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Symmetry and Symmetry Breaking in Quantum-Chromo(Flavor)-Dynamics

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Abstract: Symmetries of flavored QCD and their breaking are reinvestigated within the Lagrange-Hamilton frame as well as on the basis of general gauge quantum field theory aiming at the resolution of some shortcomings by which QCD is still burdened despite of all its undisputable successes; e.g.: The significance of $SU(3)$ (respectively $SU(?)$) flavor symmetry in the face of the hadron mass spectrum showing the structure of $SU(2) \times U(1)$ symmetry and the mass splitting at least in the meson sector asserting by no means the effect of the "medium strong" interaction responsible for the symmetry breaking to be a small one. - The origin of the "medium strong" interaction not being intrinsic to the scheme but has to be put in 'by hand'. - The vanishing of the pseudo-scalar meson masses required by axial vector conservation and the leptonic meson decay constants being compatible with their role as Goldstone states in the chiral version of QCD but in disagreement with their finite experimental values.

I. Introduction

One cornerstone for the success of the unitary symmetry scheme in the evolution of strong interaction physics leading to flavored quantum-chromodynamics consists in bringing order to the complex problems of hadron spectroscopy. The basic philosophy (see [1]-[7] and Refs. therein) of all unitary symmetry schemes from $SU(3)$ to $SU(?)$ is that in an ideal universe there is only one type of interaction, the strong interaction, invariant under some internal symmetry group such that all hadrons may be grouped into unitary multiplets, that is all members of such an unitary multiplet have the same external quantum numbers like mass, spin etc. and differ by internal quantum numbers like isospin, hypercharge and so on being the eigenvalues of a suitably chosen basis for the Cartan subalgebra of the unitary group involved. The quite rough mass splitting in the real world of hadrons is attributed to a hierarchy of weaker interactions breaking the unitary symmetry: At the top a "medium strong" one causing the splitting between isospin submultiplets of different hypercharge followed by the electro-weak interaction responsible for the splitting within an isospin submultiplet.

To derive the mass splittings between isospin submultiples it has to be assumed that the effect of the medium strong interaction Hamiltonian H_{asy} inspite of being quite stronger

than the electro-weak part is small compared to the symmetry conserving strong one, so that first order perturbation theory may be applied.

In spite of its indisputable success in bringing order to the hadron spectrum this scheme showed right from the beginning some deficiencies leaving the state of affairs a little bit artificially flavored. Some authors of textbooks state it this way: "Put more concisely, the theory would be more convincing if the symmetry breaking was 'intrinsic' to the scheme, in the sense that it emerges automatically, without it having to be put in 'by hand' ". [2, Part C, chapt. 18]. "Considering the fact that m_π is much smaller than the mass differences $m_K - m_\pi$ and $m_\eta - m_\pi$ it seems quite strange that H_{asym} can be treated as a perturbation" [1, Chapt. 12].

In the hierarchy of flavor interactions above the strongest symmetry conserving part of the Hamiltonian is invariant under all $SU(3)$ transformations, whereas the "medium strong" one is assumed to break the symmetry and the "rest symmetry group" is $SU(2) \times U(1)$ with $SU(2)$ being the isospin group. In view of the rather large mass splittings between the isotopic submultiplets within a $SU(3)$ multiplet it is not possible to assert the effect of the "medium strong" interaction to be a small one. But then the question arises at once as to the *significance* of the $SU(3)$ (respectively $SU(?)$) group. Is its role merely to specify rather roughly the multiplets or is there something more in the unitary symmetry scheme still hidden in the underground?

Indeed right at the outset the unitary scheme met the flaw, that in contrast to the baryon multiplets the masses in the splitting relations for the pseudo-scalar meson octet had - again by hand - to be replaced by their squares in order to fit the experimental data to an acceptable accuracy. Meanwhile more mesonic particles have been established by experiments [30] and it turns out, that when grouped into octets in such a way that the mass differences are all small compared to a medium mass value of the multiplet the original linear mass formulas fit the experimental values at least with the same accuracy as the quadratic ones and apart from the lowest lying pseudo-vector octet $1^{--}(\omega - \phi \text{ mixing})$ with a variation of less than 2.5%. Moreover the chiral flavor symmetry meets the further flaw that the conservation of its axial vector current in combination with either parity conservation in strong interaction [1, Chap.24.1-3] or the finite meson-decay constants in leptonic weak decays [4, Chap. 2.8] dictates in both cases zero masses for the pseudo-scalar mesons which on the one hand seems compatible with the role of Goldstone states for the spontaneous breaking of the chiral symmetry ascribed to them but on the other hand contradicts the finiteness of their experimental masses. Hence the pseudo-scalar meson octet seems to play an exceptional role for "some unknown weaker mechanism" [5] breaking the unitary flavor symmetry of the strong interaction, which still hides itself behind all plausible explanations.

In the following the flavor and color symmetry and their breaking in flavored chromodynamics is reinvestigated on the basis of Lagrangian as well as general gauge quantum field theory with color-charge confinement aiming at a resolution of the shortcomings sketched above.

In Section II the conserved energy-momentum tensors and hence the Hamilton densities are derived which arise via Noether's Theorem from the form invariance of the usual

flavored QCD-Lagrangian with respect to space-time translations, if besides the familiar standard representation of the translation group for the basic quark and gluon fields more general one dimensional ones affiliated to the flavor and color degrees of freedom are admitted. The result is a continuum of admitted energy-momentum tensors parametrized by five arbitrary four-vectors, one of which is affiliated to the color degrees of freedom and the remaining four to the flavor ones. The Hamilton density corresponding to each of them except the standard one is a sum of the standard canonical one, the conserved Noether current generating the color charge and last but not least the "medium strong" and unitary symmetry breaking conserved flavor current, usually added 'by hand'. On the basis of these Noether currents the symmetry breaking is reconsidered in Section III. The global $SU(3)$ flavor symmetry is on the algebraic level seen to be broken in the sense that the flavored CD-Lagrangian is form invariant with respect to the whole $SU(3)$ group of flavor transformations but depending on the representation chosen for the translation group its infinitesimal generators in general do not commute with the Hamilton density, that is with the generator of the time translation. However for a subclass of representations of the translation group selected from the continuum by taking one of the four flavor vectors to be a fixed multiple of a certain other one of them, the global flavor $SU(2) \times U(1)$ subgroup remains a strict symmetry with its infinitesimal generators commuting with those of the space-time translation.

On the other hand it is proved in Section VI that in the representations of the translation group with non-zero flavor (and/or color) parameters Goldstone states can occur with *non-zero* masses. Taking the pseudo-scalar mesons with their finite experimental masses as Goldstone particles affiliated to the conserved axial vector currents of the chiral $SU(3)$ flavor group then the flavor parameters of representations of the translation group and hence of Hamiltonians admitted by the chiral QCD-Lagrangian are restricted to certain Lorentz orbits within the subclass stated above, in which the $SU(2) \times U(1)$ flavor group remains a strict symmetry group commuting with the space-time-translations.

In Section IV a Poincaré covariant general gauge quantum field theory of flavored chromodynamics with color-charge confinement, which mirrors the basic structures (especially of the color confining and internal symmetry breaking representations of the subgroup of translation) obtained in Section II and III within the Lagrange-Hamilton frame, is formulated in a Hilbert space equipped with an additional indefinite metric. Poincaré invariance requires that this indefinite metric space is the direct integral of such spaces attached to every point of a Lorentz orbit in the space of the flavor parameters as well as a Lorentz orbit in the color vector describing color confinement. The direct integral is taken with respect to the invariant measures of these orbits.

In Section V the spectrum condition is formulated and the construction of one particle states is sketched. By means of them and the direct integral structure of the underlying indefinite metric space it is rather obvious that the finite meson-decay constants together with the conservation of the axial vector current do *not* imply anymore that the masses of the pseudo-scalar mesons vanish.

In Section VI finally the generalized Goldstone-Theorems are proved by means of relative locality and it is shown that the conserved vector currents of the residual $SU(2) \times U(1)$

flavor group generate a representation of this group which is isometric with respect to the indefinite metric. Hence it lifts to a unitary one in the physical Hilbert space obtained in the familiar way via factorization and completion of some subspaces. The corresponding chiral flavor group however is spontaneously broken, since the axial charges do not exist due to the presence of the pseudo-scalar mesons.

II. Noether Currents of LCD

The Lagrangian of (Quantum-)Chromodynamics is build up from two $SU(3)$ -color-triplets respectively $SU(n)$ -flavor-multiplets ($n \geq 3$) $\varphi_{a,f}^{3,n}(x)$ and $\varphi_{a,f}^{3^*,n^*}(x)$ of Dirac spinors called quark respectively antiquark fields and a $SU(3)$ -color-octet (flavor-singlet) $\varphi_a^{8,1}(x)$ of Lorentz vektors called gluon (gauge) fields [1]-[4]. Here we will restrict the flavor group $G(3)_{\mathcal{F}}$ to be $SU(3)$ respectively some subgroup of it (isomorphic to $SU(2) \times U(1)$ or $U(1) \times U(1)$), that means consider a system with three flavors (up, down, strange). However all considerations may easily be extended to a flavor group of higher degree in order to include charmed, bottom, top and so on quarks. The two "different" $SU(3)$ -groups will be marked by a suffix $\mathcal{C}$ for color respectively $\mathcal{F}$ for flavor when ever it seems necessary to avoid confusion.

For the sake of notational shortness the unitary irreducible matrix representations $D^{(p,q)}$, $(p, q) \in \mathbb{N}_0 \times \mathbb{N}_0$ of $SU(3)$ with dimension $d = 1/2[(1+p)^2(1+q) + (1+p)(1+q)^2]$ will be denoted by $R^d = D^{(p,q)}$ for $p \geq q$ respectively $R^{d^*} = D^{(p,q)}$ for $p \leq q$. Then *conventionally* for any triple (α, u, v) from the direct product $SL(2, \mathcal{D}) \times SU(3)_{\mathcal{C}} \times G(3)_{\mathcal{F}}$ the fields are supposed to transform according to:

$$\varphi_{\mu,a,f}^{m,n}(x) \rightarrow (U(\alpha, u, v)\varphi)_{\mu,a,f}^{m,n}(x) := S^{(m,n)}(\alpha)_{\mu}^{\nu} R_{\mathcal{C}}^n(u^{-1})_a^b R_{\mathcal{F}}^n(v^{-1})_f^g \varphi_{\nu,b,g}^{m,n}(\Lambda(\alpha)x) \quad (1)$$

with

$$S^{(m,n)}(\alpha)_{\nu}^{\mu} := \begin{cases} S^{1/2}(\alpha^{-1})_{\nu\mu} & \text{for } (m, n) = (3, 3) \\ \Lambda(\alpha^{-1})_{\nu}^{\mu} & \text{for } (m, n) = (8, 1) \\ S^{1/2}(\alpha)_{\mu\nu} & \text{for } (m, n) = (3^*, 3^*) \end{cases} \quad (2)$$

and

$$\begin{aligned} S^{1/2}(\alpha)^{-1} \gamma^{\mu} S^{1/2}(\alpha) &= \Lambda(\alpha)_{\nu}^{\mu} \gamma^{\nu}; \quad \gamma^0 S^{1/2}(\alpha)^+ \gamma^0 = S^{1/2}(\alpha)^{-1}; \\ \Lambda(\alpha)_{\nu}^{\mu} g_{\mu\sigma} \Lambda(\alpha)_{\kappa}^{\sigma} &= g_{\nu\kappa} \end{aligned} \quad (3)$$

The Dirac matrices are assumed to have the hermiticity properties $(\gamma^{\mu})^+ = (\tilde{\gamma}^{\mu})^t = g^{\mu\mu} \gamma^{\mu}$, $(\gamma^0)^t = \gamma^0$. With respect to the group $SU(3)$ and its representations we adopt the conventions and (apart from minor changes) the notations of de Swart [5]-[7]. The

color/flavor indices are triples $a = (I, I^3, Y)$ running over the index set Γ_d defined by the Okubo-rule [6]:

$$\begin{aligned} \Gamma_d(p, q) := \{ & (I, I^3, Y) | I = 1/2(\lambda - \rho); I^3 = -I, \dots, I; \\ & Y = \lambda + \rho - 1/3(2p + 4q); \lambda, \rho \in \mathbb{N}_0; 0 \leq \rho \leq q \} \end{aligned} \quad (4)$$

$I \dots$ is interpreted to be the total "color/flavor isospin", $I^3 \dots$ its third component and $Y \dots$ the "color/flavor hypercharge". An immediate consequence of the definition of Γ_d is:

$$(I, I^3, Y) \in \Gamma_d(\Gamma_{d^*}) \Leftrightarrow (I, -I^3, -Y) \in \Gamma_{d^*}(\Gamma_d) \quad (5)$$

In accordance with this relation the negative of an index triple $a = (I, I^3, Y)$ is by definition the triple $-a := (I, -I^3, -Y)$. Hence the "color/flavor charge"

$$Q(a) \dots = Q(I, I^3, Y) \dots := I^3 \dots + \frac{1}{2} Y \dots \quad (6)$$

satisfies $Q(-a) \dots = -Q(a) \dots$. Finally there exists an important connection between the complex conjugate representation $\overline{R^d}(R^{d^*})$ and the representation $R^{d^*}(R^d)$. Adopting the notational convention $d^{**} = (d^*)^* = d$ it reads for all d and d^* :

$$R^{d^*}(u)_a^b = (-1)^{Q(b)-Q(a)} \overline{R^d(u)_{-a}^{-b}} \quad (7)$$

$a, b \in \Gamma_{d^*}, u \in SU(3)$.

In case of the adjoint (octet) representation as well as all other ("diagonal") representations with $R^d = D^{(p,p)} = R^{d^*}$ the condition (7) is an equation between different elements of the same matrix, which is called self-conjugateness. As an immediate consequence the gluon fields form a self-conjugate color octet:

$$\overline{\varphi_{\mu,a}^{8,1}(x)} = (-1)^{Q(a)c} \varphi_{\mu,-a}^{8,1}(x) \quad (8)$$

Dirac indices will often be dropped by the usual conventions and $\varphi_{a,f}^{3,3}$ is considered to be a column whereas $\varphi_{a,f}^{3^*,3^*}$ a row. Then the Dirac adjoint of $\varphi_{a,f}^{3,3}$ is connected to $\varphi_{a,f}^{3^*,3^*}$ by:

$$\tilde{\varphi}_{a,f}^{3,3}(x) := \varphi_{a,f}^{3,3}(x)^+ \gamma^0 = (-1)^{-Q(a)c - Q(f)f - 2/3} \varphi_{-a,-f}^{3^*,3^*}(x) \quad (9)$$

In terms of these fields the Lagrangian of chromodynamics reads:

$$\begin{aligned} L(\varphi(x), B(x)) = & -1/4 (-1)^{Q(a)c} G_{\mu\nu,-a}(x) G^{\mu\nu,-a}(x) + \tilde{\varphi}_{a,f}^{3,3}(x) \left[\frac{i}{2} \gamma^\mu \{ \delta_a^b \tilde{\partial}_\mu + ig(\tau_c)_a^b \varphi_{\mu,c}^{8,1}(x) \} \right. \\ & \left. - \delta_a^b g_f \right] \delta_f^e \varphi_{b,e}^{3,3}(x) + (-1)^{Q(a)c} [\varphi_{\mu,-a}^{8,1}(x) \partial^\mu B_a^{8,1}(x) - \frac{\alpha}{2} B_{-a}^{8,1}(x) B_a^{8,1}(x)] \end{aligned} \quad (10)$$

with the gluon field tensor

$$G_{\mu\nu,a}(x) := \partial_\mu \varphi_{\nu,a}^{8,1}(x) - \partial_\nu \varphi_{\mu,a}^{8,1}(x) + ig\sqrt{3} \begin{pmatrix} 8 & 8 & 8_A \\ & b & c & a \end{pmatrix} \varphi_{\mu,b}^{8,1}(x) \varphi_{\nu,c}^{8,1}(x) \quad (11)$$

being obviously a self-conjugate color octet (flavor singlet).

$\tau_a, a \in \Gamma_8$ denote the traceless (3,3)-matrices of the self-conjugate fundamental representation of the (complex) Lie-algebra $su(3)$ satisfying the relations [5]-[7]:

$$[\tau_a, \tau_b]_- = -2\sqrt{3} \begin{pmatrix} 8 & 8 & 8_A \\ a & b & c \end{pmatrix} \tau_c, \quad \text{tr}(\tau_a \cdot \tau_b^+) = 2\delta_{ab}, \quad \tau_a^+ = (-1)^{Q(a)} \tau_{-a} \quad (12)$$

The gauge fixing field $B_a^{8,1}$ is related to the gluon octet via the Euler-Lagrange field equation $\partial^\nu \varphi_{\nu,a}^{8,1}(x) + \alpha B_a^{8,1}(x) = 0$.

