Exact Schwarzschild-Like Solution for SU(N) Gauge Theory

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Abstract

In this paper we extend our previously discovered exact solution for an SU(2) gauge theory coupled to a massless, non-interacting scalar field, to the general group SU(N+1). Using the first-order formalism of Bogomolny, an exact, spherically symmetric solution for the gauge and scalar fields is found. This solution is similar to the Schwarzschild solution of general relativity, in that the gauge and scalar fields become infinite at a radius, $r_0 = K$, from the origin. It is speculated that this may be the confinement mechanism that has long been sought for in non-Abelian gauge theories, since any particle which carries the SU(N+1) charge would become permanently trapped once it entered the region $r < r_0$. The energy of the field configuration of this solution is calculated.

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I. THE SU(N+1) SCHWARZSCHILD-LIKE SOLUTION

In a previous paper [1] we exploited the connection between general relativity and Yang-Mills theory to find an exact Schwarzschild-like solution for an SU(2) gauge theory coupled to a massless scalar field. In the present paper we wish to show that a similar solution can be found for the general group SU(N+1). Instead of using the Euler-Lagrange formalism which leads to coupled second-order, nonlinear equations, we will use the Bogomolny approach [2] to derive our field equations. Bogomolny obtained his first-order version of the Yang-Mills field equations by requiring that the gauge and scalar fields produce an extremum of the canonical Hamiltonian. The field equations obtained in this way are first-order, but their solutions are also solutions to the second-order Euler-Lagrange equations.

The model which we consider here is an SU(N+1) gauge field coupled to a massless scalar field in the adjoint representation. The Lagrangian for this theory is

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F_{\mu\nu}^{a} + \frac{1}{2} D_{\mu}(\Phi^{a}) D_{\mu}(\Phi^{a}) \]  

where

\[ F_{\mu\nu}^{a} = \partial_{\mu} W_{\nu}^{a} - \partial_{\nu} W_{\mu}^{a} + g f^{abc} W_{\mu}^{b} W_{\nu}^{c} \]  

and

\[ D_{\mu} \Phi^{a} = \partial_{\mu} \Phi^{a} + g f^{abc} W_{\mu}^{b} \Phi^{c} \]

where \( f^{abc} \) are the structure constants of the gauge group and \( a, b, c = 1, 2, \ldots, N, N+1 \).

The canonical Hamiltonian obtained from Eq. (1) is

\[ \mathcal{H} = \int d^{3}x \left[ \frac{1}{4} F_{ij}^{a} F^{aij} - \frac{1}{2} F_{a0}^{a} F^{a0i} + \frac{1}{2} D_{i} \Phi^{a} D^{i} \Phi^{a} - \frac{1}{2} D_{0} \Phi^{a} D^{0} \Phi^{a} \right] \]  

We wish to find gauge and scalar fields which produce an extremum of \( \mathcal{H} \). First we rescale the scalar field (i.e. \( \Phi^{a} \rightarrow \lambda \Phi^{a} \)). This is done so that later on it will be simple to examine the pure gauge case by setting \( \lambda = 0 \). Next we specify that all the fields are time independent, and that the time component of the gauge fields are proportional to the scalar fields (i.e
\( W_0^a = C \Phi^a \), where \( \Phi^a \) is the rescaled scalar field. The time component of the gauge fields act like an additional Higgs field except its kinetic term appears with the opposite sign in the Lagrangian [3]. Using these two requirements and the antisymmetry of \( f^{abc} \) we find that \( D_0 \Phi^a = 0 \) and \( F_{0i}^a = C(D_i \Phi^a) \), so that the Hamiltonian becomes

\[
\mathcal{H} = \int d^3 x \left[ \frac{1}{4} \left( F_{ij}^a - \epsilon_{ijk} \sqrt{A^2 - C^2} D^k \Phi^a \right) \left( F^{aij} - \epsilon_{ijk} \sqrt{A^2 - C^2} D^i \Phi^a \right) + \frac{1}{2} \epsilon_{ijk} \sqrt{A^2 - C^2} F^{aij} D^k \Phi^a \right]
\]

