AUGUSTIN et al., Phys. Letters 28 B, 519 (1969).]
   b) Excitation curve of $\phi \rightarrow \pi^+ \pi^- \pi^0$. [Ref. as for Fig. 16a.]
3.4 The pion form factor

3.4.1 The time-like region

We will again derive a cross-section for the leptonic formation of the $\rho$ vector meson. This time, however, we will rigorously derive it from Feynman rules, describing the unknown $\gamma\pi\pi$ vertex as a form factor. We hereby introduce the form factor also for the "time-like" region, i.e. for the square of the four-momentum transfer $t > 0$:

\[
\begin{align*}
\frac{d\sigma}{d\Omega} &= \frac{1}{\nu_{\text{rel}}} \frac{2\pi}{16} \left| M_{F_1} \right|^2 D \\
\nu_{\text{rel}} &= \frac{\mathbf{P}_e}{E} \approx 2 \\
t &= (2E)^2 \\
M_{F_1} &= \langle \bar{u}_i \gamma_{\mu} u_j \rangle \frac{4\pi e^2}{t} F_{\pi}(q^2) (p_1 + p_2) \\
D &= \frac{1}{(2\pi)^3} \frac{E^2 p}{2E} \, d\Omega \\
&= \frac{\alpha^2}{32E^2} \beta^2 \left| F_{\pi}(4E^2) \right|^2 \sin^2 \frac{\Theta}{2} \\
\sigma &= \frac{\pi \alpha^2}{12E^2} \beta^2 \left| F_{\pi} \right|^2
\end{align*}
\]

(73a)

Experimental result (Fig. 17): $\left| F_{\pi}(m^2) \right|^2 = 55 \pm 6$ (AUGUSTIN, 1969).

3.4.2 The space-like region

We now derive the pion form factor in the "space-like" region $t < 0$ from the knowledge in the "time-like" region $t > 0$ by means of a dispersion relation (Appendix III):

\[
F(t) = \frac{1}{\pi} \int \frac{\text{Im} \ F(z)}{z - t} \, dz .
\]
Fig. 17  Pion form factor as a function of energy. The curve is the best fit with a two parameter \((m_\rho, \Gamma_\rho)\) analytic expression for the pion form factor. The errors shown on this plot are statistical only. [AUGUSTIN et al., Phys. Letters 28 B, 511 (1969).]

We further assume "pole dominance", i.e. we invoke that the absorptive states are completely represented by the vector mesons, of which only the \(\rho\) is known to couple to the \(\pi\pi\) state. We parametrize the \(\rho\) by the exact expression for a Breit-Wigner resonance with an amplitude \(F_0\)

\[
|F_n(2E)|^2 = \frac{F_0^2 m_\rho^2 \Gamma^2}{(4E^2 - m_\rho^2)^2 + m_\rho^2 \Gamma^2}
\]

(74a)
of which the usual Breit-Wigner formula is an approximation for \(2E \approx m_\rho\). We check the consistency of this parametrization by evaluating Eq. (74a) for \(E = 0\), which should yield \(F_n(0) = 1\) because of the unity charge of the pion:

\[
|F_n(0)|^2 = \frac{F_0^2 \Gamma^2}{m_\rho^2 + \Gamma^2}.
\]

(74b)

For \(F_0^2 = 55\), \(m_\rho = 765\) MeV, \(\Gamma = 104\) MeV, the normalization condition is satisfied.

We now evaluate the dispersion integral in a very rough and approximative way by assuming that

i) the form factor is imaginary at \(2E = m_\rho\); \(\text{Im} F(m_\rho) = F_0\);

ii) the Breit Wigner shape can be replaced by a square box of height \(F_0\) and width \(\Gamma\);

iii) the variation of \(E\) over the resonance can be neglected.