Usually there is added a further term to the Lagrangian describing FP -ghosts [8]. In order to keep the explicite formulas as short as possible we dropped it, since it would only add some further terms to the explicite expression of the Noether currents in terms of the basic fields without being in any way tangent to the general structure of these currents which we are going to use in our considerations.

The members of the set $\{\varphi_{\mu,a,f}^{3,3}, \tilde{\varphi}_{\mu,a,f}^{3,3}, \varphi_{k,b}^{8,1}, B_b^{8,1} | \mu = 0, \dots, 3; k = 1, 2, 3; a, f \in \Gamma_3; b \in \Gamma_8\}$ are taken as independent canonical variable.

By construction the Lagrangian is form invariant under the global color group $SU(3)_c$ and the corresponding eight conserved Noether-currents read ($c \in \Gamma_8$):

$$\begin{aligned} j_c^{8,\mu}(x) &:= -\frac{1}{2} \{ \tilde{\varphi}_{a,f}^{3,3}(x) \gamma^\mu (\tau_c)_a^b \varphi_{b,f}^{3,3}(x) + i2\sqrt{3} (-1)^{Q(a)c} \begin{pmatrix} 8 & 8 & 8_A \\ c & b & a \end{pmatrix} \times \\ &\quad [G_{-a}^{\mu c}(x) \varphi_{\epsilon,b}^{8,1}(x) + \varphi_{\lambda,-a}^{8,1}(x) g^{\mu\lambda} B_b^{8,1}(x)] \} = \\ &\quad \frac{1}{g} (-1)^{Q(a)c} \{ \delta_c^a \partial_\lambda G_{-a}^{\mu\lambda}(x) + g^{\mu\lambda} [\delta_c^a \partial_\lambda + ig\sqrt{3} \begin{pmatrix} 8 & 8 & 8_A \\ c & d & a \end{pmatrix} \varphi_{\lambda,d}^{8,1}(x)] B_{-a}^{8,1}(x) \} \end{aligned} \quad (13)$$

In the last row the Euler-Lagrange equations have been used.

Here we are especially interested in the combination $j_{(1,0,0)}^{8,\mu}(x) + \frac{1}{\sqrt{3}} j_{(0,0,0)}^{8,\mu}(x)$ because it will reoccur in the Noether energy momentum tensor below. From the explicite connection between the real self-conjugate traceless matrices $\tau_c, c \in \Gamma_8$ and the hermitian traceless matrices $\lambda_j, j = 1, \dots, 8$ of Gell-Mann

$$\begin{aligned} \tau_{(0,0,0)} &= \lambda_8, \quad \tau_{(1,0,0)} = \lambda_3, \quad \tau_{(1,1,0)} = -\frac{1}{\sqrt{2}}(\lambda_1 + i\lambda_2) \\ \tau_{(1/2,1/2,1)} &= -\frac{1}{\sqrt{2}}(\lambda_4 + i\lambda_5), \quad \tau_{(1/2,-1/2,1)} = -\frac{1}{\sqrt{2}}(\lambda_6 + i\lambda_7) \end{aligned}$$

respectively multiplying the commutation relations (12) by $(\tau_c)^+$ and taking the trace it follows at once:

$$\begin{aligned} (\tau_{(1,0,0)} + \frac{1}{\sqrt{3}} \tau_{(0,0,0)})_a^b &= 2Q(a) \delta_a^b \\ 2\sqrt{3} [\begin{pmatrix} 8 & 8 & 8_A \\ (1,0,0) & b & a \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 8 & 8 & 8_A \\ (0,0,0) & b & a \end{pmatrix}] &= 2Q(a) \delta_{ba} \end{aligned}$$

and therefore:

$$j_{(1,0,0)}^{8,\mu}(x)c + \frac{1}{\sqrt{3}}j_{(0,0,0)}^{8,\mu}(x)c = -\{Q(a)c\tilde{\varphi}_{a,f}^{3,3}(x)\gamma^\mu\varphi_{a,f}^{3,3}(x) + i(-1)^{Q(a)c}Q(a)c[G_{-a}^{\mu,\epsilon}(x)\varphi_{c,a}^{8,1}(x) + g^{\mu\epsilon}\varphi_{c,-a}^{8,1}(x)B_a^{8,1}(x)]\} \quad (14)$$

Concerning the flavor transformations the subgroup of $SU(3)$ with respect to which the CD-Lagrangian is form invariant is determined alone by the "mass" matrix of the quarks

$$\varrho = \begin{pmatrix} \varrho_u & 0 & 0 \\ 0 & \varrho_d & 0 \\ 0 & 0 & \varrho_s \end{pmatrix} = \begin{pmatrix} \varrho_{(1/2,1/2,1/3)} & 0 & 0 \\ 0 & \varrho_{(1/2,-1/2,1/3)} & 0 \\ 0 & 0 & \varrho_{(0,0,-2/3)} \end{pmatrix},$$

which is not observable, but can only be estimated indirectly. If all three quark "masses" are equal $G(3)_{\mathcal{F}}$ is of course again $SU(3)$. If $\varrho_u = \varrho_d \neq \varrho_s$ form invariance holds only for the subgroup

$$G(3)_{\mathcal{F}} = \left\{ \begin{pmatrix} \mathcal{G} & 0 \\ 0 & 0 & 1 \end{pmatrix} \exp[i\frac{t}{2\sqrt{3}}\tau_{0,0,0}] | g \in SU(2) \wedge t \in \mathbb{R} \right\}$$

isomorphic to $SU(2) \times U(1)$. Finally if all three "masses" are pairwise different their remains only the subgroup.

$$G(3)_{\mathcal{F}} = \left\{ \exp i/2(\tau_{(1,0,0)}\rho + \frac{1}{\sqrt{3}}\tau_{(0,0,0)}t) | \rho, t \in \mathbb{R} \right\}$$

isomorphic to $U(1) \times U(1)$.

Corresponding to these three cases the Noether Theorem delivers the existence of eight, four respectively two locally conserved flavor currents:

$$j_c^{8,\mu}(x)_{\mathcal{F}} = -1/2\{\tilde{\varphi}_{a,f}^{3,3}(x)\gamma^\mu(\tau_c)_f^c\varphi_{a,e}^{3,3}(x)\} \quad (15)$$

with

$$c \in \begin{cases} \Gamma_8 & \text{if } G(3)_{\mathcal{F}} = SU(3) \\ \{(0,0,0), (1,0,0), (1,1,0), (1,-1,0)\} & \text{if } G(3)_{\mathcal{F}} \sim SU(2) \times U(1) \\ \{(0,0,0), (1,0,0)\} & \text{if } G(3)_{\mathcal{F}} \sim U(1) \times U(1) \end{cases}$$

It is generally accepted [1]-[4] that $\varrho_d \approx \varrho_u \approx 0$ and $\varrho_u + 100MeV \leq \varrho_s \leq \varrho_u + 200MeV$, which reflects the mass splitting in the hadron spectrum with the splitting within an isospin submultiplet due to electromagnetism. In the last two cases, where the Noether theorem delivers only four or two instead of eight conserved currents one can define four respectively six further currents - the missing ones with respect to the full $SU(3)$ symmetry - by the right hand side of equation (18). Of course they are not conserved. Important for the following is that the two currents $j_{(1,0,0)}^{8,\mu}(x)_{\mathcal{F}}$ and $j_{(0,0,0)}^{8,\mu}(x)_{\mathcal{F}}$ which will reoccur in the Noether energy-momentum tensor are locally conserved in all three cases above.

Besides the irreducible octet operator formed by the eight generators $\tau_a, a \in \Gamma_8$ there exists a second linear independent one [7]:

$$\sigma_a := 2\sqrt{5}/3 \begin{pmatrix} 8 & 8 & 8_S \\ b & c & a \end{pmatrix} \tau_b \tau_c.$$

In terms of the total isospin operator $\mathcal{J}^2$, the hypercharge $\mathcal{Y}$ and the second order Casimir operator $\mathcal{C}^2$ (with eigenvalues $I(a)(I(a) + 1)$, $Y(a)$ and $K(d)$ respectively) its $(0, 0, 0)$ - and $(1, 0, 0)$ - components read explicitly:

$$\begin{aligned} \sigma_{(0,0,0)} &= 2[\mathcal{J}^2 - 1/4\mathcal{Y}^2 - 1/3\mathcal{C}^2] \\ \sigma_{(1,0,0)} &= 10/(3\sqrt{3})[\tau_{(1,0,0)} - \sqrt{3}/5\tau_{(0,0,0)} + 6/5(\mathcal{J}^2 + 3/4\mathcal{Y}^2 - 1/2\mathcal{C}^2)] \end{aligned} \quad (16)$$

Whatever the quark "masses" are the Lagrangian is form invariant with respect to the substitutions

$$\begin{aligned} \varphi_{a,f}^{3,3}(x) &\rightarrow \hat{\varphi}_{a,f}^{3,3}(x) = (\exp i/2\sigma_{(r,0,0)}t)_f^c \varphi_{a,e}^{3,3}(x) \\ \hat{\varphi}_{a,f}^{3,3}(x) &\rightarrow \check{\varphi}_{a,f}^{3,3}(x) = (\exp i - i/2\sigma_{(r,0,0)}t)_f^c \hat{\varphi}_{a,e}^{3,3}(x) \end{aligned} \quad (17)$$

with $r = 0, 1$ and the gluon respectively gauge fields unchanged. But then the Noether Theorem delivers two further conserved flavor currents:

$$j_{(r,0,0)}^{8,\mu}(x)_F = -\frac{1}{2}\hat{\varphi}_{a,f}^{3,3}(x)\gamma^\mu(\sigma_{(r,0,0)})_f^c \varphi_{a,e}^{3,3}(x) \quad (18)$$

with $r = 0, 1$ which also are going to reoccur in the Noether energy momentum tensor.

Next we consider the translation group $(\mathbb{R}_4, +)$. For any fixed complex four-vector $\delta \in C_4$ and any fixed real four-vectors (κ, ω, M, N) a representation $T_{\kappa,\omega}^{\delta,M,N}$ of the translation group may be defined by:

$$\begin{aligned} (T_{\kappa,\omega}^{\delta,M,N}(y)\varphi)_{\mu,a,f}^{n,d}(x) &:= \exp[Q(a)_c(\delta \cdot y)]R^d(\exp i/2[\sigma_{(1,0,0)}(N \cdot y) + \sigma_{(0,0,0)}(M \cdot y)])_f^b \times \\ &\times R^d(\exp i/2[\tau_{(1,0,0)}(\kappa \cdot y) + \frac{1}{\sqrt{3}}\tau_{(0,0,0)}(\omega \cdot y)])_b^c \varphi_{\mu,a,e}^{n,d}(x+y) \end{aligned} \quad (19)$$

with $(n, d) \in \{(3, 3), (3^*, 3^*), (8, 1)\}$, and $B_a^{8,1}$ transforming in the same way as the gluon field $\varphi_{a,e}^{8,1}$ (i.e. color octet and flavor singlet). By means of the equation (7) and (9) it follows for the Dirac adjoint spinor $\hat{\varphi}^{3,3}$:

$$\begin{aligned} (T_{\kappa,\omega}^{\delta,M,N}(y)\hat{\varphi})_{a,f}^{3,3}(x) &= (-1)^{-Q(a)_c - Q(f)_F - 2/3}(T_{\kappa,\omega}^{\delta,M,N}(y)\varphi)_{-a,-f}^{3^*,3^*}(x) \\ &= \exp[-Q(a)_c(\delta \cdot y)]\exp i(-i/2)[\sigma_{(1,0,0)}(N \cdot y) + \sigma_{(0,0,0)}(M \cdot y)]_f^b \\ &\times (\exp(-i/2)[\tau_{(1,0,0)}(\kappa \cdot y) + \frac{1}{\sqrt{3}}\tau_{(0,0,0)}(\omega \cdot y)])_b^c \hat{\varphi}_{a,e}^{3,3}(x+y) \end{aligned} \quad (20)$$

The equations (19) are nothing else than the definition of a set of operations $\{T_{\dots}(y)|y \in \mathbb{R}_4\}$ satisfying the (Abelian) composition law $T_{\dots}(x) \circ T_{\dots}(y) = T_{\dots}(x+y)$ of the translation group and therefore generating a representation of it. (For $M = N = m = \kappa = \omega = 0$ it is just the standard "transformation law" of fields under translations well known from any textbook). The essential point comes now:

The CD-Lagrangian as given by the equations (10)-(11) is form invariant, that means $L(\hat{\varphi}(\hat{x}), \hat{B}(\hat{x})) = L(\varphi(x), B(x))$, under the following group of substitutions:

$$\begin{aligned} x &\rightarrow \hat{x} = x + y, \quad y \in \mathbb{R}_4 \\ \varphi_{\mu,a,f}^{n,d}(x) &\rightarrow \hat{\varphi}_{\mu,a,f}^{n,d}(\hat{x}) := (T_{\kappa,\omega}^{\delta,M}(-y)\varphi)_{\mu,a,f}^{n,d}(x+y), (n,d) \in \{(3,3), (3^*,3^*), (8,1)\} \\ B_a^{8,1}(x) &\rightarrow \hat{B}_a^{8,1}(\hat{x}) := (T_{\kappa,\omega}^{\delta,M}(-y)B)_a^{8,1}(x+y). \end{aligned} \quad (21)$$

The proof for the flavor part is obvious since the Lagrangian is diagonal in the flavor indices. The proof of the color part is given in the appendix of [9] and consists mainly in showing that $(\tau_c)_a^b = 0$ unless $Q(a) = Q(b) + Q(c)$.

Due to the form invariance of the Lagrangian under the (four parameter) group of translations above Noether's Theorem [10] dictates the existence of four locally conserved currents $\Theta_{\kappa,\omega}^{\delta,M,N}(x)_\nu^\mu$, ($\nu = 0, 1, 2, 3$, $\partial_\mu \Theta_{\kappa,\omega}^{\delta,M,N}(x)_\nu^\mu = 0$) which are considered as the components of the energy-momentum density tensor, provided they possess the necessary reality properties of an observable. In the case at hand they read explicitly:

$$\begin{aligned} \Theta_{\kappa,\omega}^{\delta,M,N}(x)_\nu^\mu &= \Theta(x)_\nu^\mu + i\delta_\nu Q(a)_c \hat{\varphi}_{a,f}^{3,3}(x) \gamma^\mu \varphi_{a,f}^{3,3}(x) \\ &\quad - \frac{1}{2} \hat{\varphi}_{a,f}^{3,3}(x) \gamma^\mu [\kappa_\nu(\tau_{(1,0,0)})_f^c + \frac{1}{\sqrt{3}} \omega_\nu(\tau_{(0,0,0)})_f^c] \varphi_{a,e}^{3,3}(x) \\ &\quad - \frac{1}{2} \hat{\varphi}_{a,f}^{3,3}(x) \gamma^\mu (N_\nu \sigma_{(1,0,0)} + M_\nu \sigma_{(0,0,0)})_f^c \varphi_{a,e}^{3,3}(x) \\ &\quad - Q(a)_c (-1)^{Q(a)_c} [G_{-a}^\mu(x) \varphi_{c,a}^{8,1}(x) + g^{\mu c} \varphi_{c,-a}^{8,1}(x) B_a^{8,1}(x)] \delta_\nu \end{aligned} \quad (22)$$

Here $\Theta(x)_\nu^\mu = \Theta_{0,0}^{0,0,0}(x)_\nu^\mu$ is the (standard) *canonical* energy-momentum density tensor

$$\begin{aligned} \Theta(x)_\nu^\mu &= -\delta_\nu^\mu L(\varphi(x), B(x)) + \frac{i}{2} \hat{\varphi}_{a,f}^{3,3}(x) \gamma^\mu \overset{\bullet}{\partial}_\nu \varphi_{a,f}^{3,3}(x) \\ &\quad - (-1)^{Q(a)} [G_{-a}^\mu(x) \partial_\gamma \varphi_{c,a}^{8,1}(x) + g^{\mu c} \varphi_{c,-a}^{8,1}(x) \partial_\gamma B_a^{8,1}(x)]. \end{aligned}$$

Especially $\Theta(x)_0^0$ is just the *canonical* Hamiltonian density.

