PASSIVE COMPENSATION OF LONGITUDINAL SPACE CHARGE EFFECTS

IN CIRCULAR ACCELERATORS: THE HELICAL INSERT

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Passive Compensation of Longitudinal Space Charge Effects

In Circular Accelerators: The Helical Insert

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Abstract

The longitudinal electric field associated with a particle beam having azimuthal variation in charge density and moving within a helix which is surrounded by an outer sheath of arbitrary electrical properties, is investigated theoretically. It is shown that for density variations having a wavelength long compared to the helix radius, the magnitude and sign of the electric field may be set to any desired value, by appropriate choice of the helix and sheath parameters. In particular, conditions on the parameter of an helical insert of short circumferential length are given, for which there is complete compensation of the longitudinal space charge forces at one particle energy. The insert consists of a highly conducting helix surrounded by a ferrite sheath. Such a device should permit acceleration of a large line density, through the phase transition of an alternating gradient accelerator, without a decrease in longitudinal phase space density.

1. INTRODUCTION

It has been pointed out by Briggs and Neil, 1 that the negative mass instability 2 may be suppressed by suitable choice of the vacuum chamber walls surrounding the beam; in particular, the walls must be inductive. A recent paper by Neil and Briggs 3 suggests, for this purpose, the use of a helix, previously 1 they had suggested dielectric layers on the walls.

For not too relativistic particles, the amount of dielectric required is excessive in terms of aperture, and considerable interest is therefore concentrated on their newest proposal, which indicates that the inductive effect of a helix (of given radius) may (by appropriate choice of the pitch angle) be made arbitrarily large. Their calculation 3, however, is for a helix without an outside conducting sheath -- which will always be the case in practice and which might, at first sight, have a major effect on the properties of the helix. It is important, therefore, to compute the properties of the helix when surrounded by a conductor -- which is the purpose of this paper. We find, in fact, that in practical situations the helix has very different properties than is suggested in Ref. 3.

However, we propose the use of an outer sheath of ferrite and show that it is possible in this way to recover the advantageous effects of the helix. In particular, it is possible to have an insert, of small azimuthal extent, which dominates the space charge effects from the complete circumference.

The stabilization of the negative mass instability is important in many accelerators, of equal importance, is the passage through transition energy, without loss in phase space density, of a large line density of charge 4. A number of methods for accomplishing this are given in Ref. 4; a particularly convenient method is afforded by the helical insert. Furthermore, the insert would operate equally well at all levels of intensity.
1.1. Outline

In Section 2, we first employ the equivalent circuit methods of a previous report\(^5\), to calculate the longitudinal electric field associated with a beam having azimuthal charge density variation and moving within an infinitely long helix surrounded by a conducting sheath. For a highly conducting sheath the results are very different from that of Ref. 3, but criteria on the sheath resistance are given such as to recover the strong inductive effect of an isolated helix.

In Section 2.3, and 2.4, we discuss the helical insert concept, employing in the analysis the results of Section 2.1. In Section 3, we give a rigorous derivation -- starting from Maxwell's equations -- of the properties of a model\(^6\), namely an anisotropically conducting sheath within a (second) poor conducting sheath. In the appropriate limit (long wavelength of the perturbed charge) we recover the results of Section 2, thus demonstrating the validity of the circuit methods employed in that section.

1.2. Summary

The important result of this report is that it is possible to design a practically realizable helical insert which will compensate space charge effects near transition and/or remove the negative mass instability. Furthermore, the helix insert is of small azimuthal extent, simple to construct, and will not necessitate a decrease in useable aperture. Section 2.3 and 2.4 contain the design criteria and a sample set of parameters suitable for the CERN PS.

2. EQUIVALENT CIRCUIT ANALYSIS

The formalism of Ref. 5 has been designed precisely to facilitate the analysis of problems similar to the one of interest here. The computation requires only a few lines, and is given in Section 2.1 (the length of which section is primarily due to the fact that we repeat -- for the reader's convenience -- some of the formulas and concepts of Ref. 5). In Section 2.2 we explore two limiting cases; and, in particular, find criteria for the most interesting regime -- namely, that in which the longitudinal field is readily adjusted in value and not too sensitive to particle energy and design parameters. Section 2.3 contains a discussion of an helical insert with surrounding sheath so designed as to compensate the space charge forces from the whole circumference. Section 2.4 contains a detailed numerical example; namely an insert designed to remove space charge effects at transition in the CERN PS.

2.1. Computation of the Longitudinal Electric Field

The physical situation we consider is that of a beam of constant cross section (radius a) and uniform transverse density, moving within a helix having pitch angle \(\phi\) and radius b, all surrounded by a sheath at radius d. Fig. 1 shows the situation.
The beam is assumed, exactly as in Ref. 5, to have perturbed charge per unit length \( \lambda \):

\[
\lambda = \lambda_0 e^{j(kz - \omega t)},
\]

where \( k \) is the wave number of the perturbation, and \( v = \beta_0 = \omega/k \) is the phase velocity of the perturbation. The longitudinal electric field at \( r = \infty \), \( E_z(\infty) \), may be written in the form:

\[
E_z(\infty) = -\frac{\beta_0}{2\pi k} Z ;
\]

expression for the impedance \( Z \) may be found in Ref. 5.

In the analysis of longitudinal space charge phenomena, it is usually adequate to only investigate situations in which \( k^{-1} \gg b \), vacuum chamber minor radius \( b \). In this situation the field \( E_z(\infty) \) is proportional to \( \lambda \) and \( \frac{\partial \lambda}{\partial z} \); for perfectly conducting walls we only have the out of phase term \( \sqrt{\lambda} \) and it has been customary to introduce a geometrical factor, \( g \), by the relation:

\[
E_z(\infty) = -g \frac{\partial \lambda}{\partial z}.
\]

We can employ this relation -- in its Fourier transformed form -- in the general situation by letting \( g \) be a function of wave number, and thus we obtain from (2.1), (2.2) and (2.3):

\[
g(k) = -\frac{4\beta_0 k}{2\pi \alpha k}
\]

We could of course, write expressions for the quantities \( U \) and \( V \) of Ref. 7, by using Section Ib of Ref. 5. We shall not pursue, in detail, the questions of instabilities, and so Eq. (2.4) suffices for our purpose. We need only recall that if \( g = 0 \), then the longitudinal space charge forces are completely compensated; and, if \( \text{Re}[g] < 0 \) then we are in the region of positive mass (when operating above the transition energy) and the fast negative mass instability in not present -- although a (usually slower), resistive instability is still present.