Using the fact that

\[
\frac{1}{2} \epsilon_{ijk} F^{aij} D^k \Phi^a = \partial^i \left( \frac{1}{2} \epsilon_{ijk} F^{ajk} \Phi^a \right)
\]

and the requirement that the solutions we are looking for are only functions of \( r \) we find

\[
\mathcal{H} = \sqrt{A^2 - C^2} \int_S (\Phi^a B_i^a) dS^i \\
+ \int d^3 x \left[ \frac{1}{4} \left( F_{ij}^a - \epsilon_{ijk} \sqrt{A^2 - C^2} D^k \Phi^a \right) \left( F^{aij} - \epsilon_{ijk} \sqrt{A^2 - C^2} D^i \Phi^a \right) \right]
\]

For the total divergence term we have used the definition of the non-Abelian magnetic field in terms of the field strength tensor (i.e. \( B_i^a = \frac{1}{2} \epsilon_{ijk} F^{ajk} \)), and used Gauss’s Law to turn the volume integral into a surface integral. The lower limit of this Hamiltonian can be found by requiring

\[
F_{ij}^a = \epsilon_{ijk} \sqrt{A^2 - C^2} D^k \Phi^a
\]

or

\[
B_i^a = \sqrt{A^2 - C^2} D_i \Phi^a
\]

To get the second expression we have again used the definition of the non-Abelian magnetic field. These are the Bogomolnıy equations [2]. Wilkinson and Goldhaber [4] have given a generalized ansatz for the gauge and scalar fields

\[
W_i = \frac{\epsilon_{ijk} (T^b - M^b(r))}{g r^2} \\
\Phi^a = \frac{\Phi(r)}{g}
\]
$W_i$ are three $(N+1) \times (N+1)$ matrices of the gauge fields. $M_b(r)$ and $\Phi(r)$ are four $(N+1) \times (N+1)$ matrices whose elements are functions of $r$, and in terms of which the Bogomolny equations will be written. $T_b$ are three $(N+1) \times (N+1)$ matrices which generate the maximal embedding of SU(2) in SU(N+1). Because of the spherical symmetry requirement one can look at Eq. (8) along any axis [5] [6]. Taking the positive $\hat{z}$ axis the Bogomolny equations become $\sqrt{A^2 - C^2}(D_\phi \Phi^a) = B^a_3$ and $\sqrt{A^2 - C^2}(D_\phi \Phi^a) = B^a_{\pm}$, or in terms of the ansatz of Eq. (9)

$$r^2 \sqrt{A^2 - C^2} \frac{d\Phi}{dr} = [M_+, M_-] - T_3$$

$$\frac{dM_{\pm}}{dr} = \mp \sqrt{A^2 - C^2}[M_{\pm}, \Phi]$$ (10)

Taking the third “component” of the maximal SU(2) embedding into SU(N+1) as

$$T_3 = \text{diag} \left[ \frac{1}{2} N, \frac{1}{2} N - 1, \ldots, -\frac{1}{2} N + 1, -\frac{1}{2} N \right]$$ (11)

it has been shown [4] that the matrix functions, $M_+(r)$ and $\Phi(r)$, can be taken as

$$\Phi = \frac{1}{2} \begin{pmatrix} \phi_1 \\ \phi_2 - \phi_1 \\ \vdots \\ \phi_N - \phi_{N-1} \\ -\phi_N \end{pmatrix}$$ (12)

$$M_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & a_1 \\ 0 & 0 & a_2 \\ \vdots \\ 0 & 0 & 0 & a_N \end{pmatrix}$$ (13)