Comparing the right hand side of the equation (22) with the equations (14) and (15) we see that due to the relation (14) the terms occurring in addition to the canonical part are just special components of the locally conserved flavor respectively color currents. Hence the final result for the locally conserved Noether tensor obtained from translational invariance reads:

$$\begin{aligned}\Theta_{\kappa,\omega}^{\delta,M,N}(x)_\nu^\mu &= \Theta(x)_\nu^\mu - i \delta_\nu [j_{(1,0,0)}^{8,\mu}(x)c + \frac{1}{\sqrt{3}}j_{(0,0,0)}^{8,\mu}(x)c] \\ &+ \kappa_\nu j_{(1,0,0)}^{8,\mu}(x)_\mathcal{F} + \frac{1}{\sqrt{3}}\omega_\nu j_{(0,0,0)}^{8,\mu}(x)_\mathcal{F} + N_\nu j_{(1,0,0)}^{8_S,\mu}(x)_\mathcal{F} + M_\nu j_{(0,0,0)}^{8_S,\mu}(x)_\mathcal{F}\end{aligned}\quad (23)$$

with $\delta \in C_4$ and $M, N, \kappa, \omega \in \mathbb{R}_4$.

All further considerations and results will be based on this formula without any reference to the dependence of the currents on the basic quark and gluon fields with the exception of their properties under translations originating from them. They follow directly from the equations (15), (16), (18)-(21) for $\rho = 8, 8_S$:

$$\begin{aligned}(T_{\kappa,\omega}^{\delta,M,N}(y)j)_a^{\rho,\mu}(x)_\mathcal{F} &= R^8(\exp.[i/2(y \cdot \not{p})])_a^b j_b^{\rho,\mu}(x+y)_\mathcal{F} \\ \not{p} &:= \tau_{(1,0,0)}\kappa + 1/\sqrt{3}\tau_{(0,0,0)}\omega + \sigma_{(1,0,0)}N + \sigma_{(0,0,0)}M,\end{aligned}\quad (24)$$

which in turn leads to:

$$\begin{aligned}(T_{\kappa,\omega}^{\delta,M,N}(y)j)_a^{\rho,\mu}(x)_\mathcal{F} &= \exp.[i(p(a) \cdot y)]j_a^{\rho,\mu}(x+y)_\mathcal{F} \\ p(a) &:= I(a)(I(a)+1)(M+1/\sqrt{3}N) + I_3(a)(\kappa+10/(3\sqrt{3})N) \\ &+ 1/2Y(a)(\omega-2/\sqrt{3}N) - 1/4Y(a)^2(M-2\sqrt{3}N) - (M+\sqrt{3}N)\end{aligned}\quad (25)$$

In the remainder of this section the dependence of the fields respectively currents on the parameters $(\kappa, \omega, \delta, M, N)$ is not written out explicitly but should be kept in mind (especially in connection with relativistic covariance).

In classical field theories as well as in quantized ones using a non covariant gauge (Coulomb gauge for instance) which admits a (positive definite) Hilbert space $(\mathcal{H}, (\cdot|\cdot))$ spanned by physical states the components $\Theta_{\kappa,\omega}^{\delta,M,N}(x)_\nu^0, \nu = 0, \dots, 3$ are interpreted as the energy momentum density. Hence $\Theta_{\kappa,\omega}^{\delta,M,N}(x)_\nu^0$ has to be real respectively a hermitian operator. Therefore κ, ω, M, N have to be real and δ_ν purely imaginary. In other words all representations of the translation group which do not meet these conditions would be excluded via physical interpretation. Besides loosing explicite Lorentz-covariance and Einstein-causality (locality) in the intermediate steps within these gauges the fact that only color singlets are realized in nature and hence admitted in the physical Hilbert space has to be put in as an additional condition "by hand".

The situation is different (for the color coupling δ_ν) in a covariant gauge of the Gupta-Bleuler type and especially in the case of QCD at hand with colored basic fields and only color singlets admitted to the physical Hilbert space. In such a gauge [11]-[22] the physical Hilbert space $(\mathcal{H}_{phys}, <\cdot, \cdot>)$ is detached from a Hilbert space $(\mathcal{H}, (\cdot|\cdot))$ in which the basic fields, symmetry transformations, currents etc. act as operators. It is obtained via factorization H/H_0 of two subspaces $H_0 \subset H \subset \mathcal{H}$ generated by an indefinite metric $<\cdot, \cdot> = (\cdot|\eta\cdot)$ on $\mathcal{H}$ which is positive semi-definite on H and zero on $\mathcal{H}_0 \supset H_0 = \mathcal{H}_0 \cap H$. Moreover the scalar product on $\mathcal{H}_{phys} = \overline{H/H_0}$ is again given by the metric $<\cdot, \cdot>$ and

of course is positive definite on this space. The local gauge invariance of the theory is reflected in the requirement [13],[21]

$$\langle \Psi, (j_a^{8,\mu}(x)_c - \partial_\nu G_a^{\mu\nu}(x))\Phi \rangle = 0 \quad (26)$$

for all Φ from a dense subspace of H and all $\Psi \in H$.

This is the situation in case $\delta = 0$. For $\delta \neq 0$ Poincaré-covariance is again lost for the intermediate steps in the unphysical Hilbert space $(\mathcal{H}, (\cdot|\cdot))$ but invariance is restored via color confinement in the physical Hilbert space $\mathcal{H}_{phys}$, provided δ is real [9], [11].

On the other hand the generally accepted physical picture behind the confinement mechanism, namely that the forces between the quarks grow in strength with the distance between them [1]-[4] demands the translation group to be represented by non-isometric operators in the non-physical Hilbert space $(\mathcal{H}, (\cdot|\cdot))$ because otherwise there would arise a contradiction to the cluster properties of the Wightman functions [22]. However it is well known [11, Theorem 4.2] that in gauge quantum field theories with the vacuum being an eigenstate of the metric operator an η -isometric representation of the translation group is non-unitary in $(\mathcal{H}, (\cdot|\cdot))$ if and only if $\text{Re } \delta \neq 0$. (We note in passing, that the vacuum being an eigenstate of the metric operator is only a sufficient condition but not a necessary one).

As will be shown in section IV in order to achieve both color confinement and besides Poincaré-invariance in the physical Hilbert space also Poincaré-covariance everywhere in the intermediate steps the Hilbert space $(\mathcal{H}, (\cdot|\cdot))$ with metric operator η respectively indefinit metric $\langle \cdot, \cdot \rangle$ and corresponding subspaces H and H_0 have to be replaced by a set of such objects $(\mathcal{H}(m), (\cdot|\cdot)_m), \eta_m, \langle \cdot, \cdot \rangle_m, H(m), H_0(m)$ with m running through some Lorentz orbit in the parameter space $\{(\delta, \kappa, \omega, M, N)\}$ of the representations of the translation group. The physical Hilbert space $\mathcal{H}_{phys}$ is then given by the direct integral of the completed factor spaces $(\overline{H(m)}/\overline{H_0(m)}, \langle \cdot, \cdot \rangle_m)$ with respect to the invariant measure of this Lorentz orbit.

In contrast to the imaginary color coupling $i\delta$ the flavor couplings have to be real, because otherwise the same mechanism as in the case of color confinement would ban all charged flavor states from $\mathcal{H}_{phys}$.

In accordance with the discussion above in the following $\delta_\nu \neq 0$ will be taken real (for the covariant gauges). Then the energy density $E_{\kappa,\omega}^{\delta,M,N}(x) := \Theta_{\kappa,\omega}^{\delta,M,N}(x)_0^0$ reads:

$$\begin{aligned} E_{\kappa,\omega}^{\delta,M,N}(x) = & H(x) - i \delta_0 [j_{(1,0,0)}^{8,0}(x)_c + \frac{1}{\sqrt{3}} j_{(0,0,0)}^{8,0}(x)_c] + \\ & + \kappa_0 j_{(1,0,0)}^{8,0}(x)_F + \frac{\omega_0}{\sqrt{3}} j_{(0,0,0)}^{8,0}(x)_F + N_0 j_{(1,0,0)}^{8,0}(x)_F + M_0 j_{(0,0,0)}^{8,0}(x)_F \end{aligned} \quad (27)$$

with $\delta_0, \kappa_0, \omega_0, M_0, N_0 \in \mathbb{R}$ and $H(x)$ the *canonical* Hamiltonian density.

III. Symmetry Breaking.

Of particular interest are the four flavor $SU(3)$ -octet terms which occur on the right hand side of (27) in addition to the flavor singlet canonical Hamiltonian and the flavor singlet color terms in the first row. They are just identical with the flavor symmetry breaking "medium strong interaction Hamiltonian" ($J_{(0,0,0)}^{8,0}$ and $J_{(0,0,0)}^{8s,0}$) respectively the electromagnetic interaction $J_{(1,0,0)}^{8,0}$ and $J_{(1,0,0)}^{8s,0}$ which have been successfully used to calculate the mass-splitting within hadron multiplets [1]-[7] and especially derive the famous Gell-Mann-Okubo mass formulas [24],[25]. Up to now either these terms have to be added by hand to the flavor singlet canonical Hamiltonian or the first few terms have to be selected again by hand from the expansion of a mass operator into " $SU(3)$ -multipoles" which preserve the selection rules for isospin and hypercharge. The medium strong flavor terms $J_{(0,0,0)}^{8,0}(x)\mathcal{F}$ do not touch total isospin and hypercharge conservation and they are responsible for the mass splitting between the different isospin multiplets within a flavor multiplet, whereas $J_{(1,0,0)}^{8,0}$ leaves only the third component of the isospin besides the hypercharge conserved and could cause the splitting within an isospin multiplet.

However it seems rather curious that the electromagnetic mass splitting within an isospin multiplet could be achieved without an electro-magnetic interaction term showing up in the Lagrangian. Indeed it will be shown below that the Goldstone Theorem with the pseudo scalar meson octet as Goldstone particles implies $\kappa = -16/\sqrt{27}N$. This in turn enforces (via equation (16)) the mutually compensation of the isospin breaking terms ($\sim \tau_{(1,0,0)}$) in the two flavor currents $J_{(1,0,0)}^{8,\mu}$ and $J_{(1,0,0)}^{8s,\mu}$ on the right hand side of the equations (23) and (26). Hence the mass splitting within an isospin submultiplet is not due to a genuine Goldstone mechanism.

The first main new result of our investigation is that the form invariance of the CD -Lagrangian with respect to (non-standard) representations of the translation group produces exactly these mass splitting (flavor symmetry breaking) terms in the corresponding conserved Noether current (energy momentum density tensor); in short they are derived from translational invariance. Hence the "unknown medium strong interaction Hamiltonian" has come to be known (to those who want to know it).

Since the representations of the (abelian) translation group occurring in the equation (19) are the most general one dimensional ones which leave the CD -Lagrangian form invariant our result explains why the octet model, i.e. the assumption that all the higher flavor tensor operators $J_{(0,0,0)}^{27}, J_{(0,0,0)}^{64}, \dots$ respectively $J_{(1,0,0)}^{27}, J_{(1,0,0)}^{64}, \dots$ do not contribute to the hadron mass splitting, works so well. In other words the symmetry breaking is intrinsic to the scheme, in the sense that it emerges automatically, without having to be put in by hand.

One may even turn the argument around and take the hadron mass splitting as an indication that nature prefers to select (for the strong interaction) a representation with $(\kappa_0, \omega_0, M_0, N_0)$ unequal to zero instead of the standard one. Because whenever one adds the flavor symmetry breaking interaction Hamiltonian - by hand or otherwise - to the strong symmetry conserving canonical one, the resulting states of the system can not transform any more with the standard representation of the translation group.

From the equation (27) all the hadron mass formulas known in the literature [1]-[7], [24],[25] can easily be reproduced by means of the usual and generally accepted rules of the game to be found in any textbook on this topic. They consist in taking the expectation value for the various one particle hadron states and adding the following two assumptions: (i) The flavor $SU(3)$ breaking interaction Hamiltonian, that means in the present scheme the two flavor currents in equation (27), are members of an irreducible octet tensor operator of the $SU(3)$ flavor group. (ii) The one particle hadron states may be grouped together into and approximately treated as exact $SU(3)$ flavor multiplets.

Now a second result within our scheme is that the property (i) must not be put in as an additional assumption but follows via Noether's Theorem provided all three quark "masses" in the CD-Lagrangian (10) are equal ($\rho_u = \rho_d = \rho_s$). However similarly to the case of the physical hadrons above the observed splitting of the phenomenological quark masses is obtained via equation (27) starting with equal and even zero quark mass parameters in the CD-Lagrangian and taking the expectation values for the (color- and flavor-) quark triplet with the (positive definite) scalar product $(\cdot|\cdot)_m$ in the unphysical Hilbert space $\mathcal{H}(m)$. (In every representation with $Im.\delta_0 \neq 0$ there emerges also a contribution from the color charge density $j_{(1,0,0)}^{8,0}(x)c + \frac{1}{\sqrt{3}}j_{(0,0,0)}^{8,0}(x)c$.) Hence there exists no physical reason which can prevent one from starting with a CD-Lagrangian which is $SU(3)$ form invariant with respect to both the color and the flavor group and even with vanishing quark mass parameters ($\rho_u = \rho_d = \rho_s = 0$).

The physical hadron mass splitting as well as the phenomenological quark masses then emerge via a *(algebraic) symmetry breaking of the global flavor group in the sense that the CD-Lagrangian and hence the Euler-Lagrange field equations are form invariant with respect to these groups but their infinitesimal generators τ_a with $a \neq (1,0,0), (0,0,0)$ may not commute with the energy operator (i.e. the total Hamiltonian consisting of the canonical plus the symmetry breaking one)*. Since they do not commute with the generator of the time translations the corresponding charges may not be conserved in time, if they exist at all. For $\delta \neq 0$ the equation (27) implies a symmetry breaking in the same sense (at least on the Lagrange-Hamilton level) for the global subgroup of the $SU(3)$ local gauge color group.

Hence in general the form invariance of the CD-Lagrangian under the substitution rules of the basic fields with respect to $SU(3)$ transformations do not evoke the corresponding $SU(3)$ symmetry of the physical system. Depending critically on the representation $T_{\kappa,\omega}^{\delta,M,N}$ of the translation group selected out of all those which themselves leave the Lagrangian respectively the Euler-Lagrange field equations form invariant the $SU(3)$ symmetry is broken in the sense above with only a rest-symmetry with respect to some subgroup being left. For instance in all representations with $\kappa = -10/\sqrt{27}N$ due to mutual compensation all terms containing $\tau_{(1,0,0)}$ disappear from the exponentials of the flavor matrices R^m in the substitution rules (19), (20) respectively (24), (25) and hence from the Hamilton density (27). Therefore all generators of the $SU(2)$ subgroup commute with the translations respectively the Hamiltonian. Furthermore in view of equation (16) the generator $\tau_{(0,0,0)}$ (hypercharge) of the unitary subgroup $U(1)$ obviously commutes with the translations

and of course with all generators of $SU(2)$. Therefore in all representations with $\kappa = -10/\sqrt{27}N$ the subgroup $SU(2) \times U(1)$ remains a symmetry group on the Lagrange-Hamilton or field algebra level. Of course in the standard representation the full flavor $SU(3)$ symmetry holds.

The (algebraic) symmetry breaking discussed so far should not be confused with the spontaneous symmetry breaking via a Goldstone mechanism. This eventually occurs in the next step, when the representation of the field algebra in an inner product space or Hilbert space is taken into account, giving rise to a further reduction of the symmetry of the system. As will be shown below the existence of the pseudoscalar meson octet with their finite physical masses evokes the spontaneous symmetry breaking of the *chiral* flavor symmetry via a genuine Goldstone mechanism and at the same time ruling out all representations $T_{\kappa,\omega}^{\delta,M,N}$ of the translation group with $\kappa \neq -10/\sqrt{27}N$ from quantum flavor dynamics.

