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HADRON-NUCLEUS FORWARD DISPERSION RELATIONS

T.E.O. Ericson and M.P. Locher

G E N E V A
1969

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1. INTRODUCTION

In several fields of physics, dispersion relations have proved to be of great importance, not only as a direct reflection of the basic concepts of causality and analyticity but also as a powerful phenomenological framework for the correlation and interpretation of experimental data on scattering processes.

Since complex nuclei have been extensively investigated by the scattering of nucleons and other strongly interacting particles, it might have been expected that dispersion relations would be in extensive use. Surprisingly, this is not the case. An early and isolated application of forward dispersion relations to nuclei (apart from the precursors to this article by our group) is the model-independent evaluation of the meson contribution to the photo-nuclear sum rule by Gell-Mann, Goldberger and Thirring¹⁾. We see the reason for this situation in the existence of highly successful, phenomenological models for analysing nuclear elastic scattering at specific energies, and in the nature of present nuclear reaction theories which emphasize limited energy intervals. The very success of these have inhibited the more general, but in part less detailed, questions that can be raised by the dispersion approach. On the other hand, nuclei have acquired a formidable reputation of complexity among those high-energy physicists interested in transposing their methods to nuclei, both because of the large radius and because of the many bound and excited states. These technical difficulties do exist, but they are largely exaggerated due to unfamiliarity. This is illustrated by our recent exploration of pion and nucleon forward dispersion relations for nuclei²⁾. Another example is the successful use of dispersion techniques in the discussion of the corrections to soft-pion theorems in nuclei³⁾.

There are numerous reasons why the application even of forward dispersion relations to nuclei are of great interest. We mention two examples. Low-energy nuclear physics is normally considered as a problem of non-relativistic nucleons from which the explicit mesonic and antibaryonic degrees of freedom have been eliminated. It is notoriously difficult to discuss the mesonic degrees of freedom and relativistic effects in this framework. Forward dispersion relations provide a relativistic frame in which effects of antinucleons and mesons enter naturally, so that their contributions can be isolated. As another example, the dispersion relations provide a very natural way of explicitly displaying the great importance of exchange in nucleon scattering on nuclei in low-energy nuclear physics. The exchange contributions in nucleon elastic scattering normally do not appear in any explicit fashion in the customary phenomenological optical model analyses. By the explicit display of the exchange amplitude, the dispersion relations provide a means of studying properties of very deeply bound nuclear states. Finally, from a more general point of view, the application of dispersion relations to nuclei is an instructive testing ground for exploring the specific effects that occur in composite systems, so that strength and weakness of the dispersive approach can be better understood.

The present article is concerned with the systematic discussion of forward dispersion relations for hadrons (in practice pions and nucleons) on nuclei. Our objective is first to present a systematic background to their use in a form suitable to the low-energy nuclear physicist and accessible to the experimentalist. It is further to illustrate some of the specific effects associated with the large nuclear radius on a rigorous model, for which we have chosen scattering by a central potential. In the discussion of the dispersion

properties of the potential, it is in particular made clear how deeply bound states play a central role in low-energy scattering. The later part of this paper is devoted to actual applications of the dispersion relations to concrete situations for pions and nucleon. In the case of pions, it is shown in detail how effective π -nuclear coupling constants can be derived, and how these provide direct measurements of the asymptotic pion field outside the nucleus. The relation between the effective coupling constants and the nucleons as sources of the pion field is discussed. The effective coupling constant is shown to be related to general sum rules. For nucleons we demonstrate the importance of the crossed or exchange channel. We give an estimate of the contributions of physical antinucleon processes to slow neutron scattering.

Section 2 contains the general background information on the dispersion relations. Most of the material can be found in various places in the literature. Section 3 contains a self-contained discussion of potential scattering and its relation to dispersion relations. We show, in particular, that deeply bound states are exponentially enhanced by the size of the system. The role of particle exchange in scattering from an external potential is explored. Section 4 contains an analysis and discussion of π -nuclear scattering with particular emphasis on the properties of the effective coupling constants. A general set of sum rules for a many-body system is derived. Section 5 deals similarly with nucleon-nuclear scattering. It also contains a non-relativistic dispersion sum rule to N-nuclear elastic scattering. In Section 6 we finally summarize some of the results, and indicate the advantages and limitations of the dispersive approach in nuclei. Some suggestions for future experiments are made.

2. BASIC FEATURES OF FORWARD DISPERSION RELATIONS^{*)}

Consider the elastic forward scattering of the projectile m on a target M as illustrated in Fig. 1. This process will be discussed mainly in the lab. system, in which the target is at rest. Although nuclei usually are discussed in a non-relativistic framework, we will not

Fig. 1 Notation used for forward scattering.

^{*)} This section introduces the notation used in the following. It also gives a very brief survey of basic concepts and relations of forward dispersion theory, presented heuristically for the benefit of the experimental physicist unfamiliar with the topic. Other readers are advised to go directly to Section 3.

impose this unnecessary restriction: dispersion relations automatically provide a relativistic framework, whether targets are nuclei or not. The principal physical quantities are defined in Table 1.

Table 1

Notation for main physical quantities
($c = 1$ frequently used in relativistic cases)

	Projectile	Target
Mass	m	M
Lab. energy	ω	Mc^2
Lab. kinetic energy	$E = \omega - mc^2$	0
Three-momentum	$\vec{k}$	0
Four-momentum	k_μ	p_μ
Isospin	$\vec{t}$	$\vec{I}$
<p>Relativistic c.m. energy W: $W^2 = s = (k+p)^2 = m^2 + M^2 + 2M\omega$</p> <p>Lab. forward scattering amplitude: $f(\omega)$</p> <p>Forward differential lab. cross-section: $(d\sigma/d\Omega)_0 = f(\omega) ^2$</p> <p>Total cross-section: $\sigma(\omega)$</p> <p>Energy-momentum relation: $\vec{k} = (\omega^2 - m^2)^{1/2}$</p>		

The forward amplitude $f(\omega)$ is a function of the energy ω only. It is useful to consider this functional dependence also for complex energies ω .

2.1 Basic assumptions of forward dispersion relations

First hypothesis (analyticity): the amplitude $f(\omega)$ is an analytic function of the complex energy ω for all ω , apart from certain singularities. [The analytic behaviour of $f(\omega)$ is obvious whenever $f(\omega)$ can be described in convergent perturbation theory.]

Second hypothesis (causality): there are no singularities in $f(\omega)$ for $\text{Im } \omega > 0$.

Comment: the second hypothesis is believed to be generally valid, although a proof exists only for a few special cases such as πN scattering. The proofs do not strictly use causality, but the related though not equivalent concept of local commutativity^{4,5}).

An elementary and intuitive physical "derivation" of this hypothesis can be given:

Consider a sharp wave packet which interacts with a target. The scattered wave gives a linear response function $R(t)$ at a time t in a (distant) detector. Causality requires that there is no response before a time $t = 0$ defined by the arrival of the unscattered wave. The scattering amplitude is in essence the Fourier transform of the response: $f(\omega) = \int_{-\infty}^{\infty} R(t) e^{i\omega t} dt$ which is $\int_0^{\infty} R(t) e^{i\omega t} dt$ by causality. As this integral exists for real ω it also exists for complex ω with $\text{Im } \omega > 0$, since the integrand is more convergent. It follows that $f(\omega)$ is analytic in the upper half ω -plane.

2.2 The basic dispersion relation

By assumption, the scattering amplitude $f(\omega)$ is analytic^{*)} for $\text{Im } \omega > 0$ and we can apply Cauchy's theorem. Integrating along the real axis and closing the contour at infinity above the real axis (see Fig. 2) we have for a point $(\omega + i\varepsilon)$

Fig. 2 The path in the complex ω -plane for the integration in Eq. (1).

$$f(\omega + i\varepsilon) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - \omega - i\varepsilon} dz . \quad (1)$$

If we make the special convergence assumption that the integrand is small enough at infinity, so that the distant contribution is negligible, there will only be contributions from the real axis to the integral. In the limit $\varepsilon \rightarrow 0$, we separate the real and imaginary part of $f(\omega)$ and have

$$\text{Re } f(\omega) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\text{Im } f(x)}{(x - \omega)} dx . \quad (2)$$

This relation between the real and imaginary amplitude is the basic dispersion relation. It is characteristic that the relation involves both positive and negative frequencies ω . To use this relation, in practice one has now to express $\text{Im } f(\omega)$ in terms of measurable or calculable physical quantities for all real ω . This is the principal remaining step.

2.3 Subtractions

It is not infrequent that the relation (2) manifestly does not converge because the distant contribution does not vanish. Any function of $f(\omega)$ that is analytic in the upper half-plane could have been used in the place of $f(\omega)$ in Eq. (2). We can therefore choose a more convergent expression such as $(\omega - \omega_0)^{-1} [f(\omega) - f(\omega_0)]$, where ω_0 is an arbitrary but fixed energy. For this function relation (2) becomes:

$$\text{Re } \left\{ \frac{f(\omega) - f(\omega_0)}{(\omega - \omega_0)} \right\} = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\text{Im } \{f(x) - f(\omega_0)\} / (x - \omega_0)}{(x - \omega)} dx . \quad (2')$$

This is referred to as a subtraction. Usually Eq. (2') is written in the equivalent form

$$\text{Re } f(\omega) = \text{Re } f(\omega_0) + \frac{(\omega - \omega_0)}{\pi} P \int_{-\infty}^{+\infty} \frac{\text{Im } f(x)}{(x - \omega_0)(x - \omega)} dx , \quad (3)$$

which is also obtained by formally subtracting $f(\omega_0)$ from Eq. (2)

Further convergence may be obtained by repeated subtractions, if necessary. The threshold for scattering is often used as subtraction energy, but other energies may be suitable for practical or theoretical reasons. The price to be paid for convergence is that $f(\omega_0)$ has to be known

*) For non-elastic reactions of break-up type it has been shown that complex singularities⁶⁾ can occur above the real axis. For elastic scattering this phenomenon does not exist.

2.4 The crossed reaction

The scattering amplitude $f(\omega)$ for the particle m is related to the scattering amplitude $\bar{f}(\omega)$ of its antiparticle on the same target (see Fig. 3) by the crossing relation^{*)}:

$$f(-\omega) = \bar{f}^*(\omega) . \quad (4)$$

(If spins are present, they should be reversed.)

Fig. 3 General graphs for (a) direct (s-channel, particle) scattering and (b) for exchange (u-channel, antiparticle) scattering.

This relation therefore links positive and negative frequency amplitudes for particles and antiparticles. Note, in particular, that $\text{Im } f(-\omega) = -\text{Im } \bar{f}(\omega)$.

The crossed channel, Fig. 3b^{**)} , plays a central role even in non-relativistic scattering of nucleons on nuclei; the interpretation of direct contributions to the antiparticle amplitude, Fig. 4b, is that of an exchange amplitude in the scattering channel, Fig. 4a (and vice versa). For example let diagram (a) represent $n^4\text{He}$ scattering and choose $M_n = {}^3\text{He}$ as intermediate state. Apart from the dynamical content implied by the ${}^3\text{He}$ propagator or pole, clearly diagram (a) induces the exchange of the external neutron with one of the neutrons in the α -particle, which explains our nomenclature. Quite apart from any special application, diagrams (a) and (b) are topologically equivalent and are therefore described by one and the same amplitude (substitution rule). Of course the physical regions of the argument ω are different for diagrams (a) and (b), and are in fact non-overlapping.

Fig. 4 Direct intermediate state in the antiparticle channel [diagram (b)]; and the equivalent exchange graph in the s-channel [diagram (a)].

*) The complex conjugation in Eq. (4) is characteristic of the causal amplitude that satisfies dispersion relations (see Ref. 5, p. 663).

**) Generally we read the graphs from left to right. The notation $\bar{m}$ is shorthand for the antiparticle of mass m having an inverted four-vector $(-k, -\omega)$.

2.5 Determination of amplitude (physical region)

In the region of positive kinetic energy for particle or antiparticle [i.e. $|\omega| > m$], the complex amplitude that enters the dispersion relation (2) is directly measurable. The imaginary amplitude is usually by far most accurately determined from the total cross-section for particle and antiparticle by the optical theorem:

$$\text{Im } f(\omega) = \frac{k}{4\pi} \sigma(\omega), \quad \omega > m. \quad (5a)$$

For an antiparticle of energy $\bar{\omega} = -\omega > m$ and momentum $\bar{k}$:

$$\text{Im } \bar{f}(\bar{\omega}) = \frac{\bar{k}}{4\pi} \bar{\sigma}(\bar{\omega}) \quad (5b)$$

so that

$$\text{Im } f(\omega) = - \frac{k}{4\pi} \bar{\sigma}(-\omega), \quad \omega < -m. \quad (5c)$$

The relation (5c) follows from the crossing relation (4).

In the limit of $|\omega| \rightarrow \infty$, we will assume the validity of the Pomeranchuk limit of equality of particle and antiparticle cross-sections:

$$\lim_{\omega \rightarrow \infty} \sigma(\omega) = \lim_{\omega \rightarrow \infty} \bar{\sigma}(\omega) = \text{const}. \quad (6)$$

There are three direct ways of obtaining the real amplitude:

- i) phase-shift analysis, which often exists in the neighbourhood of the threshold;
- ii) Coulomb interference⁷⁾ for charged particle scattering: the small-angle differential cross-section is approximately:

$$\left(\frac{d\sigma}{d\Omega} \right) = |f_C(\theta) + f_N(\theta)|^2,$$

where $f_C(\theta)$ is the known Coulomb amplitude and $f_N(\theta)$ the nuclear amplitude. The Coulomb and nuclear amplitudes interfere appreciably at very small angles where they are of the same magnitude. From the interference it is possible to extract $\text{Re } f(0^\circ)$ rather accurately;

- iii) from the forward differential cross-section for neutral particles. Since

$$\left(\frac{d\sigma}{d\Omega} \right)_{0^\circ} = \text{Re } f(0^\circ)^2 + \text{Im } f(0^\circ)^2 = \text{Re } f^2 + \left(\frac{k}{4\pi} \sigma(\omega) \right)^2, \quad (7)$$

it is possible to deduce $\text{Re } f(\omega)$ (but not its sign) from Eq. (7) if $\text{Re } f$ is reasonably large compared to $\text{Im } f$.

2.6 Singularities in the unphysical region: Poles, unphysical cuts, and anomalous thresholds

The imaginary part of the amplitude of Eq. (2) usually differs from zero only when direct or crossed reactions are possible with conservation laws formally fulfilled (in par-

Fig. 5 Direct intermediate state M_n in the s-channel.

ticular energy-momentum conservation). In the unphysical region (i.e. the region of negative kinetic energy both for incident particle and antiparticle $-mc^2 < \omega < +mc^2$) this is a condition on the mass M_n of the direct intermediate state both in the direct channel (s-channel) Fig. 5, and in the exchange channel (u-channel) Fig. 4b:

$$(M - m)^2 < M_n^2 < (M + m)^2 . \quad (8)$$

Of course the system M_n has usually different quantum numbers in the s- and u-channel. In the case of $n^4\text{He}$, for example, M_n has baryon number five in the s-channel, whereas the u-channel has baryon number three.

2.6.1 Poles

If the intermediate system is just one particle of mass M_n (all selection rules respected), the amplitude is singular at just one energy, given by the condition that the invariant energy s or u is equal to M_n^2 :

$$\omega_n = \mp \frac{(m^2 + M^2 - M_n^2)}{2M} c^2 , \quad \text{where } |\omega_n| < m . \quad (9)$$

The signs refer to the direct and exchange poles, respectively. Of course, a pole can be included formally into the spectrum of $\text{Im } f(\omega)$ by a term proportional to $\delta(\omega - \omega_n)$. The role of pole residues in perturbation theory is discussed in Appendix to Section 5. In Section 3, their connection to bound states in potential scattering is given. The non-relativistic limit of Eq. (9) for the nuclear system is of special interest. We have to distinguish two cases: (a) $m = \text{baryon}$; (b) $m = \text{meson or photon}$.

For case (a) we write the mass of the intermediate system

$$M_n = M \pm (m - B_n/c^2) , \quad (10)$$

where the sign refers to direct and crossed channels. Furthermore, B_n is the positive energy required to break up the bound system into free constituents for the processes $M_n + \{B_n\} \rightarrow m + M$ (direct channel) and $M + \{B_n\} \rightarrow M_n + m$ (exchange channel). We assume $|B_n|/Mc^2 \ll 1$. The singularity obviously occurs at a negative kinetic energy E_n :

$$E_n^{(\text{baryon})} = \omega_n - mc^2 \approx -(1 \pm m/M)B_n . \quad (11)$$

In case (b), the intermediate state M_n has a mass

$$M_n = M + \epsilon_n/c^2 . \quad (12)$$

The non-relativistic energy ω_n is now

$$\omega_n^{(\text{meson})} \approx \pm \left(-\frac{m^2 c^2}{2M} + \epsilon_n \right). \quad (13)$$

(Note that this expression is relativistic in particle m , but non-relativistic for M and M_n !)