In Fig. 2 we show the equivalent circuit corresponding to Fig. 1. The outer impedance per unit length is \( \tilde{Z}_0 \), \( \tilde{Y}_2 \) is the admittance per unit length between the helix and the sheath, \( \tilde{Z}_1 \) is the impedance per unit length of the helix. We may write explicit formulas for these quantities, first by defining appropriate resistances, inductances and capacitances:

\[
\tilde{Z}_0 = \tilde{R}_0 - j \omega \tilde{L}_0,
\]

\[
\tilde{Z}_1 = \tilde{R}_1 - j \omega \tilde{L}_1,
\]

\[
\tilde{Y}_2 = -j \omega \tilde{C}_2.
\]
and then by writing formulas for the various terms in (2.5).

The resistive terms are of the form

\[ R_0 = \frac{2 \rho_L}{cd}, \quad \text{and} \quad R_1 = \frac{2 \rho_R}{cb \sin^2 \psi}, \]

(2.6)

where in terms of the sheath conductivity, \( \sigma_s(z) \), the sheath dielectric constant \( \varepsilon_s(z) \); and the helix conductivity, \( \sigma_h \), we may write:

\[ \rho_h = \left( \frac{\omega}{\pi \sigma_h} \right)^{1/2}, \quad \kappa_h = \frac{\omega \mu_0}{\pi \sigma_h \mu_0} \left( \frac{\omega \mu_0}{\beta d k \varepsilon_h} \right)^{1/2}. \]

(2.7)

We have assumed \( 4\pi \sigma_h \gg \varepsilon_h \) in writing the expression for \( \rho_h \).

\[ \varepsilon_h \text{ is the dielectric constant of the helix.} \]

The inductance per unit length of the outer sheath, \( L_0 \), and the capacitive coupling per unit between the helix and the outer sheath, \( C_2 \), are:

\[ \tilde{C}_2 = \frac{1}{2 \ln \frac{\delta}{\beta}}, \quad \text{and} \quad \tilde{L}_0 = \frac{2 \ln \frac{\delta}{\beta}}{c^2} + \frac{2 \kappa_h z}{\beta d k \varepsilon_h}, \]

(2.8)

where the second term in \( \tilde{L}_0 \) is the inductive skin effect term corresponding to the real term of (2.6).

The final coefficient that we need is the helix inductance per unit length. It is the obvious term, reduced by diamagnetic shielding \( \rho_h \):

\[ \tilde{L}_1 = \frac{1}{c^2 \tan^2 \psi} \left[ 1 - \alpha \left( \frac{b^2}{d^2} \right) \right]; \]

(2.9)

where

\[ \alpha = \frac{1}{1 + (1 + j) Q_{s,h}^{-1}}, \quad Q_{s,h} = \frac{\omega L_t}{R_t}, \]

\[ R_t = \frac{8 \pi^2 d \kappa_h}{c} \quad \text{and} \quad L_t = \frac{4 \pi^2 d^2}{c^2}. \]

(2.10)

From Ref. 5, (V 7a) we have an explicit formula for the beam impedance \( Z \):
\[ Z = 2\pi R \sim \left[ \frac{\sim 2g/k + k/\sim Y_2}{\sim Z_0 + Z_1} /k + k/Y_2 \right] \]  \hfill (2.11)

but we must remember to add to \( Z \) the term from a perfectly conducting helix; namely:

\[ Z' = \frac{2\pi R k}{-j\beta_0 Y_2} \left[ 1 + 2 \ln \frac{b}{a} \right] \]  \hfill (2.12)

Inserting (2.5) with (2.6), (2.7), (2.8), (2.9) and (2.10) into (2.11) we have:

\[
\begin{align*}
Z &= -j \frac{2\pi}{\gamma^2} \left[ 6 \ln \frac{a}{b} - \frac{\beta^2 \frac{K_z}{dk}}{\tan^2 \psi} \right] - \frac{\beta^2}{\tan^2 \psi} \left[ \frac{1}{1 - j/Q_s} \right] \left[ 1 - \gamma^2 \left( \frac{b^2}{d^2} \right) \left( 1 + \frac{j+1}{Q_h} \right) \right] \\
&= \frac{\gamma^2}{\tan^2 \psi} \left[ 6 \ln \frac{a}{b} - \frac{\beta^2 \frac{K_z}{dk}}{\tan^2 \psi} \right] - \frac{\beta^2}{\tan^2 \psi} \left[ \frac{1}{1 - j/Q_s} \right] \left[ 1 - \gamma^2 \left( \frac{b^2}{d^2} \right) \left( 1 + \frac{j+1}{Q_h} \right) \right]
\end{align*}
\]  \hfill (2.13)

where \( \gamma^2 = 1 - \beta^2 \), and

\[
Q_h = \frac{\omega_0}{k_1}, \quad Q_s = \frac{k}{\omega c_0} \frac{\gamma}{R_n/k}
\]  \hfill (2.14)

Combining \( Z \) and \( Z' \), we obtain, from (2.4) an expression for \( g(k) \):

\[
\begin{align*}
g(k) &= \frac{1}{\gamma^2} \left[ 1 + 2 \ln \frac{b}{a} \right] \\
&= \frac{\beta^2}{\tan^2 \psi} \left[ \frac{1}{1 - \gamma^2 \left( \frac{b^2}{d^2} \right) \left( 1 + \frac{j+1}{Q_h} \right)} \right] \\
&= \frac{\beta^2}{\tan^2 \psi} \left[ \frac{1}{1 - \gamma^2 \left( \frac{b^2}{d^2} \right) \left( 1 + \frac{j+1}{Q_h} \right)} \right]
\end{align*}
\]  \hfill (2.15)
Equation (2.15) is the basic result of this paper, in agreement with the result obtained in section 3 (3.43) by an exact calculation.

