Initial Value Problems and Signature Change

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Abstract

We make a rigorous study of classical field equations on a signature changing spacetime using the techniques of operator theory. Boundary conditions at the signature changing surface are determined by forming self-adjoint extensions of the Schrödinger Hamiltonian. We show that the initial value problem for the Klein–Gordon equation on this spacetime is ill-posed in the sense that its solutions are unstable. Furthermore, if the initial data is smooth and compactly supported away from the surface of signature change, the solution has divergent $L^2$-norm after finite time.

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1 Introduction

A natural generalisation of general relativity is obtained by relaxing the restriction that the metric tensor is everywhere Lorentzian. This subject has produced a long debate on the nature of the junction conditions which must be satisfied by the metric tensor and matter fields at the surface of signature change. Although there seems to be some room for manoeuvre, depending upon what one regards as pathological, it is generally accepted that, the second fundamental form must vanish at the surface of signature change [1], and similarly the momenta of all fields must vanish [2, 3]. With this choice of junction conditions, the gravity-matter field equations are free of distributional terms.

In this paper we will follow the work of [4]. We fix the signature changing spacetime to be flat (and hence the second fundamental form is trivially zero), and examine classical fields that propagate, with no interaction, on the background spacetime. In this situation, there are no pre-ordained junction conditions for the matter fields at the surface of signature change. Writing the matter field equations in Hamiltonian form, we use Hilbert space techniques to investigate their solutions. Thus a matter field Hamiltonian is regarded as an operator on a Hilbert space and any junction conditions which are applied to the matter field are equivalent to boundary conditions satisfied by functions in the domain of its Hamiltonian.

It was shown in [4] that a wide range of field behaviour could be generated by an appropriate choice of boundary conditions. Thus a method for choosing one set of boundary conditions in preference to others is required. The usual approach would be to choose the boundary conditions which best modelled the physical situation, however, in the signature change scenario, the physical situation is unknown (ie. the scattering properties of a signature change surface are unknown). In such a situation, it makes sense to choose boundary conditions which provide the most well-behaved system. If, with such a choice, the system is still poorly defined then one has grounds for concluding that the situation which is being modelled is unphysical. If, on the other hand, the system is fairly well-defined, one should obtain some reasonable physical predictions, which can then be used to reassess the theory. In the
signature change scenario, a particularly simple choice is to assume that the matter field Hamiltonian operator is self-adjoint. This choice proves to be highly successful, as it singles out one particular set of boundary conditions for the matter fields. Moreover, self-adjointness allows us to construct a complete set of (generalised) eigenfunctions, which facilitates the construction of $L^2$-solutions.

In section 2 we briefly review the necessary operator theory, before proceeding in section 3 to examine the properties of the Dirac, Schrödinger and Klein–Gordon Hamiltonians on a model signature changing spacetime. Using these properties, we explain why our methods cannot be applied to the Dirac Hamiltonian. In section 4 we determine the boundary conditions corresponding to self-adjoint extensions of the Schrödinger Hamiltonian. Solutions of the Schrödinger and Klein–Gordon equations are constructed and analysed in section 5. We show that the initial value problem for the Klein–Gordon equation is ill-posed in the sense that its solutions are unstable, and that the evolution of reasonable initial data leads to runaway solutions.

2 Operator Theory

Physical systems are often modelled using operators on a Hilbert space. If such a system exhibits singular points (where the formal expression of an operator does not apply in an obvious way), it is often initially convenient to define the action of the operators on a space of ‘nice’ functions supported away from the singular points. In our case of interest, this occurs at the signature changing surface; although it is clear how to define our operators away from this surface, it is not \textit{a priori} clear how to define them on the surface. The initial choice of domain is not generally sufficient to effectively model the physical situation, in particular, the operator may not be self-adjoint when restricted to this domain. If self-adjointness is required, one can try to extend the domain and action of our operator in order to determine a rigorously defined self-adjoint operator, which is called a \textit{self-adjoint extension} of the original operator. For differential operators, this is usually equivalent to the specification of appropriate boundary conditions at singular points (in our case, at the surface of signature change).
2.1 Preliminary definitions

Although they are standard (see eg. [5]), the following definitions have been included so that the work presented here is self-contained.

Let $A$ be an operator defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot | \cdot \rangle$.

Definition. If for some $\phi \in \mathcal{H}$ there exists $\eta$ such that $\langle \phi | A \psi \rangle = \langle \eta | \psi \rangle$ for all $\psi \in \mathcal{D}$, then $\phi$ is said to be in the domain of the adjoint $A^*$ of $A$, and $A^*$ is defined on $\phi$ by $A^*\phi = \eta$.

Definition. Let $A'$ be any operator whose domain includes $\mathcal{D}$, then $A'$ will be called an extension of $A$ if $A'\psi = A\psi$ for all $\psi \in \mathcal{D}$. $A$ is said to be symmetric if $A^*$ is an extension of $A$, (ie. if $\langle \psi | A \phi \rangle = \langle A \psi | \phi \rangle$ for all $\psi, \phi \in \mathcal{D}$).

Definition. $A$ is said to be self-adjoint if $A = A^*$, (ie. if $A$ is symmetric and has the same domain as its adjoint).

Definition. If the domain of $A^*$ is dense, then we define the closure $\tilde{A}$ of $A$ by $\tilde{A} = A^{**}$. $A$ is said to be essentially self-adjoint if $\tilde{A}$ is self-adjoint. Moreover if $A$ is essentially self-adjoint then $\tilde{A} = A^*$, and $A^*$ is the unique self-adjoint extension of $A$.

2.2 Deficiency indices

The von Neumann theory of deficiency indices [5] can be used to determine whether a symmetric operator $A$ with domain $\mathcal{D}$ has any self-adjoint extensions, and if so, to construct them.

Define the deficiency subspaces $\mathcal{K}^+$ and $\mathcal{K}^-$ by

$$\mathcal{K}^+ = \ker(A^* \mp i),$$

(2.1)

ie. $\mathcal{K}^+$ is spanned by $L^2$-solutions of equation $A^*\Psi = \pm i\Psi$.

