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Abstract

Conformal field theory is applied to the study of quantum critical phenomena. By making use of Abelian bosonization procedure, we obtain a Coulomb gas picture of continuum limit of one-dimensional Hubbard model. It is shown that semi-direct product of two Virasoro algebras ($c = 1$) denotes symmetry of excitations of the Hubbard model. A systematic study of modular invariant partition function for the Hubbard model is presented after recognizing the finite size scaling. Relevant correlation functions, charge-charge and spin-spin correlators, are also shown explicitly in the scheme.

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Recent developments in the conformal field theory of critical phenomena may be regarded as a continuation and specialization of the study of critical scaling[1]. Conformal field theory methods have been applied with great success to a large variety of classical critical phenomena problems[2]. By now, most of these problems have been understood in great detail. The frontier area presently appears to be quantum critical phenomena. A large variety of interesting quantum effects, include the fractional quantum Hall effect, heavy fermions and high T_c superconductors, have been studied intensively in recent years. For one-dimensional quantum systems with short-range interactions there is no phase transition at finite temperatures, and hence the correlation functions decay exponentially at long distance. Critical phenomena can take place only at absolute zero, $T = 0$. The low-energy gap-less excitations then follow the linear dispersion relation. Consequently the long-distance behavior of correlation functions is characterized by the power-law decay. Therefore conformal field theory shall play an important role in the study of quantum critical phenomena.

The single-band Hubbard model[3] is argued as the appropriate model for the high T_c superconductivity[4]. One dimensional Hubbard model can be solved exactly by Bethe ansatz, leading to decoupled charge and spin excitations[5]. Although the spectrum has been known for many years, the calculation of the partition function and correlation functions proved to be a delicate problem. After early attempts to determine numerically the correlation functions[6], an analytic approach has become possible only recently. In the large- U limit and for special filling of the band it is possible to determine some correlation functions by analyzing the properties of the ground-state wave function[7]. It becomes possible[8] to calculate the correlation function exponents for different U and band filling n by making use of the Luttinger-liquid picture[9]–[11]. Assuming that the spectrum of excitations is described in terms of a semi-direct product of two Virasoro algebras, then conformal invariance can be used to determine the correlation functions of the Hubbard model[12][13].

In this paper, we use Abelian bosonization procedure to map the continuum limit of one-dimensional Hubbard model into a Coulomb gas picture. Then it is clearly shown

that the semi-direct product of two Virasoro algebras denotes symmetry of the Hubbard model. After recognizing the finite size scaling, a systematic study of partition function of the model is presented. Spin-spin and charge-charge correlation function exponents are also obtained in the scheme.

The free Hamiltonian of the Hubbard model is of the form

$$H_0 = -\epsilon \sum_{j,\sigma} (c_{j\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j\sigma}) , \quad (1)$$

where ϵ is the hopping matrix element and the operators $c_{j\sigma}$ ($c_{j\sigma}^\dagger$) ($j = 1, 2, \dots, L$) describe electrons on lattice and satisfy the anti-commutation relation,

$$\begin{aligned} \{c_{j\sigma}, c_{k\sigma'}\} &= 0 , \\ \{c_{j\sigma}, c_{k\sigma}^\dagger\} &= \delta_{jk} \delta_{\sigma\sigma'} , \\ \{c_{j\sigma}^\dagger, c_{k\sigma}^\dagger\} &= 0 . \end{aligned} \quad (2)$$

In Heisenberg picture ($\hbar = 1$), the equations of motion of the free system are

$$\begin{aligned} i \frac{\partial c_{j\sigma}(t)}{\partial t} &= [c_{j\sigma}(t), H_0] \\ &= -\epsilon (c_{j+1,\sigma}(t) + c_{j-1,\sigma}(t)) . \end{aligned} \quad (3)$$

By making use of the Fourier transformation

$$c_{j\sigma} = \frac{1}{\sqrt{L}} \sum_k e^{ikja} b_{k\sigma} , \quad (4)$$

we get a solution of the equations of motion,

$$b_{k\sigma}(t) = \exp(2i\epsilon \cos ka t) b_{j\sigma}(0) , \quad (5)$$

where a denotes lattice spacing.