Under these assumptions we write:
\[ F_\pi(t) = \frac{1}{\pi} \int_{\omega n^2} \text{Im} \left( \frac{F_0 m_\pi \Gamma}{z - m_\rho^2 - i m_\rho \Gamma} \right) \frac{dz}{z - t} \]  \hspace{1cm} (75a)

\[ z = 4E^2 \approx m_\rho^2 \]

\[ dz = 8E \, dE \approx 2m_\rho \Gamma \]  \hspace{1cm} (75b)

\[ F_\pi(t) = \frac{1}{\pi} F_0 \frac{2m_\rho \Gamma}{m_\rho^2 - t} \]

\[ = 0.64 \frac{m_\rho^2}{m_\rho^2 - t} \]  \hspace{1cm} (75c)

where \( F_0 = 7.4 \), \( m_\rho = 765 \text{ MeV} \), \( \Gamma = 104 \text{ MeV} \). We observe that at \( t = 0 \)

\[ F_\pi(0) = 0.64, \]

i.e. in this rough approximation 64% of the pion form factor is contributed by the \( \rho \)-meson.
3.5 The kaon and nucleon form factors

We have seen that the form factor of the charged pion can be derived (in the pole dominance approximation) from one absorptive state, the ρ-meson. This is because only the ρ-meson can decay into two charged pions. For the other stable hadrons the situation is more complex. In the table below, we show the systematics of vector mesons and form factors, as they can be constructed from strong interaction quantum numbers.

<table>
<thead>
<tr>
<th>Particle</th>
<th>Time-like process</th>
<th>Form factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>π ±</td>
<td></td>
<td>ρ</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ω</td>
</tr>
<tr>
<td></td>
<td></td>
<td>φ</td>
</tr>
<tr>
<td>π 0</td>
<td></td>
<td>allowed</td>
</tr>
<tr>
<td></td>
<td></td>
<td>forbidden by G-parity</td>
</tr>
<tr>
<td></td>
<td></td>
<td>none</td>
</tr>
<tr>
<td></td>
<td></td>
<td>π 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>π 0</td>
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<tr>
<td></td>
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<td>φ</td>
</tr>
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<td>K ± 0</td>
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<tr>
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<td></td>
<td>ρ</td>
</tr>
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<td>allowed</td>
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<tr>
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<td></td>
<td>φ</td>
</tr>
</tbody>
</table>

We learn that the π 0 does not couple to any vector particle, or that its form factor is zero. This fact is equivalent to the π 0 being its own antiparticle, which prohibits any different distribution of positive and negative charge, since this would be inverted by charge conjugation, and therefore would cause π 0 ≠ π 0.

Since K 0 ≠ K 0, the K 0 may have a form factor. We see that the kaon and nucleon form factors are represented by all three vector mesons; we further remember from Section 2.4.3 that this dispersion representation does not explain the experimental results for the nucleon form factors.
3.6 Vector mesons and unitary symmetry (KROLL, 1967)

So far we have considered the contribution of the individual vector mesons as free parameters; and the sum of the contributions was only constrained by normalization at \( t = 0 \) and by the asymptotic condition at \( t = \infty \).

Unitary symmetry gives additional relations between the different vector mesons. We look at two of them in a superficial way:

a) \( \omega - \phi \) mixing. In SU\(_3\) the physical \( \omega \) and \( \phi \) are composed of the SU\(_3\) eigenstates \( \omega_1 \) and \( \omega_8 \)

\[
\begin{equation}
|\omega\rangle = - \sin \theta |\omega_8\rangle + \cos \theta |\omega_1\rangle \tag{76a}
\end{equation}
\]

\[
\begin{equation}
|\phi\rangle = \cos \theta |\omega_8\rangle + \sin \theta |\omega_1\rangle \tag{76b}
\end{equation}
\]

where the \( \omega_8 \) is a member of the octet that couples to the e.m. current, \( \omega_1 \) is a singlet, and \( \theta \) is the \( \omega - \phi \) mixing angle.