2.2. Limiting Cases

In this section we examine the behaviour of (2.15') in two limiting cases:

Consider first the situation of perfectly conducting elements:

\[ \sigma_{\alpha} \rightarrow \infty, \quad R_{1} \rightarrow 0. \]

Then \( \alpha = 1 \) and (2.15) becomes:

\[
g(k) = \frac{1}{2} \left[ 1 + 2 \ln \frac{b}{a} \right] - \frac{\beta^{2}}{\gamma^{2}} \tan^{2} \frac{\frac{1}{2} \ln \frac{d}{b}}{\frac{2}{\gamma^{2}} \ln \frac{d}{b} - \frac{\beta^{2}}{\tan^{2} \gamma} \left( 1 - \frac{b^{2}}{d^{2}} \right)} - \frac{\frac{1}{2} \ln \frac{d}{b}}{\frac{2}{\gamma^{2}} \ln \frac{d}{b} - \frac{\beta^{2}}{\tan^{2} \gamma} \left( 1 - \frac{b^{2}}{d^{2}} \right)}
\]

\[ (Q_{g}, Q_{h} \gg 1), \quad (2.16) \]

which, after defining

\[
\Delta = \frac{2d \beta \ln \frac{d}{b}}{(d^{2} - b^{2})}, \quad (2.17)
\]

can be written in the form:

\[
g = \frac{1 + 2 \ln \frac{d}{a} - \frac{\Delta \tan^{2} \gamma}{\frac{\beta^{2}}{2} \frac{d}{b} \left( 1 + 2 \ln \frac{b}{a} \right)}}{\frac{\gamma^{2}}{\left( 1 - \frac{\Delta \tan^{2} \gamma}{\frac{\beta^{2}}{2} \frac{d}{b}} \right)}}, \quad (Q_{g}, Q_{h} \gg 1); \quad (2.18)
\]

Since the resonant structure of the device, Eq. (2.18) has the feature that \( g \) assumes all values between \(-\infty\) and \(+\infty\) as the helix parameters are varied. The dependence upon energy (\( \gamma \)) and helix parameters is very sensitive, and this case is of much interest in practice.

A second limiting case that in which the sheath resistivity in \( z \) direction is high, the helix resistivity is low, and with no assumption on the resistivity in \( A \) direction. Thus when

\[ Q_{h} \gg 1 \quad (2.19) \]
\[ \frac{2 \ln \frac{d/\phi}{\gamma^2}}{a} \ll \frac{2 \mathcal{E}_0}{dk} \quad , \quad (2.20) \]

then \( Q_s \to -1 \) and (2.15) becomes

\[ g(k) = \frac{1}{\gamma^2} \left[ 1 + 2 \ln \frac{b/a}{l} \right] - \frac{\beta^2}{\tan^2 \phi} \left( \frac{1}{1+\gamma} \right) \left( \frac{-2 \mathcal{E}_0 z}{dk} \right) \left( 1 - \alpha \frac{b^2}{d^2} \right) \]

\( (Q_h \gg 1, Q_s \to -1) \quad (2.21) \]

If furthermore,

\[ \frac{2 \mathcal{E}_0}{dk} \gg \frac{\beta^2}{\tan^2 \phi} \left( 1 - \alpha \frac{b^2}{d^2} \right) \quad , \quad (2.22) \]

then

\[ g = \frac{1}{\gamma^2} \left[ 1 + 2 \ln \frac{b/a}{l} \right] - \frac{\beta^2}{\tan^2 \phi} \left( 1 - \alpha \frac{b^2}{d^2} \right) , \quad (Q_h \gg 1, Q_s \to -1, \quad \text{and 2.22, satisfied).} \]

\( (2.23) \]

The result (2.23) is most interesting; provided only that we can satisfy (2.19), (2.20) and (2.22) -- which means only that the helix be a good conductor and the sheath be a good insulator -- we can, by suitable choice of \( \phi \), make \( g \) take any desired value. Making the sheath a dielectric would not, however, be acceptable since then the fields would penetrate the sheath: We must make the skin depth less than the sheath thickness. Curiously, (2.23) is essentially the result of Ref. 3 -- but here obtained under the condition of a highly resistive sheath.

2.3. The Helical Insert

If the helix structure only extends over a length \( L \) of the accelerator circumference, somewhat as shown in Fig. 3, then we have for the effective geometrical factor
\[ \varepsilon_{\text{eff}} = \frac{1}{\gamma^2} \left[ 1 + 2 \ln \frac{a}{d} \right] \left( \frac{2\pi - L}{2\pi R} \right) + \]
\[ + \frac{1}{\gamma^2} \left[ 1 + 2 \ln \frac{b}{a} \right] - \left( 1 - \frac{b^2}{d^2} \right) \frac{\beta^2}{\tan \theta} \cdot \frac{L}{2\pi R} \quad (2.24) \]

This expression has not been derived, but would seem to be correct: the longitudinal phase motion of particles is slow and (just as with RF acceleration where a continuous travelling wave can be imagined to be accelerating the particles, but the cavity can be at one azimuthal position) we need not compensate the longitudinal space charge forces at all azimuthal positions, but once a turn will suffice.

From Fig. 3, we can see the importance of having the outer sheath non-conducting and, at the same time, obtain further insight into the operation of the compensating helix. The beam image currents travel down the normal vacuum chamber (while having a small capacitive influence on the beam -- the positive \( \frac{1}{\gamma^2} \) dependent term in \( g \)) and suddenly encounter the helix. They are forced down (the beam is like a current generator) the helix, since all other paths have a larger impedance; and, in so moving generate a large back e.m.f., the magnitude of which depends on the tightness of the helix spiral and whose effect on the beam is inductive.

The arguments of the last paragraph may be made quantitative: The \( g \)-dependent term in \( g \) is derived by equating the back e.m.f. of a helix per unit length (generated by a current, equal to the beam current flowing through the helix to the longitudinal field \( E_z(o) \)). The inequality (2.22) is simply the requirement that the sheath resistance be larger than the helix impedance.

Constructing the sheath of an insulator would not be effective, as the rf fields would penetrate it; and an ordinary poor-conductor is equally ineffective since the skin depth and conductivity are related in such a way that the sheath thickness is excessive for the required high resistivity. Ferrites, however, exhibit high resistivity and small skin depth.