In taking the continuum limit, it is necessary to take account of the fact that the time derivative of fields on even (odd) sites is related to the values of the fields themselves on odd (even) sites. This property may be preserved by introducing two functions $\psi_e(x, t)$ and $\psi_o(x, t)$ such that

$$\begin{aligned} c_{2j,\sigma}(t) &\longrightarrow i^{2j} \sqrt{2a} \psi_e(x, t) , \\ c_{2j+1,\sigma}(t) &\longrightarrow i^{2j+1} \sqrt{2a} \psi_o(x, t) , \end{aligned} \quad (6)$$

where $2ja \rightarrow x$.

Under such a continuum limiting process, we have

$$\begin{aligned} [c_{j\sigma}, c_{l\sigma'}^\dagger]_+ &= \delta_{jl} \delta_{\sigma\sigma'} \longrightarrow [\psi_{\alpha\sigma}(x), \psi^\dagger(y)_{\beta\sigma'}]_+ = \delta_{\alpha\beta} \delta_{\sigma\sigma'} \delta(x-y) , \\ c_{j+1,\sigma} + c_{j-1,\sigma} &\longrightarrow i^{2j} (\psi_{\alpha\sigma}(ja) - \psi_{\alpha\sigma}((j-1)a)) . \end{aligned} \quad (7)$$

And then equations of motion of the free system become

$$\begin{aligned} \frac{\partial \psi_e(x, t)}{\partial t} &= -v_F \frac{\partial \psi_o(x, t)}{\partial x} , \\ \frac{\partial \psi_o(x, t)}{\partial t} &= -v_F \frac{\partial \psi_e(x, t)}{\partial x} , \end{aligned} \quad (8)$$

where $v_F = 2ta$ is the Fermi velocity.

Defining

$$\psi_1(x, t) = \frac{\psi_e(x, t) + \psi_o(x, t)}{\sqrt{2}} , \quad \psi_2(x, t) = \frac{\psi_e(x, t) - \psi_o(x, t)}{\sqrt{2}} ,$$

we have

$$\begin{aligned} \frac{\partial \psi_1(x, t)}{\partial t} &= -v_F \frac{\partial \psi_1(x, t)}{\partial x} , \\ \frac{\partial \psi_2(x, t)}{\partial t} &= v_F \frac{\partial \psi_2(x, t)}{\partial x} . \end{aligned} \quad (9)$$

It is easy to see that the above equations are just those describing fermions moving to the right and left respectively with velocity v_F . At the continuum limit, energy spectra are $\pm v_F k$ instead of $2\epsilon \cos ka$ as found in the lattice model.

The continuum form of the free Hamiltonian is

$$H_0 = -iv_F \sum_\sigma \int dx \left(\psi_{1\sigma}^\dagger \frac{\partial \psi_{1\sigma}}{\partial x} - \psi_{2\sigma}^\dagger \frac{\partial \psi_{2\sigma}}{\partial x} \right) . \quad (10)$$

By making use of the Fourier transformation

$$\begin{aligned} a_{k\sigma} &= \frac{1}{\sqrt{L}} \int dx e^{-ikx} \psi_{1\sigma}(x) , \\ b_{k\sigma} &= \frac{1}{\sqrt{L}} \int dx e^{-ikx} \psi_{2\sigma}(x) , \end{aligned}$$

we can write the free Hamiltonian (up to a chemical potential term) of the form

$$H_0 = \sum_{k>0} v_F (k - k_F) a_{k\sigma}^\dagger a_{k\sigma} + \sum_{k<0} v_F (-k - k_F) b_{k\sigma}^\dagger b_{k\sigma} . \quad (11)$$

The interaction Hamiltonian of the Hubbard model is

$$H_{\text{int}} = U \sum_j c_{j\uparrow}^\dagger c_{j\uparrow} c_{j\downarrow}^\dagger c_{j\downarrow} ,$$

where U is the on-site Coulomb interaction.