Defining the deficiency indices by $n^* = \dim(\mathcal{K}^*)$ then there are three cases
as follows:

(1) \( n^+ \neq n^- \iff A \) has no self-adjoint extensions.

(2) \( n^+ = n^- = 0 \iff A \) has a unique self-adjoint extension.

(3) \( n^+ = n^- = N > 0 \iff \exists \) a unique family of self-adjoint extensions of \( A \) labelled by \( U(N) \).

In case (2), \( A \) is essentially self-adjoint and its unique self-adjoint extension is equal to \( A^* \). In case (3), which will be our case of interest, the self-adjoint extensions are parametrised as follows: representing \( U(N) \) as the space of unitary maps from \( K^+ \) to \( K^- \), we define for each \( U \in U(N) \) the domain \( \mathcal{D}_U \) by

\[
\mathcal{D}_U = \{ \psi + \chi^* + U\chi^* | \psi \in \mathcal{D}, \chi^* \in K^+ \}.
\]  

(2.2)

The operator \( A^*[\mathbb{H}_U] \) is clearly an extension of \( A \), moreover, \( A^*[\mathbb{H}_U] \) is essentially self-adjoint with unique self-adjoint extension \( A_U \) given by

\[
A_U = (A^*[\mathbb{H}_U])^*.
\]  

(2.3)

The full set of self-adjoint extensions of \( A \) is then provided by the family \( \{ A_U | U \in U(N) \} \).

3 Hamiltonian Form of the Classical Wave Equations

We will consider a simple two-dimensional model in which the spacetime signature changes from \((+-)\) to \((+++)\). Since we are identifying ‘+’ with time and our operators (in the Lorentzian region) are the usual Hamiltonians, this type of signature change has more similarity with four-dimensional Lorentzian to Kleinian signature change [4], than with four-dimensional Riemannian to Lorentzian signature change.

To begin our investigation we will examine discontinuous signature change with the flat metric

\[
ds^2 = g_{ab}dx^a dx^b = dt^2 + \text{sign}(z)dz^2.
\]  

(3.1)
3.1 The Dirac Hamiltonian

In ordinary Minkowski space one can easily define a self-adjoint Dirac Hamiltonian [6]. We now show that this is not possible in the flat signature changing spacetime (3.1).

For \( z < 0 \) define the gamma matrices \( \gamma^a_L \) by \( \{ \gamma^a_L, \gamma^b_L \} = 2g^{ab} \), and similarly, for \( z > 0 \) define the Kleinian gamma matrices \( \gamma^a_K \) by \( \{ \gamma^a_K, \gamma^b_K \} = 2g^{ab} \). Choose \( \gamma^0_K = \gamma^0_L \), then there are two possibilities for \( \gamma^1_K \), given by \( \gamma^1_K = \pm i \gamma^1_L \). We will use the standard representation, in which

\[
\gamma^0_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^1_L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The Dirac equation for the signature changing spacetime is

\[
(\gamma^a_L \partial_a + im)\Psi = 0 \quad z < 0, \quad (3.3)
\]

\[
(\gamma^a_K \partial_a + im)\Psi = 0 \quad z > 0. \quad (3.4)
\]

Defining

\[
\xi(z) = \begin{cases} 
1 & z < 0 \\
\pm i & z > 0
\end{cases},
\]

the Dirac equation may be written in the Hamiltonian form

\[
i \frac{\partial}{\partial t} \Psi = D \Psi
\]

where we will call

\[
D = \gamma^0_L \left[ m - i\xi(z)\gamma^1_L \frac{\partial}{\partial z} \right]
\]

the Dirac Hamiltonian. Initially, we define \( D \) on the dense domain \( D \) of smooth spinors compactly supported away from the origin. Note that \( D \) is a subset of the Hilbert space of \( L^2 \)-spinors.

A short integration by parts argument shows that

\[
D^*\big|_D = \gamma^0_L \left[ m - i\xi(z)\gamma^1_L \frac{\partial}{\partial z} \right], \quad (3.8)
\]
and since $D^\ast |_D \neq D$ we find that $D$ is not symmetric, and therefore does not admit any self-adjoint extensions. Since the local Dirac Hamiltonian should certainly contain $D$ in its domain, we conclude that the spacetime (3.1) does not admit a self-adjoint Dirac Hamiltonian.

3.2 The Schrödinger Hamiltonian

Consider the formal square of the Dirac Hamiltonian, denoted $D^2$. Since we are only concerned with functions that are compactly supported away from the origin, we need not concern ourselves with derivatives of $\xi(z)$ at $z = 0$. It is then easy to show that

$$D^2 = H \otimes 1$$

where

$$H = m^2 + \text{sign}(z) \frac{\partial^2}{\partial z^2}$$ (3.9)

on $C_0^\infty(\mathbb{R}\setminus \{0\}) \subset L^2(\mathbb{R})$ will be referred to as the Schrödinger Hamiltonian. Since $-\frac{\partial^2}{\partial z^2}$ is unbounded from above on $L^2(\mathbb{R}^+)$ and $\frac{\partial^2}{\partial z^2}$ is unbounded from below on $L^2(\mathbb{R}^+)$, we note that $H$ is unbounded from both above and below.

The fact that $H$ is symmetric can be demonstrated with an integration by parts argument, and hence we can attempt to find self-adjoint extensions of $H$ using the von Neumann theory of deficiency indices.

3.3 The Klein–Gordon Hamiltonian

The Klein–Gordon equation on the discontinuously signature changing spacetime (3.1) is

$$\left( g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} + m^2 \right) \phi = 0 .$$ (3.11)
Let

\[ \Phi(t) = \begin{pmatrix} \dot{\phi} \\ \phi \end{pmatrix} \]

(3.12)

where \( \dot{\phi} = \frac{\partial \phi}{\partial t} \), then we may write the Klein–Gordon equation in ‘first-order form’ as

\[ \frac{\partial \Phi}{\partial t} = K \Phi \]

(3.13)

where the operator

\[ K = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix} \]

(3.14)

on \( C_0^\infty(\mathbb{R}\backslash\{0\}) \oplus L^2(\mathbb{R}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) will be called the Klein–Gordon Hamiltonian.