Postponing the Goldstone problem for a moment, let us shortly consider the chiral flavor symmetry. With all $\rho_f, f = u, d, s$ being zero the quark triplet may be decomposed into a left-handed $\varphi_{\dots}^{3,3,L} := (1 + \gamma_5)\varphi_{\dots}^{3,3}$ and a right-handed $\varphi_{\dots}^{3,3,R} := (1 - \gamma_5)\varphi_{\dots}^{3,3}$ one ($\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$). Then the Lagrangian (10) reads:

$$\begin{aligned} L(\varphi(x), B(x)) = & -1/4(-1)^{Q(a)}G_{\mu,\nu,-a}(x)G^{\mu\nu}_{,a}(x) + \\ & \sum_{h=L,R} \bar{\varphi}_{a,f}^{3,3,h}(x) \frac{i}{2} \gamma^\mu \{ \delta_a^b \bar{\partial}_\mu + i g(\tau_c)_a^b \varphi_{\mu,c}^{8,1}(x) \} \varphi_{b,f}^{3,3,h}(x) \\ & + (-1)^{Q(a)} [\varphi_{\mu,-a}^{8,1}(x) \partial^\mu B_a^{8,1} - \frac{a}{2} B_{-a}^{8,1}(x) B_a^{8,1}(x)] \end{aligned} \quad (28)$$

and is now form invariant under the chiral flavor group $SU(3)_{\mathcal{F},L} \times SU(3)_{\mathcal{F},R}$. Performing everywhere the replacements

$$\varphi_{\dots}^{3,3}(x) \rightarrow \varphi_{\dots}^{3,3,h}(x), \quad j_{\dots}^{8,\mu}(x)_{\mathcal{F}} \rightarrow j_{\dots}^{8,h,\mu}(x)_{\mathcal{F}}, \quad j_{\dots}^{8_S,\mu}(x)_{\mathcal{F}} \rightarrow j_{\dots}^{8_S,h,\mu}(x)_{\mathcal{F}} \quad (29)$$

and summing in the right hand side of the equations (13),(15),(22)-(24) and (27) all terms carrying the new chiral index over $h = L, R$, everything that has been said before remains true also in the chiral case with the groups $SU(3)$ respectively $SU(2) \times U(1)$ replaced by their chiral counterparts.

The implicate assumption that left and right handed quark fields transform under translations with the same representation ($\kappa_L = \kappa_R, \dots$) is enforced by parity conservation, since in this way only vector currents $j_{\dots}^{\dots V} := j_{\dots}^{\dots L} + j_{\dots}^{\dots R}$ occur in the Noether energy momentum tensor (23) respectively (27) but no axial ones $j_{\dots}^{\dots A} := j_{\dots}^{\dots L} - j_{\dots}^{\dots R}$. A second reason is, that only in this case the vector and axial vectors transform covariantly with respect to translations.

The spontaneous symmetry breaking mechanism and especially the Goldstone problem deserves a detailed and rigorous reinvestigation, since some changes concerning the mass of the Goldstone particles are to be expected due to the difference in the representations of the translation group considered here compared to the usually applied (standard) one. In

the standard case $\delta = \omega = \kappa = M = N = 0$. the generalizations of the Goldstone theorem from classical (non-gauge) quantum field theories to gauge theories with indefinite metric (see e.g. Refs. [26]-[29] respectively [13],[21] and references therein) state, that the spontaneous breaking of an algebraic (global) symmetry group whose generators are associated to conserved currents is necessarily accompanied by the occurrence of $\delta(p^2)$ singularities (Goldstone particles), which are created by the corresponding conserved currents from the vacuum and are pushed via the local gauge condition (26) into the unphysical sections of the Hilbert space in case the (global) symmetry group is the subgroup with constant elements of a local gauge group like the color group in QCD [13],[21]. In case the spontaneously broken (global) symmetry group is not considered to be a subgroup of a local gauge group like the flavor group in QCD the massless Goldstone states may be lying either in the unphysical sections or in the physical one $H \setminus H_0$ and then occur as real particles in $\mathcal{H}_{phys}$ (genuine Goldstone mechanism).

However as far as the zero mass of the Goldstone states is concerned these results may not hold true anymore in the case at hand, since they are strongly tied to the standard representation of the translation group (with $\omega = \kappa = \delta = M = N = 0$) and the four-momentum spectrum of the states allowed in both the physical and the unphysical sections of the total Hilbert space $\mathcal{H}$ (For details see Refs. [13],[21],[28]). Obviously the exponential factors in the representations (19) and (20) respectively (24) which make the difference to the standard one will displace and even deform the support of the spectral functions in the Jost-Lehmann-Dyson representation of the commutator matrix elements involved, thereby moving the light cone of the Goldstone particles and even edge them away from the light cone.

Indeed our second main result is, that the Goldstone particles accompanying the spontaneous symmetry breaking of the chiral flavor group possess the finite (squared) masses $p(a)^2, a \in \Gamma_8$ with $p(a)$ given in equation (25). Hence they may be identified with the corresponding members of the pseudo scalar meson octet 0^- — as physicists want it since two or three decades and claim it verbally in lectures and written up in any text book:

$$\begin{aligned}
p(1/2, +/ - 1/2, +/ - 1)^2 &= m(K_{+/-})^2 \\
p(1/2, -/ + 1/2, +/ - 1)^2 &= m(K_{0/0})^2 \\
p(1, +/ - 1, 0)^2 &= m(\pi_{+/-})^2 \\
p(1, 0, 0)^2 &= m(\pi_0)^2 \\
p(0, 0, 0)^2 &= m(\eta)^2
\end{aligned} \tag{30}$$

The proof follows in almost the classical way [13],[21],[27]-[29] from a generalized Jost-Lehmann-Dyson representation for the commutator matrix element of the current $j_a^{8,h,\mu}(x)_{\mathcal{F}}$ involved and an element $\phi(f)$ from the polynomial $+$ -algebra generated by the basic fields which incorporates automatically besides locality (Einstein causality) and the spectrum condition also the current conservation. The proof is given in section VI below.

The identification of the Goldstone particles with the 0^- -octet fixes in part the representation of the translation group which seems to be selected by nature for QCD. Solving the

equations (30) with equal masses of particles and antiparticles we arrive at the following eight conditions to be fulfilled by the representation parameters:

$$\begin{aligned}
(M + 1/\sqrt{3}N)^2 &= m(\pi_0)^2 \\
(\kappa + 10/\sqrt{27}N)^2 &= m(\pi_{+/-})^2 - m(\pi_0)^2 \\
((\kappa + 10/\sqrt{27}N) \cdot (M + 1/\sqrt{3}N)) &= 0 \\
((\kappa + 10/\sqrt{27}N) \cdot (\omega - 2/\sqrt{3}N)) &= m(K_{+/-})^2 - m(K_{0/0})^2 \\
((\kappa + 10/\sqrt{27}N) \cdot M) &= ((\omega - 2/\sqrt{3}N) \cdot M) = 0 \\
M^2 + (\omega - 2/\sqrt{3}N)^2 + (\kappa + 10/\sqrt{27}N)^2 &= 2m(K_{+/-})^2 + 2m(K_{0/0})^2 \\
(M + \sqrt{3}N)^2 &= m(\eta)^2
\end{aligned} \tag{31}$$

As already stated at the beginning of this section the first three equations possess a solution if and only if $m(\pi_{+/-})^2 - m(\pi_0)^2 \leq 0$ in contrast to the experimental facts. This means that the mass splitting within the isospin submultiplets cannot be due to a genuine Goldstone-mechanism. It may only be covered by the introduction of the usual electro-magnetic interaction term $j_{el}^\mu \cdot A_\mu$ into the Lagrangian.

With the electro-magnetic mass differences equal to zero ($m(\pi_{+/-})^2 = m(\pi_0)^2$ and $m(K_{+/-})^2 = m(K_{0/0})^2$) the unique solution of the second, third and fourth of the equations (31) is:

$$\kappa = -10/\sqrt{27}N \tag{32}$$

leading to the compensation of the isospin breaking terms in the Hamilton density (27). In other words: *The Goldstone Theorem together with the ("wrong") sign of the experimental pion mass difference imply the SU(2)-subgroup to be an exact symmetry in (pure) quantum flavor dynamics.*

There are left four equations of the system (31) to be solved which can be conveniently rewritten as:

$$\begin{aligned}
N^2 &= M^2 + 1/2(m(\eta)^2 - 3m(\pi)^2) \\
(\omega - 2\sqrt{3}N)^2 + M^2 &= 4m(K)^2 \\
\sqrt{3}(N \cdot M) + 2M^2 &= -1/2(m(\eta)^2 - 9m(\pi)^2) \\
((\omega - 2/\sqrt{3}N) \cdot M) &= 0
\end{aligned} \tag{33}$$

For non-vanishing meson masses and $m(\eta) > 3m(\pi)$ no solutions exist for $0 \leq M^2 < 4m(K)^2$ but infinitely many solutions for every M with $M^2 \geq 4m(K)^2$ or $M^2 < 0$. In a suitable Lorentz frame (restframe in the former case) they can be parametrized by the direction of the Euklidean three vectors N and ω (except for $8M^2 = -m(\eta)^2 + 9m(\pi)^2$).

Every solution of the equations (33) fixes an admissible representation of the translation group and hence an admissible Hamiltonian for chiral flavor dynamics with spontaneously broken chiral symmetry. A subclass of them is selected via the mass relations following from the experimental data :

$$\begin{aligned} m(\eta) &= 4m(\pi) \\ 4m(K)^2 &= 3m(\eta)^2 + 4m(\pi)^2 \end{aligned} \quad (34)$$

Obviously they are satisfied if and only if:

$$\begin{aligned} N^2 &= M^2 + 13/2m(\pi)^2 \\ (\omega - 2/\sqrt{3}N)^2 &= 52m(\pi)^2 - M^2 \end{aligned} \quad (35)$$

For fixed time-like $M^2 \geq 4m(K)^2 = 52m(\pi)^2$ a complete system of reference vectors each of which generates via Lorentz-transformations an equivalence class or so called Lorentz orbit $\mathcal{O}_{M^2}^\pm(\theta)$ of solutions of the equations (35) is given by:

$$M^2 = (\pm\sqrt{M^2}, 0) \quad (36)$$

$$N_0 = -(\text{sign } M_0)2(3M^2)^{-1/2}(M^2 + 7/8m(\pi)^2) \quad (37)$$

$$\mathbf{N} = (0, 0, (3M^2)^{-1/2}[(M^2 - 25/4m(\pi)^2)^2 - 36m(\pi)^4]^{1/2})$$

$$\omega - 2/\sqrt{3}N = (M^2 - 4m(K)^2)^{1/2}(0, 0, \sin \theta, \cos \theta) \quad (38)$$

θ denotes the angle between the Euklidean three vectors $\omega - 2/\sqrt{3}\mathbf{N}$ and $\mathbf{N}$ with $0 \leq \theta \leq \pi$.

The stability or little group of the orbits $\mathcal{O}_{M^2}^\pm(0)$ and $\mathcal{O}_{M^2}^\pm(\pi)$ is the unitary group $U(1)$ of rotations around $\mathbf{N}$ whereas for all other orbits $\mathcal{O}_{M^2}^\pm(\theta)$ with $\theta \neq 0, \pi$ (and $M^2 > 4m(K)^2$) it is the trivial subgroup of the Lorentz group. Hence on every one of them there exists an invariant measure $d_{M^2}^{\pm, \theta}\mu$ which in the latter cases is determined by the Haar measure of the Lorentz group.

The Lorentz orbits $\mathcal{O}_{M^2}(\dots)$ with $M^2 < 0$ can be parametrized in a similar way. Since one has to distinguish between several regions or subcases of M^2 we abstain from writing them down here.

Due to the occurrence of the conserved color currents $i\delta_0(j_{(1,0,0)}^{8,0}(x)c + 1/\sqrt{3}j_{(0,0,0)}^{8,0}(x)c)$ in the Hamilton density (27) in case of confinement ($\delta \neq 0$) also the $SU(3)$ -color symmetry is broken on the algebraic level and the remaining color symmetry group is the subgroup

$$G(3)_c = \left\{ \exp \left[it/2(\tau_{(1,0,0)} + 1/\sqrt{3}\tau_{(0,0,0)}) \right] \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathcal{G} \\ 0 & \end{pmatrix} \mid t \in \mathbb{R}, g \in SU(2) \right\} \quad (39)$$

isomorphic to $U(1) \times SU(2)$. It will be shown in section VI that this rest symmetry is not spontaneously broken unless there exist states of discrete mass $\kappa(a)^2 = -Q(a)_c^2 \delta^2 \geq 0$ created by the conserved color current $j_a^{8,0}(x)c$ from the vacuum. However due to the local gauge condition (26) they are confined in the unphysical section of the Hilbert space.

IV Poincaré-Covariance and Confinement

In view of the considerations and results of the foregoing sections the transformation laws (1) of the basic fields which are clued to the standard representation ($w = M = N = \delta = 0$) of the translation group (that is to a (Noether-)Hamiltonian without symmetry breaking and confinement) have to be reinvestigated and modified according to what is prescribed by the symmetry breaking dynamics of QCD mirrored in the choice of the representation for the (time) translation group or equivalently the Hamiltonian.

Furthermore as already stated in section III in the general representation (19) the basic quark and gluon fields depend in addition to the space time variable x on the "color vector" δ and the "flavor vector" triplet (M, N, w) specifying the representation. For the sake of notational shortness it has been dropped before but gains decisive importance in the construction of a Poincaré covariant quantum theory. According to the equations (19), (24), (25) and (32) the dependence on the flavor triplet $\mathcal{M} := (M, N, w)$ comes via the matrix

$$\not{p}(\mathcal{M}) := \sigma_{(0,0,0)}M + (\sigma_{(1,0,0)} - 10/\sqrt{27}\tau_{(1,0,0)})N + 1/\sqrt{3}\tau_{(0,0,0)}w \quad (40)$$

with eigenvalues of its representatives in the representation R^a :

$$\begin{aligned} 2 \cdot p_{\mathcal{M}}(u, d) &= 2[I(a)(I(a) + 1)(M + 1/\sqrt{3}N) + 1/2Y(a)(w - 2\sqrt{3}N) \\ &\quad - 1/4Y(a)^2(M - 2\sqrt{3}N) - 1/9(d + 1)(M + \sqrt{3}N)], \quad a \in \Gamma_d \end{aligned} \quad (41)$$

For every fixed quartet $m = (\delta, \mathcal{M}) = (\delta, M, N, w)$ of four vectors from the direct product $\mathcal{O}_{g, M^2}^\theta := \mathcal{O}_g^\theta \times \mathcal{O}_{M^2}^\theta(\theta)$ with $\mathcal{O}_g^\theta$ being one of the Lorentz orbits $\mathcal{O}_g^\pm = \{\delta \in \mathbb{R}_4 | \delta^2 = g, \delta_0 \gtrless 0\}$ respectively $\mathcal{O}_g = \{\delta \in \mathbb{R}_4 | \delta^2 = -g\}, g \geq 0$ besides the smeared fields

$$\varphi_{\nu, a, b}^{n, d}(f)_m := \int d^4x f(x) \varphi_{\nu, a, b}^{n, d}(x)_m \quad (42)$$

and conserved currents (vector and axial vector indices are dropped in the following)

$$j_a^{S, \nu}(f)_{m \in \Gamma_d} := \int d^4x f(x) j_a^{S, \nu}(x)_{m, \mathcal{F}/\mathcal{C}} \quad (43)$$

two- and three-fold composite quark fields [from which the one particle meson- respectively baryon-states are constructed] are defined by:

$$\begin{aligned} \varphi_{\nu_1 \nu_2, a_1 a_2, b_1 b_2}^{3^* 3, 3^* 3}(f)_m &:= \int d^4x_1 d^4x_2 f(x_1, x_2) \times \\ &\quad (-1)^{-Q(a_1) - Q(b_1) - 2/3} \varphi_{\rho, -a_1, -b_1}^{3^*, 3^*}(x_1)_m (\gamma^0)_{\nu_1}^{\rho} \varphi_{\nu_2, a_2, b_2}^{3, 3}(x_2)_m \end{aligned} \quad (44)$$

respectively for $n \in \{3, 3^*\}$:

$$\begin{aligned} \varphi_{\nu_1 \nu_2 \nu_3, a_1 a_2 a_3, b_1 b_2 b_3}^{nnn, nnn}(f)_m &:= \int d^4x_1 d^4x_2 d^4x_3 \times \\ &\quad \times f(x_1, x_2, x_3) \varphi_{\nu_1, a_1, b_1}^{n, n}(x_1)_m \dots \varphi_{\nu_3, a_3, b_3}^{n, n}(x_3)_m \end{aligned} \quad (45)$$

Here f is from the countably normed space $S(\mathbb{R}_{4k})$, $k = 1, 2, 3$ of strongly decreasing $\mathcal{C}^\infty$ -functions in all variables. With the abbreviations $|x|_k^2 := \sum_{i=1}^k \sum_{j=0}^3 (x_i^j)^2$ and $\partial_x^{|r|}$ for the partial differential operator in the components x_i^j of total order $|r| = \sum_{i=1}^k \sum_{j=0}^3 r_i^j$, $r_i^j \in \mathbb{N}_0$ the norms $\|f\|_n = \sqrt{[f, f]_n}$, $n \in \mathbb{N}_0$ are defined via the scalar products [30, Chap I, §3.6]:

$$[f, h]_n = \int d^4x_1 \dots d^4x_k (1 + |x|_k)^{2n} \sum_{\substack{r \in \mathbb{N}_0^{4k} \\ |r| \leq n}} (\partial_x^{|r|} f(x)) \partial_x^{|r|} h(x) \quad (46)$$