The perturbation form of the interaction Hamiltonian can be cast into the form

$$\begin{aligned} H_{\text{int}} &= Ua \sum_\sigma \int dx \left(\psi_{o\sigma}^\dagger \psi_{o\sigma} \psi_{o,-\sigma}^\dagger \psi_{o,-\sigma} + \psi_{e\sigma}^\dagger \psi_{e\sigma} \psi_{e,-\sigma}^\dagger \psi_{e,-\sigma} \right) \\ &= \frac{Ua}{4} \sum_\sigma \int dx \left(\psi_{1\sigma}^\dagger \psi_{1\sigma} \psi_{1,-\sigma}^\dagger \psi_{1,-\sigma} + \psi_{2\sigma}^\dagger \psi_{2\sigma} \psi_{2,-\sigma}^\dagger \psi_{2,-\sigma} \right. \\ &\quad \left. + 2\psi_{1\sigma}^\dagger \psi_{1\sigma} \psi_{2,-\sigma}^\dagger \psi_{2,-\sigma} + \left(\psi_{1\sigma}^\dagger \psi_{2\sigma} \psi_{2,-\sigma}^\dagger \psi_{1,-\sigma} + h.c. \right) \right. \\ &\quad \left. + \left(\psi_{2\sigma}^\dagger \psi_{1\sigma} \psi_{1,-\sigma}^\dagger \psi_{2,-\sigma} + h.c. \right) \right) \\ &= H_{\text{LM}} + H_{\text{BS}} + H_{\text{US}} , \end{aligned} \quad (12)$$

where

$$\begin{aligned} H_{\text{LM}} &= \frac{Ua}{4} \sum_\sigma \int dx \left(\psi_{1\sigma}^\dagger \psi_{1\sigma} \psi_{1,-\sigma}^\dagger \psi_{1,-\sigma} + \psi_{2\sigma}^\dagger \psi_{2\sigma} \psi_{2,-\sigma}^\dagger \psi_{2,-\sigma} + 2\psi_{1\sigma}^\dagger \psi_{1\sigma} \psi_{2,-\sigma}^\dagger \psi_{2,-\sigma} \right) , \\ H_{\text{BS}} &= \frac{Ua}{4} \sum_\sigma \int dx \left(\psi_{1\sigma}^\dagger \psi_{2\sigma} \psi_{2,-\sigma}^\dagger \psi_{1,-\sigma} + h.c. \right) , \\ H_{\text{US}} &= \frac{Ua}{4} \sum_\sigma \int dx \left(\psi_{2\sigma}^\dagger \psi_{1\sigma} \psi_{1,-\sigma}^\dagger \psi_{2,-\sigma} e^{iG} + h.c. \right) , \end{aligned} \quad (13)$$

here G is a reciprocal lattice vector, for a half filled band $G = 4k_F$.

By making use of the Fourier transformation (4), we have

$$\begin{aligned} H_{\text{LM}} &= \frac{Ua}{4} \sum_{k_1, k_2, p, \sigma} \left(2a_{k_1, \sigma}^\dagger b_{k_2, -\sigma}^\dagger b_{k_2+p, -\sigma} a_{k_1-p, \sigma} \right. \\ &\quad \left. + a_{k_1, \sigma}^\dagger a_{k_2, -\sigma}^\dagger a_{k_2+p, -\sigma} a_{k_1-p, \sigma} + b_{k_1, \sigma}^\dagger b_{k_2, -\sigma}^\dagger b_{k_2+p, -\sigma} b_{k_1-p, \sigma} \right) . \end{aligned} \quad (14)$$

By making use of Abelian bosonization, we express the dynamical processes represented by electron field operators, $\psi_{i\sigma}$ ($i = 1, 2$) by those of a boson field $\phi_{i\sigma}$, as follows

$$\begin{aligned} \psi_{1\sigma}(x, t) &\rightarrow \frac{1}{\sqrt{2\pi\alpha}} \exp(-ik_F x - \phi_{1\sigma}(x, t)) \equiv O_{1\sigma}(x, t) , \\ \psi_{2\sigma}(x, t) &\rightarrow \frac{1}{\sqrt{2\pi\alpha}} \exp(ik_F x + \phi_{2\sigma}(x, t)) \equiv O_{2\sigma}(x, t) , \end{aligned} \quad (15)$$

where α is the cutoff parameter to be set $\alpha \rightarrow 0$ whenever possible and

$$\phi_{i\sigma} = \frac{2\pi}{L} \sum_q \frac{1}{q} \exp\left(-\frac{1}{2}\alpha|q| - iqx\right) \rho_{i\sigma}(q) . \quad (16)$$