where $\phi_m$ and $a_m$ are real functions of $r$ and $M_- = (M_+)^T$. Substituting Eqs. (12), (13) into the first order field equations of Eq. (10) the field equations become [5] [6]
\[ r^2 \frac{d\phi_m}{dr} = \frac{1}{\sqrt{A^2 - C^2}} \left[ (a_m)^2 - m\bar{m} \right] \]
\[ \frac{da_m}{dr} = \sqrt{A^2 - C^2} \left( -\frac{1}{2} \phi_{m-1} + \phi_m - \frac{1}{2} \phi_{m+1} \right) a_m \] (14)

where \( 1 \leq m \leq N \), \( \bar{m} = N + 1 - m \) and \( \phi_0 = \phi_{N+1} = 0 \). Exact solutions have been found to Eq. (14) [5] [6] which are generalizations of the well known Prasad-Sommerfield solution [7] for SU(2). The Prasad-Sommerfield solution and their generalizations satisfy the boundary condition that the gauge and scalar fields are finite at the origin. If one does not require that the fields be finite at the origin then the coupled equations for \( \phi_m(r) \) and \( a_m(r) \) can be solved by

\[ \phi_m(r) = \frac{1}{\sqrt{A^2 - C^2}} \frac{K m \bar{m}}{r(K - r)} \]
\[ a_m(r) = \frac{r \sqrt{m \bar{m}}}{K - r} \] (15)

\( K \) is an arbitrary constant with the dimensions of distance. This is the generalization of a similar solution which we found for SU(2) using the second-order Euler-Lagrange formalism [1]. The reason for wanting to generalize our solution to SU(N+1) is to give a possible explanation for the confinement mechanism in QCD whose gauge group is SU(3). Inserting the functions \( \phi_m(r) \) and \( a_m(r) \) of Eq. (15) into the fields of Eq. (9) we find that these fields become infinite at a finite radius of

\[ r_0 = K \] (16)

The non-Abelian “electric” \( (E^a_i = F^a_{0i}) \) and “magnetic” fields \( (B^a_i = \frac{1}{2} \epsilon_{ijk} F^{jk} a) \) calculated for this solution also become infinite at \( r_0 \). Thus any particle which carries an SU(N+1) charge would either never be able to penetrate beyond \( r_0 \) (when the SU(N+1) charges are repulsive) or once it passed into the region \( r < r_0 \) it would never be able to escape back to the region \( r > r_0 \). One has in a sense a color charge black hole.

Both the present solution and our SU(2) solution were found by using the connection between Yang-Mills theory and general relativity [8], and trying to find the Yang-Mills equivalent of the Schwarzschild solution. The objects in general relativity which correspond
to the gauge fields are the Christoffel coefficients, $\Gamma^\alpha_{\beta\gamma}$. Examining a few of the Christoffel symbols of the Schwarzschild solution we find

$$\Gamma^r_{rt} = \frac{2GM}{2r(r - 2GM)}$$
$$\Gamma^r_{rr} = -\frac{2GM}{2r(r - 2GM)}$$

(17)

where $2GM$ the equivalent of the constant $K$ from the Yang-Mills solution. The similarity between these Christoffel coefficients and the gauge and scalar fields that result from the solutions, $\phi_m(r)$ and $a_m(r)$ of Eq. (15), is striking. The most important similarity from the point of explaining confinement is the existence in both solutions of an event horizon, from which particles which carry the appropriate charge can not escape once they pass into the region $r < r_0$. For general relativity the appropriate “charge” is mass-energy so that nothing can climb back out of the Schwarzschild horizon, while in the Yang-Mills case only particles carrying an SU(N+1) charge will become confined.