Clearly \( K \) is not symmetric and therefore does not admit any self-adjoint extensions. Despite this, the special form of \( K \) and the fact that \( H \) has self-adjoint extensions will allow us to construct rigorously defined solutions to (3.13).

4 Boundary Conditions

In this section we construct self-adjoint extensions of the Schrödinger Hamiltonian \( H \), and study the boundary conditions which provide these extensions. There are many possible sets of boundary conditions for a matter field \( \phi \), however, we are interested in the subset which describe how \( \phi \) and \( \frac{\partial \phi}{\partial z} \) behave as they cross the surface of signature change. Since we have no data to guide us, a reasonable physical assumption is that the field \( \phi \) is everywhere continuous. This leaves the behaviour of \( \frac{\partial \phi}{\partial z} \) to be determined. Thus we are looking for boundary conditions of the form

\[ \phi(0^-) = \phi(0^+) \],

(4.1)

\[ \frac{\partial \phi}{\partial z}(0^-) = \omega \frac{\partial \phi}{\partial z}(0^+) \]

for some \( \omega \in \mathbb{C}, \omega \neq 0 \).
4.1 Self-adjoint extensions

We will now construct self-adjoint extensions of \( H \). In this analysis, it suffices to treat the massless case \((m = 0)\) since \( m \) only appears as an additive constant in \( H \). Applying the von Neumann theory of deficiency indices we try to find \( L^2 \)-bases for \( \mathcal{K}^+ \) and \( \mathcal{K}^- \). We find that \( \mathcal{K}^+ \) is spanned by

\[
\chi^+_i = \begin{cases} 
2 \frac{i}{z} \exp(-i \frac{3\pi}{4} z) & z < 0 \\
0 & z > 0
\end{cases} \quad \chi^+_2 = \begin{cases} 
0 & z < 0 \\
2 \frac{i}{z} \exp(-i \frac{3\pi}{4} z) & z > 0
\end{cases}
\tag{4.2}
\]

and \( \mathcal{K}^- \) is spanned by

\[
\chi^-_i = \begin{cases} 
2 \frac{i}{z} \exp(+i \frac{3\pi}{4} z) & z < 0 \\
0 & z > 0
\end{cases} \quad \chi^-_2 = \begin{cases} 
0 & z < 0 \\
2 \frac{i}{z} \exp(+i \frac{3\pi}{4} z) & z > 0
\end{cases}
\tag{4.3}
\]

where the normalisation \( ||\chi^+_i|| = ||\chi^-_i|| = ||\chi^+_2|| = ||\chi^-_2|| = 1 \) has been implemented. The deficiency indices are thus \( n^+ = n^- = 2 \), and hence there is a family of self-adjoint extensions of \( H \) labelled by \( U(2) \).

As described in section 2, for each \( U \in U(2) \) we have a unique self-adjoint extension \( H_v \) of \( H \) given by

\[
H_v = (H^*|_{D_v})^*
\]

where

\[
D_v = \{ \phi + \chi^+ + U \chi^+ | \phi \in C^\infty_0(\mathbb{R}\{0\}), \chi^+ \in \mathcal{K}^+ \} .
\tag{4.5}
\]

We now construct the boundary conditions which correspond to \( H_v \). For any \( \rho, \psi \in \mathcal{D}(H^*) \) it is possible to show that

\[
\langle \rho | H^* \psi \rangle = \langle H^* \rho | \psi \rangle + \lim_{z \to 0^-} \left[ \frac{\partial \rho^+}{\partial z} \psi - \rho^+ \frac{\partial \psi}{\partial z} \right] + \lim_{z \to 0^+} \left[ \frac{\partial \rho^+}{\partial z} \psi - \rho^+ \frac{\partial \psi}{\partial z} \right] .
\tag{4.6}
\]

We note in passing that if we were studying \(-\frac{\partial}{\partial z}\) on \( C^\infty_0(\mathbb{R}\{0\}) \) instead of \( H = sign(z) \frac{\partial^2}{\partial z^2} \) on \( C^\infty_0(\mathbb{R}\{0\}) \), then the boundary terms in the analogue to (4.6) would differ by a relative sign [7].

The domain of \( H_v \) consists of those \( \rho \in \mathcal{D}(H^*) \) for which the boundary terms in (4.6) vanish for all \( \psi \in \mathcal{D}_v \). Clearly for \( \psi \in C^\infty_0(\mathbb{R}\{0\}) \) the boundary
terms are zero, hence it suffices to consider $\psi \in \{ \chi^\pm + U \chi^\pm | \chi^\pm \in \mathcal{K}^\pm \}$. By choosing such $\psi$ we are able to determine the boundary conditions which $\rho^\dagger$ and $\frac{\partial \rho^\dagger}{\partial z}$ must satisfy at $z = 0$.

Choosing the basis (4.2) for $\mathcal{K}^+$ and the basis (4.3) for $\mathcal{K}^-$, the unitary map $U$ can be written as a matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$aa + bb = 1, \quad cc + dd = 1, \quad ac + bd = 0.$$  \hspace{2cm} (4.8)

By substituting for $\psi$ in (4.6), it can now be shown that the boundary terms vanish if

$$A \begin{pmatrix} \frac{\partial \rho^\dagger}{\partial z}(0^-) \\ \frac{\partial \rho^\dagger}{\partial z}(0^+) \end{pmatrix} = e^{i\pi/4}B \begin{pmatrix} \rho^\dagger(0^-) \\ \rho^\dagger(0^+) \end{pmatrix}$$  \hspace{2cm} (4.9)

where the matrices $A$ and $B$ are given by

$$A = \begin{pmatrix} a + 1 & c \\ b & d + 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a - i & ic \\ b & id - 1 \end{pmatrix}.$$  \hspace{2cm} (4.10)

Thus for each unitary matrix $U$ there is a self-adjoint extension $H_u$ of $H$ with a corresponding set of boundary conditions (4.9).