Here the Fourier components of electron densities are used,

$$\begin{aligned} \rho_{1\sigma}(q) &= \sum_{k>0} a_{k+q, \sigma}^\dagger a_{k\sigma} , \\ \rho_{1\sigma}(-q) &= \sum_{k>0} a_{k\sigma}^\dagger a_{k+q, \sigma} , \\ \rho_{2\sigma}(q) &= \sum_{k<-q} b_{k+q, \sigma}^\dagger b_{k\sigma} , \\ \rho_{2\sigma}(-q) &= \sum_{k<-q} b_{k\sigma}^\dagger b_{k+q, \sigma} . \end{aligned} \quad (17)$$

It is easy to check that $\rho_{i\sigma}(q)$ satisfy

$$\begin{aligned} [\rho_{1\sigma}(-q), \rho_{1\sigma'}(q')] &= [\rho_{2\sigma}(q), \rho_{2\sigma'}(-q')] \\ &= \frac{qL}{2\pi} \delta_{qq'} \delta_{\sigma\sigma'} . \end{aligned} \quad (18)$$

The equivalence of the operators $\psi_{i\sigma}$ and $O_{i\sigma}$ is due to the fact that they yield the same commutations with respect to $\rho_{i\sigma}(q)$; e.g. for $\rho_{1\sigma}(q)$

$$\begin{aligned} [\psi_{1\sigma}(x, t), \rho_{1\sigma}(q)] &= e^{-iqx} \psi_{1\sigma}(x, t) , \\ [O_{1\sigma}(x, t), \rho_{1\sigma}(q)] &= e^{-iqx} O_{1\sigma}(x, t) . \end{aligned} \quad (19)$$

And then for any operator, which is a functional of $\rho_{i\sigma}(q)$, $\psi_{i\sigma}(x, t)$ and $O_{i\sigma}(x, t)$ act equivalently.

It is straightforward to verify that

$$\begin{aligned} [H_0, \rho_{1\sigma}(q)] &= v_F q \rho_{1\sigma}(q) , \\ [H_0, \rho_{2\sigma}(q)] &= -v_F q \rho_{1\sigma}(q) . \end{aligned} \quad (20)$$

It is clear that the same equation is satisfied by the boson representation of H_0 ,

$$H_0^B = \frac{2\pi v_F}{L} \sum_{q>0} \sum_\sigma (\rho_{1\sigma}(q) \rho_{1\sigma}(-q) + \rho_{2\sigma}(-q) \rho_{2\sigma}(q)) , \quad (21)$$

In terms of $\rho_{i\sigma}(q)$, the Hamiltonian H_{LM} is of the form[14]—[16]

$$\begin{aligned} H_{\text{LM}} &= \frac{Ua}{2L} \sum_{q>0} ((\rho_1(q) \rho_2(-q) + \rho_1(-q) \rho_2(q)) \\ &\quad - (\sigma_1(q) \sigma_2(-q) + \sigma_{1\sigma}(-q) \sigma_2(q)) \\ &\quad + (\rho_1(q) \rho_1(-q) + \rho_2(-q) \rho_2(q)) \\ &\quad - (\sigma_1(q) \sigma_1(-q) + \sigma_2(-q) \sigma_2(q))) , \end{aligned} \quad (22)$$

where

$$\begin{aligned}\rho_i(q) &= \frac{1}{\sqrt{2}} (\rho_{i1}(q) + \rho_{i2}(q)) , \\ \sigma_i(q) &= \frac{1}{\sqrt{2}} (\rho_{i1}(q) - \rho_{i2}(q)) .\end{aligned}$$