We are now able to replace the coupling constants \( \gamma_\omega \) and \( \gamma_\phi \) by a generalized coupling constant \( \gamma \) and the mixing angle \( \theta \).

\[
\begin{equation}
\gamma_\omega = - \gamma \cdot \sin \theta^{-1} \tag{77a}
\end{equation}
\]

\[
\begin{equation}
\gamma_\phi = \gamma \cdot \cos \theta^{-1} \tag{77b}
\end{equation}
\]

The leptonic decay width

\[
\Gamma(V^0 \to \xi^+\xi^-) = \frac{\alpha^2}{12} \left( \frac{\gamma^2}{4\pi} \right)^{-1} m_v \tag{68b}^\text{**}
\]

transforms to

\[
\begin{align}
\Gamma(\omega \to \xi^+\xi^-) &= \frac{\alpha^2}{12} \left( \frac{\gamma^2}{4\pi} \right)^{-1} m_v \sin^2 \theta \\
\Gamma(\phi \to \xi^+\xi^-) &= \frac{\alpha^2}{12} \left( \frac{\gamma^2}{4\pi} \right)^{-1} m_\phi \cos^2 \theta 
\end{align} \tag{78a}
\]

\[
\frac{\Gamma(\omega \to \xi^+\xi^-)}{\Gamma(\phi \to \xi^+\xi^-)} = \frac{m_\phi}{m_v} \tan^2 \theta \tag{79}
\]

So the leptonic decay widths of \( \omega \) and \( \phi \) yield a determination of the \( \omega - \phi \) mixing angle:

\[
\theta = 40.8 \pm 3.5^\circ \quad \text{(AUGUSTIN, 1969)}
\]

b) Sum rules. Considering the photon as a U-spin singlet SU\(_3\) octet, one can write with the appropriate SU\(_3\) Clebsch-Gordan coefficients the sum rule

\[
\sqrt{3} \gamma_\phi^{-1} = \cos \theta \gamma_\omega^{-1} - \sin \theta \gamma_\omega^{-1} \tag{80}
\]
This relation, together with Eq. (68b) gives the following prediction:

\[
\frac{1}{3} \frac{m_{\omega}}{m_{\rho}^2} \Gamma [\rho \rightarrow \pi^+ \pi^-] = m_\phi \cos^2 \Theta \Gamma [\phi \rightarrow \pi^+ \pi^-] + m_u \sin^2 \Theta \Gamma [\omega \rightarrow \pi^+ \pi^-]
\]

(81)

which is graphically represented by a triangular diagram (Sakurai plot).

We can apply the same relation on the kaon (nucleon) form factor to specify the contribution of the individual vector mesons

\[
F(t) = \left[ \frac{m_\rho^2}{m_\rho^2 - t} \ g_{\rho KK} + \frac{1}{\sqrt{3}} \left( - \frac{m_\omega^2}{m_u^2 - t} \sin \Theta \ g_{\omega KK} + \frac{m_\phi^2}{m_u^2 - t} \sin \Theta \ g_{\phi KK} \right) \right]
\]

(82)

where the \( g_{\nu KK} \) are the vector-meson coupling constants to the KK(NN) system.

* * *

Acknowledgement

The author thanks Dr. W. Rühl for comments and discussions.
The cross-section of a two-body reaction is in any reference frame

\[ \frac{d\sigma}{\nu_{\text{rel}}} = \frac{1}{16 \pi E_{i_1} E_{i_2} E_{f_1} E_{f_2}} |M_{fi}|^2 D \]  

(1.1)

where \( \nu_{\text{rel}} = |v_{i_1} - v_{i_2}| \) is the relative velocity of incident particles;

\( E_{i_1}, E_{i_2} \) is the energy of incident particles;

\( E_{f_1}, E_{f_2} \) is the energy of final particles;

\( M_{fi} \) is the transition matrix element;

\( D \) is the density of final states.

The transition matrix element for the first-order Born approximation of the S-matrix

\[ S = e^{iT} \approx 1 + iT + ... \]  

(1.2)

is defined by

\[ M_{fi} = \langle f | T | i \rangle . \]  

(1.3a)

Here \( T \) is obtained in perturbation theory by

\[ T = -H \]  

(1.4a)

\[ = - \int H(x) \, d^4x \]  

(1.4b)

\[ = - \int j_\mu(x) A_\mu(x) \, d^4x , \]  

(1.4c)

where Eq. (1.4c) is valid for the electromagnetic interaction of a current \( j_\mu(x) \) with a potential \( A_\mu(x) \), where \( j(x) \) and \( A_\mu(x) \) are four-dimensional quantities. The incoming and outgoing states are described by plane waves

\[ \psi = u \exp \left( i px - i Et \right) , \]  

(1.5)

so that we obtain for the matrix element

\[ M_{fi} = -u^\mu \, O_\mu \, u_i \int e^{i(p_i - p_f)^x} \, A_\mu(x) \, d^4x \int e^{i(E_f - E_i)} \, dt \]  

(1.5b)

\[ = - \delta(p_\mu p_i) \times A(q) \times \delta(E_f - E_i) \]  

(1.5c)

In the following we skip the \( \delta \) function. \( O_\mu \) is an operator containing \( \gamma \) matrices.