### 4.2 Determining the boundary conditions

We wish to find self-adjoint extensions of $H$ with corresponding boundary conditions of the form (4.1). In order to complete this task, it is clear from (4.9) that $A$ and $B$ must be singular, since if either $A$ or $B$ is invertible, we will obtain boundary conditions which relate the field to its derivative. In the appendix we prove the surprising result that there is only one self-adjoint extension satisfying the above restrictions. Its corresponding boundary conditions are

$$\rho^\dagger(0^-) = \rho^\dagger(0^+),$$
This is the set of boundary conditions which we will adopt in section 5 for the construction of wave equation solutions.

In particular attention was devoted to the boundary conditions $\rho^t$ continuous at $z = 0$, and $\frac{\partial \rho^t}{\partial z}(0^-) = \frac{\partial \rho^t}{\partial z}(0^+)$ or $\frac{\partial \rho^t}{\partial z}(0^-) = \pm i \frac{\partial \rho^t}{\partial z}(0^+)$; it is worth emphasising that no self-adjoint extension corresponds to such boundary conditions with corresponding boundary conditions.

4.3 Continuous signature change

Choosing coordinates $\{t, \zeta\}$, let us briefly consider the continuous signature change metric

$$ (g^{ab}) = d t \, d g(1, \zeta) . \tag{4.12} $$

We have chosen $g^{ab}$ rather than $g_{ab}$ to be continuous, since it is $g^{ab}$ which appears in the matter field equations.

The region $z < 0$ for the discontinuous signature change metric (3.1) is diffeomorphic to the region $\zeta < 0$ for the continuous signature change metric (4.12). Similarly, the regions $z > 0$ and $\zeta > 0$ are diffeomorphic. Thus, if there is a physical difference between the two spacetimes, it occurs at $z = \zeta = 0$. However, we note that the analysis of the Schrödinger Hamiltonian on a spacetime with metric (3.1) begins by considering functions compactly supported away from the origin $z = 0$. Thus it seems likely that there is no physical difference between analysis on (3.1) and analysis on (4.12).

Taking the positive definite measure $\sqrt{|g|} d\zeta$ and proceeding exactly as before, we find that, once again there is a family of self-adjoint extensions parametrised by $U(2)$. The boundary terms vanish if

$$ A \begin{pmatrix} \sqrt{|g|} \frac{\partial \rho^t}{\partial \zeta}(0^-) \\ \sqrt{|g|} \frac{\partial \rho^t}{\partial \zeta}(0^+) \end{pmatrix} = B \begin{pmatrix} \rho^t(0^-) \\ \rho^t(0^+) \end{pmatrix} . \tag{4.13} $$

\begin{equation}
\frac{\partial \rho^t}{\partial z}(0^-) = -\frac{\partial \rho^t}{\partial z}(0^+) .
\end{equation}
where the matrices $A$ and $B$ are once again given by (4.10). Thus we see that the analysis for continuous signature change is much the same as for the discontinuous case.

5 Wave Equation Solutions

In this section we construct solutions of the Schrödinger and Klein–Gordon equations on the signature changing spacetime (3.1), satisfying the boundary conditions (4.11). Throughout this section $H_v$ will represent the self-adjoint extension of $H$ corresponding to these boundary conditions, and $\mathcal{H}$ will denote the Hilbert space $L^2(\mathbb{R}, dz)$.

5.1 Solutions of the Schrödinger equation

For our choice of self-adjoint extension, the Schrödinger equation on the signature changing spacetime (3.1) is

$$i \frac{\partial \Psi}{\partial t} = H_v \Psi,$$

and has general solution $\Psi(t) = e^{-iH_v t} \Psi(0)$. The evolution operator $e^{-iH_v t}$ is most conveniently constructed using the spectral representation of $H_v$, which we now study.

Firstly, solving the time-independent Schrödinger equation

$$H_v \Psi = E \Psi,$$

as an ODE subject to the boundary conditions (4.11), we obtain the generalised eigenfunctions (mode solutions)

$$\tilde{\Psi}(z, E) = a(E) \left\{ (1 - i) \theta(-z) \theta(-E) \exp(\sqrt{-E} z) + (1 - i) \theta(+z) \theta(-E) \left[ \cos(\sqrt{-E} z) - \sin(\sqrt{-E} z) \right] + (1 + i) \theta(-z) \theta(+E) \left[ \cos(\sqrt{+E} z) + \sin(\sqrt{+E} z) \right] + (1 + i) \theta(+z) \theta(+E) \exp(-\sqrt{+E} z) \right\},$$

(5.3)
where \( \theta \) is the usual step function and the function \( a(E) \) is

\[
a(E) = \frac{1}{\sqrt{2\sqrt{|E|}}}.
\] (5.4)

Next, we define an integral transform \( \mathcal{M} : L^2(\mathbb{R}, dE) \to L^2(\mathbb{R}, dz) \) by

\[
(\mathcal{M}f)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(E) \tilde{\Psi}(z, E) dE.
\] (5.5)

In Appendix B, we show that \( \mathcal{M} \) is unitary. Heuristically, this implies that the mode solutions obey the continuum normalisation

\[
\int_{-\infty}^{\infty} dz \tilde{\Psi}(z, E)^* \tilde{\Psi}(z, E') = 2\pi \delta(E - E'),
\] (5.6)

and

\[
\int_{-\infty}^{\infty} dE \tilde{\Psi}(z, E)^* \tilde{\Psi}(z', E) = 2\pi \delta(z - z').
\] (5.7)

A simple calculation shows that \( \mathcal{M} E \mathcal{M}^{-1} \) agrees with \( H \) on \( C_0^\infty(0, \infty) \). It is therefore one of the self-adjoint extensions of \( H \); moreover, it is clear that it must be equal to \( H_U \). Thus we have \( H_U = \mathcal{M} E \mathcal{M}^{-1} \). The spectral theorem entails that the general solution of (5.1) satisfying the boundary conditions (4.11) may be written in the initial value form as

\[
\Psi(z, t) = \mathcal{M} e^{-iEt} \mathcal{M}^{-1} \Psi(z, 0),
\] (5.8)

for any \( \Psi(z, 0) \in \mathcal{H} \). For smooth compactly supported initial data, one may show that the solutions \( \Psi(z, t) \) are \( C^\infty \) everywhere except at \( z = 0 \) where they are \( C^0 \).