For every fixed $m \in \mathcal{O}_{g, M^2}^\theta$ the smeared fields are closable linear operators with a common dense subdomain in a Hilbert space $(\mathcal{H}(m), (\cdot|\cdot)_m, \eta_m)$ with η_m a bounded symmetric and invertible operator generating the non-degenerate indefinite metric $\langle \cdot, \cdot \rangle_m := (\cdot|\eta_m \cdot)_m$ which is non-negative on a proper maximal subspace $H(m)$ and zero on the subset $\mathcal{H}_0(m)$. η_m maps the common dense subdomain of all fields into itself. The counterpart to the complex conjugate φ_m and Dirac adjoint field $\tilde{\varphi}_m$ of the classical level now is the η_m -adjoint field operator $\varphi_m^{\dots+}(f)_m := \eta_m^{-1} \varphi_m^{\dots*}(f)_m \eta_m = \eta_m^{-1} \varphi_m^{\dots}(\hat{f})_m^* \eta_m$ respectively $\tilde{\varphi}_m^{\dots}(f)_m = \varphi_m^{\dots+}(f)_m \gamma^0$ with $\hat{f}(x_1 \dots x_k) := \overline{f(x_k, \dots, x_1)}$. Accordingly the self conjugateness (8) of the gluon fields and conserved currents respectively the relation (9) between the (anti-)quark fields read on the operator level:

$$\varphi_{\mu, c}^{8,1+}(f)_m = (-1)^{Q(c)} \varphi_{\mu, -c}^{8,1}(f)_m, \quad j_{a, \dots}^{8, \mu+}(f)_m = (-1)^{Q(a)} j_{-a, \dots}^{8, \mu}(f)_m \quad (47)$$

$$\varphi_{\mu_1 \mu_2, c_1 c_2, a_1 a_2}^{3^* 3, 3^* 3+}(f)_m = \varphi_{\mu_2 \mu_1, -c_2 -c_1, -a_2 -a_1}^{3^* 3, 3^* 3}(f)_m \quad (48)$$

$$\begin{aligned} \tilde{\varphi}_{\mu_1 \dots \mu_k, c_1 \dots c_k, a_1 \dots a_k}^{nnn, nnn}(f)_m &= \varphi_{\nu_1 \dots \nu_k, c_1 \dots c_k, a_1 \dots a_k}^{\bar{n} \bar{n} \bar{n}, nnn+}(f)_m \Pi_{i=1}^k (\gamma^0)_{\mu_i}^{\nu_i} \\ &= \left(\Pi_{i=1}^k (-1)^{-Q(c_i) - Q(a_i) - 2/3} \right) \varphi_{\mu_k \dots \mu_1, -c_k \dots -c_1, -a_k \dots -a_1}^{\bar{n}^* \bar{n}^* \bar{n}^*, n^* n^* n^*}(f)_m \end{aligned} \quad (49)$$

with $n \in \{3, 3^*\}$, $k \in \{1, 3\}$.

In the subset $H(m) \setminus H_0(m)$ with $H_0(m) := H(m) \cap \mathcal{H}_0$ exists a vector $\Psi_0(m)$ with $\langle \Psi_0(m), \Psi_0(m) \rangle_m = 1$ which is cyclic for the polynomial $+$ -algebra $\pi(\varphi_m, j_m)$ spanned by the basic (including the conserved currents) and the composite field operators; that means the subspace $\mathcal{D}(m) := \pi(\varphi_m, j_m) \Psi_0(m)$ is dense in $(\mathcal{H}(m), (\cdot|\cdot)_m)$. Moreover the subspace $\mathcal{D}_H(m) := \mathcal{D}(m) \cap H(m)$ is assumed to be dense in $H(m)$. Last not least the matrix elements of the field operators with all states from $\mathcal{D}(m)$ are continuous linear functionals on $S(\mathbb{R}_{4k})$, $k = 1, 2, 3$.

On $\mathcal{D}(m)$ a representation of the translation group is obtained by linear extension of the operators $\mathcal{T}_m(y)$ defined successively via the following covariance equations on all states created by products of the field operators from $\Psi_0(m)$:

$$\mathcal{T}_m(y) \Psi_0(m) := \Psi_0(m) \quad (50)$$

$$\begin{aligned}
& \mathcal{T}_m(y) \varphi_{\nu_1 \dots \nu_k, c_1 \dots c_k, a_1 \dots a_k}^{n_1 \dots n_k, d_1 \dots d_k}(f)_m \mathcal{T}_m(-y) := \\
& \int d^4x f(x_1, \dots, x_k) \Pi_{i=1}^k (T_{-10/\sqrt{27}N, w}^{m, M, N}(y) \varphi)_{\nu_i, c_i, a_i}^{n_i, d_i}(x_i)_m = \\
& \Pi_{i=1}^k \exp[Q(c_i)(m \cdot y)] \varphi_{\nu_1 \dots \nu_k, c_1 \dots c_k, b_1 \dots b_k}^{n_1 \dots n_k, d_1 \dots d_k} \left(\begin{matrix} f_{(1, y)}^{d_1 \dots d_k} \\ a_1 \dots a_k \end{matrix} \right)_{b_1 \dots b_k}^m
\end{aligned} \tag{51}$$

with $k \in \{1, 2, 3\}$ and $(f_{(1, y)}^{\dots})_{\dots}$ defined by:

$$\begin{aligned}
& f_{(1, y)}^{d_1 \dots d_k}(x_1, \dots, x_k)_{a_1 \dots a_k}^{b_1 \dots b_k} := \\
& \left(\Pi_{j=1}^k R^{d_j}(\exp[i/2(\not{p}(\mathcal{M}) \cdot y)]_{\alpha_j}^{b_j}) \right) f(x_1 - y, \dots, x_k - y)
\end{aligned} \tag{52}$$

Here and in the following the covariance equations for the currents are not written down explicitly because they are obtained by obvious replacements from those of the gluon fields.

This representation is certainly not isometric with respect to the Hilbert scalar product $(\cdot | \cdot)_m$ [9], [11], [23] and by many authors [13]–[22] claimed to be even unbounded also in the standard case $(\mathcal{M} = (0, 0, 0), \delta = 0)$. However it has to be isometric with respect to the indefinite metric $\langle \cdot, \cdot \rangle_m$ in order that it can be lifted to a unitary representation in the physical Hilbert space [11]. In other words the metric operator η_m has to be constructed in such a way that this so called η -isometry does hold.

In the same way as before a representation of the internal symmetry group $G(3)_C \times G(3)_F$ which commutes with the translation operators $\mathcal{T}_m(y)$ is obtained from the covariance equations:

$$I_m(u, v) \Psi_0(m) := \Psi_0(m) \tag{53}$$

$$\begin{aligned}
& I_m(u, v) \varphi_{\nu_1 \dots \nu_k, c_1 \dots c_k, a_1 \dots a_k}^{n_1 \dots n_k, d_1 \dots d_k}(f)_m I_m(u^{-1}, v^{-1}) := \\
& (\Pi_{i=1}^k R^{n_i}(u^{-1})_{c_i}^{a_i} R^{d_i}(v^{-1})_{e_i}^{b_i}) \varphi_{\nu_1 \dots \nu_k, a_1 \dots a_k, b_1 \dots b_k}^{n_1 \dots n_k, d_1 \dots d_k}(f)_m
\end{aligned} \tag{54}$$

As will be shown in Section VI this representation is the η_m -isometric integral or exponential of the representation of the Lie-algebra $\mathcal{G}(3)_C \oplus \mathcal{G}(3)_F$ which is given by the space integral over the zeroth component of the η_m -selfconjugate conserved color currents $j_{\pm}^{8, \mu}(f)_{m, C} = \pm 1/2 [j_{(1, 0, 0)}^{8, \mu}(f)_{m, C} \pm \sqrt{3} j_{(0, 0, 0)}^{8, \mu}(f)_{m, C}]$, $j_{(1/2, 1/2, -1)}^{8, \mu}(f)_{m, C}$, $j_{(1/2, -1/2, 1)}^{8, \mu}(f)_{m, C}$ respectively the flavor (vector) currents $j_{a, V}^{8, \mu}(f)_{m, F}$, $a \in \{(1, 0, 0), (1, \pm 1, 0), (0, 0, 0)\}$. (In the cases where the algebraic color symmetry or part of it is spontaneously broken this is still true for the remaining unbroken subgroup and in any case for the flavor subgroup $I_m^F(v) = I_m(1, v)$, $v \in G(3)_F$).

It should be remarked in passing that the irreducible matrix representations R^d of $SU(3)$ become completely reducible when restricted to the subgroup $G(3) \sim SU(2) \times U(1)$. They

decompose into a direct sum of irreducible representations D^I of $SU(2)$ labbed by the total isospin I and each term carrying a specific hypercharge Y . Explicitly we have:

$$R^d(r, t)_{(I, I^3, Y)}^{(I, I^3, Y)} = \delta_{II} \delta_{Y Y} D^I(r)_{I^3} \exp(iYt) \quad (55)$$

with $(I, Y) \in \{(1/2, 1/3), (0, -2/3)\}$ for $d = 3$ and

$(I, Y) \in \{(1/2, 1), (1, 0), (0, 0), (1/2, -1)\}$ for $d = 8$ etc.

Concerning the Lorentz- and even more the Poincaré-group matters are more involved. It is known [23, section VII, VIII] that there does not exist a representation of the Poincaré-group in $\mathcal{H}(m)$ resembling the substitution rule (1) of section II for Lorentz transformations together with the representation $\mathcal{T}_m, m \neq 0$ of the subgroup of translations. However for every Lorentz transformation $\Lambda(\alpha), \alpha \in SL(2, \mathbb{C})$ a linear operator $\mathcal{L}(\alpha)$ from $\mathcal{D}(m) \subset \mathcal{H}(m)$ onto $\mathcal{D}(\Lambda(\alpha)m) \subset \mathcal{H}(\Lambda(\alpha)m)$ is obtained in the same way as before (successive definition on product states and linear extension) by:

$$\mathcal{L}(\alpha)\Psi_0(m) := \Psi_0(\Lambda(\alpha)m) \quad (56)$$

$$\begin{aligned} \mathcal{L}(\alpha)\varphi_{\nu_1, \dots, \nu_k}^{n_1, \dots, n_k, d_1, \dots, d_k}(f)_m \mathcal{L}(\alpha^{-1}) := \\ (\prod_{i=1}^k S^{(n_i, d_i)}(\alpha)_{\nu_i}^{\nu_i}) \varphi_{\nu_1, \dots, \nu_k}^{n_1, \dots, n_k, d_1, \dots, d_k}(f_{(\Lambda(\alpha), 0)})_{\Lambda(\alpha)m} \end{aligned} \quad (57)$$

with $f_{(\Lambda(\alpha), 0)}(x_1, \dots, x_k) := f(\Lambda(\alpha)^{-1}x_1, \dots, \Lambda(\alpha)^{-1}x_k)$ and the matrices $S^{(n, d)}(\alpha)$ given by the equations (2), (3) of section II. The operators $\mathcal{L}(\alpha)$ obviously satisfy the composition law $\mathcal{L}(\alpha_1)\mathcal{L}(\alpha_2) = \mathcal{L}(\alpha_1\alpha_2)$. η -isometry for the Lorentz-operators means for all $\Psi, \Phi \in \mathcal{D}(m)$:

$$\langle \mathcal{L}(\alpha)\Psi, \mathcal{L}(\alpha)\Phi \rangle_{\Lambda(\alpha)m} = \langle \Psi, \Phi \rangle_m \text{ or } \mathcal{L}(\alpha)^* \eta_{\Lambda(\alpha)m} \mathcal{L}(\alpha) = \eta_m \quad (58)$$

For all $y \in \mathbb{R}_4, \alpha \in SL(2, \mathbb{C})$ the linear operators $\mathcal{T}_m(y)$ and $\mathcal{L}(\alpha)$ satisfy the commutation relations $\mathcal{T}_{\Lambda(\alpha)m}(y)\mathcal{L}(\alpha)\Psi(m) = \mathcal{L}(\alpha)\mathcal{T}_m(\Lambda(\alpha^{-1})y)\Psi(m)$ and therefore

$$\mathcal{W}(\alpha, y) : \mathcal{D}(m) \rightarrow \mathcal{D}(\Lambda(\alpha)m), \Psi(m) \rightarrow \mathcal{W}(\alpha, y)\Psi(m) := \mathcal{T}_{\Lambda(\alpha)m}(y)\mathcal{L}(\alpha)\Psi(m) \quad (59)$$

defines linear operators satisfying the composition law of the Poincaré group and in addition η -isometry if the Lorentz operators $\mathcal{L}(\alpha)$ do so. The question if the η -isometry of $\mathcal{L}(\alpha)$ can be derived from the presence of a conserved tensor current (corresponding to the Noether spin tensor) as in the case of the internal flavor respectively color symmetry is a problem of its own and will be treated together with the spin of Goldstone particles in a separate note.

Due to the exponential factor in the representations (51) of the translation group the problem of cluing these representations and the Lorentz operators $\mathcal{L}$ together to form an unitary representation of the Poincaré group in a suitable Hilbert space is closely connected with the color confinement. The key to its solution is the following theorem

which states that no colored state exists in the "physical section" $H(m) \setminus H_0(m)$ for any m . They all have non-positive η_m -norm.

Theorem (Color-Confinement): a) Every state $\Psi_{c_1 \dots c_r}^{n_1 \dots n_r}(m) \in H(m)$ with total color charge $Q_C = \sum_{j=1}^r Q(c_j) \neq 0$ belongs to $H_0(m) = H(m) \cap \mathcal{H}_0(m)$, that is:

$$\langle \Psi_{c_1 \dots c_r}^{n_1 \dots n_r}(m), \Psi_{c_1 \dots c_r}^{n_1 \dots n_r}(m) \rangle_m = 0$$

b) Every state $j_c^{8,\mu}(f)_{m,c} \Psi_0(m)$ and $\varphi_{\mu,c,r}^{n,d}(f)_m \Psi_0(m)$ with color charge $Q(c) \neq 0$ is from $\mathcal{H}_0(m)$.