We consider, therefore, an outer sheath constructed of ferrite and operated in the range of frequencies in which it exhibits high magnetic losses, namely
\[ \mu_r = \mu - j\mu' = \mu (1 - j\tan \delta). \quad (2.25) \]

The wave impedance is, as \( d_z = d_A \equiv d_s \) and \( \varepsilon_z = \varepsilon_A \equiv \varepsilon_s \),
\[ z = \sqrt{\frac{\omega (\mu - j\mu')}{\epsilon_s \omega_0 s - j\omega_2 \epsilon_s}} \]  

(2.26)

The wall will have a wave impedance given by (2.26) if the wave-front in the ferrite is almost plane (see appendix) and if the thickness of the wall is infinite; however, it will be sufficient that the thickness is larger than the skin depth \( \Delta \). This is given by the formula:

\[
\Delta = \frac{1}{\left[ \left( \omega \left| \mu_r \right| 4\pi \sigma_s \right)^2 + \left( \omega^2 \epsilon - \left| \mu_r \right| \right)^2 \right]^{1/2} + \omega \mu' 4\pi \sigma_s - \omega^2 \mu \epsilon_s} \]

(2.27a)

with

\[
\left| \mu_r \right| = \mu \sqrt{1 + \tan^2 \delta} \]

(2.27b)

2.4. Design Criteria and Numerical Example

The design criteria for a compensating helical insert is that

\[ E_{\text{eff}} = 0 \]  

(2.28a)

or from (2.24) \[ \tan^2 \phi = \frac{L}{2\pi R} \frac{\beta^2 \gamma^2}{1 + 2 \ln \frac{b}{a}} \]  

(2.28b)

where we have taken the helix radius equal to the normal vacuum chamber radius.

The condition (2.22) is more stringent than (2.20); it may be written, for a ferrite sheath, in the form:

\[ E_{\text{eff}} = 0 \]

The helix with ferrite sheath has the following interesting property: the saturation of the ferrite ring can be driven by coils varying the magnetic constant \( \mu \); in so doing, we can adjust the wave impedance at the desired value in such a way that a part of the image current will flow in the wall; the back e.m.f. will be reduced, but the equation

\[ E_{\text{eff}} = 0 \]

can be satisfied for higher values of \( \gamma \).

In this way we can have, by means of an appropriate saturation program,

\[ E_{\text{eff}} = 0 \]

for all values of \( \gamma \) ranging within a specified interval.
\[
\sigma_s + \frac{\omega \varepsilon_s}{4\pi} \ll \frac{1}{2\pi n} \left( \frac{R_0}{d^2} \right) \tan^2 \psi \sqrt{1 + \tan^2 \delta} \quad (2.29)
\]

where we have set \( k = n/R \) and \( \omega = n \omega_0 \). The condition (2.19) may be expressed, employing (2.6) and (2.7), in the form

\[
\sigma_h \gg \frac{1}{2\pi n} \left( \frac{R_0}{b^2} \right) \frac{\tan^4 \psi}{\sin^4 \psi} \quad (2.30)
\]

Finally we must satisfy the condition that the sheath thickness \( T \) be larger \(^*\) than the skin depth, namely:

\[
T \gg \frac{c}{\left[ \omega \tau \right]^{-1} \sqrt{2}} \left\{ \left( \omega \varepsilon_s \right)^2 + (4\pi \sigma_s)^2 \right\}^{1/2} + 4\pi \sigma_s \left[ \frac{\mu^1}{\mu_x} - \frac{\omega \varepsilon_s}{\mu_x} \right] \left[ \left( \omega \varepsilon_s \right)^2 + (4\pi \sigma_s)^2 \right]^{1/2} \quad (2.31)
\]

If we employ the design criterion that \( \varepsilon_{eff} = 0 \), we may express the inequalities (2.29) and (2.30) in the more convenient form:

\[
\sigma_h \gg \frac{1}{(2\pi)^2 n \cdot \beta} \left( \frac{L}{b} \right)^2 \left( \frac{c}{R} \right)^2 \left( \frac{\beta^2 \gamma^2}{1 + 2 \ln \frac{b}{a}} \right)^2 \frac{1}{\sin^4 \psi} \quad (2.32)
\]

\[
\left( \sigma_s \right)^2 + \left( \frac{f \varepsilon_s}{2} \right)^2 \ll \frac{\sqrt{1 + \tan^2 \delta}}{(2\pi)^2 n} \left( \frac{L}{d} \right)^2 \left( \frac{c}{R} \right)^2 \left[ \left( \frac{b}{a} \right)^4 \right] \left( \frac{1}{1 + 2 \ln \frac{b}{a}} \right)^2 \quad (2.33)
\]

while (2.31) may be expressed in the form:

\[
\left[ \left( \sigma_s \right)^2 + \left( \frac{f \varepsilon_s}{2} \right)^2 \right]^{1/2} \ll \left[ \frac{\sigma_s \tan \delta}{2} + \left( \frac{f/2}{T} \right) \varepsilon_s \right] + \left( \frac{R}{T} \right) \frac{c}{T} \left( \frac{2\eta \beta}{\mu \left( 1 + \tan^2 \delta \right)} \right)^{1/2} \quad (2.34)
\]

\(^*\) Larger, but not much larger.
In practical applications, the stringent requirements are (2.33) and (2.34) that is to say, the sheath conductivity must be low, while at the same time -- for a reasonably thin sheath -- the skin depth must not be too large.

We consider, as an interesting numerical example, the CERN PS at its transition energy. Parameters are:

\[ \beta \approx 1, \gamma_t = 6.1 \]

\[ R = 10^4 \text{cm}, \quad b = 3.5 \text{cm} \]

\[
\left[ 1 + 2 \ln \frac{b}{a} \right] \approx 2.25,
\]

so that (2.28b) implies \( \tan \psi = 1.6 \times 10^{-2} \text{ cm}^{1/2} \text{L} \), while (2.33) and (2.34) become (with \( d = 4.0 \text{ cm} \)).

\[
\frac{4.8 \times 10^3}{n^2 \text{(cm)}} \left[ \frac{\sigma_s \tan^2 \delta}{1 + \tan^2 \delta} - \left( \frac{\tan^2 \psi}{1 + \tan^2 \psi} \right) \right] \approx \left[ \left( \sigma_s \right)^2 + \left( \frac{\sigma_s}{2} \right)^2 \right]^{1/2}
\]

\[ L^2 2.1 \times 10^5 \frac{\mu}{n} \left[ 1 + \tan^2 \delta \right] \gg \left[ \left( \sigma_s \right)^2 + \left( \frac{\sigma_s}{2} \right)^2 \right]^{1/2}
\]

choosing the ferrite Siemens 1500 M 4 which has the following properties for \( n \gg 6 \), i.e. above 3 MHz.

\[
\mu = \frac{4200}{n} \quad \frac{3860}{n^{3/4}} = \frac{4200}{n} \quad (1 - 0.92n^{1/4})
\]

\[ \sigma_s = 10^{-4} \quad \text{mhos/cm} \]

\[ 1 \ll \varepsilon_s \ll 10 \]

we can satisfy (2.36) for \( 6 \ll n \ll 250 \)

with \( T > 7 \text{ cm} \).
On the other hand (2.37) is easily satisfied with \( L = 100 \) cm for \( n \ll 180 \), with \( L = 200 \) cm for \( n \ll 500 \).