2.6.2 Unphysical cuts

In general, intermediate states are more complicated than the one-body states that give rise to poles (Figs. 5 or 4b). Two or more particles in an intermediate state have the additional freedom of a relative kinetic energy. Consequently, the position of the singularity is smeared corresponding to the different relative energies. The singularity now starts from a threshold value corresponding to zero relative energy. A simple example is that of the physical intermediate states with $M_n \geq M + m$. The total cross-section is made up from the contributions of these states, when they are on the mass shell. Therefore, they determine the imaginary part of the amplitude by the optical theorem (unitarity condition). The total cross-section contributes a continuous integral over energy to the dispersion relation, i.e. it contributes as a cut.

It is quite a common feature in particle physics that the unitarity relation has to be generalized below the threshold $\omega = m$. In NN scattering, for example, the lightest such system that can be exchanged in the $N\bar{N}$ or u-channel are two pions, extending the unphysical scattering far below $N\bar{N}$ threshold. The corresponding imaginary part is not directly accessible to measurement and theory is usually not in a position to give reliable predictions. In this situation the following approximations are customary:

- From a scattering length expansion of the complex amplitude above threshold, the behaviour of $\text{Im } f(\omega)$ below threshold is estimated by continuation of the analytic expansion. This controls the contribution of the nearby cut. We shall use such a procedure in the case of πX scattering.
- The cut that is further away from the physical region is replaced by a number of poles or resonances whose positions usually are known. Sometimes even a single effective pole is sufficient to simulate the far-away cut. A number of strength constants (coupling constants) enter the dispersion relation as phenomenological parameters. An example where the non-physical region is well described by an effective pole is $n^4\text{He}$ scattering, Section 5.

2.6.3 Anomalous thresholds

From the preceding remarks one might conclude, wrongly, that the threshold of an unphysical cut is always given by the condition of zero internal energy of the state M_n which reads, for Fig. 5:

$$s_0 = m^2 + M^2 + 2M\omega_{\text{thresh}} = M_0^2 \leq M_n^2, \quad (14)$$

where M_0 is the mass of M_n for zero internal momentum and s is the invariant energy of the channel considered. The threshold s_0 above is called a normal threshold and the unphysical cut (if any) extends from s_0 to the scattering threshold $s = (m + M)^2$. There are situations

(and this depends only on the mass ratios involved) where the threshold starts below s_0 , increasing the length and importance of the unphysical cut. Such an "anomalous" situation is frequently encountered for composite or weakly bound systems but in the present treatment of forward nuclear scattering the difficulty is essentially not present^{*)}.

* * *

To see this we could simply refer the reader to the formal literature, which derives the existence and position of the anomalous thresholds from the Landau rules⁸⁾. Although this treatment is necessary and mathematically straightforward, we feel it is intuitively not very appealing. In the present context we therefore try to illustrate the main points by reproducing a simple argument due to Bohr⁹⁾ [see also Cutkosky¹⁰⁾ and Karplus et al.¹¹⁾].

The simplest graph leading to anomalous thresholds is the triangular graph, Fig. 6:

Fig. 6 Triangular graph leading to anomalous thresholds in the momentum transfer variable t .

We do not need to specify the wavy lines (twice particle M for elastic scattering) nor the nature of the bubble. In this simple graph, only three masses, m , m_a , m_b , are assumed to be different. The anomalous threshold will occur in the momentum transfer variable t corresponding to the $m\bar{m}$ annihilation channel. (Labelling of channels is of course free, but we stick to the earlier convention that unequal mass particle scattering (Fig. 3a) defines the s -channel, whereas Fig. 3b is the u -channel.)

If $m < m_a + m_b$, particle m is stable in the usual sense. Although real decay is forbidden, a virtual decay is still possible in the well-defined mass region:

$$m_a^2 + m_b^2 < m^2 < (m_a + m_b)^2 . \quad (15)$$

To explain what this means we consider particle m in a state with energy $\omega < m$ and momentum $+i\vec{k}$ corresponding to an asymptotic wave function $\exp(-\kappa r)$ leaking out perpendicularly from

*) The remainder of this section is not essential for the non-specialist.

a plane situated at $r = 0$. A virtual decay is a decay into states a and b, which have similarly positive imaginary momentum and positive energy components. Particularly for the collinear decay, the energy-momentum vectors can be visualized in a Euclidean plane (see Fig. 7); the lengths of the vectors are then the masses ($m^2 = \omega^2 + \kappa^2$):

The virtual decay defined above requires $\theta < \pi/2$ in order to have the components of all three vectors positive. This means that $m^2 > m_a^2 + m_b^2$, which gives the range of Eq. (15). This condition is sometimes referred to as the loosely bound mass region¹⁰⁾. If m decreases further, one of the energies becomes negative so that the situation corresponds to virtual absorption rather than to virtual decay. In this case, $m^2 < m_a^2 + m_b^2$, particle m is "tightly bound" or stable in an extended sense. There is then no anomalous contribution in the dispersion relation.

Fig. 7 Vector addition of energy momentum vectors for virtually decaying states.

The simple picture above also allows one to determine¹⁰⁾ the position t_0 of the anomalous threshold. To see this we observe that particles a and b can acquire real momentum q transverse to the direction of the component $i\kappa$. The particles can escape to infinity in transverse directions leading to a "real decay" of the initial state. In the brick-wall system we shall choose zero imaginary momentum component for particle a; the momenta are as in Fig. 6. Energy-momentum conservation at the lower left-hand vertex reads:

$$\sqrt{-\kappa^2 + m^2} = \sqrt{q^2 + m_a^2} + \sqrt{q^2 - \kappa^2 + m_b^2}$$

or solved for q^2

$$q^2 = \frac{(m^2 - m_a^2 - m_b^2)^2 - 4m^2(m_b^2 - \kappa^2)}{4(m^2 - \kappa^2)} . \quad (16)$$

Further, we have the invariant momentum transfer

$$t = [(\omega, i\kappa) - (\omega, -i\kappa)]^2 = 4\kappa^2 .$$

Therefore the condition that q^2 be real defines a range in t with lower limit t_0 :

$$t_0 = 4m_b^2 - \frac{(m^2 - m_a^2 - m_b^2)^2}{m^2} \quad (17)$$

valid for m in the range Eq. (15). Equation (17) is the position of the anomalous threshold. The second term is the deviation from the ordinary threshold which is at $4m_b^2$. For the deuteron, for example, $m = m_d$, $m_a = m_n$, $m_b = m_p$, and we obtain $t_0 = 1.7 m_\pi^2$ which is far smaller than the ordinary threshold at $t = 4m_N^2 \approx 200 m_\pi^2$.

In the present context of forward dispersion relations ($t = \text{fixed} = 0$), we are not concerned with the singularities in the $m\bar{m} \rightarrow \bar{M}M$ or t -channel, and it is only there that anomalous cuts play an important role. This is immediately seen by inspection of the triangular graphs in the s - and u -channel. They are somewhat more complicated as the masses are now all different. A typical example is shown in Fig. 8.

Fig. 8 Triangular graph for π -nucleus scattering exemplified by $\pi^+{}^4\text{He}$ scattering. The corresponding graph with appropriate mass applies generally.

The important thing to note is, however, that only the $n^3\text{He}^4\text{He}$ vertex is loosely bound or "unstable" in the sense of Eq. (15). The pionic vertex is strongly bound. We have therefore a mixed situation, and it is not surprising that the correct formula for the general mass case^{*)} gives no anomalous threshold in s for the actual masses of Fig. 8. Even in the s - (or u -) channel, there occurs in certain cases, such as πD scattering, a small anomalous lengthening of the cut at the end furthest from the physical region. This is completely unimportant

from a practical point of view. A similar situation applies for nX scattering, where the s - and u -channel triangular graphs always contain a strongly bound vertex, which prevents anomalous cuts from being of any importance.

* * *

3. POTENTIAL SCATTERING AND DISPERSION RELATIONS

It is instructive to discuss the properties of the forward dispersion relations for the scattering from a potential before turning to the more general problem of scattering from physical nuclei. The advantage of the potential problem is that it is mathematically well-defined; therefore effects established for the potential are likely to appear in the general case. Secondly, elastic scattering of nucleons and even complex particles from nuclei are extensively and successfully described by optical model potentials. While these potentials are complex, they still retain a more than superficial similarity to the real potential model. The potential description will clearly be inadequate to describe the more subtle effects associated with meson-currents in nuclei. It will also be unable to reproduce the special singularities associated with the composite nature of the nucleus.

Consider the scattering of a spinless particle from the central static potential $U(x)$. The particle obeys the Schroedinger equation^{*)}

$$\left(\vec{\nabla}^2 + k^2\right) \phi(\vec{x}) = 2m U(x) \phi(\vec{x}). \quad (18)$$

*) In the general mass case¹¹⁾ of Fig. 8, the discriminant, Eq. (15), is replaced by

$$m_{3\text{He}}^2 + m_{3\text{H}} m_n < \frac{m_{3\text{H}} m_{^4\text{He}}^2 + m_n m_\pi^2}{m_{3\text{H}} + m_n}.$$

If this inequality holds (for the actual masses it does not), an anomalous threshold in s occurs. Its position can be given explicitly by a somewhat lengthy second-order equation¹¹⁾.

*) Although we talk of this wave equation as non-relativistic, this property is not really exploited in any conclusion!

The kinetic energy is $E = k^2/2m$. This equation has bound-state solutions $\phi_\beta(\vec{x})$ at energies $E_\beta < 0$. These bound states appear as negative energy resonances in the dispersion relation with residues Γ_β . The subtraction constant for potential scattering is simply the Born term ("subtraction at infinity"). The dispersion relation for a normal potential is:

$$\text{Re } f(E) = -\frac{m}{2\pi} \int U(x) d\vec{x} - \frac{m}{2\pi} \sum_\beta \frac{\Gamma_\beta}{E - E_\beta} + \frac{1}{\pi} \int_0^\infty \frac{\text{Im } f(E') dE'}{E' - E} . \quad (19)$$

The appearance of the Born term $-m/2\pi \int U(x) d\vec{x}$ is due to a sufficiently rapid decrease of $\text{Im } f(E)$ with energy, that the last term in Eq. (19) vanishes in the limit $E \rightarrow \infty^{**}$. In this limit the scattering amplitude is given by the Born approximation.

3.1 Relation of residues to wave functions

The pole residues Γ_β can be obtained easily from the Lippman-Schwinger expression for the transition operator T :

$$T = U + U \frac{1}{E - H + i\eta} U , \quad (20)$$

where the total Hamiltonian

$$H = \left(-\frac{1}{2m} \vec{\nabla}^2 + U \right)$$

[see Ref. 5, Eqs. (5.177) and (10.123)].

The plane wave expectation value of the transition operator T is proportional to the amplitude $f(E)$. The first term in U is clearly the Born term of Eq. (19). In the second term we insert a complete set of normalized eigenfunctions to the exact Hamiltonian. These contain, in particular, the bound-state wave functions $\phi_\beta(x) = R_\beta(x) Y_{\ell m}(\hat{x})$ with parity $(-)^{\ell}$. They correspond to pole terms of residue Γ_β :

$$\Gamma_\beta = \lim_{k \rightarrow i\kappa_\beta} \left\{ \left\langle e^{-i\vec{k} \cdot \vec{x}} U(x) \phi_\beta(\vec{x}) \right\rangle \left\langle \phi_\beta(\vec{y}) U(y) e^{i\vec{k} \cdot \vec{y}} \right\rangle \right\} . \quad (21)$$

Since

$$U(x) \phi_\beta(\vec{x}) = \left(\frac{1}{2m} \nabla^2 + E_\beta \right) \phi_\beta(\vec{x})$$

this can be written in terms of the Fourier transform of $\phi_\beta(\vec{x})$:

$$\Gamma_\beta = \lim_{k \rightarrow i\kappa_\beta} \left\{ (E - E_\beta)^2 \left\langle e^{-i\vec{k} \cdot \vec{x}} \phi_\beta(\vec{x}) \right\rangle \left\langle \phi_\beta(\vec{y}) e^{i\vec{k} \cdot \vec{y}} \right\rangle \right\} \equiv (-)^{\ell} \lim_{k \rightarrow i\kappa_\beta} \left\{ (E - E_\beta) \left\langle e^{i\vec{k} \cdot \vec{x}} \phi_\beta(\vec{x}) \right\rangle \right\}^2 . \quad (21')$$

The factor $(-)^{\ell}$ appears by a change of variable $\vec{x} \rightarrow -\vec{x}$ and the known parity of $\phi_\beta(\vec{x})$. This expression can be considerably simplified. For large x the wave function $\phi_\beta(\vec{x})$ will always have the Yukawa form

*) A sufficient condition¹²⁾ for this is that $\int d^3x' |U(x')| / |\vec{x} - \vec{x}'|$ exists, is continuous in x and vanishes for large x .

$$\phi_{\beta}(\vec{x}) \xrightarrow{x \rightarrow \infty} N_{\beta} Y_{\ell m}(\hat{x}) \frac{e^{-\kappa_{\beta} x}}{x}, \quad (22)$$

where the constant N_{β} describes the amount of the bound state in the asymptotic region. The Fourier transform of this Yukawa function has a pole at $(E-E_{\beta}) = 0$, which exactly cancels the factor $(E-E_{\beta})$ in Eq. (21') at this point and gives a residue proportional to N_{β}^2 . More exactly^{*)},

$$\Gamma_{\beta} = (-)^{\ell} \frac{\pi(2\ell + 1)}{m^2} N_{\beta}^2. \quad (23)$$

The change of sign of the pole with parity is important. In essence, its origin is the replacement of $k^{2\ell}$ by $(ik)^{2\ell} \propto (-)^{\ell}$ for negative kinetic energy. It is analogous to the well-known difference of sign of the pole term in scalar versus pseudoscalar meson-nucleon scattering [see Part (A) of Appendix to Section 5].

The continuum states of the complete set inserted in Eq. (20) gives the imaginary amplitude for the integral in Eq. (19). This amplitude is directly expressed by the total cross-sections using the optical theorem Eq. (5a). Collecting these results we obtain Eq. (19) in the form

$$\text{Re } f(E) = -\frac{m}{2\pi} \int U(x) d^3\vec{x} - \frac{1}{2m} \sum_{\beta} (-)^{\ell} \frac{(2\ell + 1)N_{\beta}^2}{(E - E_{\beta})} + \frac{1}{2\pi^2} \int_0^{\infty} \frac{k'^2 \sigma(k') dk'}{(k'^2 - k^2)}. \quad (24)$$

3.2 Sum rule for potential scattering

If we evaluate Eq. (24) at threshold we immediately obtain the following sum rule

$$\text{Re } f(E=0) - \frac{1}{2m} \sum_{\beta} (-)^{\ell} \frac{(2\ell + 1)N_{\beta}^2}{E_{\beta}} + \frac{m}{2\pi} \int U(x) d^3\vec{x} = \frac{1}{2\pi^2} \int_0^{\infty} \sigma(k') dk', \quad (25)$$

which must hold for any scattering amplitude as long as it is adequately described by a normal potential. Therefore there exists a linear relation between an integral over the cross-section and a sum over pole residues for a potential. We will later attempt to generalize this relation into an approximate sum rule for nucleon-nuclear scattering.

3.3 Properties of the asymptotic wave functions (qualitative)

For potential forward scattering, the pole residues are determined by the strength of the asymptotic wave function N_{β} . In complete analogy, the pole strength for forward scattering of elementary particles is related to the asymptotic wave function (or asymptotic field). The pole residues are, in the latter case, the "coupling constants" of the system. Neither in the scattering on elementary particles, nor on potentials nor on nuclei, is the shape of the wave function explored by forward elastic scattering, since the momentum transfer is

*) For particles of spin $\frac{1}{2}$, a spin orbit interaction labels the bound states according to spin J and orbital angular momentum ℓ . The corresponding residue is clearly

$$\Gamma_{\beta} = (-)^{\ell} \frac{(2J + 1)}{2} \frac{\pi}{m^2} N_{\beta}^2.$$

zero. There is still an extremely important size effect for an extended system, which occurs whenever the characteristic radius R is larger than the characteristic wavelength of the bound state $\lambda_\beta = \kappa_\beta^{-1}$. On this scale, elementary particles are relatively small with $\kappa_\beta R \lesssim 1$, while nuclei are large with $\kappa_\beta R > 1$ (for heavier elements even $\kappa_\beta R \gg 1$) for deeply bound states.

The consequences are illustrated for a small and for a large system in Fig. 9 which displays $u_\beta(x) = xR_\beta(x)$ versus x . In the small system, the exponential asymptotic form of the wave function is valid so close to the origin that the deviation from the maximum value of $u(x) = u_{\max}$, and the intercept of the exponential at the origin (g) is very small.

Fig. 9 Radial wave function $u(x)$ for a "small" [diagram (a)] and a "large" system [diagram (b)].

For a large system there is a strong deviation from the asymptotic form for a large region inside the system, so that the extrapolated value at the origin N_β differs enormously from the value $u_{\max}$. Since the wave function has to be normalized, $u_{\max}$ is not a strong function of R . We therefore see at once that pole residues N_β^2 can be very large indeed for deeply bound states in an extended system, which leads to large contributions of such states to the scattering amplitude at low energies.

The effect we have just discussed crudely is of so great importance in nuclei that we will discuss it in considerable detail for the potential case.