It would also be possible to determine the integral kernel for the evolution operator \( e^{-iHt} \). However, we will not do this here because our main focus is the Klein–Gordon equation.
5.2 Solutions of the Klein–Gordon equation

Given our choice of Schrödinger Hamiltonian $H_v$, the Klein–Gordon Hamiltonian becomes

$$K_v = \begin{pmatrix} 0 & 1 \\ -H_v & 0 \end{pmatrix}$$

(5.9)

on $\mathcal{D}(H_v) \oplus L^2(\mathbb{R}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Following [8] we observe that the first order form of the massless Klein–Gordon equation

$$\frac{\partial \Phi}{\partial t} = K_v \Phi$$

(5.10)

has a formal solution of the form

$$\Phi(t) = \mathcal{T}(t)\Phi(0) ,$$

(5.11)

where

$$\mathcal{T}(t) = \begin{pmatrix} \cos(H_v^{1/2}t) & H_v^{-1/2}\sin(H_v^{1/2}t) \\ -H_v^{1/2}\sin(H_v^{1/2}t) & \cos(H_v^{1/2}t) \end{pmatrix} .$$

(5.12)

At first sight, $\mathcal{T}(t)$ seems to depend on some choice of square root. However, as noted in [8], the formal power series for $\cos(H_v^{1/2}t)$, $H_v^{-1/2}\sin(H_v^{1/2}t)$ and $H_v^{1/2}\sin(H_v^{1/2}t)$ are independent of any definition of the square root. Indeed, we have that

$$\cos(H_v^{1/2}t) = 1 - \frac{H_v t^2}{2!} + \frac{H_v^2 t^4}{4!} - \cdots ,$$

(5.13)

$$H_v^{-1/2}\sin(H_v^{1/2}t) = t - \frac{H_v t^3}{3!} + \frac{H_v^2 t^5}{5!} - \cdots ,$$

(5.14)

$$H_v^{1/2}\sin(H_v^{1/2}t) = H_v t - \frac{H_v^2 t^3}{3!} + \frac{H_v^3 t^5}{5!} - \cdots .$$

(5.15)

Thus $\mathcal{T}(t)$ can be unambiguously defined using the spectral theorem, with domain

$$\mathcal{D}(\mathcal{T}(t)) = \mathcal{D}(H_v^{1/2}\sin(H_v^{1/2}t)) \oplus \mathcal{D}(\cos(H_v^{1/2}t)) .$$

(5.16)

Note also that if $\Phi \in \mathcal{D}(\mathcal{T}(t_1))$ and $\mathcal{T}(t_1)\Phi \in \mathcal{D}(\mathcal{T}(t_2))$ then the group property

$$\mathcal{T}(t_1 + t_2)\Phi = \mathcal{T}(t_2)\mathcal{T}(t_1)\Phi$$

(5.17)
Let $E_{[A,B]}$ denote the spectral projector of $H_v$ on $[A, B]$ (i.e. the space of position space functions with $A \leq \text{energy} \leq B$). By considering the positive and negative energy subspaces of $H_v$, then we see that $\cos(H_v^{1/2}t)$ may be written

$$\cos((H_v|E_{[-M,M]}^{1/2})t) = \cos((H_v|E_{[0,M]}^{1/2})t) = \cosh((H_v|E_{[-M,0]}^{1/2})t).$$  \hspace{1cm} (5.18)

Similar decompositions clearly exist for $H^{-1/2}_v \sin(H_v^{1/2}t)$ and $H^{1/2}_v \sin(H_v^{1/2}t)$. Note that it is the exponentially growing parts of these decompositions which force us to restrict the domain of $\mathcal{T}(t)$ to the subset of $\mathcal{H} \oplus \mathcal{H}$ given in (5.16). Thus we see that, in contrast to the bounded Schrödinger evolution $e^{-iH_v t}$, $\mathcal{T}(t)$ is unbounded.

Using (5.18) and its analogues, and dominated convergence arguments similar to the proof of Theorem VIII.7 in [9], it is possible to show that, for any $\tau > 0$ and $\Phi(0) \in \mathcal{D}(\mathcal{T}(\tau))$

\begin{enumerate}
  \item \hspace{1cm} $\mathcal{T}(t)\Phi(0) \in \mathcal{D}(K_v)$ for all $t \in [0, \tau]$,
  \item \hspace{1cm} The vector-valued function $\Phi(t) = \mathcal{T}(t)\Phi(0)$ is differentiable with respect to $t$ for all $t \in [0, \tau)$, with derivative $K_v \Phi(t)$.
\end{enumerate}

Thus $\Phi(t)$ is an $L^2$-solution of the Klein–Gordon equation (3.13) satisfying the boundary conditions (4.11) and with initial data $\Phi(0)$. Note that a different self-adjoint extension would give different solutions satisfying different boundary conditions at $z = 0$.

### 5.3 Initial Data for the Klein–Gordon equation

#### Instability

It is clear from (5.18) and the fact that $H_v$ is unbounded from below that the operator $\mathcal{T}(t)$ is unbounded. As a consequence, there exist sequences of initial data $\Phi_n(0)$ with $\Phi_n(0) \to 0$ but $\mathcal{T}(t)\Phi_n(0)$ divergent for any $t > 0$. For example, simply choose any $\{\Phi_n(0)\}$ such that $||\Phi_n(0)|| \leq \frac{1}{n}$ and

$$\Phi_n(0) \in E_{[-n,1-n]}\mathcal{H} \oplus E_{[-n,1-n]}\mathcal{H}.$$

$$\hspace{1cm} (5.19)$$
Clearly $\Phi_n(0) \in \mathcal{D}(T(t))$ for all $t \in \mathbb{R}$ and $\Phi_n(0) \to 0$ as $n \to \infty$. However,

$$||T(t)\Phi_n(0)|| \geq \cosh((n - 1)^{1/2}t),$$  \hspace{1cm} (5.20)

which diverges as $n \to \infty$. This problem is related to the fact that initial value problems for elliptic equations are ill-posed (see eg. Hadamard’s example in [10]). However, as is noted in [10], many problems which are of physical interest are actually ill-posed, and so we will proceed, but with caution.