3. Conducting Sheet Model - Careful derivation

In the present section we make a rigorous calculation -- starting from Maxwell's equation -- of a model problem; namely an anisotropic conducting sheath, within a non perfectly conducting sheath. This model should be an excellent approximation to a multi-start helix.

3.1. Rigorous Calculation

We consider a centrally located beam moving in the \( Z \) direction of a straight cylindrical pipe of radius \( d \). At radius \( b \) \( (b < d) \) there is a sheath, conducting in a direction making angle \((90^\circ - \phi)\) with field boundary conditions are, at the out sheath,

\[
\begin{align*}
E_z(d) &= -\kappa \frac{B_z(d)}{A}, \\
E_\phi(d) &= \frac{B_z(d)}{A},
\end{align*}
\]

where

\[
\frac{\kappa}{A} = (1 - j) \frac{B_z(d)}{A},
\]

at the "helix"

\[
\begin{align*}
E_z(b) &= E_z(b^-), \\
E_\phi(b) &= E_\phi(b^-),
\end{align*}
\]

\[
E_\phi(b) \cos \psi + E_z(b) \sin \psi = -\kappa \left[ H_\phi(b^+) - H_\phi(b^-) \right] / \cos \phi = \frac{\kappa}{A} \left[ H_\phi(b^+) - H_\phi(b^-) \right] / \sin \phi
\]

\[
H_z(b) \sin \phi + H_\phi(b) \cos \phi = H_\phi(b^-) \sin \phi + H_\phi(b^-) \cos \phi
\]

Equations (3.2) and (3.3) are obvious, (3.4) makes the electric fields along the conducting direction proportional to the current, while (3.5) expresses the continuity of magnetic field component along the conducting direction.
The field has a source, the beam, which is assumed to have constant cross section and uniform transverse density, but with a longitudinal variation in density. Considering only the perturbed (i.e. varying) part of the density, we have a charge density:

$$\rho(z,t) = n_o e H(z-a-\omega t)$$  \hspace{1cm} (3.6)

where $H$ is the Heaviside step function, $a$ is the beam radius, $e$ is the particle charge, $n_o$ is the number of perturbed particles per unit volume, $k$ is the wave number associated with the longitudinal density variation, and $v = \beta c = \omega/k$ is the perturbation velocity. Associated with $(z,t)$, is a current:

$$J_z(z,t) = \frac{\omega}{k} \pi a^2 \rho(z,t).$$  \hspace{1cm} (3.7)

The field components are all independent of azimuthal angle $\theta$. In the absence of the helix we may obtain all the fields $(E_\theta, E_r, E_z)$ from $H_z$. The helix mixes in other field components $(E_\phi, E_r, E_z)$ which can, however, be derived from $H_z$. Thus we have a source-free $H_z$:

$$\frac{\partial}{\partial r} \left( r \frac{\partial H_z}{\partial r} \right) - \frac{k^2}{\gamma^2} H_z = 0,$$  \hspace{1cm} (3.8a)

and a beam source for $E_z$:

$$\frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) - \frac{k^2}{\gamma^2} E_z = \frac{4\pi n_o e}{\gamma} H(r-a)$$  \hspace{1cm} (3.8b)

where $\gamma^{-2} = 1 - \beta^2$ and $H$ is the Heaviside function.

From (3.8b) and with $q = k/\gamma$, we have explicit expressions for $E_z$ in terms of Bessel functions:

$$E_z = C I_0(qr) + \frac{4\pi n_o e}{jk}$$  \hspace{1cm} (3.9)

$$E_z = C I_0(qr) + \frac{4\pi n_o e}{jk} \left[ K_1(qa) I_0(qr) + I_1(qa) K_0(qr) \right]$$  \hspace{1cm} (3.10)

$$E_z = A I_0(qr) + B K_0(qr)$$  \hspace{1cm} (3.11)
Equation (3.9) makes \( E_z \) finite at \( r = 0 \); (3.10) follows from the continuity at \( r = a \). Imposing (3.1a) and (3.2) we obtain \( A \) and \( B \) in terms of \( C \)

\[
A = - \frac{\tilde{K}_0(qd) B}{\tilde{I}_0(qd)} \quad (3.12)
\]

\[
B = \frac{C \tilde{I}_0(qb) + \gamma \tilde{I}_0(qd)}{K_0(qb) \tilde{I}_0(qd) - I_0(qb) \tilde{K}_0(qd)} \quad (3.13)
\]

where

\[
\tilde{I}_0(qd) = I_0(qd) - j \beta \tau K_z I_1(qd) \quad (3.14)
\]

\[
\tilde{K}_0(qd) = K_0(qd) + j \beta \tau K_z K_1(qd) \quad (3.15)
\]

\[
\gamma = \frac{4 \pi n \rho a e}{j k} \left[ K_1(qa) I_0(qb) + I_1(qa) K_0(qb) \right] \quad (3.16)
\]

We now compute \( H_A \) from

\[
H_A = B E_r \quad (3.17)
\]

\[
E_r = - \frac{j \gamma}{q} \frac{\partial E_z}{\partial r} \quad (3.18)
\]

Thus we find

\[
H_A(b^+) - H_A(b^-) = - j \left( \frac{B r}{q b} \right) \frac{C \tilde{I}_0(qd) + \gamma I_0(qa) K_0(qd) + I_1(qa) \tilde{I}_0(qd)}{I_0(qb) \tilde{K}_0(qd) - K_0(qb) \tilde{I}_0(qd)} \quad (3.19)
\]

where

\[
S = \frac{4 \pi n \rho a e}{j k} \quad (3.20)
\]