One slightly disturbing feature of these Schwarzschild-like SU(N+1) solutions is that they have an infinite energy due to the singularity at $r = 0$. When quantities such as the energy of this field configuration are calculated the integral must be cutoff at some arbitrary radius, $r_c$. The singularity at $r_0$ does not give an infinite energy unless one sets $r_c = K$. This singular behaviour at the origin is shared by several other classical field theory solutions. Both the Schwarzschild solution of general relativity, and the Coulomb solution in electromagnetism have similar singularities. The Wu-Yang solution [9] for static SU(2) gauge fields with no time component also blows up at the origin, leading to an infinite energy if one integrates the energy density down to $r = 0$. Just as these classical solutions are not expected to hold down to $r = 0$ so the present solution will certainly be modified by quantum corrections as $r$ approaches zero. Phenomenologically we know that the present solutions can not be correct for very small $r$, since they do not exhibit the asymptotic freedom behaviour that is a desirable consequences of the quantum corrections to QCD. It would be interesting to see if the behaviour of the fields at the origin could be modified by introducing a mass term ($m^2 \Phi^a \Phi^a$) and a self interaction term ($\frac{1}{4}(\Phi^a \Phi^a)^2$) to the scalar field part of the Lagrangian,
while still retaining the color event horizon feature of the present solution. This smoothing of the fields at the origin does happen when one compares the Prasad-Sommerfield exact solution (where there are no mass or self interaction terms for the scalar field) with the the numerical results of 't Hooft [10] or Julia and Zee [3] (where mass and self interaction terms are included). The numerical results lead to monopoles and dyons with a finite core, while the exact Prasad-Sommerfield solution leads to a point monopole (despite this their exact solution still has finite energy when the energy density is integrated down to zero, unlike our present solution). As in the case of the Prasad-Sommerfield solution, introducing mass and self interaction terms for the scalar fields would require solving the equations numerically, since we have not been able to find an analytical solution under these conditions.

To calculate the energy of the field configuration of our solution it is necessary to integrate the $T_{00}$ component of the energy-momentum tensor over all space, excluding the origin. The $T_{00}$ component of the energy-momentum tensor is similar to the Hamiltonian density of Eq. (5) except that all the terms have positive signs

$$T_{00} = \frac{1}{4} F_{ij}^a F^{aij} + \frac{1}{2} F_{a0i}^a F^{a0i} + \frac{A^2}{2} D_i \Phi^a D^i \Phi^a + \frac{A^2}{2} D_0 \Phi^a D^0 \Phi^a$$  \hspace{1cm} (18)

The energy in the fields of our solution is the integral over all space of $T_{00}$. Using the field equations of Eq. (8), and the fact that $F_{0i} = C(D_i \Phi^a)$ the energy of the fields is

$$E = \int T_{00} d^3x$$

$$E = A^2 \int d^3x D_i \Phi^a D^i \Phi^a$$  \hspace{1cm} (19)

Since the fields only have a radial dependence the angular part of the integration can be easily done. Further using the radial symmetry to evaluate the integrand along the positive $\hat{z}$ axis, and using the matrix expression for $\Phi^a$ as well as the field equations, Eq. (10), we find that the energy becomes [4]

$$E = \frac{8\pi A^2}{g^2} \int_{r_c}^\infty r^2 dr \left( Tr \left( \left( \frac{d\Phi}{dr} \right)^2 \right) + \frac{2}{A^2 - C^2} Tr \left( r^{-2} \frac{dM_+}{dr} \frac{dM_-}{dr} \right) \right)$$  \hspace{1cm} (20)

Using the solutions for the elements of the matrices, $\Phi$ and $M_\pm$ of Eq. (15) we find
\[
Tr \left( \left( \frac{d \Phi}{d r} \right)^2 \right) = \frac{K^2 (2r - K)^2}{4r^4 (A^2 - C^2)(K - r)^4} \sum_{n=0}^{N} (N - 2n)^2 \\
Tr \left( r^{-2} \frac{d M_+}{d r} \frac{d M_-}{d r} \right) = \frac{K^2}{r^2 (K - r)^4} \sum_{n=0}^{N} n (N + 1 - n)
\]