**Runaway solutions**

A reasonable experiment would be to observe the behaviour of matter which propagated through a Lorentzian region and scattered off a Kleinian region. Such an experiment is modelled by evolving data which is initially compactly supported in the Lorentzian region $z < 0$.

Consider any smooth initial data $\Phi(0)$ with compact support $[-z_1, -z_0]$ for some real constants $z_1 > z_0 > 0$. Thus $\Phi(0)$ can be written as

$$\Phi(0) = \Phi(z, 0) = \begin{cases} 
\left( \begin{array}{c}
\phi_1(z) \\
\phi_2(z)
\end{array} \right) & z \in [-z_1, -z_0] \\
0 & otherwise
\end{cases}$$ \hspace{1cm} (5.21)

for smooth compactly supported functions $\phi_1(z)$ and $\phi_2(z)$. Without loss of generality, we can assume that both $\phi_1$ and $\phi_2$ are real valued functions, since if they were complex valued the following analysis could simply be applied to their real and imaginary parts.

Define

$$\hat{\phi}_i(E) = \int_{-z_1}^{-z_0} \phi_i(z) e^{\sqrt{|E|} z} dz \quad i = 1, 2.$$ \hspace{1cm} (5.22)

With the above assumption, we can always find some $\epsilon_0 > 0$ so that both $\frac{\partial \phi_1}{\partial z}$ and $\frac{\partial \phi_2}{\partial z}$ are single signed in the region $[-z_0 - 2\epsilon_0, -z_0]$. Choose $\epsilon$ so that $0 < \epsilon < \epsilon_0$, then it is simple to show that

$$e|\phi_i(-z_0 - \epsilon)| e^{-\sqrt{|E|} |z_0 + 2\epsilon|} < \left| \int_{-z_0 - 2\epsilon}^{-z_0 - \epsilon} \phi_i(z) e^{\sqrt{|E|} z} dz \right| < e|\phi_i(-z_0 - 2\epsilon)| e^{-\sqrt{|E|} |z_0 + \epsilon|}$$ \hspace{1cm} (5.23)
from which it can be shown that the \( \hat{\phi}_i(E) \) decay at a rate immediately less than \( e^{-\sqrt{|E|}z_0} \).

Introduce the the notation \( \Phi^-(E,t) \) and \( \Phi^+(E,t) \) by

\[
\theta(-E)\Phi^-(E,t) + \theta(E)\Phi^+(E,t) = (\mathcal{M}^{-1} \oplus \mathcal{M}^{-1})\Phi(z,t) ,
\]

then for the negative energy part we find

\[
\Phi^-(E,0) = k(E) \left( \begin{array}{c} \hat{\phi}_1(E) \\ \hat{\phi}_2(E) \end{array} \right) ,
\]

where

\[
k(E) = \frac{(1 + i)\theta(-E)a(E)}{\sqrt{2\pi}} .
\]

Evolving this data in energy space gives

\[
\Phi^-(E, t) = k(E) \left( \begin{array}{c} \cosh(\sqrt{|E|t})\hat{\phi}_1(E) + \frac{1}{\sqrt{|E|}} \sinh(\sqrt{|E|t})\hat{\phi}_2(E) \\ \sqrt{|E|} \sinh(\sqrt{|E|t})\hat{\phi}_1(E) + \cosh(\sqrt{|E|t})\hat{\phi}_2(E) \end{array} \right) .
\]

From the above decay properties of the \( \hat{\phi}_i(E) \), we see that the evolved solution \( \Phi^-(E, t) \) fails to be \( L^2(\mathbb{R}, dE) \) at the time \( t = z_0 \). This time corresponds to the instant that the massless data begins to arrive at the surface of signature change. Such behaviour suggests that there is a severe back-reaction at \( t = z_0 \). Further analysis shows that the evolved data \( \Phi(z,t) \) is \( C^\infty \) everywhere in the region \( t + z < z_0 \).

### 6 Second Order Analysis

In case it is thought that the above runaway solution is simply due to the first order formalism and our use of self-adjoint extensions, we provide a second order analogue of the result using only Fourier analysis and elementary arguments. In the following, it will only be necessary to assume boundary
conditions of the general form \( (4.1) \). This allows us to consider boundary conditions for which both the field and its derivative are continuous, as well as the boundary conditions employed in section 5.

Consider the Klein–Gordon equation \( (3.11) \) in the Kleinian region \( (z > 0) \) with \( m = 0 \). Fourier transforming in \( t \) we obtain

\[
\frac{\partial^2 \tilde{\phi}}{\partial z^2} = E^2 \tilde{\phi}.
\]

(6.28)

Taking the solution which decays as \( z \to \infty \) we find

\[
\tilde{\phi}(E, z) = \tilde{\phi}(E, 0) e^{-|E|z},
\]

(6.29)

and hence

\[
\frac{\partial \tilde{\phi}}{\partial z}(E, 0^+) = -E |\tilde{\phi}(E, 0)|.
\]

(6.30)

Thus, on the Lorentzian side of the signature change surface,

\[
\frac{\partial \tilde{\phi}}{\partial z}(E, 0^-) = -\omega |E| \tilde{\phi}(E, 0),
\]

(6.31)

where \( \omega \) parametrises the boundary condition \( (4.1) \).

If \( \tilde{\phi}(E, 0) \) is analytic in \( E \) (but not identically zero) then we see that \( \frac{\partial \tilde{\phi}}{\partial z}(E, 0^-) \) is not analytic, and vice versa. Furthermore, by the Paley–Wiener theorem (see eg. [5]), we then have that it is not possible for both \( \tilde{\phi}(t, 0) \) and \( \frac{\partial \tilde{\phi}}{\partial z}(t, 0^-) \) to be elements of \( C^\infty_0(\mathbb{R}) \).