Returning to (3.8a) we can write formulas for \( H_z \):

\[
H_z = D I_0(qr) \quad 0 < r < b \quad (3.21)
\]

\[
H_z = G I_0(qr) + P K_0(qr) \quad b < r < d \quad (3.22)
\]
We now compute $E_\Delta$ from

$$E_\Delta = \beta H_r$$

$$H_r = -\frac{x}{q} \frac{\partial H_s}{\partial r}$$

(3.23)

(3.24)

namely

$$E_\Delta = j \beta \gamma D I_1(qr)$$

(3.25)

$$E_\Delta = j \beta \gamma \left[ G I_1(qr) - F K_1(qr) \right]$$

(3.26)

Imposing (3.1b) and (3.3) we obtain $G$ and $F$ in terms of $D$

$$F = -\frac{\widetilde{I}_1(qd) I_1(qb)}{K_1(qb) \widetilde{I}_1(qd) - I_1(qb) \widetilde{K}_1(qd)} D$$

(3.27)

$$G = -\frac{I_1(qb) \widetilde{K}_1(qd)}{K_1(qb) \widetilde{I}_1(qd) - I_1(qb) \widetilde{K}_1(qd)} D$$

(3.28)

where

$$\widetilde{I}_1(qd) = I_1(qd) - (\frac{\kappa_s}{j \beta \gamma} I_0(qd))$$

(3.29)

$$\widetilde{K}_1(qd) = K_1(qd) + (\frac{\kappa_s}{j \beta \gamma} K_0(qd))$$

(3.30)

Now from equation (3.4) with (3.25) and (3.10) we get

$$D = \frac{j \beta \gamma I_1(qb)}{\kappa_s I_1(qb)} \left\{ C I_0(qb) + S \left[ K_1(qa) I_0(qb) + I_1(qa) K_0(qb) \right] \right\} + \frac{j \kappa_s \Delta H_s}{\cos^2 \psi I_1(qb) \beta \gamma}$$

(3.31)

Introducing (3.31) in (3.21), (3.27) and (3.28) and (3.27) and (3.28) in (3.22), we have at $b$

$$H_z(b^+) - H_z(b^-) = \frac{j \widetilde{I}_1(qd)}{q \beta \gamma I_1(qb)}$$

$$\tan \psi \left\{ C I_0(qb) + S \left[ K_1(qa) I_0(qb) + I_1(qa) K_0(qb) \right] \right\} + \frac{\kappa_s \Delta H_z}{\cos^2 \psi}$$

(3.32a)
and, explicating \( H_z(b^+) - H_z(b^-) = \Delta H_z \)

\[
\Delta H_z = j \frac{\tilde{I}_1(qd) \gamma I_1(qb)}{\gamma I_1(qb)} C \frac{I_0(qb)}{I_1(qb)} + S \frac{K_1(qa) I_0(qb) + I_1(qa) K_0(qb)}{K_1(qb) \tilde{K}_1(qd) I_1(qb) \tilde{K}_1(qd)}
\]

(3.32b)

with

\[
\gamma = \left[ 1 - \frac{\tilde{I}_1(qd) \mathcal{K}_0}{\sqrt{\gamma I_1(qd) \cos^2 \psi I_1(qd) \tilde{K}_1(qd)}} \right].
\]

(3.33)

Finally we impose (3.5) by using (3.19) and (3.32b) to obtain the expression for \( C \)

\[
C = S \frac{I_1(qb) \beta^2 \gamma^2 \tilde{I}_1(qd) K_1(qa) \tilde{I}_1(qd) + I_1(qa) K_0(qb)}{\Delta \gamma I_0(qb)} \left[ \frac{K_1(qa) I_0(qb) + I_1(qa) K_0(qb)}{\gamma I_0(qb)} \right] - \frac{\beta^2 \gamma^2 \tilde{I}_1(qd) \tilde{I}_0(qb)}{\Delta \gamma I_0(qb)} \tilde{I}_1(qd) + \tilde{I}_1(qd) I_0(qb)
\]

(3.34)

where

\[
\Delta = \frac{K_0(qb) \tilde{I}_1(qd) - I_0(qb) \tilde{K}_1(qd)}{K_1(qb) \tilde{K}_1(qd) - I_1(qb) \tilde{K}_1(qd)}
\]

(3.35)

From (3.9), we have now an explicit expression for \( E_z(r=0) \). Finally we get the expression for \( g \).

\[
g = \frac{E_z(r=0)}{\frac{\gamma}{\gamma I_1(qb)}} = - \frac{\left( \frac{4\pi \gamma}{jk} + C \right)}{\frac{\gamma}{\gamma I_1(qb)}}
\]

\[
(\frac{\beta^2 \gamma^2}{\Delta \gamma I_0(qb)} \frac{I_1(qb) \tilde{I}_0(qb)}{\tilde{I}_1(qd) I_0(qb)} \left[ \frac{K_1(qa) - 1 + I_1(qa) \tilde{K}_0(qd)}{I_0(qb)} \right] - \left[ \frac{K_0(qa) q_1 - 1 + I_1(qa) K_0(qb)}{I_0(qb)} \right] \beta^2 \gamma^2 \tilde{I}_1(qd) \tilde{I}_0(qb) \frac{\Delta \gamma I_0(qb)}{4 \beta^2 \gamma^2 \tilde{I}_1(qd) \Delta \gamma I_0(qb) \tilde{I}_1(qd) \gamma I_0(qb)}
\]

(3.36)
3.2. Two limiting cases

There are two special cases to be considered:

1) we may let \( qd \to \infty \) and then \( q \to 0 \) so that \( qd \gg 1 \) and \( qb \ll 1 \)

2) we may let \( qd \ll 1 \)

In the first we obtain:

\[
g = \frac{1}{\gamma^2} \left[ 1 + 2 \ln \left( \frac{2}{\gamma_b q_b} \right) - \frac{\frac{2 \beta}{\gamma_k^2}}{2} \right. \\
\left. - \frac{\frac{2 \beta}{\gamma_k^2}}{2} \right] \quad \text{(3.37)}
\]

\( (qd \gg 1) \)

where \( \gamma_k \) is the Buler constant. The expression for \( g \) becomes

\[
g = \frac{1}{\gamma^2} \left[ 1 + 2 \ln \left( \frac{b}{a} \right) \right] - \frac{\frac{2 \beta}{\gamma_k^2}}{2} \quad \text{(3.38)}
\]

\( (qd \gg 1, qb \ll 1) \)

when

\[
\frac{2 \gamma_k^2}{\beta^2} \frac{\gamma_k^2}{\gamma_k^2} \ln \left( \frac{2}{\gamma_b q_b} \right) \gg 1 \quad \text{(3.39)}
\]

Recall that: \( q = \frac{k}{\gamma} = \frac{n}{\gamma R} \)

where \( n \) is the mode number. In practical applications to circular to circular accelerators \( n \) ranges from the number of equal bunches to \( \approx (2\pi R)/b \) so that (3.37) is certainly not satisfied at low \( n \) values and probably in special circumstances.