Using this in Eq. (20) and carrying through the integration the energy in the field configuration of this Schwarzschild-like solution is

\[
E = \frac{2 \pi A^2 K^2 N (N + 1)(N + 2)}{3g^2 (A^2 - C^2)} \left[ \frac{K - 2r_c}{r_c (K - r_c)^3} \right]
\]

where the sums from Eq. (21) have been done explicitly. This result can be checked against the SU(2) result [1] by taking \( N = 1 \) in Eq. (22), and the expressions for the energy do indeed agree (In Ref. [1] we required that \( A^2 - C^2 = 1 \) whereas here this factor is divided out). As it stands there are two arbitrary constants that enter the solution (\( i.e. K \) and \( r_c \)) which would have to be specified before any connection between this Schwarzschild-like solution and the real world could be carried out. As has already been mentioned \( K \) is the Yang-Mills equivalent of \( 2GM \) in general relativity. Thus it could be conjectured that \( K \) is related to the strength of the gauge interaction (\( G \) in general relativity), and the magnitude of the central charge which produces the gauge field configuration (\( M \) in general relativity).

One interesting feature which this general SU(N+1) Schwarzschild-like solution shares with our previous SU(2) solution is that scalar fields are apparently required in order to get a physically non-trivial solution. If there where no scalar fields in the original Lagrangian (\( i.e. A = 0 \)), then the field energy of Eq. (22) would be zero, and the \( W^a_0 \) component of the gauge fields would be pure imaginary. Although the pure gauge case with no scalar fields is a solution mathematically, its physical significance is dubious. Requiring that the solutions are pure real, or that the energy in the fields be non-zero would exclude the pure gauge case solution. Under either of these requirements on the solution, it can be seen that scalar fields must be present.
II. CONCLUSIONS

In this paper we have generalized our previous exact, Schwarzschild-like solution for SU(2) Yang-Mills theory to SU(N+1) by using an embedding of SU(2) into SU(N+1) [4]. This exact SU(N+1) solution was found by using the connection between general relativity and Yang-Mills theories. It was found that the Schwarzschild solution of general relativity carries over with only a little modification into an equivalent solution for an SU(N+1) gauge theory coupled to massless scalar fields. Just as the Schwarzschild solution possesses an event horizon which permanently confines any particle which carries the “charge” of the gravitational interaction (i.e. mass-energy), so the present solution also has a “color” event horizon which permanently confines any particle which carries the SU(N+1) gauge charge. This may be the confinement mechanism which has long been sought for the SU(3) gauge theory of the strong interaction, QCD. Before this claim can be made there are several important questions which must be resolved. First, under some reasonable physical assumptions about the nature of our solution it is found that scalar fields are required for a solution to exist. Normally scalar fields are not thought to play a significant role in confinement, so the physical importance of these scalar fields would need to be addressed. Second, there are several arbitrary constants which crop up in the solution (K and $r_c$). In order to make a connection with the real world these constants would have to be given. Theoretically $K$ should be related to the strength of the gauge interaction as well as the magnitude of the gauge charge which produces the Schwarzschild-like gauge fields. Experimentally $K$ should be related to the radius of the various QCD bound states (e.g. protons, pions, etc.). The other constant, $r_c$, was introduced chiefly to avoid the singularity at $r = 0$, but also because our solution does not possess the property of asymptotic freedom as $r \to 0$. This should not be too surprising since our solution is for classical Yang-Mills fields, but as $r \to 0$ quantum effects should become increasingly important. Thus $r_c$ can be thought of roughly, as marking the boundary between the classical, confining solution of this paper, and the quantum dominated asymptotic freedom regime. All this strongly suggests a bag-like structure for QCD
bound states: As a particle approaches $r = K$ from $r < K$, it feels a progressively stronger color force which confines it to remain inside the bound state. As the particle approaches $r \to 0$ it enters the asymptotic freedom regime, where it moves as if it were free.

An interesting extension of this work would be to see if other exact solutions from general relativity have Yang-Mills counterparts. In particular if a Yang-Mills equivalent of the Kerr solution could be found it might give some insight into the nature of the spin of fermions.

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REFERENCES