A simple semi-quantitative estimate of the asymptotic wave function can be obtained as follows: consider a square well of radius R . The radial function $u_\beta(x) = xR_\beta(x)$ has the asymptotic form $N_\beta e^{-\kappa_\beta x}$. It has to be matched to the inside wave function at the nuclear radius R so that

$$N_\beta = e^{\kappa_\beta R} u(R) . \quad (26)$$

We have to impose the normalization condition that

$$\int_0^\infty u^2(x) dx = \int_0^R u^2(x) dx + \frac{u(R)^2}{2\kappa_\beta} = 1 . \quad (27)$$

If it is assumed that the wave function $R_\beta(r)$ is constant inside the nucleus (Wigner estimate), then

$$u^2(R) \left[\frac{R}{3} + \frac{1}{2\kappa_\beta} \right] = 1 \quad (28)$$

and hence

$$N_\beta = \left(\frac{3}{R} \right)^{\frac{1}{2}} \frac{1}{\left(1 + \frac{3}{2} \cdot \frac{1}{\kappa_\beta R} \right)^{\frac{1}{2}}} e^{+\kappa_\beta R}. \quad (29)$$

While this estimate is rough, it brings out the crucial point quite clearly: the normalization N_β of the asymptotic wave function is proportional to a slowly varying function of R times an exponential dependence on $(\kappa_\beta R)$. This gives rise to a characteristic size enhancement factor for the "coupling constant" of deeply bound states particularly since N_β enters squared in the pole residues. These effects are sizable for light nuclei and become very large in heavier nuclei: for a nucleon bound by 40 MeV in a medium-weight nucleus of $R = 5$ f the enhancement factor $\exp \{ 2\kappa_\beta R \} \approx 3.5 \times 10^5$. It is a consequence of this that deeply bound states tend to be very important in the dispersion relations, while weakly bound states play a secondary role since they are not size enhanced.

3.4 Asymptotic wave functions (quantitative)

A direct calculation of the asymptotic wave function would obviously give the asymptotic coefficient N_β directly. Unfortunately this approach is not very instructive, since the dependence of N_β on the properties of the potential has to be numerically explored. It is very fortunate therefore that there exist highly accurate WKBJ methods that give explicit expressions for N_β with rather weak assumptions about the potential. The detailed derivation of N_β is given in the appendix to Section 3.

The essence of the method is the following: consider a potential $U(x)$ in one dimension^{*)}. Consider the bound state of energy E_β and its two turning points x_0 and x_1 (see Fig. 10). The WKBJ solution is achieved by introducing a comparison function with known solution and the same classical turning points. In our case the harmonic oscillator provides a good comparison function. With comparison functions it is possible to describe the difficult region at the turning points correctly; in the other regions the ordinary WKBJ methods work well. The wave functions can be accurately normalized analytically. This method is equally well applied to the radial case provided the potential used is taken to include the centrifugal barrier:

Fig. 10 One-dimensional potential $U(x)$ with two classical turning points.

*) We will assume that the bound states of this potential have two classical turning points only.

$$v(x) = U(x) + \frac{1}{2m} \frac{(\ell + \frac{1}{2})^2}{x^2}.$$

[Note the modification of $\ell(\ell+1)$ into $(\ell+\frac{1}{2})^2$!] The corresponding WKB solution behaves correctly like x^ℓ at the origin.

For a wave function of n nodes, the square of the asymptotic wave function coefficient is (see Part A of Appendix to Section 3):

$$N_\beta^2 = \frac{\sqrt{\pi} 2^n}{n! \kappa_\beta \left(\int_{x_0}^{x_1} d\xi/p(\xi) \right)} \left(\frac{2\sqrt{e}}{2n+1} \right)^{2n+1} \exp \left[2 \kappa_\beta x_1 + 2 \int_{x_1}^{\infty} \frac{2m v(\xi) d\xi}{|p(\xi)| + \kappa_\beta} \right]. \quad (30)$$

Here we use the notation

$$p^2(x) = -[\kappa_\beta^2 + 2m v(x)].$$

From this expression for N_β^2 we see that in practice the asymptotic wave function is determined by three quantities: 1) the binding energy; 2) the outer classical turning point; and 3) to a minor degree by the slope of the potential at the outer turning point. Any potential that reproduces these quantities well will give a good value for N_β^2 quite independent of its other properties provided it has only two classical turning points. We further note that for all but the lightest nuclei, $\kappa_\beta x_1 \gg 1$, so that the exponential factor in Eq. (30) is dominant. It is therefore possible to turn this argument around: from the determination of the binding energy E_β and the asymptotic factor N_β^2 it is possible to obtain a highly accurate value for the classical turning point of the orbit.

3.5 Crossing and exchange in a potential model

The potential we have described here is a single particle potential. The bound states appear in this case as negative energy resonances in the compound system "potential + one particle". Therefore the picture we have discussed is one in which all scattering occurs in the direct channel; there is no crossed channel^{*)}.

To see explicitly the effects of crossing, we will introduce a very simple model with this property.

Consider A identical particles (fermions or bosons) without mutual interaction. These particles are placed in a static external potential (which may be the one previously discussed), so that the A particles occupy bound states. We now scatter an additional particle identical to the previous ones from the same potential. This produces a scattering amplitude $f_1(\theta)$. We first enquire how this scattered amplitude is related to the single-particle scattering amplitude from the unoccupied potential $f(\theta)$.

Theorem: It is impossible, by a scattering experiment alone, to decide whether the bound states of the potential are occupied or not, i.e. $f_1(\theta) \equiv f(\theta)$.

*) If we consider additional interactions such as the electromagnetic one, direct bound states become observable, since there will then exist γ -transitions from the continuum states into the bound states.

Proof: Obvious, since the interaction by construction can be thought of as diagonal. There is therefore no coupling between states of different energy. Since it is irrelevant whether the bound states are occupied or not, $f_1(\theta) = f(\theta)$, whether the particles are fermions or bosons^{*)}.

The point of this idealized model is that it is possible to replace an amplitude $f_1(\theta)$, which includes exchange (between projectile and target particles), by an equivalent single particle amplitude $f(\theta)$ for which no exchange is possible.

In a more realistic picture of nucleon scattering from an actual nucleus, all mutual particle interactions appear. There will be direct and exchange contributions to the amplitude as symbolized by Fig. 11:

Fig. 11 Direct pole (bound state, "negative energy resonance") [diagram (a)] and exchange pole [diagram (b)].

By dynamical accident, graphs (a) and (b) may coincide in pole position and residue in the s-channel. As in our model, it is then not possible to distinguish phenomenologically between exchange and direct contributions^{**)} . It is thus clear that the use of a phenomenological single particle potential ("optical model potential") may simulate not only direct poles in the scattering amplitude but also exchange poles.

It is often possible to distinguish between exchange and direct contributions, but this requires extraneous information about the spectrum of bound states in the (A-1) and (A+1) channels. An example of this is the pole in our dispersive treatment of $n^4\text{He}$ scattering, which must be associated with exchange (^3He) since no bound states exist for the ^5He system.

* * *

*) On the other hand, if γ -transitions are considered, it is indeed possible to find out if the states are occupied or not. For fermions the transition from continuum to an occupied state is forbidden, while for bosons the transition rate changes with occupation number.

**) This is exact only for forward scattering. Diagrams 11a and 11b interpreted as Feynman diagrams have in general different angular and energy dependence.

Appendix to Section 3

ASYMPTOTIC WAVE FUNCTION FOR A BOUND STATE IN A POTENTIAL

In the dispersion relation (19) for potential scattering, the residues of the poles are directly related to the asymptotic wave function of the bound states by Eq. (23). We here give analytic expressions for the asymptotic wave functions using the powerful comparison version of the WKBJ method as developed by Miller and Good¹³⁾.

Consider the radial equation for a bound state

$$u'' + \left[-\kappa^2 - U(r) - \frac{\ell(\ell+1)}{r^2} \right] u = 0, \quad (\text{A3.1})$$

where $u(r) = rR(r)$. As is well-known, the relevant wave number in the WKBJ approximation is always¹⁴⁾

$$p^2(r) = -\kappa^2 - U(r) - \frac{(\ell + \frac{1}{2})^2}{r^2}. \quad (\text{A3.2})$$

We will assume that these are exactly two classical turning points r_1 and r_2 for which $p(r) = 0$. The approximate solution is obtained by comparison with the known solution for the harmonic oscillator $\phi(S)$, which also has two classical turning points:

$$\phi''(S) + (K^2 - S^2) \phi(S) = 0. \quad (\text{A3.3})$$

For the n^{th} bound state, $K_n^2 = 2n + 1$, and the turning points are $\pm K_n$. The solution for the n^{th} bound state is obtained by considering S as a function of r with the functional relation

$$\int_{r_2}^r |p(r)| dr = \int_{K_n}^{S(r)} \sqrt{S^2 - K_n^2} dS = \frac{1}{2} \left\{ S(S^2 - K_n^2)^{\frac{1}{2}} - K_n^2 \log \left[\frac{(S^2 - K_n^2)^{\frac{1}{2}} + S}{K_n} \right] \right\}. \quad (\text{A3.4})$$

The normalized wave function $u_n(r)$ is expressible in a Hermite polynomial as

$$u_n(r) \simeq \left[\frac{\sqrt{\pi}}{2^n n! \int_{r_1}^{r_2} \frac{dr}{p(r)}} \frac{[S^2(r) - K_n^2]^{\frac{1}{2}}}{|p(r)|} \right]^{\frac{1}{2}} H_n[S(r)] \exp[-S^2(r)/2]. \quad (\text{A3.5})$$

We are not interested in the full solution but only in the asymptotic form of $u_n(r)$ as $r \rightarrow \infty$, i.e. for $S \rightarrow \infty$. Since in this limit $H_n(S) \rightarrow 2^n S^n$, $|S^2 - K_n^2|^{\frac{1}{2}} \rightarrow S$, and $|p(r)| \rightarrow \kappa$ we find

$$u_n(r) \xrightarrow{r \rightarrow \infty} \left[\frac{\sqrt{\pi} 2^n}{n!} \frac{1}{\int_{r_1}^{r_2} \frac{\kappa dr}{p(r)}} \right]^{\frac{1}{2}} \exp \left[-\frac{S(r)^2}{2} + \left(n + \frac{1}{2}\right) \log S(r) \right]. \quad (\text{A3.6})$$

But from Eq. (A3.4) we see that

$$\int_{r_2}^r |p(r)| \, dr \rightarrow \frac{1}{2} \left\{ S^2 - \frac{2n+1}{2} - (2n+1) \log \frac{2S}{\sqrt{2n+1}} \right\} =$$

$$\frac{1}{2} S^2 - (n + \frac{1}{2}) \log S - (n + \frac{1}{2}) \log \frac{2e}{(n + \frac{1}{2})} . \quad (A3.7)$$

Therefore

$$u_n(r) \xrightarrow{r \rightarrow \infty} \left[\frac{\sqrt{\pi} \, 2^n}{n!} \frac{1}{\int_{r_1}^{r_2} \frac{\kappa \, dr}{p(r)}} \right]^{\frac{1}{2}} \left[\frac{(n + \frac{1}{2})}{2e} \right]^{(n+\frac{1}{2})/2} \exp \left[- \int_{r_2}^r |p(r)| \, dr \right] . \quad (A3.8)$$

Since $u_n(r) \rightarrow N_n e^{-\kappa_n r}$, we have finally

$$N_n = \left[\sqrt{\frac{\pi}{2}} \cdot \left(\frac{n + \frac{1}{2}}{e} \right)^{n+\frac{1}{2}} \cdot \frac{1}{n!} \frac{1}{\int_{r_1}^{r_2} \frac{\kappa_n \, dr}{p(r)}} \right]^{\frac{1}{2}} \exp \left\{ \kappa_n r_2 + \int_{r_2}^{\infty} [\kappa_n - |p(r)|] \, dr \right\} . \quad (A3.9)$$

* * *

4. PION-NUCLEAR FORWARD SCATTERING

The forward dispersion relations for pions on nuclei are of particular interest. The singularities in the unphysical region (particularly for the charge exchange amplitude) provide a direct determination of the pion field that asymptotically surrounds the nucleus.

4.1 The π -nuclear forward dispersion relations

We now give the detailed dispersion relations for the scattering on a nucleus X . These are very similar to the D.R. for πN scattering¹⁵⁾ of which they are generalizations. In the πN case, the unphysical region consists simply of the nucleon pole in the direct and the crossed channels (see Fig. 12).

Fig. 12 Pole graphs for $\pi^- p$ scattering.

These poles are located^{*)} at $\omega_N = \mp m^2/(2M) \approx \mp 10$ MeV (minus sign for the direct pole); their residue f^2 is the ordinary πN coupling constant $f^2 = 0.08$. There is no other intermediate state of energy smaller than $(m + M)$, which makes πN dispersion relations unusually simple.

The unphysical region for πX scattering has two complications: first there are in general many poles corresponding to nuclear excited states and second, the physical pion absorption extends as a cut below the threshold for scattering.

The pole terms correspond to the ground state and the excited states of the neighbouring nuclei X_1 with the same nucleon number but differing in charge from X according to the pion charge (see Fig. 13).

Fig. 13 Pole graphs for pion-nucleus scattering.

*) According to our conventions (Section 2, Table 1) m is the pion mass, whereas M is the nucleon or nuclear mass whichever is the target particle.

The energies of these poles are $\omega_i = \pm[(-m^2/2M) + \epsilon_i]$ according to Eq. (13), where $\epsilon_i = (M_i - M)c^2$ is the energy of M_i counted from the ground state of M . The pole terms corresponding to Fig. 13 are

$$\frac{r}{\omega - \omega_i} \quad \text{and} \quad \frac{r_i}{\omega + \omega_i} \quad (31)$$

for the direct and crossed channel, as is immediately evident from the crossing relation (4) that $f(-\omega) = \bar{f}(\omega)^*$. The real constant r_i is the dimensionless residue of the i^{th} pole. Although this residue is proportional to f_i^2 , the square of the coupling constant at the nXX_i vertex, it need not be positive. The reason is the appearance of a mass factor that can have either sign, depending on the relative parity at the vertex. This is analogous to the detailed discussion of the $n^4\text{He}^3\text{He}$ vertex given in Part A of Appendix to Section 5. In potential scattering (Section 3), the corresponding sign change with orbital angular momentum has already been discussed.

We will, in the following, neglect Coulomb effects^{*)}. It is then useful to introduce the amplitudes^{**)}

$$f^{(\pm)}(\omega) = \frac{1}{2} [f_{\pi^+X}(\omega) \pm f_{\pi^-X}(\omega)] \quad (32)$$

which are symmetric and antisymmetric under crossing:

$$f^{(\pm)}(-\omega) = \pm f^{(\pm)*}(\omega) . \quad (33)$$

We will limit the further discussion to $I = 0$ and $I = \frac{1}{2}$ nuclei. For an $I = 0$ nucleus, in the absence of Coulomb forces the π^+ and π^- amplitudes are equal, so that $f^{I=0}(\omega) = f^{(+)}(\omega)$. For an $I = \frac{1}{2}$ nucleus, the full elastic amplitude including charge exchange within the isospin multiplet is ($\vec{t}$ = pion isospin operator):

$$f(\omega) = f^{(+)}(\omega) - 2(\vec{t} \cdot \vec{I}) f^{(-)}(\omega) . \quad (34)$$

Introducing Cartesian coordinates $\alpha, \beta = 1, 2, 3$ of the initial and final pion, Eq. (34) is conventionally¹⁵⁾ written as a matrix equation

$$f_{\beta\alpha}(\omega) = \delta_{\beta\alpha} f^{(+)}(\omega) + \frac{1}{2} [\tau_\beta, \tau_\alpha] f^{(-)}(\omega) , \quad (34')$$

where $\vec{\tau} = 2\vec{I}$ are the 2×2 isospin matrices of the target.

*) In light elements, Coulomb effects are of minor importance. They enter in several ways: as corrections to total cross-sections as well as to incident pion energies to order $Z\alpha$; as shifts in pole positions by the electromagnetic mass-splittings of nuclear isospin multiplets. For the real amplitude leading order, Coulomb effects can be separated according to standard treatments⁷⁾. Inner Coulomb corrections are of less importance.

**) We assume neutron excess for the target X . Reverse the pion charge if the isospin component I_3 of X is positive (as for a proton).