For convenient we introduce the phase variables $\Phi_i(x, t)$,

$$\begin{aligned}\Phi_1(x, t) &= \frac{i}{\sqrt{4\pi}} \sum_q \frac{2\pi}{Lq} \exp\left(-\frac{1}{2}\alpha|q| - iqx\right) (\rho_1(q) + \rho_2(q)) \\ &= \frac{i}{2\sqrt{2\pi}} ((\phi_{11}(x) + \phi_{11}(x)) + (\phi_{21}(x) + \phi_{21}(x))) , \\ \Phi_2(x, t) &= \frac{i}{\sqrt{4\pi}} \sum_q \frac{2\pi}{Lq} \exp\left(-\frac{1}{2}\alpha|q| - iqx\right) (\sigma_1(q) + \sigma_2(q)) \\ &= \frac{i}{2\sqrt{2\pi}} ((\phi_{11}(x) - \phi_{11}(x)) + (\phi_{21}(x) - \phi_{21}(x))) .\end{aligned}\quad (23)$$

The canonical conjugations of the phase variables $\Phi_i(x, t)$ are

$$\begin{aligned}\Pi_1(x, t) &= \frac{\partial \Phi_1(x, t)}{\partial t} \\ &= \frac{-1}{\sqrt{4\pi}} \sum_q \frac{2\pi}{L} \exp\left(-\frac{1}{2}\alpha|q| - iqx\right) (\rho_1(q) - \rho_2(q)) , \\ \Pi_2(x, t) &= \frac{\partial \Phi_2(x, t)}{\partial t} \\ &= \frac{-1}{\sqrt{4\pi}} \sum_q \frac{2\pi}{L} \exp\left(-\frac{1}{2}\alpha|q| - iqx\right) (\sigma_1(q) - \sigma_2(q)) .\end{aligned}\quad (24)$$

By making use of Eq.(18), we have the following equal time commutation relations

$$[\Phi_j(x, t), \Pi_l(x', t)] = i\delta_{jl}\delta(x - x') . \quad (25)$$

In terms of the phase variables Φ_i and Π_i , the Hubbard Hamiltonian is of the form

$$\begin{aligned}H_0^B + H_{LM} &= \int dx \left(\left(\frac{v_F}{2} + \frac{Ua}{4\pi} \right) \frac{\partial \Phi_1}{\partial x} \frac{\partial \Phi_1}{\partial x} + \frac{v_F}{2} \frac{\partial \Phi_1}{\partial t} \frac{\partial \Phi_1}{\partial t} \right) \\ &\quad + \int dx \left(\left(\frac{v_F}{2} - \frac{Ua}{4\pi} \right) \frac{\partial \Phi_2}{\partial x} \frac{\partial \Phi_2}{\partial x} + \frac{v_F}{2} \frac{\partial \Phi_2}{\partial t} \frac{\partial \Phi_2}{\partial t} \right) , \\ H_{BS} &= \frac{Ua}{4(2\pi\alpha)^2} \int dx \cos(2\sqrt{2\pi}\Phi_2(x, t)) , \\ H_{US} &= \frac{Ua}{4(2\pi\alpha)^2} \int dx \cos((G - 4k_F)x + 2\sqrt{2\pi}\Phi_1(x, t)) .\end{aligned}\quad (26)$$

H_{BS} and H_{US} introduce non-linearity into the system. However, for repulsive interaction ($U > 0$) H_{BS} is renormalized to zero in the long-wavelength limit. And at off half-band H_{US} do not appear[14]–[16]. So that the only interested Hamiltonian is

$$\begin{aligned}H &= \int dx \left(\left(\frac{v_F}{2} + \frac{Ua}{4\pi} \right) \frac{\partial \Phi_1}{\partial x} \frac{\partial \Phi_1}{\partial x} + \frac{v_F}{2} \frac{\partial \Phi_1}{\partial t} \frac{\partial \Phi_1}{\partial t} \right) \\ &\quad + \int dx \left(\left(\frac{v_F}{2} - \frac{Ua}{4\pi} \right) \frac{\partial \Phi_2}{\partial x} \frac{\partial \Phi_2}{\partial x} + \frac{v_F}{2} \frac{\partial \Phi_2}{\partial t} \frac{\partial \Phi_2}{\partial t} \right) .\end{aligned}\quad (27)$$