Equation (3.39) is almost the result of Neil and Briggs \(^5\), (it differs only in the term \( 1/\gamma^2 \)) and is, normally, only applicable in the unphysical situation \( d \gg \gamma R \) (for \( n = 1 \)).

In the second case we have

\[
g = \frac{1}{\gamma^2} \left[ 1 + 2 \ln \left( \frac{b}{a} \right) \right]
\]
\[
\frac{1}{\gamma^2} \ln \frac{d}{b} = \frac{2 \hat{\kappa}_m \beta / k_d}{1 - j \beta kd} \left[ -\frac{\alpha^2}{\tan^2 \psi} \left( 1 - \frac{b^2}{d^2} \right) + \frac{j 2 \hat{\kappa}_h c}{\cos^2 \psi \omega b} \right]
\]

where \( \tilde{\alpha} = \frac{1 - \frac{qd}{2 \beta^2} \hat{\kappa} \Delta \ln \left( \frac{\gamma E}{2} \right)}{1 - 2 \frac{\hat{\kappa} \Delta}{\beta^2} qd} \) \hspace{1 cm} (3.41)

If further
\[
\frac{qd}{2 \beta^2} \gamma \Delta \ln \left( \frac{\gamma E}{2} \right) \ll 1
\]
(3.42)

and \( \beta kd \ll 1 \), \hspace{1 cm} (3.43)

the geometrical factor becomes
\[
g = \frac{1}{\gamma^2} \left[ 1 + 2 \ln \frac{b}{a} \right] + \]

\[
\left[ \frac{2 \ln \frac{d}{b} - 2 \frac{\hat{\kappa} \Delta}{kd}}{2 \ln \frac{d}{b} - 2 \frac{\hat{\kappa} \Delta}{kd}} + \left( 1 - \frac{\alpha b^2}{d^2} \right) + \frac{j 2 \hat{\kappa}_h c}{\cos^2 \psi \omega b} \right] \left( 1 - \frac{\hat{\kappa} \Delta}{j \beta^2 \gamma \Delta} \right) \]

where \( \alpha = \frac{1}{\left( 1 + \frac{1}{2} \frac{\hat{\kappa} \Delta}{\beta^2} qd \right)} \hspace{1 cm} (3.45) \)

and with (2.19)
\[
\alpha = \frac{1}{1 + (1-j)Q_{a,h}} \hspace{1 cm} (3.46)
\]
Taking into account (2.14), (2.6), and that
\[ \tilde{\rho}_z = \tilde{\rho}_z (1 - j) \quad \text{and} \quad \tilde{\rho}_h = \tilde{\rho}_h (1 - j) , \]
we get, from (3.44),
\[
\varepsilon = \frac{1}{\gamma^2} \left[ 1 + 2 \ln \frac{b}{a} \right] + \frac{\frac{2 \ln \frac{b}{a} - \frac{\tilde{\rho}_z}{kd}}{2} \left( 1 - \frac{1}{Q_s} \right) \frac{b^2}{\varepsilon_0 \delta^2 \psi} \left( 1 - \alpha \frac{b}{d} \right) \left( 1 + \frac{1}{Q_h} \right)}{2 \ln \frac{b}{a} - \frac{\tilde{\rho}_h}{kd} \left( 1 - \frac{1}{Q_s} \right) + \frac{b^2}{\varepsilon_0 \delta^2 \psi} \left( 1 - \alpha \frac{b}{d} \right) \left( 1 + \frac{1}{Q_h} \right)} .
\]
(3.47)

This result agrees totally with (2.15), putting on a firm basis the circuital analysis.

3.3. Conclusions From Rigorous Analysis

From the previous section it is seen that (3.42) and (3.45) agree completely with (2.15), thus proving the validity of the circuit analysis; the only difference is that now we must satisfy (3.42) and (3.43). This is easy to do for not very high wall resistances; however, it must be remembered that in reference (5) it was said that the circuit analysis is valid only for wall elements of not very high impedance, which is not contradictory with (3.42) and (3.43); further (3.42) and (3.43) give a criterion of validity of circuit analysis. Not only, but, numerically speaking, (3.43) does not bound \( \tilde{\rho}_z \) in such a way that (2.22) cannot be satisfied. In fact from (3.43) and (2.22), introducing the design criterion \( \varepsilon_{eff} = 0 \), we get
\[
\left[ 1 + 2 \ln \frac{b}{a} \right] \ll \frac{1}{\beta \pi n^2} \left( \frac{R}{d} \right) \left( \frac{L}{d} \right) \gamma^2 .
\]
(3.48)

It is easy to see that, for \( \gamma = 6.1, \ d = 4 \text{ cm}, \ R = 100 \text{ m} \) and
\[
\left[ 1 + 2 \ln \frac{b}{a} \right] = 2.25, \ (3.48) \text{ is satisfied}
\]
for \( L = 100 \text{ cm} \) by \( 1 \leq n \leq 100 \)
and for \( L = 200 \text{ cm} \) by \( 1 \leq n \leq 141 \)
APPENDIX

In section 2 and section 3 we set as wave impedance for the ferrite external and helical internal sheath the quantity

\[ \tilde{\mathcal{R}} = (1-j) \left[ \frac{\omega \mu}{\delta \sigma - j 2 \omega \varepsilon} \right]^{1/2}, \]

i.e. the wave impedance for plane waves; in the case of cylindrical waves this would not be correct if some restrictive conditions are not satisfied.