The forward elastic charge exchange cross-section for the $I = \frac{1}{2}$ target^{*)} is

$$\left. \frac{d\sigma_{c.e.}}{d\Omega}(\omega) \right|_{0^\circ} = |f^{(-)}(\omega)|^2.$$

Since $\sigma^{(+)}(\omega) = \frac{1}{2}[\sigma_{\pi^+X}(\omega) + \sigma_{\pi^-X}(\omega)] \rightarrow \text{const}$ for $\omega \rightarrow \infty$, the dispersion relation for the symmetric amplitude is divergent and must be subtracted. It is convenient to do this subtraction at threshold, as the physical amplitude is small there. Collecting the expressions (3) to (5) and (31) to (33), and using $k^2 = \omega^2 - m^2$, the dispersion relation for $f^{(+)}$ becomes

$$\text{Re } f^{(+)}(\omega) - \text{Re } f^{(+)}(m) = \sum_i \frac{2\omega_i r_i}{\omega^2 - \omega_i^2} \frac{k^2}{k_i^2} + \frac{2k^2}{\pi} P \int_{\omega_0}^{\infty} \frac{\omega' d\omega' \text{Im } f^{(+)}(\omega')}{k'^2 (\omega'^2 - \omega^2)}. \quad (35)$$

Note that the crossing relation (33) reduces the integral essentially to positive values of ω . The unphysical region in Eq. (35) is $0 \approx \omega_0 < \omega < m$. In the analogous πN case, the nucleon pole in the unphysical region gives only a very small contribution to physical scattering, and it is in fact quite useless for determining the coupling constant $r_N = f^2$ from $f_{\pi N}^{(+)}$. This feature is also characteristic of the symmetric nuclear amplitude, and it is fundamentally due to the smallness of $|\omega_1| \ll m$ in the numerator of Eq. (35). This will be discussed in considerable detail, as well as its physical significance.

The dispersion relation for the antisymmetric amplitude gives a convergent integral since $\sigma^{(-)}(\omega) = \frac{1}{2}[\sigma_{\pi^+X}(\omega) - \sigma_{\pi^-X}(\omega)] \rightarrow 0$ as $\omega \rightarrow \infty$ (Pomeranchuk theorem). Therefore no subtraction is needed, and hence

$$\text{Re } f^{(-)}(\omega) = \sum_i \frac{2\omega r_i}{\omega^2 - \omega_i^2} + \frac{2\omega}{\pi} P \int_{\omega_0}^{\infty} \frac{d\omega' \text{Im } f^{(-)}(\omega')}{\omega'^2 - \omega^2}. \quad (36)$$

We note that the pole terms are intrinsically larger than for the symmetric amplitude by a factor $\omega/|\omega_1| \gg 1$ for $\omega > m$.

4.2 Orders of magnitude of cross-sections, amplitudes, and pole terms

The symmetric πX amplitude (35) has been studied in detail for $X = {}^2D$, 4He , and ${}^{12}C$. A similar analysis for the charge-exchange amplitude (36) has been made for 9Be .

To get a feeling for the magnitudes involved, let us first survey the input data for $I = 0$ nuclei, particularly the total cross-sections that control the imaginary amplitude in the physical region.

Experimental values for symmetrical total cross-sections are given in Fig. 14. In all these cross-sections the (3,3) resonance is prominent. It is, however, appreciably broadened

*) Nuclei of $I > \frac{1}{2}$ are known experimentally to have very small amplitude for the double charge exchange reaction $\pi^+X \rightarrow \pi^-X^{++}$. It is therefore, in practice, a good approximation to write

$$f^{I>\frac{1}{2}}(\omega) \simeq f^{(+)}(\omega) - 2f^{(-)}(\omega) \vec{t} \cdot \vec{I}$$

as long as the elastic double charge exchange is neglected.

by nuclear Fermi motion. Higher πN resonances are so broadened that they are not noticed in the nuclear cross-sections. The great similarity of the shape of the total cross-sections for $A \geq 4$ is very remarkable. It can be empirically described by a scaling factor A^n with $n \approx 0.83$. After reduction by this factor, the nuclear cross-sections closely coincide (see Fig. 14), and are qualitatively very similar to the πN cross-section. For the present context we retain that the remarkable experimental similarity in shapes will extend the validity of the ${}^4\text{He}$ and ${}^{12}\text{C}$ results universally to all light nuclei, at least as far as concerns the contributions from the physical region.

At very low energies, the total cross-sections are dominated by absorptive processes. Typical examples of these are:

$$\begin{aligned} \pi^- p &\rightarrow \begin{cases} \pi^0 n \\ \gamma n \end{cases} \\ \pi D &\rightarrow \begin{matrix} 2N \\ 2N\gamma \end{matrix} \\ \pi {}^4\text{He} &\rightarrow 2N + D, \text{ etc.} \\ \pi {}^{12}\text{C} &\rightarrow 2N + {}^{10}\text{B}^*, \text{ etc.} \\ \pi {}^{16}\text{O} &\rightarrow 2N + {}^{14}\text{N}^*, \text{ etc.} \end{aligned} \quad (37)$$

Fig. 14 Total cross-sections¹⁶⁻¹⁸⁾ for pions on ${}^4\text{He}$, ${}^{12}\text{C}$, ${}^{16}\text{O}$, ${}^{27}\text{Al}$ and on nucleons¹⁹⁾. (For ${}^{27}\text{Al}$ and the nucleon the charge symmetric combination is given.) Some data at isolated energies are omitted. Black diamonds show the nuclear cross-sections scaled by a factor $A^{0.83}$ with error bars omitted. In all other cases, missing error bars mean statistical errors smaller than symbol size. Eye-guiding solid lines are drawn.

Many other absorption processes are possible in the nuclear case. All these processes correspond to cuts that extend into the unphysical region. The corresponding total cross-sections vary as v_π^{-1} with the pion velocity. It therefore diverges at threshold when $v_\pi = 0$. In spite of this, the absorption cuts appear to be weak and produce only minor effects in the dispersion relations. This can be seen as follows.

Fig. 15 Imaginary part of the amplitude for π -nuclear and charge symmetric π -proton scattering. The data for π D are from Fäldt and Ericson²⁰⁾, the remaining ones are as in Fig. 14. The imaginary scattering lengths are from Nordberg and Kinsey^{21a)} and Crowe et al.^{21b)} for ^4He , and from Part B of Appendix to Section 4 for ^{12}C . Eye guiding solid lines are drawn.

The dispersion relations involve primarily the amplitude $\text{Im } f = k\sigma(\omega)/4\pi$ and not the divergent cross-section. The imaginary amplitude corresponding to Fig. 14 is therefore given in Fig. 15, which also includes the deuteron data^{*)}. To illustrate the threshold behaviour, we have included information on the threshold value and shape of $\text{Im } f$ for ^4He and ^{12}C as obtained from the absorptive part of the s- and p-wave π -nuclear scattering lengths. By mere inspection of the curves it is immediately clear that the threshold amplitude is very small as compared to the 3,3 resonance region. In the case of ^{12}C , we have for example $\text{Im } f(m)/\text{Im } f(\omega_{33}) \approx 1/100$. For the nucleon the value is even smaller by more than a magnitude (electromagnetic origin!) as it is also for the deuteron. It is therefore very reasonable to expect that the absorption cuts have only small effects on the dispersion integrals, perhaps with the exception of the threshold region itself where the real amplitude also is very small. More detailed estimates will be made later.

Estimates of the contribution from the unphysical cut to the antisymmetric amplitude are more uncertain due to lack of experimental data. There is, however, no obvious isospin dependence²²⁾ of the 1s widths in pionic atoms at a precision of about 20% in the widths. Furthermore, no isospin dependence in 2p widths is apparent within experimental uncertainty, but it is expected to be less than for the 1s widths, as the 1s interaction has anomalously strong isospin dependence. Therefore, the absorption cut should play a minor role also in this case. We consider the argument above as being the most reasonable one at present, in the absence of any evidence to the contrary.

The second striking feature concerning magnitudes is the small value of the isoscalar amplitude at threshold. For ^{12}C for example, the scattering length $a^{(+)} = -0.26 \text{ m}^{-1}$ [see Part B, Appendix to Section 4, Eq. (A4.10a)] so that the ratio of $|a^{(+)}|/\text{Im } f(\omega_{33}) \approx 4.5 \times 10^{-4}$. Quite apart from the physical origin of this small value for $a^{(+)}$ (soft pion theorems!) ²³⁾, the important practical consequence is that the subtraction should be made at threshold where the amplitude is small. This fixes absolute normalization of the amplitude quite reliably, irrespective of the relative error in the subtraction constant. As this error is moreover small, the predictions remain meaningful near threshold. There is therefore a logical rationale for making the subtraction at threshold. Thirdly, we estimate the magnitude of the pole terms in Fig. 13. The poles are in practice all concentrated near $\omega = 0$ due to the smallness of nuclear excitation energies ϵ_i as compared to the pion mass.

A glance at Eq. (35) shows that the symmetric pole term is multiplied by the small quantity $|\omega_i| = |(-m^2/2M) + \epsilon_i|$ and therefore is heavily suppressed for $\omega > m$. A better estimate is obtained as follows. Suppose the nucleus to consist of an assembly of weakly bound nucleons, sufficiently extended that the forward π -nuclear amplitude is simply the coherent sum of the individual πN amplitudes. Then we obtain a quick estimate by adding A charge symmetric πN poles coherently, leading to

$$f_{\text{coherent pole}}^{(+)}(\omega) \approx A \frac{f^2}{M_n} \frac{k^2}{\omega^2} = 0.012 A \frac{k^2}{\omega^2} \text{ m}^{-1} \quad (38)$$

*) The πD total cross-section is excellently described even in the 3,3 region by the symmetrical πN amplitude with Fermi motion and a small additional correction for double scattering^{24,25)}.

neglecting the nucleon pole energy ω_N^2 against m^2 . This is a small number, because typical real parts are measured in units of m^{-1} (see Fig. 18 below). We can compare also to the experimental p-wave part which is

$$\text{Re } f_{\text{p-wave}}^{(+)} = 3 \text{ Re } a_1 k^2 . \quad (39)$$

As an example, $3 \text{ Re } a_1 = 0.75 \text{ m}^{-3}$ for ${}^4\text{He}^{21a}$, so that the pole contribution (38) is only 6%. The p-wave scattering length is dominated by the 3,3 resonance region. Similarly, the coherent pole term for the unsubtracted exchange amplitude is, for an excess of one neutron:

$$f_{\text{coherent pole}}^{(-)}(\omega) \simeq \frac{2f^2}{\omega} , \quad f^2 = 0.08 . \quad (40)$$

In contrast to the small values obtained for the symmetric pole term, expression (40) for $f^{(-)}$ represents about 15% of the imaginary amplitude at 200 MeV, and it gives the most important contribution at threshold. The importance of the pole term in the πN charge exchange amplitude is well known, and it gives a standard determination of the coupling constant f^2 .

A more detailed discussion is given at the end of this section. In view of the present accuracy, the symmetric pole terms have been dropped in all the applications to isospin zero nuclei.

We now turn to the explicit evaluation of the dispersion relations

4.3 Symmetric amplitude

The characteristic over-all variation of $\text{Re } f^{(+)}(\omega)$ with the energy is the following: its value at threshold is small. It rises rapidly to a maximum at about $E = 100 \text{ MeV}$. It then goes through zero at $E \approx E_{\text{res}}$. At higher energy there is little structure and $\text{Re } f^{(+)}(\omega)$ stays negative above $E \approx 800 \text{ MeV}$, the ratio $\text{Re } f/\text{Im } f$ tending slowly to zero for increasing energy. All these features follow from the dominance of the nucleon 3,3 resonance in the nuclear total cross-sections and the nearly universal shape of $\sigma_{\pi X}^{(+)}(\omega)$. In the special case of the deuteron, the weak binding permits higher πN resonances to manifest themselves by some additional structure at higher energies.

The numerical evaluation has been carried out for D, ${}^4\text{He}$, and ${}^{12}\text{C}$.

The results for πD are from Ref. 2c and shown in Fig. 16, where details of the calculation can be found. As there are no phase shifts available, the dispersive prediction for $\text{Re } f$ is compared to a Glauber model calculation which includes Fermi motion. The agreement is remarkable over the whole energy region, and the errors are of the order of different experimental sets for total cross-sections.

The results for ${}^4\text{He}$ and ${}^{12}\text{C}$ obtained from applying the dispersion relation (35) are shown in Figs. 17 and 18. They are based on experimental total cross-sections as shown in Fig. 19. Detailed information on input is collected in the figure captions. The experimentally uncertain high-energy total cross-section parameters are deduced from a Glauber multiple scattering calculation using πN amplitudes from Cline et al.²⁶). In fact, a reasonable constant would do as well, as the behaviour of $\text{Im } f$ above 2 GeV hardly affects the results of $\text{Re } f$ below 500 MeV.

Fig. 16 Real part^{2c)} of πD amplitude from dispersion relations compared to a Glauber-type calculation for Fermi motion.

The subtraction constants are from phase-shift analysis^{21a,b)} in the case of ${}^4\text{He}$, and from π -mesic atoms for ${}^{12}\text{C}$. The connection between strong interaction level shifts and widths of mesic atoms and complex scattering lengths is recapitulated in Part B of Appendix to Section 4. Mesic atoms are a very accurate and sometimes unique source of these low-energy parameters.

For the imaginary part below threshold, a simple parametrization is obtained by continuing the scattering length expansion by the replacement $k \rightarrow i|k|$, leading to the form

$$\text{Im } f(\omega) = \alpha + \beta|k| + \gamma|k|^2 + \delta|k|^3 + \epsilon|k|^4; \quad \omega < m \quad (41a)$$

where the coefficients in terms of complex s-, p-, and d-wave scattering lengths^{*)} are

$$\alpha = \text{Im } a_0; \quad \beta = -2 \text{Re } a_0 \text{Im } a_0; \quad \gamma \approx -3 \text{Im } a_1; \quad \delta \approx 0; \quad \epsilon \approx 5 \text{Im } a_2. \quad (41b)$$

This parametrization should be reliable near threshold, where we already noticed the smallness of $\text{Im } f$. In view of its small value, we use the same form down to the branch point ω_0 , which is slightly negative and close to zero [cf. Eq. (13) with $\epsilon_n = 0$], irrespective of its normal or anomalous origin (see Section 2). In fact near the unphysical energy $\omega = 0$ the imaginary part is controlled by Adler's self-consistency condition^{27,7b)} demanding that $\text{Im } f^{(+)}(0) = 0$ in the soft pion limit. But even without this restriction, the leading terms α and γ of the parametrization (41) for the unphysical region hardly contribute in the scale of Figs. 17 and 18. An increase of 50% in the parameter $|\gamma|$ is shown in Fig. 18.

*) Defined by $\lim_{k \rightarrow \infty} k^{2\ell+1} \cot \delta_\ell = a_\ell^{-1}$. The difference between c.m. and lab. momenta is neglected.

Fig. 17 $\text{Re } f(\omega)$ for $\pi^+{}^4\text{He}$ versus energy. The solid line is the dispersion prediction with the following input:

- a) $\text{Im } f_{\pi^+{}^4\text{He}}(\omega) = [0.084 + 0.064 \, k^2/m_\pi^2] \, m_\pi^{-1}$ for $0 < \omega < m_\pi + 10$ MeV from Ref. 21a using Eq. (41).
- b) A smooth polynomial fit through the imaginary parts of Fig. 15 for $\omega < 1300$ MeV.
- c) For higher energies (see text) an asymptotic expression $\sigma(\omega) = [0.071 + (2.95 \, \text{MeV}/\omega)^{1/2}]b$.
- d) Subtraction constant $\text{Re } f(m_\pi) = (-0.132 \pm 0.003)m_\pi^{-1}$ from Ref. 21a.

Continuous matching between (a) (b) (c) is enforced. The dashed line is for a 3 standard deviation increase of $\sigma(\omega)$ in the resonance region. The full dots are obtained directly from phase shifts²¹⁾.

Therefore the prominent feature of these results is the leading role of the 3,3 resonance despite its broadening and quenching in nuclei. An arbitrary increase by three standard deviations above experimental cross-sections near 180 MeV produces marked effects (dashed line).

Also shown are several real amplitudes below 100 MeV from phase shifts²¹⁾ for ${}^4\text{He}$ and from an optical model fit²⁴⁾ for ${}^{12}\text{C}$. Those form a valuable independent check of our dispersion predictions, although it is difficult to judge the inherent errors of these values. However, they confirm very nicely our assumption of small unphysical contributions for this energy region.

A recent experiment²⁵⁾ on ${}^{12}\text{C}$ reports a shift of the 3,3 maximum in the cross-section (see Fig. 19) to 150 MeV. This is not included in our dispersion calculation. Essentially

Fig. 18 $\text{Re } f(\omega)$ for $\pi^{12}\text{C}$ versus energy. Notation and input analogous to Fig. 17 with the following explicit parameters:

- a) $\text{Im } f(\omega) = [0.063 + 0.22 k^2/m_\pi^2] m_\pi^{-1}$ for $0 < \omega < m_\pi + 20$ MeV from Part B of Appendix to Section 4.
- b) Analogous. The recent values for σ_{33} given in Fig. 19 are not included.
- c) $\sigma(\omega) = [0.20 + (15.7 \text{ MeV}/\omega)^2] b$.
- d) $\text{Re } f(m_\pi) = (-0.26 \pm 0.03) m_\pi^{-1}$ from Part B of Appendix to Section 4.

The dashed-dotted line obtains for

a') $\text{Im } f(\omega) = [0.063 + 0.33 k^2/m_\pi^2] m_\pi^{-1}$.

The full points are from an independent optical model fit to differential cross-sections²⁴⁾.

it will shift the zero of the real part by a corresponding amount. Of course, it is most probable that remeasuring the cross-sections of ^4He and ^{16}O will show a similar shift, as the data in the 3,3 region of Fig. 14 mostly stem from the measurements of Ignatenko¹⁷⁾ and should have the same systematic errors. The origin of the shift in peak energy may be the double scattering term in a Glauber expansion²⁸⁾.