We write the corresponding Lagrangian into the familiar form

$$L = -\frac{g_+}{4\pi} \sqrt{1 + \frac{Ua}{2\pi v_F}} \int dx \frac{\partial \Phi_1}{\partial x^+} \frac{\partial \Phi_1}{\partial x^+} - \frac{g_-}{4\pi} \sqrt{1 - \frac{Ua}{2\pi v_F}} \int dx \frac{\partial \Phi_2}{\partial x^-} \frac{\partial \Phi_2}{\partial x^-} , \quad (28)$$

where $g_{\pm} \equiv 2\pi v_F \sqrt{1 \pm \frac{Ua}{2\pi v_F}}$, $x_0^{\pm} \equiv i\sqrt{1 \pm \frac{Ua}{2\pi v_F}}t$ and $x_1^{\pm} = x$.

It is now clear that the continuum limit of the one-dimensional Hubbard model is equivalent to the two-dimensional Coulomb gas model with only one kind of charges[17][18]. However, it should be noticed that we are now discussing a finite size system and in a finite geometry, the boundary conditions generate various constraints on the successive transformations, resulting in a modified free-field theory.

Conformal field theories in two dimensions play a central role in the study of critical phenomena. Cardy has initiated the program to relate the universal properties of conformal field theories to numerically accessible properties of the transfer matrix of a finite width strip[19]. The crucial observation is that the scaling operators and the eigenstates of the transfer matrix of the theory on a cylinder have a one-to-one correspondence through a conformal mapping.

The consistency of conformal theories defined on a torus restricts severely their operator content. The partition function of a conformal theory on a torus may be written as

$$Z = \text{tr} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right) , \quad (29)$$

where $q = e^{2i\pi\tau}$, and $\tau(\tau_R + i\tau_I) = \omega_2/\omega_1$ is the modular ratio of the torus.

This trace which runs over all states of the Hilbert space may be decomposed on the various irreducible representations of the semi-direct product of the two Virasoro algebras in the form[20][21]

$$Z = \sum_{h_{\pm}, \bar{h}_{\pm}} N_{h_{+}\bar{h}_{+}} N_{h_{-}\bar{h}_{-}} \chi_{h_{+}}(q) \chi_{\bar{h}_{+}}(\bar{q}) \chi_{h_{-}}(q) \chi_{\bar{h}_{-}}(\bar{q}) . \quad (30)$$

The non-negative integers $N_{h_{\pm}\bar{h}_{\pm}}$ represent the multiplicity of the operators of dimensions $(h_{\pm}, \bar{h}_{\pm})$ in the partition function. The character $\chi_{h_{\pm}}$ read

$$\chi_{h_{\pm}} = tr_{h_{\pm}} \left(q^{L_0 - \frac{c}{24}} \right)$$

in each irreducible representation of highest weight $h_{\pm}$.

Imposing that Z is a modular invariant function of τ then puts strong constraints on the N 's, hence on the operator content of the theory, leading ultimately to a classification of the possible partition function.

The Lagrangian (28) is characterized by the $(c=1)$ $Vir \otimes \overline{Vir} \otimes Vir \otimes \overline{Vir}$ algebra. The primary field $\phi_{n^{\pm}, m^{\pm}}$ in the theory has scaling dimension

$$\begin{aligned} h_{n^{\pm}, m^{\pm}} &= \frac{(n^{\pm})^2}{2g_{\pm}} + \frac{g_{\pm}(m^{\pm})^2}{2} + m^{\pm}n^{\pm} , \\ \bar{h}_{n^{\pm}, m^{\pm}} &= h_{n^{\pm}, -m^{\pm}} . \end{aligned} \quad (31)$$

The partition function takes the following form

$$Z = \int [D\Phi_1][D\Phi_2] \exp \left(-\frac{g_{+}}{4\pi} \int d^2x^{+} \frac{\partial\Phi_1}{\partial x_{\mu}^{+}} \frac{\partial\Phi_1}{\partial x^{+\mu}} - \frac{g_{-}}{4\pi} \int d^2x^{-} \frac{\partial\Phi_2}{\partial x_{\mu}^{-}} \frac{\partial\Phi_2}{\partial x^{-\mu}} \right) . \quad (32)$$

The x integration is performed on a torus defined by the ratio τ which lies in the upper half-plane. To evaluate the partition function, we first perform the integral over fields Φ_1 and Φ_2 periodic around the torus.