For this purpose, we consider Maxwell's equations, for instance in the case of TM waves inside the ferrite ring:

\[
\begin{align*}
\{ \frac{j k E_r}{r} - \frac{\partial E_z}{\partial r} &= \frac{j \omega \mu}{c} H_\lambda, \\
\frac{1}{r} \frac{\partial}{\partial r} (r H_\lambda) &= \frac{E_z}{c} (4\pi \sigma - j \omega \varepsilon), \\
-j k H_\lambda &= \frac{E_z}{c} (4\pi \sigma - j \omega \varepsilon)
\end{align*}
\]

(A.1)

We get the usual Bessel equation

\[
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) \right] - k^2 \left( 1 + \frac{r^2}{k^2} \right) E_z = 0,
\]

(A.4)

where

\[ r^2 = \frac{1}{c^2 (-j \omega \mu)(-j \omega \varepsilon + 4\pi \sigma)}. \]

(A.5)

Let us suppose that the thickness of the ferrite is larger than the skin depth; as a consequence there will be only progressive waves, i.e. the solution of (A.4) will be

\[ E_z = D K_0 (\tilde{q} r) \]

(A.6)

where

\[ \tilde{q} = k \sqrt{1 + \frac{r^2}{k^2}}. \]

(A.7)
and from equations (1) (2) and (3)

\[ H_z = \frac{4\pi \sigma - j \omega c}{c q} \frac{3E_z}{\partial q} = - \frac{4\pi \sigma - j \omega c}{c q} D K_1(qr) \]  \hspace{1cm} (A.8)

In this way we get the r-dependent characteristic impedance for progressive cylindrical TM waves.

\[ Z_z^+ (r) = \frac{E_z}{H_z} = - \frac{c q E_0(qr)}{(4\pi \sigma - j \omega \epsilon) K_1(qr)} \]  \hspace{1cm} (A.9a)

by means of analogous calculation the impedance for \( \lambda \)-direction (TE waves) is

\[ Z_z^+ = \frac{E_\lambda}{H_z} = - \frac{j \omega \epsilon K_1(qr)}{c q E_0(qr)} \]  \hspace{1cm} (A.9b)

so that

\[ (Z_z^+ Z_z^-)^+ = - \lambda \]  \hspace{1cm} (A.10a)

In the same way we can find the wave impedance for only regressive TM and TE waves,

\[ Z_z^- = - \frac{c q E_0(qr)}{(4\pi \sigma - j \omega \epsilon) I_1(qr)} \quad Z_\lambda^- = \frac{j \omega \epsilon I_1(qr)}{c q I_0(qr)} \]  \hspace{1cm} (A.10b)

to that in general

\[ (Z_z Z_\lambda)^+ = - \lambda \]  \hspace{1cm} (A.11)

Since we suppose only progressive waves in the ferrite at \( d^+ \) we must consider only (9.a.b).

Matching the waves on the boundary at \( d \), between vacuum and ferrite, we require the tangential components of electric and magnetic fields to be equal:

\[ E_z(d^+) = E_z(d^-) \quad E_\lambda(d^+) = E_\lambda(d^-) \]  \hspace{1cm} (A.12a)

\[ H_z(d^+) = H_z(d^-) \quad H_\lambda(d^+) = H_\lambda(d^-) \]  \hspace{1cm} (A.12b)

In this way, we have in vacuum:

\[ E_z(d^-) = Z_z^+(d) H_\lambda(d^-) \]  \hspace{1cm} (A.13a)
\[ E_{\Lambda}(d^-) = Z_{\Lambda}^+(d) H_{\Lambda}^-(d^-), \quad (A.13b) \]

where magnetic and electric fields at \( d^- \) (in vacuum) include regressive and progressive waves.

In the case of the ferrite \( \Gamma^2 \gg k^2 \), then \( \gamma^2 = \Gamma^2 \) and

\[ Z_{\Lambda}^+(d) = \left[ \frac{-1 + \omega \sigma}{4\pi \sigma - j \omega \varepsilon} \right]^{1/2} \frac{K_1(\tilde{\omega}d)}{K_0(\tilde{\omega}d)}, \quad (A.14a) \]

\[ Z_{\gamma}^+(d) = \left[ \frac{-1 + \omega \sigma}{4\pi \sigma - j \omega \varepsilon} \right]^{1/2} \frac{K_0(\tilde{\omega}d)}{K_1(\tilde{\omega}d)}, \quad (A.14b) \]

Bessel functions \( K_1(x) \) and \( K_0(x) \) have the property that they converge very rapidly to their asymptotic expression

\[ K(x) \sim (\pi/2x) e^{-x} . \]

In fact, for

\[ 0.4 \leq |\Gamma| d \leq \infty \]
\[ 1.9 \gg |K_1/K_0| \gg 1. \]

We must satisfy (2.22) with a strong inequality sign, and thus we make very little error by setting

\[ Z_{\gamma}^+(d) = Z_{\Lambda}^+ = \tilde{\mu} . \]
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References


8. The shielding factor \( \frac{1 - \alpha b^2/d^2}{\sqrt{\frac{1}{\omega}}_1} \), in \( \frac{\phi}{\mu} \), is due to the induced currents flowing in the outer sheath and perpendicular to the axis of the helix. We may derive the shielding factor -- by letting \( \frac{\phi}{\mu} \) be the flux generated by the helix. The flux \( \frac{\phi}{\mu} \) induces a current per unit length in the sheath, \( i_A \), given by:

\[
\frac{i_A}{\mu} = \frac{1}{\frac{R_t - jL_t}{\omega}} \frac{1}{\frac{R_t - jL_t}{\omega}}
\]

where the impedance per unit length has been employed, and where

\[
R_t = \frac{\omega d^2}{c}, \quad \text{and} \quad L_t = \frac{4\pi^2 d^2}{c^2}.
\]

The current \( i_A \) produces a flux \( \frac{\phi}{\mu} = L_t i_A \), of which a fraction \( \frac{b^2}{d^2} \) links the helix. Hence the total flux linking the helix is:

\[
\frac{\phi}{\mu} = \left[ \frac{\phi}{\mu} - \frac{b^2}{d^2} \frac{\phi}{\mu} \right]
\]

which, upon combining the various formulas of this footnote, yields (2.9) and (2.10).
Fig. 1. Helical structure with beam and outer sheath

Fig. 2. Equivalent circuit for the beam-helix-sheath; the outer sheath impedance is $\tilde{Z}_0$, $\tilde{Y}_2$ represents the capacitative and inductive coupling between the helix and the sheath, and $Z_1$ is the impedance of the helix.
Fig. 3. Symbolic drawing of a helical insert in a circular accelerator. The helix spiral is tapered so as to reduce reflections; the outer sheath is electrically isolated from the metal vacuum chamber, while the helix is terminated in the chamber.