A direct confirmation of the negative sign of $\text{Re } f$ above 200 MeV by Coulomb interference is desirable. Although this is quite feasible experimentally, the present angular resolution in the small-angle region ($4\text{--}12^\circ$) does not yet permit firm conclusions.

4.4 Antisymmetric amplitude and effective π -nuclear coupling

The charge exchange dispersion relation (36) makes it possible to determine the effective π -nucleus coupling strength. At present, a quantitative determination is only marginally possible in one case, ^9Be , but there is no difficulty of principle, once there exist moderately complete total cross-sections for pions of both charges in the 3,3 region.

Fig. 19 Shift²⁵⁾ in 3,3 peak for $\pi^{12}\text{C}$ (solid squares). The remaining data are as in Fig. 14 and serve as input for the calculations leading to Figs. 17 and 18.

As the poles are close to $\omega = 0$ they act as one single effective pole in the physical region $\omega > m$. Introducing the effective coupling $\sum_i r_i = r_{\text{eff}}$ and neglecting ω_i^2/m^2 , we solve Eq. (36) for r_{eff} at threshold

$$r_{\text{eff}} = m/2 \operatorname{Re} f^{(-)}(m) - \frac{m^2}{\pi} P \int_{\omega_0}^{\infty} \frac{\operatorname{Im} f^{(-)}(\omega')}{k'^2} d\omega' . \quad (42)$$

We claim, although with less certainty, that the integral is again dominated by the 3,3 region. This is certainly reasonable for the physical region, as can be verified from Fig. 20. The missing data on $\pi^+{}^9\text{Be}$ cross-sections in the resonance region are replaced by scaled cross-sections on ${}^{12}\text{C}$ [Ignatenko¹⁷⁾]:

$$\sigma_{9\text{Be}}^{(+)} = \left(\frac{9}{12} \right)^{0.83} \sigma_{12\text{C}} ,$$

Fig. 20 Total cross-sections¹⁷⁾ for $\pi^{-9}\text{Be}$ and $\pi^{\pm 12}\text{C}$ scattering. The solid line represents smoothed $\sigma_{\pi^{-9}\text{Be}}$ cross-sections, while the dashed line is $\sigma_{9\text{Be}}^{(+)}$ [Eq. (43)]. The difference (shaded area) is needed for the evaluation of Eq. (42).

which determines the needed cross-section difference:

$$\sigma_{9\text{Be}}^{(-)} = \frac{1}{2}(\sigma_{\pi^{+9}\text{Be}} - \sigma_{\pi^{-9}\text{Be}}) = \sigma_{9\text{Be}}^{(+)} - \sigma_{\pi^{-9}\text{Be}}. \quad (43)$$

This difference is given by the shaded area of Fig. 20. The corresponding contribution to r_{eff} is of the order of 0.03. By the arguments given earlier in this section, the unphysical cut should be small (of order 0.01), also for the charge exchange amplitude. At threshold, the values for $\text{Im } a_0^{(-)}$ and $\text{Im } a_1^{(-)}$ deduced from the multiple scattering analysis of M. Ericson and T.E.O. Ericson²⁹⁾ indicate a 10% correction.

The c.e. scattering length $\text{Re } f^{(-)}(m)$ is deducible from π -mesic atoms (Part B of Appendix to Section 4) using the same trick as above to obtain $\text{Re } f^{(-)}(m)$ from neighbouring $I = 0$ nuclei. This leads to

$$\text{Re } f^{(-)}(m) = (0.048 \pm 0.015)m^{-1} \quad (44)$$

in agreement with the Weinberg value²³⁾ corrected for nuclear effects^{3a)}.

Both the contributions to r_{eff} being positive we obtain without cancellations

$$r_{\text{eff}} \simeq (0.06 \pm 0.03) \quad (45)$$

as order of magnitude, which is close to the nucleon value $r_{\text{eff}} = f^2 = 0.08$. Indeed the coherent sum of πN poles as in Eq. (40) would just give this relation, the antisymmetric pole being proportional to the total isospin, which is 1/2 as for the nucleon.

In this naïve explanation the $I = 1/2$ ground state multiplet ${}^9\text{Be}(\frac{3}{2}^-) - {}^9\text{B}(\frac{3}{2}^-)$ would play the role of the np doublet. Assuming pion pole dominance (PCAC), the coupling to the pion should be proportional to the axial matrix elements of β -decay. The strength of the ${}^9\text{Be} - {}^9\text{B}$ transition is not known; however the allowed Gamow-Teller β -transition for the similar doublet ${}^{11}\text{C}(\frac{3}{2}^-) - {}^{11}\text{B}(\frac{3}{2}^-)$ has only 13% of the nucleon value. Therefore other states than the ground-state doublet are needed to saturate (45). This is discussed in the following.

4.5 Sum rules for the π -nuclear poles

In the previous estimate of the magnitude of the pole contributions using the coherent pole sum for the πN system of Eqs. (38) and (40), no explicit reference was made to the many nuclear states that contribute individual π -nuclear poles. In the further explicit evaluation of an effective π -nuclear coupling constant for ${}^9\text{Be}$, there was also no explicit reference to nuclear excited states. The underlying philosophy in both cases is that sum rules or approximate sum rules apply to the π -nuclear poles in the unphysical region. We will now explore this feature in more detail using a model.

The contributions to pion absorption by the nucleus below the threshold $\omega < m$, are visualized in Fig. 21 for transition into the nuclear state $|n\rangle$ (the intermediate state defining the individual pole position).

The first term in the expansion represents pions absorbed on a single nucleon, while the remaining terms represent multinucleon vertices. Typical of the latter is the pion absorption at threshold, which experimentally is well known to be dominated by multinucleon processes. We will in the following neglect this absorption cut at threshold, since the numerical investigation above showed that its influence is rather weak in the physical region $\omega \geq m$. In the further discussion we will retain only the single nucleon term in the expansion of Fig. 21. Since we have removed the cut at threshold, the pole terms can be written

$$\begin{aligned} f_{\text{pole}}^{(+)}(\omega) &= k^2 \frac{2}{\pi} \int_0^m \frac{\omega' \text{Im } f^{(+)}(\omega') d\omega'}{k'^2 (\omega'^2 - \omega^2)} \\ f_{\text{pole}}^{(-)}(\omega) &= \frac{2\omega}{\pi} \int_0^m \frac{\text{Im } f^{(-)}(\omega') d\omega'}{(\omega'^2 - \omega^2)} . \end{aligned} \quad (46)$$

The pole expressions (46) are direct consequences of the basic dispersion relations (35) and (36). They are written in an integral form with the discrete pole contributions conveniently included as δ -function contributions to the integrand.

Fig. 21 Single-nucleon and multinucleon processes contributing to the transition: pion + ground state $\rightarrow$ nuclear excited states $|n\rangle$.

The position of the πN poles is close to $\omega = 0$ (see Fig. 12):

$$\omega_N^2 = \frac{m^4}{4M^2} \ll m^2 .$$

In the nuclear case (46) the principal strength of $\text{Im } f(\omega)$ is also concentrated close to $\omega = 0$ with $\omega_n^2 \ll m^2$, since nuclear excitation energies are of order $\epsilon_n \sim m^2/2M_N$ for pion absorption. The momentum of the absorbed pion is thus to good approximation $k_n^2 \simeq -m^2 = \text{const}$ for such states. On the scale of a pion mass, the nuclear poles may therefore in first approximation be considered to be degenerate with the nuclear ground state. The physical region $\omega \geq m$ therefore experiences the pole terms of Eq. (46) as effective poles to a good approximation:

$$f_{\text{poles}}^{(+)}(\omega) \simeq \frac{k^2}{m^2 \omega^2} \cdot \frac{2}{\pi} \int_0^m \omega' \text{Im } f^{(+)}(\omega') d\omega' \quad (47a)$$

$$f_{\text{poles}}^{(-)}(\omega) \simeq -\frac{1}{\omega} \frac{2}{\pi} \int_0^m \text{Im } f^{(-)}(\omega') d\omega' . \quad (47b)$$

The precision of this approximation is of order 1% if the nuclear excitation energies that dominate are of order $\lesssim 10$ MeV; it is still of order 10% even in the extreme case in which all the contribution is assumed to come from nuclear states of ≈ 45 MeV of excitation.

The characteristic contribution by a nuclear state $|n\rangle$ to the imaginary part of the amplitudes in Eq. (47) are related to the matrix elements for the pion source function

$$j_\pi(\vec{x}) = (\square - m^2) \phi(\vec{x})$$

by

$$\begin{aligned} \delta^{\alpha\beta} \text{Im } f^{(+)}(\omega) &= \frac{1}{4} \sum_{|\omega_n| < m} \left[\langle 0 | j_\pi^\alpha | n; \vec{k}_n \rangle \langle \vec{k}_n; n | j_\pi^\beta | 0 \rangle + \langle 0 | j_\pi^\beta | n; \vec{k}_n \rangle \langle \vec{k}_n; n | j_\pi^\alpha | 0 \rangle \right] \delta(\omega - \omega_n) \\ \frac{1}{2} [\tau^\alpha, \tau^\beta] \text{Im } f^{(-)}(\omega) &= \frac{1}{4} \sum_{|\omega_n| < m} \left[\langle 0 | j_\pi^\alpha | n; \vec{k}_n \rangle \langle \vec{k}_n; n | j_\pi^\beta | 0 \rangle - \langle 0 | j_\pi^\beta | n; \vec{k}_n \rangle \langle \vec{k}_n; n | j_\pi^\alpha | 0 \rangle \right] \delta(\omega - \omega_n) \end{aligned} \quad (48)$$

neglecting the small nucleon recoil. The momentum $k_n^2 = \omega_n^2 - m^2 \approx -m^2$. The general equation (48) follows from the reduction formalism, as described in many text books [Ref. 30, Eq. (4.36)].

In the spirit of our previous discussion we approximate the pion source function in Eq. (48) by a sum of nucleon single-particle operators using the non-relativistic limit for the nucleons:

$$j_\pi^\alpha(\vec{x}) \approx \frac{\sqrt{4\pi} f}{m} \sum_i (\vec{\sigma}_i \cdot \vec{k}) \tau_i^\alpha \delta(\vec{x} - \vec{x}_i) \equiv J^\alpha(\vec{x}) \quad (49)$$

The commutator of this single nucleon operator is local and proportional to the non-relativistic isovector mass-density $\rho^\gamma(\vec{x}) = \sum_i \tau_i^\gamma \delta(\vec{x} - \vec{x}_i)$ for the nucleus [the component $\rho^{(3)} = (\rho_p - \rho_n)$, the difference of neutron and proton densities]:

$$[J^\alpha(\vec{x}), J^\beta(\vec{x}')] = -4\pi f^2 \delta(\vec{x} - \vec{x}') \varepsilon_{\gamma}^{\alpha\beta} \sum_i \tau_i^\gamma \delta(\vec{x} - \vec{x}_i) \equiv -4\pi f^2 \delta(\vec{x} - \vec{x}') \varepsilon_{\gamma}^{\alpha\beta} \rho^\gamma(\vec{x}) . \quad (50)$$

If we therefore substitute Eq. (48) into Eq. (47) and apply closure of the nuclear states, we have [cf. Eq. (40)]^{*}:

$$\begin{aligned} \frac{[\tau^+, \tau^-]}{\pi} \int_0^m \text{Im } f^{(-)}(\omega') d\omega' &= \frac{1}{4\pi} \iint \langle 0 | [J^+(\vec{x}), J^-(\vec{x}')] | 0 \rangle e^{i\vec{k}(\vec{x}-\vec{x}')} d\vec{x} d\vec{x}' \\ &= f^2 \iint \delta(\vec{x} - \vec{x}') \rho^{(3)}(\vec{x}) d\vec{x} d\vec{x}' = f^2 (Z - N) \end{aligned} \quad (51)$$

for π^+ and π^- elastic scattering.

Consequently the antisymmetric pole term is:

$$f_{\text{pole}}^{(-)} \approx \frac{2 f^2}{\omega} \quad (52)$$

in agreement with the coherent pole sum and universally valid for all nuclei inside our approximations. The fundamental reason for this universality is that the commutator (50) is local, since the single nucleon terms are local. This eliminates the dependence on nuclear size. Note that multinucleon operators that are part of the general pion source function do not have this local property. Therefore they introduce an explicit dependence of the effective pole on more detailed nuclear properties.

The model (49) can also be applied to the symmetric pole term (47a). It clearly shows that this pole is related to time variations of the one-nucleon source $J^\alpha(\vec{x})$. We first note that the nuclear excitation energies $\varepsilon_n \approx \omega_n$, since the nuclear recoil energy $m^2/2 A M_n \ll m^2/2 M_n$, the recoil energy of a free nucleon. For simplicity of isospin algebra we will discuss elastic π^0 scattering, which occurs only by the symmetric amplitude.

^{*}) The case treated is a special case of the general sum rule 2 in Part A of Appendix to Section 4.

Consider now the time derivative of the source function

$$\frac{1}{i} \dot{J}^0(\vec{x}) = [H, J^0(\vec{x})] , \quad (53)$$

where $H = T + V$ is the full Hamiltonian for nucleons in the nucleus. The commutator of this time derivative with the source function has a ground-state expectation value which satisfies the following identity:

$$\begin{aligned} \frac{1}{i} \langle 0 | [J^0(\vec{x}), \dot{J}^0(\vec{x}')] | 0 \rangle &\equiv \langle 0 | [J^0(\vec{x}), [H, J^0(\vec{x}')]] | 0 \rangle \\ &\equiv \sum_n \epsilon_n \langle 0 | J^0(\vec{x}) | n \rangle \langle n | J^0(\vec{x}') | 0 \rangle \end{aligned} \quad (54)$$

as is seen simply by inserting a complete set $|n\rangle$ of nuclear states.

Substitution of Eqs. (48) and (49) into the pole term (47a) gives now (neglecting terms of order A^{-1})

$$\begin{aligned} f_{\text{poles}}^{(+)} &= \frac{k^2}{m^2 \omega^2} \frac{1}{\pi} \sum_n \epsilon_n \iint \langle 0 | J^0(\vec{x}) | n \rangle \langle n | J^0(\vec{x}') | 0 \rangle e^{i\vec{k}(\vec{x}-\vec{x}')} d\vec{x} d\vec{x}' \\ &= \frac{k^2}{m^2 \omega^2} \frac{1}{\pi} \iint \langle 0 | [J^0(\vec{x}), \dot{J}^0(\vec{x}')] | 0 \rangle e^{i\vec{k}(\vec{x}-\vec{x}')} d\vec{x} d\vec{x}' , \end{aligned} \quad (55)$$

where we used Eq. (54) in reverse.

From Eq. (55) we can heuristically see the following features of the symmetric pole:

- i) Since nucleons in the nucleus do not have large velocities, their positions change slowly with time. It is hence to be expected that the symmetric pole term is "small" in accordance with experiments.
- ii) Provided that the nuclear nucleons change their positions smoothly during nuclear interactions, the commutator should be nearly local in the sense that it only should contribute when $\vec{x}$ and $\vec{x}'$ are close to the position $\vec{x}_i$ of a nucleon. If so, one might expect the commutator to have the free nucleon value in this neighbourhood, and hence that the total commutator is just a coherent sum of single nucleon pole terms.

We prove these approximate results for the model (49) in Part A of Appendix to Section 4. We note, however, that a realistic nucleus has exchange forces, by which a proton changes non-locally in an interaction to a neutron and vice versa due to the elimination of the mesonic degrees of freedom. Because of this, the assumption of "smooth" nucleon trajectories is only approximate.

With these assumptions the integral over the commutator in Eq. (55) can be explicitly evaluated as a sum rule. We find from Eq. (A4.6) that

$$f_{\text{poles}}^{(+)}(\omega) = \frac{k^2}{\omega^2} f^2 \frac{A}{M_n} . \quad (56)$$

It is striking that this simple result, that the pole term is a coherent sum of poles, is obtained under quite general assumptions as a dynamical result both for $f^{(-)}$ and for $f^{(+)}$. The excited states of the nucleus play a crucial role in these relations: the sum rules will not even be approximately satisfied if only the ground state isospin multiplet is used to attempt a saturation. In the specific case of the symmetric amplitude, its smallness makes it possible that it may show important deviations from the coherent value although it still stays small. Some corrections come from nuclear exchange forces. By analogy to the photonuclear sum rule we expect this effect to be of order of 25% only, simply by invoking nuclear SU_4 symmetry: in the absence of the spin term, our sum rule is in all essentials exactly the classical photonuclear sum rule. A more important correction to both our sum rules will come from multinucleon terms in the source function. The effects of these are at present hard to evaluate.

To summarize: in the approximation that the dominant part of the nuclear source function for the pion field comes from individual nucleons, we find that the entire nuclear energy spectrum of excited states can be considered degenerate with the nuclear ground state as a giant supermultiplet. These nuclear states produce an effective nuclear pole. This pole is in this approximation universal for the antisymmetric pole [Eq. (40)]:

$$f_{\text{poles}}^{(-)}(\omega) = \frac{2f^2}{\omega}$$

independent of target size. For the symmetric term, the pole residue is again universal in the absence of nuclear exchange forces in the non-relativistic approximation with [Eq. (38)]

$$f_{\text{pole}}^{(+)}(\omega) = \frac{A}{M_n} f^2 \frac{k^2}{\omega^2} .$$

Its smallness reflects that nucleons are nearly static.