For a free field on a torus $\mathcal{T}$, with the Lagrange (28) integrated over $\mathcal{T}$, a properly renormalized expression for the partition function

$$Z_1(g_{\pm}) = \int_{\Phi_1, \Phi_2 \text{ periodic}} [D\Phi_1][D\Phi_2] \exp \left(-\frac{g_{+}}{4\pi} \int d^2x^{+} \frac{\partial\Phi_1}{\partial x_{\mu}^{+}} \frac{\partial\Phi_1}{\partial x^{+\mu}} - \frac{g_{-}}{4\pi} \int d^2x^{-} \frac{\partial\Phi_2}{\partial x_{\mu}^{-}} \frac{\partial\Phi_2}{\partial x^{-\mu}} \right) \quad (33)$$

is

$$Z_1(g_{\pm}) = \frac{\sqrt{g_{+}g_{-}}}{\tau_I \eta^2(q) \bar{\eta}^2(\bar{q})} \quad (34)$$

η is the Dedekind function

$$\eta(q) = q^{\frac{1}{24}} \prod_{N=1}^{\infty} (1 - q^N) . \quad (35)$$

For a variation of angle equation to $2\pi m_{\pm}$ ($2\pi m'_{\pm}$) along ω_1 (ω_2), i.e., $\delta_1\Phi_1 = 2\pi m_{+}$, $\delta_2\Phi_1 = 2\pi m'_{+}$, $\delta_1\Phi_2 = 2\pi m_{-}$, $\delta_2\Phi_2 = 2\pi m'_{-}$, the corresponding partition function is readily evaluated using the classical solution (such that $\Delta\Phi_1 = \Delta\Phi_2 = 0$),

$$\begin{aligned} Z_{m'_{\pm}, m_{\pm}}(g_{\pm}) &= Z_1(g_{\pm}) \exp \left(-\pi g_{+} \frac{m_{+}^{\prime 2} + m_{+}^2 (\tau_R^2 + \tau_I^2) - 2\tau_R m_{+} m'_{+}}{\tau_I} \right) \\ &\cdot \exp \left(-\pi g_{-} \frac{m_{-}^{\prime 2} + m_{-}^2 (\tau_R^2 + \tau_I^2) - 2\tau_R m_{-} m'_{-}}{\tau_I} \right) . \end{aligned} \quad (36)$$

The physical partition function is obtained by summing over $Z_{m'_{\pm}, m_{\pm}}$. After a Poisson transformation, one finds

$$Z = \frac{1}{(\eta\bar{\eta})^2} \sum_{e_{\pm}, m_{\pm} \in Z} q^{\frac{(e_{+}/\sqrt{g_{+}} + m_{+}/\sqrt{g_{+}})^2}{4}} \bar{q}^{\frac{(e_{+}/\sqrt{g_{+}} - m_{+}/\sqrt{g_{+}})^2}{4}} q^{\frac{(e_{-}/\sqrt{g_{-}} + m_{-}/\sqrt{g_{-}})^2}{4}} \bar{q}^{\frac{(e_{-}/\sqrt{g_{-}} - m_{-}/\sqrt{g_{-}})^2}{4}} . \quad (37)$$

The partition function enjoys the important property of modular invariance, i.e., invariance under the modular group

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} , \quad a, b, c, d \in Z, \quad ad - bc = 1 . \quad (38)$$

Besides the partition function discussed in the above, the charge-charge and spin-spin correlation functions are of special interested for physicists. Local density fluctuations of charge and spin varying slowly in space are given by

$$\begin{aligned} n(x, t) &= \sqrt{\frac{2}{\pi}} \nabla \Phi_1(x, t) , \\ S_z(x, t) &= \sqrt{\frac{1}{2\pi}} \nabla \Phi_2(x, t) . \end{aligned} \quad (39)$$

On the other hand the electron density operator with wave number around $\pm 2k_F$ is given by

$$\rho_{\pm 2k_F, \sigma}(x, t) = \frac{1}{2\pi\alpha} \exp(\pm i(2k_F x + \Phi_1(x, t) + \sigma\Phi_2(x, t))) . \quad (40)$$