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Proof: Since $\mathcal{D}_H(m)$ is dense in $H(m)$ and the metric $\langle \cdot, \cdot \rangle_m$ is semi-positive on $H(m)$ it suffices to prove the validity only for all states from $\mathcal{D}_H(m)$ which are created from $\Psi_0(m)$ by pure or mixed products of the basic (again including the conserved currents) and composite field operators a typical example reads:

$$\Psi_{\dots, c_1, \dots, c_r, a_1, \dots, a_r}^{n_1, \dots, n_r, d_1, \dots, d_r}(m) = \prod_{j=1}^r \varphi_{\dots, c_j, a_j}^{n_j, d_j}(f_j)_m \Psi_0(m)$$

Denoting by $|\eta_m|$ the operator norm of the metric operator η_m it follows by means of the η_m -isometry of $\mathcal{T}_m$ for all $y \in R_4$:

$$\begin{aligned} & | \langle \prod_{k=1}^r \varphi_{\dots, c_k, a_k}^{n_k, d_k}(f_k)_m \Psi_0(m), \prod_{j=1}^r \varphi_{\dots, c_j, a_j}^{n_j, d_j}(f_j)_m \Psi_0(m) \rangle_m | = (\prod_{i=1}^r \exp[2Q(c_i)(\delta \cdot y)]) \times \\ & \times | (\prod_{k=1}^r \varphi_{\dots, c_k, b_k}^{n_k, d_k}((f_k^{d_k}(1,y))_{a_k}^{b_k})_m \Psi_0(m) | \eta_m \prod_{j=1}^r \varphi_{c_j, a_j}^{n_j, d_j}((f_j^{d_j}(1,y))_{a_j}^{b_j})_m \Psi_0(m) |_m | \\ & \leq |\eta_m| \exp[2Q_C(\delta \cdot y)] \sum_{b_1, c_1 \in \Gamma_{d_1}} \dots \sum_{b_r, c_r \in \Gamma_{d_r}} \times \\ & \times | (\Psi_0(m) | [\prod_{k=r}^1 \varphi_{\dots, c_k, c_k}^{n_k, d_k}((\hat{f}_k^{d_k}(1,y))_{a_k}^{c_k})_m] \prod_{j=1}^r \varphi_{\dots, c_j, b_j}^{n_j, d_j}((f_j^{d_j}(1,y))_{a_j}^{b_j})_m \Psi_0(m))_m | \end{aligned}$$

Since every matrix element in the $2r$ -fold sum on the right hand side is a continuous linear functional on the $2r$ -fold tensor product of the countably normed nuclear space $S(\mathbb{R}_4)$ it is of finite order $n_0(c_1 \dots c_r, b_1 \dots b_r)$ [31, Vol. 2]. Therefore there exists a positive, real number A and a natural number $n_0 = \max\{n_0(c_1 \dots c_r, b_1 \dots b_r) | c_i, b_i \in \Gamma_{d_i}, i = 1, \dots, r\}$ such that:

$$\begin{aligned} & | \langle \prod_{k=1}^r \varphi_{\dots, c_k, d_k}^{n_k, d_k}(f_k)_m \Psi_0(m), \prod_{j=1}^r \varphi_{\dots, c_j, a_j}^{n_j, d_j}(f_j)_m \Psi_0(m) \rangle_m | \leq \\ & A \cdot |\eta_m| \exp[2Q_C(\delta \cdot y)] \sum_{b_1, c_1 \in \Gamma_{d_1}} \dots \sum_{b_r, c_r \in \Gamma_{d_r}} \times || [\prod_{k=1}^r (\hat{f}_k^{d_k}(1,y))_{a_k}^{c_k}] \prod_{j=1}^r (f_j^{d_j}(1,y))_{a_j}^{b_j} ||_{n_0} \\ & \leq A \cdot |\eta_m| \exp[2Q_C(\delta \cdot y)] \sum_{b_1, c_1 \in \Gamma_{d_1}} \dots \sum_{b_r, c_r \in \Gamma_{d_r}} \prod_{j=1}^r || (f_j^{d_j}(1,y))_{a_j}^{c_j} ||_{n_0} || (f_j^{d_j}(1,y))_{a_j}^{b_j} ||_{n_0} \end{aligned}$$

The elements of the diagonal unitary matrix $R^d(\exp[1/2(\not{p} \cdot y)])_b^a = \delta_b^a \exp[i(p_{\mathcal{M}}(a, d)(m \cdot y))]$ with $\not{p}$ and $p_{\mathcal{M}}(a, d)$ given in the equations (40)-(41) together with all derivatives with

respect to the components of $\mathcal{M} = (M, N, w)$ are bounded by polynomials in y . Therefore the equations (52) and (46) imply the whole $2r$ -fold sum on the right hand side of the last inequality to be bounded by a polynomial $\Pi_m(y)$ in y . Thus we arrive at the bound:

$$\begin{aligned} | < \Pi_{k=1}^r \varphi_{\dots, c_k, a_k}^{n_k, d_k} (f_k)_m \Psi_0(m), \Pi_{j=1}^r \varphi_{\dots, c_j, a_j}^{n_j, d_j} (f_j)_m \Psi_0(m) >_m | \\ \leq A |\eta_m| \Pi_m(y) \exp[2Q_C(\delta \cdot y)]. \end{aligned}$$

which proves the theorem since for every $\delta \neq 0$ and $Q_C \neq 0$ there is a direction in $\mathbb{R}_4$ in which the exponential goes to zero for y going to infinity. Part b) follows in the same way since in case of these states positive semi-definiteness needs not be applied. $\square$

Now it is rather obvious how to proceed in order to construct a carrier space for an isometric representation of the Poincaré group resembling the color confining representations of the translation group as well as the Lorentz transformations $\mathcal{L}$ given by the equations (50)-(52) respectively (56)-(58). Let $d\mu_g^\pm(\delta) = \Theta(\pm\delta_0)(\delta^2 - g)d^4\delta$ and $d\mu_g(\delta) = (\delta^2 + g)d^4\delta$ denote the invariant measures of the color orbit $\mathcal{O}_g^\pm$ respectively $\mathcal{O}_g$, $d\mu_{g, M^2}^\theta(m) := d\mu_g^\pm(\delta)d\mu_{M^2}^{\pm, \theta}(\mathcal{M})$ its product with the invariant measure of the flavor orbit $\mathcal{O}_{M^2}^\pm(\theta)$ and $\mathcal{V}_H$ the *vector space* of vector valued mappings

$$\psi : \mathcal{O}_{g, M^2}^\theta \rightarrow \cup_{m \in \mathcal{O}_{g, M^2}^\theta} \mathcal{D}_H(m), m \rightarrow \Psi(m) \in \mathcal{D}_H(m) \subseteq H(m) \quad (60)$$

with the property

$$\int_{\mathcal{O}_{g, M^2}^\theta} d\mu_{g, M^2}^\theta(m) < \Psi(m), \Psi(m) >_m < +\infty$$

In $\mathcal{V}_H$ vector addition and multiplication by complex numbers are defined in the usual way:

$$\psi + \phi : \mathcal{O}_{g, M^2}^\theta \rightarrow \cup_{m \in \mathcal{O}_{g, M^2}^\theta} \mathcal{D}_H(m), m \rightarrow (\Psi + \Phi)(m) = \Psi(m) + \Phi(m)$$

$$\alpha\psi : \mathcal{O}_{g, M^2}^\theta \rightarrow \cup_{m \in \mathcal{O}_{g, M^2}^\theta} \mathcal{D}_H(m), m \rightarrow (\alpha\Psi)(m) = \alpha\Psi(m)$$

The *vector space* $\mathcal{V}_H$ is just a subspace of the Cartesian product $\times_{m \in \mathcal{O}_{g, M^2}^\theta} H(m)$. Since the metric $< \cdot, \cdot >_m$ is positive semi-definite on $H(m)$ and hence also on $\mathcal{D}_H(m)$ a positive semi-definite inner product on $\mathcal{V}_H$ is given by:

$$< \psi, \phi > := \int_{\mathcal{O}_{g, M^2}^\theta} d\mu_{g, M^2}^\theta(m) < \Psi(m), \Phi(m) >_m \geq 0 \quad (61)$$

Moreover the invariance of the measure μ_{g, M^2}^θ implies $< \psi, \psi > = 0$ if and only if $< \Psi(m), \Psi(m) >_m = 0$ almost every where on $\mathcal{O}_{g, M^2}^\theta$. Hence an isometric representation $\mathcal{P} : ISL(2, \mathcal{C}) \rightarrow Aut.\mathcal{V}_H$, $(\alpha, y) \rightarrow \mathcal{P}(\alpha, y)$ of the Poincaré group on the inner product

space $(\mathcal{V}_H, \langle \cdot, \cdot \rangle)$ which leaves the subspace $\mathcal{V}_H^0 := \{\psi \in \mathcal{V}_H \mid \langle \psi, \psi \rangle = 0\}$ invariant is given by:

$$\begin{aligned} \mathcal{P}(\alpha, y) : \mathcal{V}_H &\rightarrow \mathcal{V}_H, \quad \psi \rightarrow \mathcal{P}(\alpha, y)\psi \\ \mathcal{P}(\alpha, y)\psi : \mathcal{O}_{g, M^2}^\theta &\rightarrow \bigcup_{m \in \mathcal{O}_{g, M^2}^6} \mathcal{D}_H(m), \quad m \rightarrow (\mathcal{P}(\alpha, y)\psi)(m) := \\ \mathcal{T}_{\Lambda(\alpha)m}(y)\mathcal{L}(\alpha)\Psi(m) &= \mathcal{L}(\alpha)\mathcal{T}_m(\Lambda(\alpha^{-1})y)\Psi(m), \end{aligned} \quad (62)$$

where $\mathcal{T}_m(y)$ and $\mathcal{L}(\alpha)$ are the η -isometric operators defined via the equations (50)-(52) respectively (56)-(57).

In the same way an isometric representation of the internal symmetry group $G(3)_C \times G(3)_F$ and therefore also of its direct product with the Poincaré group is obtained by

$$\begin{aligned} \mathcal{I}(u, v) : \mathcal{V}_H &\rightarrow \mathcal{V}_H, \quad \psi \rightarrow \mathcal{I}(u, v)\psi \\ \mathcal{I}(u, v)\psi : \mathcal{O}_{g, M^2}^\theta &\rightarrow \bigcup_{m \in \mathcal{O}_{g, M^2}^6} \mathcal{D}_H(m), \quad m \rightarrow (\mathcal{I}(u, v)\psi)(m) = I_m(u, v)\Psi(m) \end{aligned} \quad (63)$$

with the η -isometric operators $I_m(u, v)$ defined via the equations (53)-(55).

It is the triple $(\mathcal{V}_H, \langle \cdot, \cdot \rangle, \mathcal{V}_H^0)$ from which the physical Hilbert space is obtained via factorisation and completion $(\mathcal{H}_{phys}, \langle \cdot, \cdot \rangle) = (\overline{\mathcal{V}_H / \mathcal{V}_H^0}, \langle \cdot, \cdot \rangle)$ and the isometric representations on $\mathcal{V}_H$ (respectively η -isometric ones in $\mathcal{H}(m)$) are lifted to unitary representations in $\mathcal{H}_{phys}$.

A direct way to the physical Hilbert space and the unitary representations of the (rest) symmetry groups in it which shortcuts the introduction of the vector space $\mathcal{V}_H$ is to take the direct integral of the completed factor spaces $(\overline{H(m) / H_0(m)}, \langle \cdot, \cdot \rangle_m)$ over the orbits $\mathcal{O}_{g, M^2}^\theta$:

$$\mathcal{H}_{phys}' = \int_{\bigotimes} d\mu_{g, M^2}^\theta(m) \overline{H(m) / H_0(m)}$$

with its scalar product again given by the equation (61) [where $\Psi(m)$ and $\Phi(m)$ now denote classes from $\overline{H(m) / H_0(m)}$]. Obviously $\mathcal{H}_{phys}'$ is unitary equivalent to $\mathcal{H}_{phys}$.

In any case due to the direct product structure required by Poincaré' invariance, there are only three real parameters left free in the theory: $M^2 \geq 4m(K)^2$ ($M^2 < 0$) which may be interpreted as the coupling constant or strenght of the flavor interaction, $\delta^2 = \pm g \geq 0$ which may be considered as the strenght of the color confinement and the angle θ (which eventually may be fixed via the masses of the axial-vector meson octet and replace the $w - \phi$ -mixing).

V One Particle Hadron States

For the remainder of our investigations and especially the construction of one particle hadron states a spectrum condition on the fields and translation operators has to be specified which ensures that the physical Hilbert space $\mathcal{H}_{phys}$ respectively the "physical

section" $\mathcal{V}_H \setminus \mathcal{V}_H^0$ of the inner product space $\mathcal{V}_H$ contains only states carrying non-negative energy and non-negative momentum (mass) squares. Such a general *spectrum condition* taking into account the non-isometry of the translation operators reads:

$$\begin{aligned} \int d^4y \exp.[-i(q \cdot y)] < \Phi(m), \mathcal{T}_m(y) \Psi(m) >_m &= 0 \\ \text{for all } q \notin \overline{V}_+^0 = \{q \in \mathbb{R}_4 | q^0 \geq 0, q^2 \geq 0\} & \text{ (closed forward cone),} \\ \text{all } \Psi(m) \in \mathcal{D}_H(m) = \mathcal{D}(m) \cap H(m), \text{ all } \Phi(m) \in \mathcal{H}(m) & \\ \text{and almost all } m \in \mathcal{O}_{g, M^2}^\theta & \end{aligned} \quad (64)$$

Concerning its domain of validity this condition is slightly weaker than the spectrum condition generally postulated for local quantum gauge theories [13]-[22]. The integration in equation (64) is to be understood as Fourier transform of generalized functions. Precisely the equation (64) means:

$$\int d^4y h(y) < \Phi(m), \mathcal{T}_m(y) \Psi(m) >_m = 0 \quad (65)$$

for all testfunctions h from the Schwartz space $S(\mathbb{R}_4)$ whose Fourier transforms

$$\tilde{h}(q) = (2\pi)^{-3/2} \int d^4y \exp.[-i(q \cdot y)] h(y)$$

vanish in a neighbourhood of the closed backward cone $\overline{V}_-^0 = \{q \in \mathbb{R}_4 | q^0 \leq 0, q^2 \geq 0\}$.

In quantum field theories with unitary translation operators the states of discrete mass κ arise from a $\delta(q^2 - \kappa^2)$ in the spectral support of the translation operators. For the case of non-isometric translation operators at hand the occurrence of a state ψ_κ with discrete mass κ may be characterized via the existence of a sufficiently small neighbourhood $\mathcal{N}_\kappa$ of the hyperboloid $q^2 = \kappa^2$ in the backward cone V_-^0 such that for all $\Phi(m) \in \mathcal{H}(m)$ and almost all $m \in \mathcal{O}_{g, M^2}^\theta$

$$\int d^4y < \Phi(m), \mathcal{T}_m(y) \Psi_\kappa(m) >_m (\Box_y + \kappa^2) h_\kappa(y) = 0 \quad (66)$$

for all $h_\kappa \in S(\mathbb{R}_4)$ with $\text{supp. } \hat{h}_\kappa \subseteq \mathcal{N}_\kappa$, but

$$\int d^4y < \Psi_\kappa(m), \mathcal{T}_m(y) \Psi_\kappa(m) >_m \neq 0 \quad (67)$$

for at least one $h_\kappa \in S(\mathbb{R}_4)$ with $\text{supp. } \hat{h}_\kappa \subseteq \mathcal{N}_\kappa$.

Once the existence of discrete mass states is established by the spectrum condition the explicite construction of the corresponding one particle states in $\mathcal{V}_H \setminus \mathcal{V}_H^0$, that means relativistic color singlet states with definite spin, (flavor-)isospin I and (flavor-)hypercharge Y satisfying the Klein-Gordon-respectively Dirac-equation etc. with mass parameter $\kappa(I, Y)$ is easily performed applying suitable linear combinations of the composite fields (44) respectively (45) or the currents (43) with suitably restricted testfunctions to the cyclic

state $\Psi_0(m)$ in $H(m)$. Since this construction is well known from the formation of so called "one particle excitation operators" in the Haag-Ruelle scattering theory of classical Wightman quantum field theories (see e.g.[29], [32] and references therein) we may drop all details. Furthermore we restrict ourselves to the meson states especially (axial-)vector and (pseudo-)scalar states. The barion states are obtained from the threefold products of quark fields (45) in the same way as the meson states from the twofold ones (44).

I.) *(Pseudo-)Scalar Mesons:* In order to obtain (pseudo-)scalar meson multiples we take for $\mu_i \in \{0, \dots, 3\}$, $c_i \in \Gamma_3$, $a_i \in \Gamma_3$, $\alpha \in \Gamma_d$ and $d \in \{1, 8\}$ the special functions

$$h_{p_{\mathcal{M}}(d,\lambda),0\pm,d}^{[\mu_1\mu_2,c_1,c_2,a_1,a_2]\lambda}(x_1,x_2) = (-1)^{Q(c_1)+Q(a_1)-Q(\lambda)} \times \\ \times \begin{pmatrix} 3^* & 3 & 1 \\ -c_1 & -c_2 & 0 \end{pmatrix} \begin{pmatrix} 3^* & 3 & d \\ -a_1 & -a_2 & -\lambda \end{pmatrix} E_{0\pm}^{\mu_1\mu_2} h_{\kappa(i(\lambda),Y(\lambda))}^{0\pm,d}(x_1,x_2,p_{\mathcal{M}}(d,\lambda)) \quad (68)$$

with E_{0+} the unit matrix $E_{0-} = \gamma_5$ and $h_{\kappa(\dots)}^{0\pm,d}(x_1,x_2,q)$ a Lorentz invariant function from $S(\mathbb{R}_{12})$ and consider the states created in $\mathcal{D}_H(m)$ from the cyclic state $\Psi_0(m)$ by the translated twofold composite field (44) smeared with these special functions:

$$\Psi^\lambda(x,m,h_{\kappa(\dots)}^{0\pm,d}) := T_m(x) \varphi_{\mu_1\mu_2,c_1,c_2,a_1a_2}^{3^*3,3^*3}(h_{p_{\mathcal{M}}(d,\lambda),0\pm,d}^{[\mu_1\mu_2,c_1,c_2,a_1,a_2]\lambda})_m \Psi_0(m) \\ = (-1)^{Q(c_1)+Q(a_1)-Q(\lambda)} \begin{pmatrix} 3^* & 3 & 1 \\ -c_1 & -c_2 & 0 \end{pmatrix} \begin{pmatrix} 3^* & 3 & d \\ -a_1 & -a_2 & -\lambda \end{pmatrix} \\ \int d^4x_1 d^4x_2 h_{\kappa(\dots)}^{0\pm,d}(x_1-x,x_2-x,p_{\mathcal{M}}(d,\lambda)) \exp[-i(x \cdot p_{\mathcal{M}}(d,\lambda))] \times \\ \times \varphi_{\mu_1,c_1,a_1}^{3,3}(x_1)_m E_{0\pm}^{\mu_1\mu_2} \varphi_{\mu_2,c_2,a_2}^{3,3}(x_2)_m \Psi_0(m)_m \quad (69)$$

For the second equality the transformation properties (50)-(52) and the relation

$$R^{3^*}(v)_{a_1}^{b_1} R^3(v)_{a_2}^{b_2} \begin{pmatrix} 3^* & 3 & d \\ a_1 & a_2 & \lambda \end{pmatrix} = R^d(v)_\lambda^\rho \begin{pmatrix} 3^* & 3 & d \\ b_1 & b_2 & \rho \end{pmatrix} \quad (70)$$

have been used together with the fact that $R^d(\exp[-i/2(\not{p}(\mathcal{M}) \cdot x)])$ is a diagonal matrix. The color exponential of equation (51) has disappeared due to the first Clebsch-Gordan coefficient in equation (68) which at the same time ensures that these states are color singlets and thus escape the fate of confinement.