These sum rules involve many excited states in the nucleus and they reflect explicitly the dynamical nature of the nuclear pion field. Their saturation requires, in general, nuclear excitations of the order of $m^2/2M_n \approx 10$ MeV. The operators occurring in these sum rules are essentially Gamow-Teller matrix elements.

The sum rules are therefore closely related to β -decay sum rules, μ -capture sum rules, as well as magnetic sum rules in photo reactions. Present pion experiments indicate that the symmetric pole term is indeed small, and that the antisymmetric pole of ${}^9\text{Be}$ has $(f^2)_{\text{eff}} = 0.06 \pm 0.03$ consistent with universality.

* * *

A. NUCLEAR SUM RULES RELEVANT TO πX SCATTERING

We prove the following general sum rule for an interacting N-body system:

Sum rule 1: Consider a system of N particles with a Hamiltonian $H = T + V$, where the kinetic energy $T = \sum_i p_i^2/2m$. Consider two one-body operators:

$$\begin{aligned} R(\vec{x}) &= \sum_i r_i(\vec{x}) \delta(\vec{x} - \vec{x}_i) \\ S(\vec{x}) &= \sum_i s_i(\vec{x}) \delta(\vec{x} - \vec{x}_i) \end{aligned} \quad (A4.1)$$

for which neither $r_i(\vec{x})$ nor $s_i(\vec{x})$ depend on the position coordinate of the i^{th} particle. If the operator $S(\vec{x})$ commutes with V , i.e. if

$$[S(\vec{x}), V] = 0, \quad (A4.2)$$

then

$$\begin{aligned} \sum_n \epsilon_n \iint \langle 0 | R(\vec{x}) | n \rangle \langle n | S(\vec{x}') | 0 \rangle d\vec{x} d\vec{x}' &= \frac{1}{i} \iint \langle 0 | [R(\vec{x}), \dot{S}(\vec{x}')] | 0 \rangle d\vec{x} d\vec{x}' = \\ &= \frac{1}{m} \int \langle 0 | \sum_i \left[(\vec{\nabla} r_i(\vec{x})) \cdot (\vec{\nabla} s_i(\vec{x})) \right] \delta(\vec{x} - \vec{x}_i) | 0 \rangle d\vec{x}, \end{aligned}$$

which is the coherent sum of the contributions from the individual nucleons.

Proof:

$$\begin{aligned} \frac{1}{i} \iint \langle 0 | [R(\vec{x}), \dot{S}(\vec{x}')] | 0 \rangle d\vec{x} d\vec{x}' &= \iint \langle 0 | [R(\vec{x}), [H, S(\vec{x}')]] | 0 \rangle d\vec{x} d\vec{x}' \\ &= \iint \langle 0 | [R(\vec{x}), [T, S(\vec{x}')]] | 0 \rangle d\vec{x} d\vec{x}' \quad [\text{by Eq. (A4.2)}] \\ &= \frac{1}{2m} \iint \langle 0 | [R(\vec{x}), [\sum_i \vec{p}_i^2, S(\vec{x}')]] | 0 \rangle d\vec{x} d\vec{x}' \\ &= \frac{1}{m} \iint \langle 0 | \sum_i \left[R(\vec{x}), \left(\vec{p}_i \delta(\vec{x}' - \vec{x}_i) \right) \vec{p}_i \right] s_i(\vec{x}') | 0 \rangle d\vec{x} d\vec{x}' \\ &= \frac{(-)}{m} \iint \langle 0 | \sum_i r_i(\vec{x}) s_i(\vec{x}') \left(\vec{p}_i \delta(\vec{x} - \vec{x}_i) \right) \cdot \left(\vec{p}_i \delta(\vec{x}' - \vec{x}_i) \right) | 0 \rangle d\vec{x} d\vec{x}'. \end{aligned} \quad (A4.3)$$

We can now transfer $\vec{p}_i = -i\vec{\nabla}_i$ onto the variables $\vec{x}, \vec{x}'$ and partially integrate Eq. (A4.3)

$$\begin{aligned} \frac{1}{i} \iint \langle 0 | [R(\vec{x}), \dot{S}(\vec{x}')] | 0 \rangle d\vec{x} d\vec{x}' &= \frac{1}{m} \iint \langle 0 | \sum_i r_i(\vec{x}) s_i(\vec{x}') \left(\vec{\nabla} \delta(\vec{x} - \vec{x}_i) \right) \cdot \left(\vec{\nabla}' \delta(\vec{x}' - \vec{x}_i) \right) | 0 \rangle d\vec{x} d\vec{x}' \\ &= \frac{1}{m} \iint \langle 0 | \sum_i \left(\vec{\nabla} r_i(\vec{x}) \right) \cdot \left(\vec{\nabla}' s_i(\vec{x}') \right) \delta(\vec{x} - \vec{x}_i) | 0 \rangle \delta(\vec{x} - \vec{x}') d\vec{x} d\vec{x}' \\ &= \frac{1}{m} \int \langle 0 | \sum_i \left(\vec{\nabla} r_i(\vec{x}) \right) \cdot \left(\vec{\nabla} s_i(\vec{x}) \right) \delta(\vec{x} - \vec{x}_i) | 0 \rangle d\vec{x}. \quad \text{Q.E.D.} \end{aligned}$$

A very important special case is the following:

Sum rule 1': If the product $r_i(\vec{x})s_i(\vec{x}) = r(\vec{x})s(\vec{x})$ independent of the particle i , then:

$$\frac{1}{i} \iint \langle 0 | [R(\vec{x}), \dot{S}(\vec{x}')] | 0 \rangle d\vec{x} d\vec{x}' = \frac{1}{m} \int (\vec{\nabla} r(\vec{x}) \cdot \vec{\nabla} s(\vec{x})) \rho(\vec{x}) d\vec{x} , \quad (A4.4)$$

provided $[S(\vec{x}), V] = 0$. In this relation the density

$$\rho(\vec{x}) = \langle 0 | \sum_i \delta(\vec{x} - \vec{x}_i) | 0 \rangle .$$

Examples:

a) $r_i(\vec{x}) = s_i(\vec{x}) = z \tau_{iz} .$

Since $\tau_{iz}^2 = 1$, sum rule 1' applies with $r(\vec{x}) = s(\vec{x}) = z$:

$$\sum_n \epsilon_n |\langle 0 | \sum_i \tau_{iz} z_i | n \rangle|^2 = \frac{1}{m} \int (\vec{\nabla} \cdot z) (\vec{\nabla} \cdot z) \rho(\vec{x}) d\vec{x} = A/m$$

(Thomas-Reiche-Kuhn sum rule).

b) $r_i(\vec{x}) = s_i^*(\vec{x}) = e^{i\vec{q} \cdot \vec{x}} \tau_{iz}$ with $r(\vec{x}) = s^*(\vec{x}) = e^{i\vec{q} \cdot \vec{x}}$

$$\sum_n \epsilon_n |\langle 0 | \sum_i \tau_{iz} e^{i\vec{q} \cdot \vec{x}_i} | n \rangle|^2 = A \frac{q^2}{m} .$$

c) $r_i(\vec{x}) = s_i^*(\vec{x}) = \sigma_{iz} \tau_{iz} e^{i\vec{q} \cdot \vec{x}}$

$$\sum_n \epsilon_n |\langle 0 | \sum_i \tau_{iz} \sigma_{iz} e^{i\vec{q} \cdot \vec{x}_i} | n \rangle|^2 = A \frac{q^2}{m} .$$

We now apply these sum rules to Eq. (55) for the symmetric pole terms with

$$\begin{aligned} R(\vec{x}) &= e^{i\vec{k} \cdot \vec{x}} J^0(\vec{x}) \\ S(\vec{x}) &= e^{-i\vec{k} \cdot \vec{x}} J^0(\vec{x}) . \end{aligned} \quad (A4.5)$$

According to Eq. (49) we have

$$\begin{aligned} r_i(\vec{x}) &= \frac{\sqrt{4\pi f}}{m} (\vec{\sigma}_i \cdot \vec{k}) \tau_i^0 e^{i\vec{k} \cdot \vec{x}} ; \\ s_i(\vec{x}) &= \frac{\sqrt{4\pi f}}{m} (\vec{\sigma}_i \cdot \vec{k}) \tau_i^0 e^{-i\vec{k} \cdot \vec{x}} \end{aligned}$$

with $\vec{k}^2 = -m^2$. The operators are thus essentially those of example (c) above, so that the sum rule in the absence of exchange forces reads:

$$\begin{aligned} f_{\text{poles}}^{(+)}(\omega) &= \frac{k^2}{m^2 \omega^2} \frac{1}{\pi i} \iint \langle 0 | [J^0(\vec{x}), J^0(\vec{x}')] | 0 \rangle e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} d\vec{x} d\vec{x}' \\ &= \frac{k^2}{\omega^2} f^2 \frac{A}{M_n} . \end{aligned} \quad (A4.6)$$

Sum rule 2: Consider in an interacting N particle system the operators $R(\vec{x})$ and $S(\vec{x})$ of Eq. (A4.1), which are sums of one-body operators. Then

$$\sum_n \iint [\langle 0 | R(\vec{x}) | n \rangle \langle n | S(\vec{x}') | 0 \rangle - \langle 0 | S(\vec{x}) | n \rangle \langle n | R(\vec{x}') | 0 \rangle] d\vec{x} d\vec{x}' = \langle 0 | \sum_i [r_i(\vec{x}_i), s_i(\vec{x}_i)] | 0 \rangle \quad (A4.7)$$

= coherent sum of contributions from the individual particles.

Proof:

By closure, $\sum_n$ in Eq. (A4.7) is the commutator

$$\begin{aligned} \iint \langle 0 | [R(\vec{x}), S(\vec{x}')] | 0 \rangle d\vec{x} d\vec{x}' &= \iint \delta(\vec{x} - \vec{x}') \langle 0 | \sum_i [r_i(\vec{x}), s_i(\vec{x}')] \delta(\vec{x} - \vec{x}_i) | 0 \rangle d\vec{x} d\vec{x}' \\ &= \langle 0 | \sum_i [r_i(\vec{x}_i), s_i(\vec{x}_i)] | 0 \rangle. \end{aligned} \quad (A4.8)$$

Examples of sum rule 2 are:

a) $r_i(\vec{x}) = z\tau_i^+; s_i(\vec{x}) = z\tau_i^-$

$$\sum_n \{ \langle 0 | \sum_i \tau_i^+ z_i | n \rangle \langle n | \sum_j \tau_j^- z_j | 0 \rangle - \text{c.c.} \} = \langle 0 | \sum_i z_i^2 \tau_{iz} | 0 \rangle = Z \langle 0 | z_p^2 | 0 \rangle - N \langle 0 | z_n^2 | 0 \rangle;$$

b) $r_i(\vec{x}) = e^{i\vec{q} \cdot \vec{x}} \tau_i^+; s_i(\vec{x}) = e^{-i\vec{q} \cdot \vec{x}} \tau_i^- = r_i^*(\vec{x})$

$$\sum_n \{ \langle 0 | \sum_i e^{i\vec{q} \cdot \vec{x}_i} \tau_i^+ | n \rangle \langle n | \sum_j e^{-i\vec{q} \cdot \vec{x}_j} \tau_j^- | 0 \rangle - \text{c.c.} \} = (Z - N);$$

c) $r_i(\vec{x}) = \sigma_{iz} e^{i\vec{q} \cdot \vec{x}} \tau_i^+; s_i(\vec{x}) = \sigma_{iz} e^{-i\vec{q} \cdot \vec{x}} \tau_i^- = r_i^*$

$$\sum_n \{ \langle 0 | \sum_i \sigma_{iz} e^{i\vec{q} \cdot \vec{x}_i} \tau_i^+ | n \rangle \langle n | \sum_j \sigma_{jz} e^{-i\vec{q} \cdot \vec{x}_j} \tau_j^- | 0 \rangle - \text{c.c.} \} = (Z - N).$$

B. SCATTERING LENGTHS AND MESIC ATOMS

The electromagnetic levels of mesic atoms are disturbed by the strong π -nuclear interaction, resulting in a shift of the Bohr energy levels and in line broadening due to strong absorption processes. These effects, if measured, can be transposed into low-energy pion-nuclear scattering parameters (scattering lengths) and vice versa. For our present needs it is sufficient to use the leading linear term³¹⁾ connecting the complex energy shifts $\Delta E = \text{Re } \Delta E + i(\Gamma/2)$ of the Bohr level E to the s - and p -wave scattering lengths a_0 and a_1 (Z = charge, α = fine structure constant):

$$\begin{aligned} a_0 &\approx \frac{1}{4Z\alpha} \frac{\Delta E_{1s}}{E_{1s}} m_\pi^{-1} \\ a_1 &\approx \frac{2}{3(Z\alpha)^3} \frac{\Delta E_{2p}}{E_{2p}} m_\pi^{-3}. \end{aligned} \quad (A4.9)$$

From the mesic data of Backenstoss et al.³²⁾, the s -wave $\pi^{12}\text{C}$ scattering length is

$$a_0 = [(-0.26 \pm 0.03) + i(0.063 \pm 0.006)] m_\pi^{-1}. \quad (A4.10a)$$

For the p-wave only the imaginary part can be deduced³³⁾ at present from X-ray attenuation. As $\Gamma_{2p} = 0.6 \pm 0.2$ eV, we have

$$\text{Im } a_1 = (0.07 \pm 0.2) m_\pi^{-3} . \quad (\text{A4.10b})$$

Similar information exists for other light nuclei.

* * *

5. NUCLEON-NUCLEUS FORWARD SCATTERING

While the fundamental dispersion relations for pion and nucleon scattering are formally similar, in practice there are profound differences. At non-relativistic energies the nucleon-nuclear interaction exhibits a wealth of structure. A reflection of this is the many bound states that typically occur, as well as a complicated resonance structure in nucleon-nuclear scattering at low energies. It is quite well known that the elastic scattering is on the average well represented by absorptive potentials with shapes closely related to the nuclear matter distribution. These important non-relativistic features of the nucleon-nuclear interaction do not in any direct way refer to the mesonic degrees of freedom of the nucleus, which are implicit in the effective interaction of the system. The relativistic dispersion relation, which explicitly involves mesons and antinucleons, must be able to yield these non-relativistic aspects as a special case. The relativistic nature of the dispersion relation is by itself of great interest, since nuclei can seldom be discussed in a relativistic framework. The particular role that mesons play is illustrated by a comparison with the potential dispersion relations of Section 3. For a potential model the total cross-section goes to zero at high energy. The potential is, however, incapable of describing meson production and meson exchange. Above threshold for meson production the total cross-section has a very important component from these channels, which makes it a constant at high energies. Mesons and antiparticles also enter into the dispersion relation, by the relation between antinucleon amplitudes and the scattering amplitude at negative frequencies (crossing). Since the singularities associated with physical antinucleon scattering are very far from the physical region, the true nature of their contributions becomes apparent only when the amplitudes are studied over energy intervals of several hundreds of MeV. Although these effects are of great interest for the structure of actual interactions, the proper experimental material for detailed studies is at present very scarce.

In the present section we will first give the dispersion relations in an appropriate form. Then we will briefly discuss the qualitative behaviour of total cross-sections and forward scattering amplitudes as known at present for light elements. Some qualitative estimates of relativistic effects in low-energy scattering are given. We discuss in some detail numerical applications to ${}^4\text{He}$ and ${}^{12}\text{C}$ scattering, which emphasize the enormous role that exchange amplitudes play in the low-energy region, and we comment on the significance of this for nuclear reaction theory. Further we give a determination of the volume integral of the equivalent potential.

5.1 The nucleon-nuclear dispersion relation

For simplicity, we will discuss only nucleon scattering from a target of spin and isospin 0 (examples ${}^4\text{He}$, ${}^{12}\text{C}$, ${}^{16}\text{O}$, ...), since there is then only one forward scattering amplitude. The unsubtracted dispersion relation (2) is then^{*)}:

$$\text{Re } f(\omega) = \sum_{\beta} \frac{r_{\beta}}{\omega_{\beta} - \omega} + \frac{1}{\pi} \left[\int_m^{\infty} + \int_{-\infty}^{\omega_b} \right] \frac{\text{Im } f(\omega')}{\omega' - \omega} d\omega' , \quad (57)$$

*) Note that the incident particle m is the nucleon in this section.

where r_β are the pole residues. The branch point ω_b is below the scattering threshold $\omega = m$ by essentially the nucleon binding energy. From the crossing relation (4) we have that $\text{Im } f(-\omega) = -\text{Im } \bar{f}(\omega)$.