The Poisson equation allows the electromagnetic potential \( A(q) \) to be decomposed into a photon propagator and its source, which is again a four-dimensional current:
Poisson equation
\[ \Delta^2 A_\mu(x) = -4\pi j_\mu(x) \]  
(I.6a)

Green's formula
\[ \int (\nabla \cdot \nabla V - V \nabla^2) U \, dx = 0 \]  
(I.6b)

Fourier transformation
\[ A_\mu(q) = \frac{4\pi}{q^2} j_\mu(q) \]  
(I.7)

So we obtain finally
\[ M_{1\mu} = j_\mu(p) \frac{4\pi}{q^2} j_\mu(p) \]  
(I.3d)

where \( j_1 \) and \( j_2 \) are the currents of the scattering and the scattered particle, respectively.

The currents \( j_{\mu}(p) \) for the different particles or sources are:

<table>
<thead>
<tr>
<th>Current</th>
<th>Current</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coulomb potential</td>
<td>- ie</td>
</tr>
<tr>
<td>spin zero boson</td>
<td>- ie (( p_1 + p_2 ))</td>
</tr>
<tr>
<td>Dirac electron (proton)</td>
<td>- ie [( \bar{u}(p_2)\gamma_\mu u(p_1) )]</td>
</tr>
<tr>
<td>proton (neutron)</td>
<td>- ie ( \bar{u}(p_2) [\gamma_\mu F_1 + (i\sigma_{\mu\nu}/2\hbar) q^\nu F_2] u(p_1) )</td>
</tr>
</tbody>
</table>

The normalization of the Dirac spinors is
\[ |\bar{u}_1|^2 = (2E)^2 \]

The density of final states is for a two-body state
\[ D = \frac{1}{(2\pi)^3} \frac{E_1 E_2}{E_{tot} p_1^2 - E_1 (\vec{p}_{tot} \cdot \vec{p}_1)} p_1^3 \, d\Omega_1 \]  
(I.8)

which reduces for the c.m. system with \( \vec{p}_{tot} = 0 \) and \( E_{tot} = E_1 + E_2 \):
\[ D_{c.m.} = \frac{1}{(2\pi)^3} \frac{E_1 E_2}{E_1 + E_2} E_1^3 \, d\Omega_1 \]  
(I.9)

If the polarization is not measured, the cross-sections have to be averaged over the incoming spin states, and the final spin states have to be summed up. For electrons
\[ \sigma = \frac{1}{2} \sum_i \sum_f |\bar{u}(p_f) F_\mu u(p_i)|^2 \]  
(I.10a)

\[ = \frac{1}{2} \sum_i \sum_f (\bar{u}_f F_\mu u_i) (\bar{u}_f F_\mu u_i) \]  
(I.10b)

where \( F_\mu \) is an operator which may contain \( \gamma \) matrices. The double sums can be evaluated with the conventional spur technique.
RADIATIVE CORRECTIONS (MO, 1969)

Electromagnetic interactions are always accompanied by the emission of extra, soft photons. Consider the following radiative modifications to Mott scattering:

a) the flux of incoming electrons of energy $E_i$ is reduced;

b) the yield of outgoing electrons of energy $E_f$ is reduced;

c) the energy of the electron during the interaction is reduced (second-order process).

 Whereas process (c) modifies the cross-section, but not the kinematics, the processes (a) and (b) modify the line spectrum of outgoing electrons from that given by two-body kinematics. The true spectrum is affected with a radiative tail:

![Graph showing the spectrum with a radiative tail.]