Let us again model the experiment described in the section 5. Choose the initial data \( \tilde{\phi}(0, z) \) and \( \frac{\partial \tilde{\phi}}{\partial t}(0, z) \) to be in \( C^\infty_0(-\infty, 0) \), and assume that at least some of the matter propagates towards the Kleinian region at \( z = 0 \). For massless particles this would give data \( \tilde{\phi}(t, 0) \) and \( \frac{\partial \tilde{\phi}}{\partial z}(t, 0^-) \) on the surface of signature change which are both in \( C^\infty_0(\mathbb{R}) \), but such data will not provide a solution of the Klein–Gordon equation in the Kleinian region which decays as \( z \to \infty \). Hence the second order formalism also leads to divergent solutions from smooth compactly supported initial data.
7 Conclusion

We have seen that the Schrödinger Hamiltonian on the flat signature changing spacetime (3.1) admits self-adjoint extensions. The additional restrictions that (i) the boundary conditions corresponding to the self-adjoint extension have the form of junction conditions for the matter field and its first derivative, and (ii) that the matter field is continuous, were shown to pick out one particular set of boundary conditions (4.11). Using these results, we were able to construct solutions of the Schrödinger and Klein–Gordon matter field equations. The Dirac Hamiltonian is non-symmetric and it is highly likely that the Dirac equation is only satisfied by solutions of the type found in [4].

The boundary conditions (4.11) provide a well-behaved system to study, and furthermore, since a single set of boundary conditions are selected, the requirement of self-adjointness provides a possible way of solving the dilemma of which boundary conditions to choose for signature change. The boundary conditions (4.11) are physically very similar to what might be regarded as the ‘natural boundary conditions’ where both the field and its derivative are continuous, since on applying them to mode solutions (as in [4]) we find that a scalar field is completely reflected, whilst the Dirac field is completely absorbed.

A reasonable physical experiment would be to observe the result of throwing matter at a Kleinian region. In our model, this type of experiment corresponds to taking smooth compactly supported initial data in the Lorentzian region \( z < 0 \) at \( t = 0 \). The results for the Schrödinger equation were very promising. Smooth compactly supported initial data in the Lorentzian region could be evolved indefinitely. However, the more physically relevant Klein–Gordon solutions gave very different results. Although we could define such solutions, in section 5 we saw that smooth compactly supported initial data could only be evolved for a finite time. Indeed, for a massless scalar field this time was equal to the earliest possible time that classical matter could reach the surface of signature change. As soon as matter reaches the signature change surface the energy momentum tensor of that matter field becomes unbounded and our assumption that the matter fields do not interact with the background spacetime breaks down.
Such severe behaviour suggests that if interaction were allowed, the back-
reaction of matter fields on the metric might annihilate the region of signa-
ture change, and thus regions of signature change would be unable to form.
These results are consistent with what could be called the “Signature Pro-
tection Conjecture”, or in other words, the hypothesis that fluctuations in
the signature of the spacetime metric are suppressed. In light of the fact that
even this ‘well-behaved’ signature change system predicts its own downfall,
it may be prudent to reassess the inclusion of signature changing metrics in
quantum gravity theories.
Appendix A

The general form of the boundary conditions for the Schrödinger Hamiltonian are given by equation (4.9). Let $A$ and $B$ be the matrices defined by (4.10), then we are interested in the subset of boundary conditions given by the restriction that both $A$ and $B$ are singular. For such $A$ and $B$, the of the boundary conditions are of the form

$$\omega_1 \rho^\dagger(0^-) = \omega_2 \rho^\dagger(0^+) ,$$  
(A.1)

$$\omega_3 \frac{\partial \rho^\dagger}{\partial z}(0^-) = \omega_4 \frac{\partial \rho^\dagger}{\partial z}(0^+) ,$$

for some $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{C}$. We also wish to exclude any cases for which any of the $\omega_i$ are zero.

We can calculate the general form of the unitary matrix $U$ which satisfies $\det(A) = 0, \det(B) = 0$ and the constraints (4.8) on $a, b, c$ and $d$. We find the two parameter family

$$U = \begin{pmatrix} 
-\frac{1}{2}(1 - i)(1 + ie^{i\theta}) & \sqrt{\sin(\theta)} \ e^{i\kappa} \\
-i\sqrt{\sin(\theta)} \ e^{i(\theta-\kappa)} & -\frac{1}{2}(1 + i)(1 + ie^{i\theta}) 
\end{pmatrix}$$  
(A.2)

for any $\theta \in [0, \pi]$ and $\kappa \in [0, 2\pi]$.

Except for two special cases (when $\theta = 0$ and $\theta = \pi$), the boundary conditions corresponding to this $U$ are

$$\rho^\dagger(0^-) = \left( \frac{1 + \frac{1}{2}(1 + i)(1 + ie^{i\theta})}{\sqrt{\sin(\theta)} \ e^{i\kappa}} \right) \rho^\dagger(0^+) ,$$  
(A.3)

$$\frac{\partial \rho^\dagger}{\partial z}(0^-) = \left( \frac{\frac{1}{2}(1 + i)(1 + ie^{i\theta}) - 1}{\sqrt{\sin(\theta)} \ e^{i\kappa}} \right) \frac{\partial \rho^\dagger}{\partial z}(0^+)$$  
(A.4)

In the special case $\theta = 0$, the boundary conditions are

$$\rho^\dagger(0^-) = 0 ,$$

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and the special case $\theta = \pi$ gives the boundary conditions

$$\rho^+(0^+) = 0 ,$$

$$\frac{\partial \rho^+}{\partial z}(0^-) = 0 .$$

These special cases correspond to setting $\omega_2 = \omega_3 = 0$ and $\omega_1 = \omega_4 = 0$ respectively, and hence we will ignore them.