By straight forward however tedious Clebsch-Gordan calculations it follows from the transformation properties of the quark fields, the Lorentz invariance of $h_{\kappa(\dots)}^{0\pm,d}$ together with the equation (70) and the special structure (55) of the representation $\bar{R}^d$ restricted to the subgroup $SU(2) \times U(1)$ that the states $\Psi^\lambda(x,m,h_{\kappa(\dots)}^{0\pm,d})$ transform under the QCD (rest) symmetry group $ISL(2,\mathcal{C}) \times G(3)_C \times G(3)_F$ according to:

$$\begin{aligned}
\mathcal{U}((\alpha, y), u, v)_m \Psi^\lambda(x, m, h_{\kappa(I(\lambda), Y(\lambda))}^{0\pm, d}) &:= \\
\mathcal{T}_{\Lambda(\alpha)m}(y) \mathcal{L}(\alpha) \mathcal{I}_m(u, v) \Psi^\lambda(x, m, h_{\kappa(I(\lambda), Y(\lambda))}^{0\pm, d}) &= \\
R^d(v)_\rho^\lambda \Psi^\rho(\Lambda(\alpha)x + y, \Lambda(\alpha)m, h_{\kappa(I(\rho), Y(\rho))}^{0\pm, d}) &
\end{aligned} \tag{71}$$

Finally taking into account only Lorentz invariant functions $h_{\kappa(\dots)}^{0\pm, d}(x_1, x_2, q)$ such that the support of their Fourier transform

$$\tilde{h}_{\dots}(k_1, k_2, q) = (2\pi)^{-3} \int d^4x_1 d^4x_2 \exp[-i(x_1 k_1 + x_2 k_2)] h_{\dots}(x_1, x_2, q)$$

in the variable $k_1 + k_2 + q$ is contained in the neighbourhood $\mathcal{N}_{\kappa(I(\lambda), Y(\lambda))}$ of the backward branch of the mass shell $(k_1 + k_2 + q)^2 = \kappa(I(\lambda), Y(\lambda))^2$ specified in the spectrum condition before then the resulting states $\Psi^\lambda(x, m, h_{\kappa(\dots)}^{0\pm, d})$ are positive frequency solutions of the Klein-Gordon equation

$$(\Box_x + \kappa(I(\lambda), Y(\lambda))^2) \Psi^\lambda(x, m, h_{\kappa(I(\lambda), Y(\lambda))}^{0\pm, d}) = 0$$

This is most easily seen by rewriting the equation (69) in terms of Fourier transforms with respect to the space-time variables and using the spectrum condition (66).

But then also the states generated in the inner product space $(\mathcal{V}_H, \langle \cdot, \cdot \rangle)$ by the above ones (in $\mathcal{D}_H(m)$) :

$$\Psi_{\kappa(I(\lambda), Y(\lambda))}^{0\pm, d; \lambda}(x) : \mathcal{O}_{g, M^2}^\theta \rightarrow \cup_{m \in \mathcal{O}_{g, M^2}^\theta} \mathcal{D}_H(m), m \rightarrow \Psi^\lambda(x, m, h_{\kappa(I(\lambda), Y(\lambda))}^{0\pm, d}) \tag{72}$$

are positive frequency solutions of the Klein-Gordon equation with mass parameter $\kappa(I(\lambda), Y(\lambda))$ transforming under the QCD (rest) symmetry group according to:

$$\mathcal{U}((\alpha, y), u, v) \psi_{\kappa(I(\lambda), Y(\lambda))}^{0\pm, d; \lambda}(x) = R^d(v)_\rho^\lambda \psi_{\kappa(I(\lambda), Y(\lambda))}^{0\pm, d; \rho}(\Lambda(\alpha)x + y) \tag{73}$$

with R^d the d -dimensional reducible representation of the flavor subgroup $SU(2) \times U(1)$ of $SU(3)$ explicitly given in equation (55). Hence they describe (pseudo-)scalar, color singlet and $SU(2) \times U(1)$ flavor one particle multiplets with isospin $I(\lambda)$, hypercharge $Y(\lambda)$ and mass $\kappa(I(\lambda), Y(\lambda))$.

II.) (*Axial-Vector Mesons*): The (axial-)vector mesons could be obtained in the same way as before from the two-fold composite quark field by special choices of (pseudo-)vectorial testfunctions. However since it is rather complicated to build in the transversality condition it is much simpler to use the conserved flavor (axial-)vector currents $j_{a, \mathcal{F}}^{8, \mu, V(A)}(f)_m$ of equation (43) for the flavor octets, which a priori satisfy this condition due to current conservation.

Let $h_{\kappa(I(\lambda), Y(\lambda))}(x, q)$ be a Lorentz invariant function from $S(\mathbb{R}_8)$ and the support of its Fourier transform $\tilde{h}(k, p_M(8, \lambda))$ in the variable $k + p_M(8, \lambda)$ be contained in the neighbourhood

$\mathcal{N}_{\kappa(I(\lambda), Y(\lambda))}$ of the backward branch of the mass shell $(k + p_{\mathcal{M}}(8, \lambda))^2 = \kappa(I(\lambda), Y(\lambda))^2$. Then in exactly the same way as before it follows that the states

$$\begin{aligned} \Psi_{8, V(A)}^{\nu, \lambda}(x, m, h_{\kappa(I(\lambda), Y(\lambda))}) &:= \mathcal{T}_m(x) j_{\lambda, \mathcal{F}}^{8, \nu, V(A)+}(h_{\kappa(I(\lambda), Y(\lambda))})_m \Psi_0(m) \\ &= (-1)^{Q(\lambda)} \int d^4x \, h_{\kappa(I(\lambda), Y(\lambda))}(x, p_{\mathcal{M}}(8, \lambda)) \times \\ &\times \exp[-i(x \cdot p_{\mathcal{M}}(8, \lambda))] j_{-\lambda}^{8, \nu, V(A)}(x)_{m, \mathcal{F}} \Psi_0(m) \end{aligned} \quad (74)$$

are positive frequency solutions of the Klein-Gordon-equation with mass parameter $\kappa(I(\lambda), Y(\lambda))$ satisfying the transversality condition

$$\partial_\nu \Psi_{8, V(A)}^{\nu, \lambda}(x, m, h_{\kappa(I(\lambda), Y(\lambda))}) = 0$$

and transforming under the $QCD(\text{rest})$ symmetry group according to

$$\begin{aligned} \mathcal{U}((\alpha, y), u, v)_m \Psi_{8, V(A)}^{\nu, \lambda}(x, m, h_{\kappa(I(\lambda), Y(\lambda))}) &= \\ R^8(v)_\rho^\lambda \Lambda(\alpha^{-1})_\mu^\nu \Psi_{8, V(A)}^{\mu, \rho}(\Lambda(\alpha)x + y, \Lambda(\alpha)m, h_{\kappa(I(\lambda), Y(\lambda))}) \end{aligned} \quad (75)$$

But then the states $\psi_{\kappa(I(\lambda), Y(\lambda))}^{8, V(A); \mu, \lambda}(x)$ arising from them in $\mathcal{V}_H \setminus \mathcal{V}_H^0$ share the analogous properties. Hence they describe colorless (pseudo-)vector $SU(2) \times U(1)$ flavor one particle octets with spin 1, isospin $I(\lambda)$, hypercharge $Y(\lambda)$ and mass $\kappa(I(\lambda), Y(\lambda))$.

VI Generalized Goldstone Theorems

In this section we prove the generalization of the Goldstone Theorem for the spontaneously broken symmetries to the general representations of the translation group. The proof rests on a Jost-Lehmann-Dyson representation for the commutator of the conserved currents $j_{a, \dots}^{8, \mu}(f)_{m, \dots}$ with the elements of the polynomial field algebra $\pi(\varphi_m, j_m)$ which incorporates locality, spectrum condition and current conservation. Besides the ingredient that the flavor currents create from $\Psi_0(m)$ only states in $H(m)$ and thus due to the spectrum condition (64) only states with non-negative energy and non-negative four momentum squares a locality or causality condition is needed. Each current is assumed to commute with all basic fields and currents (inclusively itself) at space-like separated points (*relative Einstein causality*):

$$\begin{aligned} [j_{a, \dots}^{8, \mu}(x)_{m, \dots}, \varphi_{\dots}(y)_m]_- &= \\ [j_{a, \dots}^{8, \mu}(x)_{m, \dots}, j_{\dots}(y)_{m, \dots}]_- &= 0 \quad \text{for } (x - y)^2 < 0 \end{aligned} \quad (76)$$

Without loss of generality it is assumed that $\langle \Psi_0(m), j_{\dots}(f)_{m, \mathcal{F}/c} \Psi_0(m) \rangle_m = 0$.

Finally let $f_R(|\mathbf{x}|)$ be from $S(\mathbb{R}_3)$ where $f_R(|\mathbf{x}|)$ besides being equal to 1 for $0 \leq |\mathbf{x}| \leq R$

and zero for $|\mathbf{x}| > R + \Delta R$ satisfies the bound $|R \frac{d}{dr} f_R(r)| \leq c < +\infty$ for all $R > 0$. Then the Goldstone Theorem [13], [21], [26]-[29] reads:

(Generalized Flavor-) Goldstone Theorem: The flavor symmetry generated by the conserved current $j_{a,\dots}^{8,\mu}(h)_{m,\mathcal{F}}$ is not spontaneously broken, that is

$$\langle \Psi_0(m), \int d^3 \mathbf{x} f_R(|\mathbf{x}|) [j_{a,\dots}^{8,0}(x)_{m,\mathcal{F}}, \phi(f)]_- \Psi_0(m) \rangle_m = 0 \quad (77)$$

for all $|x^0| < \epsilon$, $\phi(f) \in \pi(\varphi_m, j_m)$ with testfunctions f of compact support and all $R > R(\epsilon, f)$, unless there exists a particle of mass $\kappa(a)^2 = p_{\mathcal{M}}(8, a)^2$ created from $\Psi_0(m)$ by $j_{a,\dots}^{8,0}(f_R)_{m,\mathcal{F}}$.

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Proof: Following the pattern of Wightman [33], Vladimirov [34] and especially the version of Araki, Hepp and Ruelle [32] it can be shown that in consequence of Einstein causality, spectrum condition and translational covariance the vacuum expectation value of the commutator may be represented by the following Jost-Lehmann-Dyson representation:

$$\begin{aligned} \langle \Psi_0(m), [j_{a,\dots}^{8,\mu}(x)_{m,\mathcal{F}}, \phi(f)]_- \Psi_0(m) \rangle_m &= \exp\{-i(p(a)x)\} \times \\ &\times \int d^3 \mathbf{u} \int_0^\infty ds [\rho_a^\mu(\mathbf{u}, s, m)_1 \Delta_s(x^0, \mathbf{x} - \mathbf{u}) + \rho_a^\mu(\mathbf{u}, s, m)_2 \frac{\partial}{\partial x^0} \Delta_s(x^0, \mathbf{x} - \mathbf{u})], \end{aligned} \quad (78)$$

with $p(a) := p_{\mathcal{M}}(8, a)$ and

$$\Delta_s(x) = \frac{1}{i(2\pi)^3} \int d^4 k \exp\{-i(kx)\} \epsilon(k^0) \delta(k^2 - s) \quad (79)$$

being the solution of the initial value problem

$$(\Box_x + s) \Delta_s(x) = 0, \quad \Delta_s(x)|_{x^0=0} = 0, \quad \frac{\partial}{\partial x^0} \Delta_s(x)|_{x^0=0} = -\delta(\mathbf{x}), \quad s \geq 0 \quad (80)$$

The structure of this representation is determined by the spectrum condition (77) via a solution of a five dimensional wave equation in configuration space. The spectralfunctions $\rho_{a,\dots}^\mu(\mathbf{u}, s, m) \dots$ are strongly decreasing distributions in the variable s and $\mathcal{C}^\infty$ -functions in $\mathbf{u}$ with compact support as a consequence of Einstein causality. The exponential factor in equation (78) arises from translational covariance.

Remark: The unitarity of the translation operator used in Ref. [32] is not a necessary precondition to establish the representation (78). It can be dropped for the price of some gymnastics in (tempered) distribution theory along the lines of [36].

The most important point for the following is now the application of current conservation $\partial_\mu j_{a,\dots}^{8,\mu}(x)_{m,\mathcal{F}} = 0$. By straight forward calculations it leads to the following two relations between the spectral functions:

$$\begin{aligned}\rho_a^0(\mathbf{u}, s, m)_1 &= ip_0(a)\rho_a^0(\mathbf{u}, s, m)_2 + \left(\frac{\partial}{\partial u^j} + ip_j(a)\right)\rho_a^j(\mathbf{u}, s, m)_2 \\ \left(\frac{\partial}{\partial u^j} + ip_j(a)\right)[\rho_a^j(\mathbf{u}, s, m)_1 + ip_0(a)\rho_a^j(\mathbf{u}, s, m)_2] &= (\Delta\mathbf{u} - s + p_0(a)^2)\rho_a^0(\mathbf{u}, s, m)_2\end{aligned}\quad (81)$$

Here and in the following summation over the Latin double index j from 1 to 3 is understood. Introducing instead of $\rho_a^j(\mathbf{u}, s, m)_1$ the new spectral function $\sigma_a^j(\mathbf{u}, s, m)$ via

$$\rho_a^j(\mathbf{u}, s, m)_1 + ip_0(a)\rho_a^j(\mathbf{u}, s, m)_2 = -\left(\frac{\partial}{\partial u^j} - ip^j(a)\right)\rho_a^0(\mathbf{u}, s, m)_2 + \sigma_a^j(\mathbf{u}, s, m) \quad (82)$$

the second of the equations (81) reduces to the much simpler one

$$(s - p(a)^2)\rho_a^0(\mathbf{u}, s, m)_2 = \left(\frac{\partial}{\partial u^j} + ip_j(a)\right)\sigma_a^j(\mathbf{u}, s, m) \quad (83)$$

with the most general solution [35]

$$\rho_a^0(\mathbf{u}, s, m)_2 = \left(\frac{\partial}{\partial u^j} + ip_j(a)\right)(s - p(a)^2)^{-1}\sigma_a^j(\mathbf{u}, s, m) + \delta(s - p(a)^2)E_a(\mathbf{u}, m) \quad (84)$$

$E_a(\mathbf{u})$ is a $\mathcal{C}^\infty$ -function with compact support since all other spectral functions share these properties. Feeding the equations (82) and (84) together with the first one of the system (81) back into the representation (78) and integrating by parts over the variables u_j we end up with the following representation for the zeroth component of the commutator matrix element:

$$\begin{aligned}< \Psi_0(m), [j_{a,\dots}^{8,0}(x)_{m,\mathcal{F}}, \phi(f)]_- \Psi_0(m) >_m = \exp\{-i(p(a)x)\} \times \\ &\times \int d^3\mathbf{u} \{E_a(\mathbf{u}, m)(ip_0(a) + \frac{\partial}{\partial x^0})\Delta_{p(a)^2}(x^0, \mathbf{x} - \mathbf{u}) - \\ &\int_0^\infty ds [\sigma_a^j(\mathbf{u}, s, m)_1 + \sigma_a^j(\mathbf{u}, s, m)_2 \frac{\partial}{\partial x^0}] (\frac{\partial}{\partial x^j} - ip_j(a))\Delta_s(x^0, \mathbf{x} - \mathbf{u})\}\end{aligned}\quad (85)$$