The acceptance window $\Delta E$ of the detector will always lose a fraction $\delta$, with approximately

$$\delta \approx \frac{\int_0^{\Delta E} N(E) \, dE}{\int_0^{E_f} N(E) \, dE}$$  \hspace{1cm} (II.1)

For electron-proton scattering a good expression for $\delta$ is

$$\delta_{\text{RC}} = \frac{\alpha}{\pi} \left\{ \left( \ln \frac{E_f^2}{E_i \Delta E} + \ln \frac{E_f}{\Delta E} - \frac{13}{6} \right) \left( \ln \frac{q^2}{m^2} - 1 \right) + \frac{17}{18} \right\}. \hspace{1cm} (II.2)$$
An additional radiative correction arises from the energy loss of the electrons by normal bremsstrahlung on target nuclei other than the scattering one. Here the lost fraction is

$$
\delta_{BR} = X \cdot \ln \left( \frac{E_f}{\Delta E} \right) / \ln 2,
$$

where $X$ is the target thickness in units of radiation lengths.

The true Coulomb cross-section is obtained from the measured cross-section as follows:

$$
\frac{d\sigma}{d\Omega}_{\text{true}} = \frac{d\sigma}{d\Omega}_{\text{meas}} e^{\delta_{RC} + \delta_{BR}} \quad (II.4a)
$$

$$
\approx \frac{d\sigma}{d\Omega}_{\text{meas}} \left( 1 + \delta_{RC} + \delta_{BR} \right) \quad (II.4b)
$$

where Eq. (II.4b) is valid for $\delta \ll 1$.

The corrected spectral function $N(E)$ can be obtained by unfolding the radiative correction from the measured distribution. For that purpose the measured distribution is binned, and, starting at the highest bin, the radiative correction is applied differentially, correcting the number of events in that bin by

$$
N_{\text{true}}(E_i) = N_{\text{meas}}(E_i) e^\delta
$$

and subtracting the appropriate function of the correction in all lower bins, and so on.

Note: radiative corrections are different for different processes:
Consider an arbitrary complex function \( f(t) \) which meets the conditions:

a) analytic in the complex plane, except on the positive real axis;

b) \( \lim_{t \to \infty} f(t) = 0 \).

Then \( f(t) \) is represented by Cauchy's integral for \( t \) inside \( C \):

\[
f(t) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - t} \, dz
\]

(III.1)

For \( t \) outside \( C \) the integral gives zero. Now we evaluate the integral by the following purely mathematical steps:

i) replace \( C \) by \( C' \) and apply (b)

\[
f(t) = \frac{1}{2\pi i} \int_{C'} \frac{f(x)}{x - t} \, dx
\]

(III.2)

ii) \( t = x_0 + \epsilon \) real

\[
f(t) = \frac{1}{2\pi i} \int_{x_0 - \epsilon}^{x_0 + \epsilon} \frac{f(x)}{x - x_0 - \epsilon} \, dx
\]

(III.3)

iii) \( t = x_0 - \epsilon \)

\[
0 = \frac{1}{2\pi i} \int_{x_0 - \epsilon}^{x_0 + \epsilon} \frac{f(x)}{x - x_0 + \epsilon} \, dx
\]

(III.4)

iv) C.C.

\[
0 = \frac{1}{2\pi i} \int_{x_0 - \epsilon}^{x_0 + \epsilon} \frac{f(x)}{x - x_0 - \epsilon} \, dx
\]

(III.5)

v) (ii-iv)

\[
f(t) = \frac{1}{2\pi i} \int_{x_0 - \epsilon}^{x_0 + \epsilon} \frac{f(x) - f(x)}{x - x_0 - \epsilon} \, dx
\]

(III.6a)

\[
= \frac{1}{2\pi i} \int_{x_0 - \epsilon}^{x_0 + \epsilon} \frac{2 \text{ Re } f(x)}{x - x_0 - \epsilon} \, dx
\]

(III.6b)

vi)

\[
\text{Re } f(t) = \frac{1}{\pi} \int_{x_0 - \epsilon}^{x_0 + \epsilon} \frac{\text{Im } f(x)}{x - x_0 - \epsilon} \, dx
\]

(III.7)
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