We can therefore identically rewrite Eq. (57) introducing the antinucleon amplitude in such a way that the second term closely resembles the symmetric π -nucleon dispersion integral ^{*)}

$$\begin{aligned} \text{Re } f(\omega) = & \sum_{\beta} \frac{r_{\beta}}{(\omega_{\beta} - \omega)} + \frac{1}{\pi} \int_m^{\infty} \text{Im } f(\omega') \left[\frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right] d\omega' \\ & + \frac{1}{\pi} \int_{-\omega_b}^m \frac{\text{Im } \bar{f}(\omega')}{(\omega' + \omega)} d\omega' + \frac{1}{\pi} \int_m^{\infty} \frac{[\text{Im } \bar{f}(\omega') - \text{Im } f(\omega')]}{(\omega' + \omega)} d\omega' , \end{aligned} \quad (58)$$

where the arguments of $\text{Im } \bar{f}$ are energies in the exchange channel.

We now introduce total cross-sections in Eq. (58) by the optical theorem (5) and subtract the relation at threshold. With the kinetic energy $E = \omega - m$ and the momentum $k^2 = \omega^2 - m^2$ introduced explicitly, we finally have

$$\begin{aligned} \text{Re } f(E) = & f(0) + \sum_{\beta} \frac{E r_{\beta}}{(E_{\beta} - E)E_{\beta}} + \frac{k^2}{2\pi^2} \int_0^{\infty} \frac{\sigma(k') dk'}{(k'^2 - k^2)} \\ & + \frac{E}{\pi} \int_{-2m}^{E_b} \frac{\text{Im } f(E') dE'}{(E' - E)E'} - \frac{E}{4\pi^2} \int_0^{\infty} k' dE' \frac{\bar{\sigma}(E') - \sigma(E')}{(E' + 2m)(E' + E + 2m)} . \end{aligned} \quad (59)$$

Note that by the Pomeranchuk theorem $[\bar{\sigma}(E) - \sigma(E)] \rightarrow 0$ for $E \rightarrow \infty$ so that the last integral in Eq. (59) is convergent.

5.2 Qualitative behaviour of cross-sections and amplitudes

The typical behaviour of the total cross-section with energy for lighter nuclei (not necessarily of $J = I = 0$) is illustrated in Figs. 22 and 23. The first 10 to 20 MeV above threshold (Fig. 22) exhibit high average cross-sections of detailed resonance structure. The cross-section then falls off rapidly (Fig. 23) by over a magnitude for the lightest elements, by less for heavier elements (a factor of 4 for ^{16}O), to a minimum at $E \sim 200$ MeV where the cross-section is essentially geometrical. At approximately 200 MeV, the mesonic degrees of freedom begin to manifest themselves. Meson production sets in and produces a slightly increasing cross-section with a very broad maximum at ~ 1600 MeV. At still higher energies there is seemingly some broad structure, and the cross-section falls off very slowly approaching (presumably) a constant asymptotic value. We note, in particular, that the conventional low-energy nuclear physics occurs in the low-energy "spike" of the cross-section, with $E \sim 200$ MeV as the physical separation point between the non-relativistic

^{*)} We could, in principle, have used the crossing even and odd amplitudes $f^{(\pm)} = \frac{1}{2}[f \pm \bar{f}]$ in the dispersion relations, as for π -nuclear scattering. Since little is known about $\bar{f}$, the relation (58) is in practice the most useful one.

Fig. 22 Typical examples of total cross-sections of neutrons on nuclei at low energies. The data for ^4He and ^{12}C are from Stehn et al.³⁴).

Fig. 23 High-energy behaviour of nX total cross-sections. The solid lines mean abundant data³⁴⁾, whereas an eye-guiding dashed line is drawn through the data from the compilation of Barashenkov et al.³⁵⁾.

and relativistic régime. The real amplitude has also a characteristic behaviour (cf. Figs. 27 and 30 below). In the low-energy region $E < 10$ MeV, its typical value is of the order of the nuclear radius, but strong local deviations occur due to resonances. Its sign is typically negative. At energies around 15 MeV the real amplitude becomes positive with a very flat maximum at about 100 MeV. It then falls off slowly with energy and crosses the real axis again³⁶⁾ at about 400 MeV and presumably stays negative. The NN amplitude averaged over spin and isospin has a similar behaviour³⁷⁾, changing sign from positive to negative in the same energy region. The origin of this effect is mainly due to vector meson exchange. We must therefore associate this behaviour with mesonic effects on nuclei too. For energies above 150 MeV, only fragmentary experimental information exists for the real amplitude. Reliable qualitative information on its expected behaviour at higher energies can be obtained from the NN amplitudes via multiple scattering theories.

Little is known about antinucleon cross-sections on nuclei. The main information below 1 GeV is that the ratios of antiproton to proton cross-sections for several elements³⁸⁾ is 1.4 around 500 MeV and about 2.5 at 140 MeV³⁹⁾. Further, since antinucleons are absorbed by nuclei at threshold, their cross-sections must vary like v^{-1} for low velocities of the antinucleon (but $\text{Im } \tilde{f}(0) = \text{const}$).

5.3 Unphysical region of dispersion relation

In addition to the physical scattering of nucleons and antinucleons, the dispersion relation (59) has several types of contributions in the unphysical region (cf. the discussion at the end of Section 2). There are first the bound states β in the direct channel, which are simple "compound states" formed by the target and the incident nucleon (see Fig. 5). Apart from these pole singularities, direct channel unphysical cuts rarely occur, as the nuclear binding of light nuclei is mostly not strong enough to produce a two-particle system of mass less than $m + M$. In particular, such cuts do not occur for ${}^4\text{He}$ and ${}^{12}\text{C}$. Secondly there are poles and cuts from the exchange or antiparticle channel with two types of contributions:

Fig. 24 "Non-relativistic" exchange contribution to nX scattering [diagram (a)] and the equivalent antiparticle channel direct intermediate state [diagram (b)].

- i) The non-relativistic exchange contributions are illustrated by Fig. 24. Diagram (a) can be considered as a "pick-up" reaction of the projectile in which one nucleon of the target continues and the rest reforms the target particle with the projectile. The exchanged system X^* has $A-1$ nucleons and is either a single bound system (excited or not) leading to poles, or a continuum many-particle state leading to cuts. It is obvious that a large number of states can contribute to diagram 24b or equivalently to 24a. In practice, the complicated contribution of the cut is usually simulated by a number of narrow resonances or effective poles.
- ii) The second type of contributions from the unphysical region of the crossed channel is associated with mesonic processes as illustrated in Fig. 25; the dashed line denotes any kind of meson:

Fig. 25 Intermediate states involving mesons leading to the "relativistic" exchange contribution (a), and the equivalent antiparticle channel pole (b).

These are basically relativistic contributions. Their magnitude can be estimated from models, but we will not do so here.

5.4 Estimate of physical antinucleon effects

It is possible to obtain model-independent estimates of the influence of antinucleon reactions on the nucleon-nuclear scattering in the non-relativistic region. The effect of the antinucleon scattering enters into the nucleon amplitude by the last integral in Eq. (59) which we denote by $\delta f_{\bar{N}}$. The asymptotic nucleon and antinucleon cross-section tend to the same constant. Furthermore, we have the already quoted experimental information³⁸⁾

$$\left(\frac{\bar{\sigma} - \sigma}{\sigma} \right)_{500 \text{ MeV}} = \alpha \approx 0.4 .$$

We will make the following crude estimate: we assume a step function for the cross-section difference

$$(\bar{\sigma} - \sigma) = \frac{m}{k} \alpha \sigma_{500 \text{ MeV}} \quad \text{for} \quad E \leq m \quad \text{and} \quad (\bar{\sigma} - \sigma) = 0 \quad \text{for} \quad E > m . \quad (60)$$

This has the right k^{-1} threshold behaviour. Furthermore, $m/k \sim 1$ near $E = 500 \text{ MeV}$, so that Eq. (60) satisfies experiment there. Neglecting E' against $2m$ in the denominator of the integrand, we obtain immediately

$$\delta f_{\bar{N}} \approx - \frac{E}{4\pi^2} \frac{\alpha \sigma_{500 \text{ MeV}} m^2}{(2m)(E + 2m)} , \quad (61)$$

which can be simplified observing that $\sigma_{500 \text{ MeV}} \approx 2\pi R^2$ in terms of the nuclear radius R .

Hence

$$\delta f_{\bar{N}} \approx - \frac{E}{E + 2m} \frac{\alpha R^2 m}{4\pi} \sim - 1.5 \times 10^{-4} A^{\frac{2}{3}} E (\text{fermi}) , \quad (62)$$

where E is in MeV. The last expression is valid for $E \ll 2m$ and $A =$ nucleon number.

The effect of antinucleons shows up mainly in the energy variation of the amplitude. For ${}^4\text{He}$ the contribution (62) is of the order of $-7.5 \times 10^{-4} E (\text{fermi})$, which even for

$E = 100$ MeV is still less than 2% of the observed amplitude. As expected, we therefore find that the physical region of the antinucleon channel has negligible effects in non-relativistic nucleon-nuclear scattering.

5.5 Evaluation of forward dispersion relations

5.5.1 $n^4\text{He}$

For the case of $n^4\text{He}$ scattering^{2b)} the situation is particularly simple. There are no direct channel poles or resonances below threshold and there is only one exchange pole, due to the exchange of ^3He at a position [(Eq. (11))]

$$E_0 = -15.4 \text{ MeV} . \quad (63)$$

The residue r_0 must be positive, the relative parity between (^4He , $n^3\text{He}$) being positive. The dependence of sign with parity is shown in Part A of Appendix to Section 5 in the context of Lagrangian formalism.

The next close singularity is a branch point at

$$E_b = -19.4 \text{ MeV} \quad (64)$$

corresponding to $p + d$ exchange. Three-particle exchange (ppn) begins at -21.4 MeV, whereas ($^3\text{He}\pi^0$) starts at -120 MeV, etc. The corresponding spectrum of the imaginary part in Eq. (57) is shown in Fig. 26:

Fig. 26 Structure of singularities in the kinetic energy E for $n^4\text{He}$ scattering.

The total cross-sections used in the evaluation of Eq. (59) have been previously displayed in Figs. 22 and 23. We then make the natural test for the importance of the unphysical region evaluating Eq. (59) without poles and cuts [second and fourth terms in Eq. (59)]. As has been shown, the last term $\propto (\bar{\sigma} - \sigma)$ is negligible. The real part from the physical region thus obtained is shown in Fig. 27, solid line. Fortunately enough, there exist accurate $n^4\text{He}$ phase shifts⁴⁰⁾ for $0 < E < 50$ MeV. The corresponding independently determined real parts^{*}) are shown as full points connected by a dashed line to guide the eye. It is quite evident that tremendous contributions from the unphysical region are missing. This is in sharp contrast to the $\pi^4\text{He}$ case. The nature of these contributions is seen by plotting in Fig. 28 the difference

$$\Delta = (\text{Re } f_{\text{phase-shifts}} - \text{Re } f_{\text{physical region}}) . \quad (65)$$

If the single ^3He pole is the dominant contribution, the quantity E/Δ should be linear. Indeed the dashed line in Fig. 28 is remarkably linear over a large energy region. The cor-

*) We thank Mr. W. Klein for numerical help in this context.

Fig. 27 The real part of the $n^4\text{He}$ amplitude. The solid line is the contribution to the dispersion prediction of Eq. (59) from the physical region only. The solid "experimental" points are obtained from phase shifts⁴⁰). The wiggle at 22 MeV is a resonance (omitted in the dispersion calculation), whereas the four points at 40 MeV show the inherent error from using different sets of phase shifts.

responding effective pole parameters can be read off directly:

$$E_0^{\text{eff}} = -10 \text{ MeV} ; \quad r_0^{\text{eff}} = +1.3 ; \quad \left(\frac{r_0}{-E_0} \right)^{\text{eff}} = 25 \text{ f} . \quad (66)$$

The pole position is not at $E_0 = -15 \text{ MeV}$. This is astonishing since no singularity occurs between the ^3He pole and the scattering region (Fig. 26). Therefore there must exist contributions from the cut having opposite sign, i.e. negative parity. This is not a probable effect of the nearby pd cut. At low pd relative momentum, the pd system should be s -wave and hence of same parity as ^3He . A possible and natural candidate for the required negative contribution is the $\pi^3\text{He}$ cut. Introducing a second effective pole at $E_1 = -120 \text{ MeV}$ the $\pi^3\text{He}$ branch point, a two-pole expression for Δ is:

$$\Delta = \frac{r_0 E}{(E_0 - E)E_0} + \frac{r_1 E}{(E_1 - E)E_0} . \quad (67)$$

In Fig. 29 we show the quantity

$$L = \Delta(E_0 - E)(E_1 - E)/E , \quad E_0 = -15.4 \text{ MeV} , \quad (68)$$

Fig. 28 The contribution Δ [Eq. (65)] from the unphysical region and test for one-pole hypothesis (linearity of E/Δ). Solid points as in Fig. 27.

which is linear for $E_1 = -120$ MeV, showing that Eq. (67) is still an adequate approximation, whereas for $E_1 = -250$ MeV, L develops curvature and the fit is no longer acceptable. The fits improve for E_1 going closer to E_0 and are excellent over a broad energy region of E_1 . Limiting values for r_0 and r_1 are given in the figure caption. The values for r_0 are higher than the value (66) from the one-pole expression. This is supported by a recent value⁴¹⁾ $r_0 \approx 2.3$ from (p,2p) experiments.

To the extent that the ^3He pole can be disentangled, the information on r_0 is directly related to the tail of the nucleon wave function for the α -particle, providing an independent although not yet practical test on r_0 [the relation to $\phi_{^4\text{He}}(x \rightarrow \infty)$ is given in Part B of Appendix to Section 5].

5.5.2 $n^{12}\text{C}$

We briefly investigate the corresponding situation for ^{12}C for which marginally useful experimental information is available. The input data for total cross-sections are given in Figs. 22 and 23. For neutron energies below 15 MeV there are a number of nuclear resonances. The $\text{Re } f$ therefore varies very rapidly with energy in this region. The experimental information on this quantity is sparse and in part inconsistent. Apart from the accurately known scattering length, the phase-shift analysis below 4 MeV⁴²⁾ is not very helpful. At higher energies, forward differential cross-sections⁴³⁾ in principle give $|\text{Re } f|$. Unfortunately, the optical theorem is violated at a number of energies below 15 MeV, in some cases grossly. The corresponding $\text{Re } f$ would be imaginary! As a compromise we have used a calculated $\text{Re } f$ obtained from an optical model potential⁴⁴⁾. These should on the average be a reasonable approximation to the actual value.

Fig. 29 Tests for the significance of a two-pole approximation for Δ . The quantity L is defined in Eq. (68). E_0 is fixed at -15.4 MeV (${}^3\text{He}$ pole) whereas E_1 is varied. For $E_1 = -120$ MeV the residues are $r_0 = 3.1$, $r_1 = -24.5$, whereas for $E_1 = -20$ MeV we obtain $r_0 = 6.6$, $r_1 = -6.3$.

Table 2

Direct poles below threshold in $n^{12}\text{C}$ scattering

κR	${}^{13}\text{C}$ state, $E_{\text{c.m.}}$ MeV	θ^2	r	$-r/E_{\text{c.m.}}$ (fermi)
2.12	$(\frac{1}{2}^-) \ell = 1, -4.95$	0.042	-0.3	-12
1.27	$(\frac{1}{2}^+) \ell = 0, -1.86$	0.19	0.18	20
1.05	$(\frac{3}{2}^-) \ell = 1, -1.27$	0.013	-0.006	-0.5
0.98	$(\frac{5}{2}^+) \ell = 2, -1.10$	0.073	0.004	0.7

Each pole $r/(E_{\text{c.m.}} - E)$ contributes $(-r/E_{\text{c.m.}})(E/E - E_{\text{c.m.}})$ to the subtracted amplitude (or asymptotically $-r/E_{\text{c.m.}}$). The residue is estimated by

$$r = (-1)^\ell (2\ell + 1) \gamma^2 R \left| \kappa R h_\ell^{(1)}(i\kappa R) \right|^{-2}$$

where $h_\ell^{(1)}$ is the spherical Hankel function of the first kind, $\kappa^2 = 2m_{\text{red.}} E_{\text{c.m.}}$ and $\gamma^2 = 3\theta^2/(2m_{\text{red.}} R^2)$ is the reduced width. A typical set of values for θ^2 has been taken from Ref. 47, p. 576 (with $R = 4.5$ fermi) (see also Ref. 48); the uncertainties are of order 50%. [Our simple estimate of Eq. (29) would give the right order $\theta^2 = 0.35$ for the $\frac{1}{2}^+$ state.] Note that large cancellations occur due to alternating parity.

The real amplitude obtained in this way is given in Fig. 30 where the physical dispersion integral is also shown as a solid curve. The empirical contribution Δ from the unphysical region is shown in Fig. 31. Since there are four negative energy resonances in ^{13}C , these have to be subtracted from Δ in order to establish the importance of the exchange amplitude. The properties of these states are given in Table 2. The main contributors are the $(2s_{1/2})$ and the $(1p_{1/2})$ single-particle states. These two poles contribute about 8 fermi to Δ . Since Δ at 50 MeV is about +25 fermis, a large amount must be ascribed to positive parity exchange as in $n^4\text{He}$ scattering. The principle candidate is the $(1s_{1/2})^{-1}$ hole state seen in $(p,2p)$ experiments as bound by 34 MeV, and corresponding to an excitation energy of about 17 MeV in ^{11}C . If the exchange contribution from this state could be regarded as arising from potential scattering, one would be led to the conclusion that the radius of the state (classical turning-point) is considerably smaller than the r.m.s. for ^{12}C and probably of the order of the ^4He r.m.s., since the dependence of the residue with κR is exponential. This is consistent with the broad momentum distribution seen for this state in $(e,e'p)$ and $(p,2p)$ experiments^{45,46}. An alternative possibility is an anomalously small reduced width. Unfortunately, the uncertainty of present experiments is so large that a more quantitative investigation of this point is not yet possible. A direct experimental determination of real parts for $E > 15$ MeV, where the variation with energy is smooth, either by phase shifts or by $p^{12}\text{C}$ Coulomb interference is badly needed.