It seems reasonable to assume that the matter field remains continuous across the surface of signature change. In this case, using (A.3) to eliminate $\kappa$ from (A.4) we find that the derivative boundary condition becomes

$$\frac{\partial \rho^+}{\partial z}(0^-) = \left( \frac{\sin(\theta)}{\cos(\theta) - 1} \right) \frac{\partial \rho^+}{\partial z}(0^+)$$

Finally, we can employ something particular to this signature change system. We want every mode solution to satisfy the boundary conditions. Furthermore, for $E > 0$ and $z > 0$ the only bounded solution of $H\rho^+ = E\rho^+$ is

$$\rho^+(z) = \rho^+(0^+) e^{-\sqrt{E} z}$$

and hence we have the additional information that

$$\frac{\partial \rho^+}{\partial z}(0^+) = -\sqrt{E} \rho^+(0^+) .$$

Using equations (A.7) and (A.9), equation (4.9) may then be rewritten as

$$-\sqrt{E} A \begin{pmatrix} \frac{\sin(\theta)}{\cos(\theta) - 1} \\ 1 \end{pmatrix} = e^{i\pi/4} B \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

(A.10)
but this must hold for all $E > 0$ and so we must have

$$A \begin{pmatrix} \frac{\sin(\theta)}{\cos(\theta) \cdot 1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (A.11)$$

These equations are consistent only if $\theta = \frac{\pi}{2}$.

Thus, setting $\theta = \frac{\pi}{2}$ in (A.7), we have shown that the only set of boundary conditions of the form (4.1), which provide a self-adjoint extension of the Schrödinger operator on the signature changing spacetime (3.1), are

$$\rho^i(0^-) = \rho^i(0^+) , \quad (A.12)$$

$$\frac{\partial \rho^i}{\partial z}(0^-) = - \frac{\partial \rho^i}{\partial z}(0^+) .$$
Appendix B

Here, we prove that the integral transform \( \mathcal{M} \) is unitary. It is convenient to work with the operator \( \tilde{\mathcal{M}} \) defined by

\[
\tilde{\mathcal{M}} = U^* \mathcal{M} V
\]

where \( U : L^2(\mathbb{R}, dE) \to L^2(\mathbb{R}^+, dk) \otimes \mathbb{C}^2 \) and \( V : L^2(\mathbb{R}, dz) \to L^2(\mathbb{R}^+, dz) \otimes \mathbb{C}^2 \) are unitary operators given by

\[
(Uf)(k) = \begin{pmatrix} (2k)^{1/2} f(k^2) \\ (2k)^{1/2} f(-k^2) \end{pmatrix}
\]

and

\[
(Vf)(z) = \begin{pmatrix} f(z) \\ f(-z) \end{pmatrix}.
\]

Explicitly, \( \tilde{\mathcal{M}} : L^2(\mathbb{R}^+, dk) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}^+, dz) \otimes \mathbb{C}^2 \) takes the matrix form

\[
\tilde{\mathcal{M}} = \begin{pmatrix} (1 + i)A & (1 - i)B \\ (1 + i)B & (1 - i)A \end{pmatrix},
\]

where \( A, B : L^2(\mathbb{R}^+, dk) \to L^2(\mathbb{R}^+, dz) \) are defined by

\[
(Af)(z) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{-kz} f(k) dk
\]

and

\[
(Bf)(z) = \frac{1}{2} ((C - S)f)(z),
\]

and \( C \) and \( S \) are the cosine and sine transforms defined by

\[
(Cf)(z) = \left( \frac{2}{\pi} \right)^{1/2} \int_0^\infty dk f(k) \cos k z
\]

and

\[
(Sf)(z) = \left( \frac{2}{\pi} \right)^{1/2} \int_0^\infty dk f(k) \sin k z.
\]

In order to prove that \( \tilde{\mathcal{M}} \) (and hence \( \mathcal{M} \)) is unitary, it now suffices to establish the following identities:

\[
2(A^*A + B^*B) = 1 \quad 2(AA^* + BB^*) = 1, \quad A^*B + B^*A = 0 \quad AB^* + BA^* = 0.
\]

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Let \( f \in C_0^\infty(0, \infty) \). Then

\[
(A^* B f)(k) = \frac{1}{2\pi} \int_0^\infty dz e^{-kz} \int_0^\infty dk' (\cos k' z - \sin k' z) f(k)
= \frac{1}{2\pi} \int_0^\infty dk' f(k') G(k, k') ,
\]

where

\[
G(k, k') = \int_0^\infty dz e^{-kz} (\cos k' z - \sin k' z) = \frac{k - k'}{k^2 + k'^2} ,
\]

and Fubini’s theorem \cite{9} has been employed. Moreover, an analogous argument (with \( z \) and \( k \) interchanged) shows that

\[
(B^* A f)(k) = \frac{1}{2\pi} \int_0^\infty dk' f(k') G(k', k) .
\]

Thus, because \( G(k, k') = -G(k', k) \), we conclude that \( A^* B + B^* A \) vanishes on \( C_0^\infty(0, \infty) \) and hence on the whole of \( L^2(\mathbb{R}^+, dk) \). An identical argument shows that \( AB^* + BA^* = 0 \).

Next, note that \( 2B^* B = 1 - \frac{i}{2} (S^* C + C^* S) \). Thus we need to show that \( \frac{i}{2} (S^* C + C^* S) = 2A^* A \). Let \( f \in C_0^\infty(0, \infty) \). Then Fubini’s theorem may be used to show that

\[
2(A^* A f)(k) = \frac{1}{\pi} \int_0^\infty dk' \frac{f(k')}{k + k'} ,
\]

and also that

\[
\frac{i}{2} ((S^* e^{-\epsilon z} C + C^* e^{-\epsilon z} S)) f(k) = \frac{1}{\pi} \int_0^\infty dk' \frac{f(k')}{\epsilon^2 + (k + k')^2} \rightarrow \frac{L^2}{\pi} \int_0^\infty dk' \frac{f(k')}{k + k'}
\]

as \( \epsilon \to 0^+ \). Because \( S^* e^{-\epsilon z} C + C^* e^{-\epsilon z} S \to S^* C + C^* S \) strongly as \( \epsilon \to 0^+ \), we have \( 2(A^* A + B^* B) = 1 \) on \( C_0^\infty(0, \infty) \) and hence on \( L^2(\mathbb{R}^+, dk) \). An analogous argument shows that \( 2(AA^* + BB^*) = 1 \).
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References