Consequently, the total density operators of charge and spin, $n(x, t)$ and $S_z(x, t)$, are given as follows

$$\begin{aligned} n(x, t) &= \sqrt{\frac{2}{\pi}} \nabla \Phi_1(x, t) - \frac{1}{2\pi\alpha} \left(e^{i(2k_F x + \Phi_1)} + e^{-i(2k_F x + \Phi_1)} \right) \cdot \left(e^{i\Phi_2} + e^{-i\Phi_2} \right) , \\ S_z(x, t) &= \sqrt{\frac{1}{2\pi}} \nabla \Phi_2(x, t) - \frac{1}{2\pi\alpha} \left(e^{i(2k_F x + \Phi_1)} - e^{-i(2k_F x + \Phi_1)} \right) \cdot \left(e^{i\Phi_2} - e^{-i\Phi_2} \right) . \end{aligned} \quad (41)$$

It is clear that the density operators of charge and spin, $n(x, t)$ and $S_z(x, t)$, are sums of primary fields of the $(c = 1)$ $Vir \otimes \overline{Vir} \otimes Vir \otimes \overline{Vir}$ algebra. Thus the charge and spin density correlators can be read out directly in the conformal field theory scheme as follows,

$$\begin{aligned} \langle n(x, 0) n(0, 0) \rangle &= \frac{2}{\pi g_+} \cdot \frac{1}{x^2} + \frac{2i}{\pi\alpha} \cos(2k_F x) \frac{1}{x^{\frac{1}{g_+} + \frac{1}{g_-}}} , \\ \langle S_z(x, 0) S_z(0, 0) \rangle &= \frac{1}{2\pi g_-} \cdot \frac{1}{x^2} + \frac{2i}{\pi\alpha} \cos(2k_F x) \frac{1}{x^{\frac{1}{g_+} + \frac{1}{g_-}}} . \end{aligned} \quad (42)$$

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References

- [1] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.
- [2] See for a collection of reprints, *Conformal Invariance and Applications in Statistical Mechanics*, ed. C. Itzykson, H. Saleur and J. -B. Zuber, (World Scientific, Singapore, 1988).
- [3] J. Hubbard, Proc. R. Soc. London, **A276** (1963) 238.
- [4] P. W. Anderson, Science **235** (1987) 1196.
- [5] E. Lieb and F. Y. Wu, Phys. Rev. Lett. **20** (1968) 1445.
- [6] M. Ogata and H. Shiba, Phys. Rev. **B41** (1990) 2326.
- [7] A. Parola and S. Sorella, Phys. Rev. Lett. **64** (1990) 1831.
- [8] H. J. Schult, Phys. Rev. Lett. **64** (1990) 2831.
- [9] S. Tomonaga, Prog. Theor. Phys. **5** (1950) 349.
- [10] J. M. Luttinger, J. Math. Phys. **15** (1963) 609.
- [11] F. D. M. Haldane, Phys. Rev. Lett. **45** (1980) 1358; **47** (1980) 1840.
- [12] N. Kawakami and S. K. Yang, Phys. Lett. **A148** (1990) 359.
- [13] H. Frahm and V. E. Korepin, Phys. Rev. **B42** (1990) 10553.
- [14] J. Sólyom, Adv. Phys. **28** (1979) 201.
- [15] V. J. Emery, in *Highly Conducting One-Dimensional Solids*, ed. J. Devreese, R. Evrad, and V. van Doren (Plenum, New York, 1979).
- [16] H. Fukuyama and Takayama, in *Electronic Properties of Inorganic Quasi-One-Dimensional Compounds*, ed. P. Monceau (D. Reidel Publishing Company, 1985).
- [17] L. P. Kadanoff, J. Phys. **A11** (1978) 1399.
- [18] B. Nienhuis, J. Stat. Phys. **34** (1984) 731.
- [19] J. L. Cardy, J. Phys. **A17** (1984) L385.
- [20] P. di Francesco, H. Saleur and J. B. Zuber, J. Stat. Phys. **49** (1987) No.1/2.
- [21] S. K. Yang, Nucl. Phys. **B285** [FS19] (1987) 183.