5.6 Consequences of the results for $n^4\text{He}$ and $n^{12}\text{C}$

The previous discussion for $(n\alpha)$ and $(n^{12}\text{C})$ scattering emphasizes the enormous importance of non-relativistic exchange scattering in nuclear reactions. Since the individual properties of these nuclei were not essential for this result, we conclude that it is a general feature which extends also to elastic scattering of deuterons, ^3He , T , α -particles, etc., on nuclei.

We noticed previously that there exists a sum rule for potential scattering [Eq. (25)]. This sum rule links the volume integral of the potential to a sum of pole residues and the integrated cross-section. Since the asymptotic cross-section relativistically becomes constant instead of going to zero as for a potential, the sum rule can only be approximately valid in the relativistic case. If for a moment we imagine that all non-relativistic nuclear properties are kept constant, but that the meson masses are made very large, all the mesonic effects in forward dispersion relations can be absorbed into the subtraction constant. Further, there will be a natural separation between non-relativistic and mesonic phenomena in the total cross-section, since this quantity will decrease with energy until the high energy at which production processes set in. This suggests that the integration limits for the imaginary amplitude should be taken about a π -meson mass around the physical region for nucleon scattering; this is indeed the approximate energy at which $\sigma(E)$ has a minimum. Provided the non-relativistic pole residues have been determined, we then have from Eqs. (25) and (57)

$$\frac{m}{2\pi} \int U \, d\vec{x} = -\text{Re } f(0) + \frac{1}{2\pi^2} \int_0^{E_{\min}} \sigma(k') \, dk' + \sum_{\beta} \frac{r_{\beta}}{E_{\beta}} . \quad (69)$$

If this is applied to $(n^4\text{He})$ scattering as previously discussed with the effective pole $E_0 = -10$ MeV as observed, we find

$$\frac{m}{2\pi} \int U d\vec{x} = - 8.8 f . \quad (70)$$

This is in good agreement with the value $-9.8 f$ found by Satchler et al.⁴⁹⁾ in a phenomenological description of $n^4\text{He}$ differential scattering at low energy. This relation indicates that the relativistic and mesonic aspects of the nuclear interaction can be absorbed into a subtraction term which is a constraint on the equivalent potential. For heavier elements, the sum rule is probably not more than a useful constraint between pole residues and the integrated cross-section up to meson production threshold.

Finally, it is interesting to note that the general forward dispersion relation we have considered, also throws new light on nuclear reaction theories for non-relativistic energies. These theories use a local dispersion structure of the amplitude. In addition, they contain a background amplitude which varies slowly with energy. This term is usually ascribed to "far away direct resonances".

Fig. 30 $\text{Re } f$ for $n^{12}\text{C}$ scattering versus kinetic energy. Notation as in Fig. 27. Solid line: dispersion prediction from physical region. The full dots are from phase shifts⁴²⁾, the circles are from forward cross-sections⁴³⁾ [cf. Eq. (7)] with signs guessed by continuity; the dashed line is an optical model fit⁴⁴⁾ to differential scattering.

Fig. 31 The contributions Δ from the unphysical region for $n^{12}\text{C}$ scattering [cf. Eq. (65)]. Notation as in Fig. 30. The strong scattering of points below 5 MeV is almost certainly due to inconsistencies in the experimental input in the region of nuclear resonances (great sensitivity to energy calibration, etc.).

Our discussion makes it quite clear that the background amplitude has, in addition to this origin, a very important non-relativistic component from exchange reactions which are not to be associated with direct resonance poles in the amplitude. Further, the relativistic contributions, although smaller, are far from negligible in these background terms. An alternative to the simple way of accounting for these background terms by a phenomenological potential is therefore to regard non-relativistic reaction theories as subtracted dispersion relations.

* * *

Appendix to Section 5

A. SIGN OF RESIDUES IN LAGRANGIAN THEORY

In Section 5 we exploited the fact that the residue of a pole in the unphysical region has a definite sign⁵⁰⁾ depending on the relative parity of the particles involved^{*)}. For the particular case of the ${}^3\text{He}$ exchange pole in $n{}^4\text{He}$ scattering, we shall illustrate this relation by graph techniques^{**)}. The result will be quite general for any spin $\frac{1}{2}$ pole, in spin 0-spin $\frac{1}{2}$ scattering. The residue r turns out to be the product of a Lagrangian coupling constant squared and a known function of the particle masses which can have either sign.

Consider the graph, Fig. A5.1

Fig. A5.1 Notation for ${}^3\text{He}$ exchange in $n{}^4\text{He}$ scattering.

The simplest Lagrangian^{***)} to describe the vertices involved is

$$g \bar{\psi}_{3\text{He}} \psi_n \phi_\alpha^* + \text{c.c.}, \quad (\text{A5.1})$$

where $\bar{\psi}_{3\text{He}}$ denotes the antiparticle. Working in the $\bar{s}$ exchange channel $n\bar{4}\text{He} \rightarrow \bar{4}\text{He} n$ and in the lab. system of M , the basic expression^{†)} for the spin averaged amplitude is simply

$$\bar{f}(\bar{\omega}) = - \frac{1}{8\pi M} \frac{g^2}{\bar{s} - M_3^2} \frac{1}{2} \text{tr} (\not{k} + m)(\not{q} - M_3) \quad (\text{A5.2})$$

where $\not{k} = k^\mu \gamma_\mu \equiv k \cdot \gamma$ and conventionally (in the forward direction)

$$\begin{aligned} \bar{s} = \bar{q}^2 &= (k + \bar{p})^2 = (k - p)^2 = -s + 2m^2 + 2M^2 \\ s &= (k + p)^2 = m^2 + M^2 + 2\omega M q; \quad \bar{\omega} = -\omega. \end{aligned} \quad (\text{A5.3})$$

*) For the case of potential scattering, the sign of r goes with parity (or angular momentum) as stated in Eq. (23), provided the pole is within the domain of holomorphy of the S -matrix⁵¹⁾.

**) We thank Professors D. Amati and H. Leutwyler for a discussion of this point.

***) Note that introduction of derivative couplings would not affect the sign question.

†) The conventions are those from Feynman⁵²⁾.

As the trace is equal to $4m^2 - 4k \cdot p - 4m M_3$, the amplitude reduces to

$$\bar{f}(\bar{\omega}) = f(\omega) = \frac{m}{8\pi M^2} \frac{g^2}{\omega - \omega_0} \left(m - \frac{M}{m} \omega_0 - M_3 \right) = \frac{r(M_3, M, m)}{\omega - \omega_0} . \quad (A5.4)$$

Here we have used the crossing relation (4) and introduced the pole position ω_0 in the s-channel lab. energy ω . The value of ω_0 is obtained from

$$\begin{aligned} \bar{s} - M_3^2 &= -2M(\omega - \omega_0) = 0 \\ \text{with } \omega_0 &= \frac{m^2 + M^2 - M_3^2}{2M} . \end{aligned} \quad (A5.5)$$

The residue $r(M_3)$ of Eq. (A5.4) is simplified by introducing the explicit expression (A5.5) for ω_0 into $r(M_3)$

$$r(M_3) = (16\pi M^2)^{-1} g^2 [(m - M_3)^2 - M^2] . \quad (A5.4')$$

The sign of this residue is determined by the parabolic mass factor, which has real zeros at $M_3 = m \pm M$ as shown in Fig. A5.2.

Fig. A5.2 The sign of the residue for a u-channel pole in the unphysical region $(M - m) < M < m + M$. The function $r(M_3)$ applies for ${}^3\text{He}(\frac{1}{2}^+)$ exchange, whereas $r(-M_3)$ holds for $(\frac{1}{2}^-)$ exchange.

The correctness of the over-all sign in Eq. (A5.4) is easily checked by moving our exchange pole into the physical region for the exchange pole, i.e. $M_3 > m + M$, producing a resonance pole at $\bar{\omega} = \omega_R$ of the form

$$\bar{f}_R(\bar{\omega}) = \frac{r}{\omega_R - \bar{\omega}} = \frac{r}{\omega_R + \omega}$$

(in zero width approximation). Here the residue r must be > 0 by unitarity, for any relative parity. Indeed, Eq. (A5.4) has $r(\pm M_3)$ positive for $M_3 > m + M$ (note that $\omega_R = -\omega_0$). In the unphysical region $r(M_3)$ becomes negative (see Fig. A5.2), whereas $r(-M_3)$ stays positive.

Actually the basic expression (A5.2) shows that the sign of M_3 in the unphysical region has two sources:

- i) The quantity $\bar{q} \pm M_3$ is, apart from normalization, a projector on positive and negative energy intermediate states, respectively. (In our particular example, Fig. A5.1, the minus sign applies.)
- ii) The relative parity at the $(m M_3 M)$ vertex is equivalent to a sign flip in the mass of M_3 : Indeed, if the relative parity in our example were negative, a γ_5 would be necessary between the spinors of the Lagrangian (A5.1). As γ_5 and γ^μ anticommute, the γ_5 can be dropped for the price of a negative sign. (This is, of course, the familiar statement that the parity of antifermions is negative.)

Therefore everything else being fixed, the sign of r goes with parity (the sign of M_3) throughout the unphysical region $M-m < M_3 < m+M$. These results remain valid if we allow for derivative couplings in the Lagrangian and are therefore independent of the particular form (A5.1).

In our example the actual ${}^3\text{He}$ pole in the $n\alpha$ amplitude is $r/(\omega-\omega_0)$ with $r < 0$ [this means r_β positive for the sign convention of Eq. (57), which is adapted to s -channel resonances]. This amounts after subtraction to the positive contribution $r_\beta E/E_0(E_0-E)$ in Eq. (59), as it is observed approximately (see Fig. 28).

Finally, we remark the connection to the Λ exchange pole in K^+p scattering (Fig. A5.3):

Fig. A5.3 Hyperon exchange in K^+p scattering.

The intermediate state is now of positive energy, but the corresponding sign flip is compensated by the pseudoscalarity of K . Therefore again $r(+M_3)$ applies and Fig. A5.2 is valid with $M = M_K$. The same situation occurs in pion-nucleon or pion-nucleus scattering. Only the parabolae in Fig. A5.2 have to be shifted according to the varying external masses.

B. RESIDUE AND ASYMPTOTIC WAVE FUNCTION

To discuss the connection of the residue (A5.4) with the asymptotic wave function^{53,9)} of the α -particle, we consider the related process of $n{}^3\text{He}$ scattering. The α -particle pole (Fig. A5.4) dominates the interaction for large separations $\vec{x}$ of n and ${}^3\text{He}$. The Lagrangian

Fig. A5.4 ${}^4\text{He}$ pole in $n{}^3\text{He}$ scattering.

(A5.1) of Part A of this appendix leads to the following c.m. amplitude for graph (A5.4):

$$f_{n{}^3\text{He}}^{\text{c.m.}}(s') = \frac{1}{\sqrt{2}} \frac{r(M_3, M, m)}{\sqrt{s'} - M}, \quad (\text{A5.6})$$

where

$$s' = (p_n + p_{3\text{He}})^2.$$

The residue is the same as in Eq. (A5.4'). The factor $1/\sqrt{2}$ comes from the spin average, which extends over two external spin $\frac{1}{2}$ particles, whereas there was only one, the neutron, in the $n{}^4\text{He}$ scattering (Fig. A5.1).

The relation to the bound-state pole of Section 3 and its residue is now easily established by comparing Eq. (A5.6) with Eq. (19). The amplitude in Eq. (19) is to be read as the $n{}^3\text{He}$ potential scattering amplitude^{*}) (averaged over spin). Identifying $(E - E_\beta)$ by $(\sqrt{s'} - M)$ we obtain

$$\frac{\mu}{2\pi} |\Gamma_\beta| = |r(M_3, M, m)|, \quad (\text{A5.7})$$

where signs are omitted as we need the relation to the normalization constant N_β^2 of the asymptotic wave function^{**)} $\phi(x)$ for large x [see Eq. (22)], where $\phi(x)$ is normalized to unity. Making a comparison with Eq. (23) one immediately obtains the desired relation

$$N_\beta^2 = \frac{\mu^2}{\pi} |\Gamma_\beta| = 2\mu |r(M_3, M, m)| = \frac{g^2 \mu}{8\pi M^2} [(m - M_3)^2 - M^2]. \quad (\text{A5.8})$$

* * *

*) In the Schrödinger equation (18) we interpret x more generally as the relative coordinate and m as the reduced mass $\mu = m M_3 / (m + M_3)$.

**) The corresponding coefficient of $\Psi(r=\infty)$ in Ref. 26 implies the unfortunate convention $\mu = 1$ or an unusual normalization.

6. CONCLUSION

In the previous sections, the application of forward dispersion relations to complex nuclei has clearly shown that the method can be applied in a controllable and useful way, at least to light elements. The technical complications that, in principle, could occur both from the numerous states in direct and exchange channels are controllable; those due to the effects of weakly bound systems (anomalous thresholds) do not really enter for the forward scattering problem. The use of relativistic forward dispersion relations for nuclei is therefore no more difficult in essence than their familiar application in high-energy physics.

Although the forward dispersion relations only deal with the variation of the forward amplitude with energy, and therefore only probe nuclear properties to a limited extent, a number of interesting and outstanding features appeared both for pion and neutron elastic scattering.

For pion scattering, the principle features and conclusions are as follows:

The unphysical region gives only small, nearly negligible contributions to π scattering on light $T = 0$ nuclei (D, ${}^4\text{He}$, ${}^{12}\text{C}$). The subtracted dispersion relation is therefore, in this case, expected to be given accurately by the physical region only. Experimental tests are not yet completely conclusive, as they are restricted to energies below 100 MeV. The smallness of the unphysical region is associated with the smallness of pion absorption and it is analogous to the small contributions from the nucleon pole to the symmetric πN amplitude. It is further favoured by Adler's condition.

The antisymmetric (charge-exchange) amplitude has important contributions from the unphysical region, which can be described in good approximation by an effective π -nuclear pole close to $\omega = 0$. A crude determination of $(f^2)_{\text{eff}}$ for ${}^9\text{Be}$ indicates an experimental value close to that for the nucleon. In fact a sum rule predicts the nucleon value $f^2 = 0.08$ to be universal in the absence of rescattering phenomena. The summation extends over a large number of nuclear excited states. The extent to which this universality is valid is an interesting question, but experimental information is exceedingly scarce. It is therefore important to obtain good data both on the difference of π^+, π^- cross-sections for nuclei of $I \neq 0$ as well as on differences in scattering amplitudes of various energies, including the threshold region. Furthermore, charge exchange amplitudes inside the isospin multiplet would give similar information. An important nucleus for such investigations is in particular ${}^3\text{He}$, but also other light elements.

For nucleon scattering the main conclusions are the following:

The contributions from the unphysical region are major ingredients of low-energy scattering. The interpretation of these contributions is that they are due to exchange processes; for example, in the case of $n^4\text{He}$ they can be traced to the ${}^3\text{He}$ exchange pole and exchange cuts. Important exchange contributions of this kind (which might be thought of as "heavy particle pick-up") should occur quite generally in nucleon forward scattering as well as in elastic scattering of complex projectiles on nuclei. These terms must be major contributors to the so-called potential scattering amplitude in resonance scattering. It is, in fact, quite interesting that a deliberate non-relativistic approximation to the dispersion relation, leaving all explicit meson contributions out, leads to a subtraction constant for $n^4\text{He}$ scattering that is in very close agreement with the volume integral for the potentials that describe the very low energy scattering.

It is an interesting feature of the bound states in an extended system that their residues are experimentally enhanced by κR . The deep-lying bound states therefore should play a much more important role in the dispersion relations than might have been naïvely expected, for these factors can be very large.

Some evidence of this can also be seen in ^{12}C , where the contributions of the $1s$ state bound by 34 MeV are quite sizeable. The experimental situation on real amplitudes is, however, extremely poor with both Coulomb interference and forward cross-sections unmeasured or very inaccurate. Therefore the possibility of studying such states by this method must await experiments. For the present analysis of $n^{12}\text{C}$ (or $p^{12}\text{C}$) we would need urgently Re f both for low and medium energies.

An interesting possibility to study meson effects and other relativistic effects can at present be gleaned. While contributions from real antinucleon scattering is unimportant (and can be evaluated), the contributions from meson exchange such as ρ exchange are not. Even with the poor data currently available, they can be seen to contribute rather importantly to low-energy scattering amplitudes. Determination of their contributions requires real scattering amplitudes in the region 100-1000 MeV, where they can be measured by Coulomb interference.

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