Here the abbreviations

$$\begin{aligned}\sigma_a^j(\mathbf{u}, s, m)_1 &:= \rho_a^j(\mathbf{u}, s, m)_2 + ip_0(a)(s - p(a)^2)^{-1}\sigma_a^j(\mathbf{u}, s, m) \\ \sigma_a^j(\mathbf{u}, s, m)_2 &:= (s - p(a)^2)^{-1}\sigma_a^j(\mathbf{u}, s, m)\end{aligned}\quad (86)$$

have been used and for the partial integration the compact support of the spectral functions. By means of a further partial integration we get from equation (85)

$$\begin{aligned}< \Psi_0(m), \int d^3\mathbf{x} f_R(|\mathbf{x}|) [j_{a,\dots}^{8,0}(x)_{m,\mathcal{F}}, \phi(f)]_- \Psi_0(m) >_m = \int d^3\mathbf{u} E_a(\mathbf{u}, m) \times \\ &\times \int d^3\mathbf{x} f_R(|\mathbf{x}|) \exp\{-i(p(a) \cdot x)\} (ip_0(a) + \frac{\partial}{\partial x^0}) \Delta_{p(a)^2}(x^0, \mathbf{x} - \mathbf{u}) = \\ &\int d^3\mathbf{x} \left(\frac{\partial}{\partial x^j} f_R(|\mathbf{x}|)\right) \exp\{-i(p(a) \cdot x)\} \times \\ &\times \int d^3\mathbf{u} \int_0^\infty ds [\sigma_a^j(\mathbf{u}, s, m)_1 + \sigma_a^j(\mathbf{u}, s, m)_2 \frac{\partial}{\partial x^0}] \Delta_s(x^0, \mathbf{x} - \mathbf{u})\end{aligned}\quad (87)$$

Now the further arguments are the same as in the standard case. For all $|x^0|$ smaller than some $\epsilon > 0$ the right hand side vanishes for all R larger than some fixed $R_0(\epsilon)$ as a consequence of the support properties of the spectral functions and the commutator function $\Delta_s(x)$, since the derivative of f_R is unequal to zero only in the ring region between R and $R + \Delta R$. By the same reasons the second term on the left hand side becomes independent from f_R for $R > R_0(\epsilon)$, that is only the ball $|\mathbf{x}| \leq R_0(\epsilon)$ contributes to the x -integrations. Hence the only contribution to the left hand side of (77) comes from the one particle intermediate state in the commutator with the internal quantum numbers $a = (I, I^3, Y)$ of the current and the mass $p(a)^2$. $\square$

From the initial values (80) we obtain for $x^0 = 0$ and all $R > R_0(\epsilon)$

$$\begin{aligned} & \langle \Psi_0(m), \int d^3 \mathbf{x} f_R(|\mathbf{x}|) [j_{a,\dots}^{8,0}(x, m)_{m,\mathcal{F}}, \phi(f)]_- \Psi_0(m) \rangle_{m|x^0=0} = \\ & = - \int d^3 \mathbf{u} E_a(\mathbf{u}, m) \exp[i(\mathbf{p}(a) \cdot \mathbf{u})] = - \dot{E}_a(\mathbf{p}_{\mathcal{M}}(a, \delta), m) \end{aligned} \quad (88)$$

The remaining considerations rest essentially on the identity

$$\begin{aligned} & \langle \Psi_0(m), [\mathcal{T}_m(y) j_{a,\dots}^{8,0}(x^0, f_R)_{m,\mathcal{F}} \mathcal{T}_m(-y), \phi(f)]_- \Psi_0(m) \rangle_m = \\ & = \langle \Psi_0(m), [j_{a,\dots}^{8,0}(x^0, f_R)_{m,\mathcal{F}}, \mathcal{T}_m(y) \phi(f) \mathcal{T}_m(-y)]_- \Psi_0(m) \rangle_m \end{aligned} \quad (89)$$

which is easily obtained from the η_m -isometry of the translation operators and the invariance of $\Psi_0(m)$ under the translations (equation (59)).

Let $\pi(\varphi_m, j_m)_{SL}$ denote the subalgebra of $\pi(\varphi_m, j_m)$ defined by taking only testfunctions of compact support in the smeared fields and $\mathcal{D}(m)_{SL} \subset \mathcal{D}(m)$ the corresponding dense subspace of $\mathcal{H}(m)$. Then in the same way as in the case of a Wightman theory [27], [37], [38] for every $y \in \mathbb{R}_4$ an octet of current generated charges may be defined on $\mathcal{D}(m)_{SL}$ by:

$$\begin{aligned} \mathbf{Q}_a^{\dots}(y)_{m,\mathcal{F}} \phi(f) \Psi_0(m) := \\ [\mathcal{T}_m(y) j_{a,\dots}^{8,0}(f_\epsilon, f_R)_{m,\mathcal{F}} \mathcal{T}_m(-y), \phi(f)]_{- \{R > R(\epsilon, f)\}} \Psi_0(m) \end{aligned} \quad (90)$$

Here f_ϵ is an arbitrary C^∞ -function with support contained in $[-\epsilon, \epsilon]$ and normalized according to $\int dx^0 f_\epsilon(x^0) = 1$.

Due to the Goldstone Theorem, the nondegeneracy of η_m , the η_m -isometry of the translation operators and the relations (47)-(49) it follows by almost literally the same arguments as in [27]:

I.) $\mathbf{Q}_a^{\dots}(y)_{m,\mathcal{F}}$ is a well defined linear operator on $\mathcal{D}(m)_{SL}$ independent of the details of f_ϵ and f_R if and only if the equation (77) of the Goldstone Theorem holds. Therefore the vector charges $\mathbf{Q}_a^V(y)_{m,\mathcal{F}}$ exist, since no discrete scalar mesons are known in nature,

whereas the axial vector charges $\mathbf{Q}_a^A(y)_{m,\mathcal{F}}$ do not exist as linear operators due to the presence of the pseudo-scalar meson octet. Hence the chiral symmetry is spontaneously broken.

II.) The vector charges $\mathbf{Q}_a^V(y)_{m,\mathcal{F}}$ are closable, η_m -self conjugate and translational covariant operators mapping $\mathcal{D}(m)_{SL}$ into itself.

$$\begin{aligned}\mathbf{Q}_a^V(y)_{m,\mathcal{F}}^* \eta_m \Psi &= \eta_m (-1)^{Q(a)} \mathbf{Q}_{-a}^V(y)_{m,\mathcal{F}} \Psi \\ \mathbf{Q}_a^{V+}(y)_{m,\mathcal{F}} &= \eta_m^{-1} \mathbf{Q}_a^V(y)_{m,\mathcal{F}}^* \eta_m = (-1)^{Q(a)} \mathbf{Q}_{-a}^V(y)_{m,\mathcal{F}} \\ \mathcal{T}_m(y) \mathbf{Q}_a^V(x)_{m,\mathcal{F}} \mathcal{T}_m(-y) &= \mathbf{Q}_a^V(x+y)_{m,\mathcal{F}} \\ \mathbf{Q}_a^V(y)_{m,\mathcal{F}} \mathcal{D}(m)_{SL} &\subseteq \mathcal{D}(m)_{SL}\end{aligned}\tag{91}$$

Up to this point besides the spectrum condition, relative Einstein-causality and translational covariance only current conservation and the non-existence of discrete scalar particles have been exploited. In order to proceed we have to tighten the bonds between the currents $j_{a,V}^{8,\mu}$ at hand and the Noether currents arising from the flavor transformations in Lagrangian field theory (Section II) by imposing the commutation relations [27], [28] which in Lagrangian field theory define the infinitesimal flavor transformations

$$\begin{aligned}[j_{a,V}^{8,0}(f_\epsilon, f_R)_{m,\mathcal{F}}, \varphi_{\dots b}^{n,d}(f)_{m,\mathcal{F}}]_{-|R>R(\epsilon,J)} \Psi &= i \frac{d}{dt} (R^d(\exp. - i/2\tau_a t)_b^a \varphi_{\dots c}^{n,d}(f)_{m,\mathcal{F}} \Psi)_{|t=0} \\ \text{for } a &\in \{(1, \pm 1, 0), (1, 0, 0), (0, 0, 0)\}, \Psi \in \mathcal{D}(m)_{SL},\end{aligned}\tag{92}$$

and the same ones with $\varphi_{\dots}$ replaced by the flavor currents $[(n, d) = (1, 8)]$ respectively the color currents $[(n, d) = (8, 1)]$. The corresponding commutation relations with the composite fields (44), (45) are easily derived from the above ones.

Assuming for a moment that the commutation relations hold for all eight flavor currents then it follows by a straight forward calculation from the equations (50)-(52) and (92) that the charges $\mathbf{Q}_a^V(x)_{m,\mathcal{F}}$ commute with the translations if and only if $[\not{p}(\mathcal{M}), \tau_a]_- = 0$ with $\not{p}(\mathcal{M})$ explicitly given in equation (40), that is for $a \in \{(1, \pm 1, 0), (1, 0, 0), (0, 0, 0)\}$. In other words the charges representing the Lie-generators of the $SU(2)_{\mathcal{F}} \times U(1)_{\mathcal{F}}$ flavor subgroup and only these are constant in space and time:

$$\mathbf{Q}_a^V(0)_{m,\mathcal{F}} = \mathbf{Q}_a^V(x)_{m,\mathcal{F}} \quad \text{for } a \in \{(1, \pm 1, 0), (1, 0, 0), (0, 0, 0)\}\tag{93}$$

Due to the commutation relations (92) and the analogous ones for the currents and composite fields the charges operate on $\mathcal{D}(m)_{SL}$ via finite dimensional matrices acting on the space of indices. Therefore $\mathcal{D}(m)_{SL}[\eta_m \mathcal{D}(m)_{SL}]$ is a dense set of analytic vectors for the charge operators $\mathbf{Q}_a^V(0)_{m,\mathcal{F}}[\mathbf{Q}_a^V(0)_{m,\mathcal{F}}^*]$ since the exponential power series of them (respectively of the finite dimensional matrices corresponding to them) converge on $\mathcal{D}(m)_{SL}$ in the Hilbert norm of $\mathcal{H}(m)$ [27], [29] giving rise to one parameter groups of η_m -isometric

linear operators which map $\mathcal{D}(m)_{SL}[\eta_m \mathcal{D}(m)_{SL}]$ into itself and can be lifted to unitary ones on $\overline{H(m)}/H_0(m)$:

$$\begin{aligned} I_a(t)_{m,\mathcal{F}} \phi(f) \Psi_0(m) &:= \sum_{n=0}^{\infty} \frac{1}{n!} (it \mathbf{Q}_a^V(0)_{m,\mathcal{F}})^n \phi(f) \Psi_0(m) \\ I_a^*(t)_{m,\mathcal{F}} \eta_m \phi(f) \Psi_0(m) &= I_a(-t)_{m,\mathcal{F}}^* \eta_m \phi(f) \Psi_0(m) \\ &= \eta_m I_a^+(t)_{m,\mathcal{F}} \phi(f) \Psi_0(m), \quad t \in \mathbb{R}, \phi(f) \in \pi(\varphi_m, j_m)_{SL} \end{aligned} \quad (94)$$

In view of the commutation relations (92) the flavor charge operators $\mathbf{Q}_a^V(0)_{m,\mathcal{F}}$ form a representation $\mathcal{Q}_m$ of the Lie-algebra $\mathcal{G}(3)_{\mathcal{F}}$ of the flavor group $G(3)_{\mathcal{F}} \sim SU(2) \times U(1)$ on $\mathcal{D}(m)_{SL}$. The final step in our program is the integration or exponentiation of this representation, that means to establish the existence of a η_m -isometric representation $I_m^{\mathcal{F}}$ of the flavor group $G(3)_{\mathcal{F}}$ on $\mathcal{D}(m) \supset \mathcal{D}(m)_{SL}$ such that the restriction of its differential $\partial I_m^{\mathcal{F}}$ to $\mathcal{D}(m)_{SL}$ is equal to $\mathcal{Q}_m$

$$\partial I_m^{\mathcal{F}}(\tau_a) | \mathcal{D}(m)_{SL} = \mathbf{Q}_a^V(0)_{m,\mathcal{F}} \quad (95)$$

If η_m commutes with $\mathbf{Q}_a^V(0)_{m,\mathcal{F}}$ then the one parameter groups (94) consist of strongly continuous operators on $\mathcal{D}(m)_{SL}$ and hence admit an unique unitary extension onto $\mathcal{H}(m)$. According to criterions of Simon [40] and Jørgensen [41] this establishes the integrability of $\mathcal{Q}_m$. The difficulties arising in the general case rest in the fact that the charge operators $\mathbf{Q}_a^V(0)_{m,\mathcal{F}}$ are not symmetric with respect to the Hilbert scalar product $(\cdot | \cdot)$ and therefore do not establish the strong continuity of the operators $I_a^{(*)}(t)_{m,\mathcal{F}}$. Hence none of the classical integrability criterions (known to the author) is directly applicable.

However due to the commutation relations (92) the representation $\mathcal{Q}_m$ on $\mathcal{D}(m)_{SL}$ is directly connected with a representation of $\mathcal{G}(3)_{\mathcal{F}}$ in the underlying countably normed testfunction space. For the domain $\mathcal{D}(m)$ is the linear hull of states created from $\Psi_0(m)$ by polynomials of the Wightman-Gårding fields [11] with respect to the flavor group:

$$\phi_{\mu,c}^{n,d}(\mathbf{f})_m := \sum_{b \in \Gamma_d} \phi_{\mu,c,b}^{n,d}(f^b)_m \quad (96)$$

with $\phi_{\mu,c}$ denoting any one of the basic fields or currents and $f^b \in S(\mathbb{R}_4)$. The corresponding expressions for the composite fields are obvious. For the sake of shortness we do not write them down explicitly neither here nor in the following. Now the equations (90) - (92) imply for all f_i^b with compact support and $a \in \{(1, \pm 1, 0), (1, 0, 0), (0, 0, 0)\}$

$$\begin{aligned} \mathbf{Q}_a^V(0)_{m,\mathcal{F}} \Pi_{i=1}^k \phi_{\mu_i, c_i}^{n_i, d_i}(\mathbf{f}_i)_m \Psi_0(m) = \\ \sum_{j=1}^k (\Pi_{i=1}^{j-1} \phi_{\mu_i, c_i}^{n_i, d_i}(\mathbf{f}_i)_m) \phi_{\mu_j, c_j}^{n_j, d_j}(\delta^{d_j}(\tau_a) \mathbf{f}_j)_m \Pi_{i=j+1}^k \phi_{\mu_i, c_i}^{n_i, d_i}(\mathbf{f}_i)_m \Psi_0(m) \end{aligned} \quad (97)$$

where δ^d is defined for all $\mathbf{f} = (f^1, \dots, f^d) \in S(\mathbb{R}_4, \mathbb{C}_d)$ by

$$(\delta^d(\tau_a)\mathbf{f})^\epsilon := \sum_{b \in \Gamma_d} i \frac{d}{dt} R^d(\exp. - i/2\tau_a t)_b^c f|_{t=0}^b \quad (98)$$

Obviously the $\delta^d(\tau_a), \tau_a \in \mathcal{G}(3)_{\mathcal{F}}$ as defined in equation (98) make up a basis for a d-dimensional representation Δ^d of $\mathcal{G}(3)_{\mathcal{F}}$ on the testfunction space $S(\mathcal{H}_4, \mathcal{C}_d)$ equipped with the countable set of norms $\|\mathbf{f}\|_n^d, n \in \mathbb{N}_0$ derived from the scalar products $[\mathbf{f}, \mathbf{f}]_n^d = \sum_{a \in \Gamma_d} [f^a, f^a]_n$. The representatives once again act via d-dimensional matrices only on the space of indices. Therefore every element of $S(\mathcal{H}_4, \mathcal{C}_d)$ is an analytic vector (with respect to every norm) for the representation Δ^d . In addition $\delta^d(\tau_a)$ is skew symmetric with respect to every scalar product $[\mathbf{f}, \mathbf{f}]_n^d, n \in \mathbb{N}_0$. But then due to the general theory of unbounded representations of symmetry groups in gauge quantum field theories [23], [41] there exists an unique strongly continuous representation $R^d : G(3)_{\mathcal{F}} \rightarrow \text{Aut}.S(\mathcal{H}_4, \mathcal{C}_d), v \rightarrow R^d(v)$ on the countably normed space $S(\mathcal{H}_4, \mathcal{C}_d)$ the differential ∂R^d of which is equal to Δ^d [Ref.[42], Sec. IV, Theorem 4.1] and moreover a η_m -isometric representation $I_m^{\mathcal{F}} : G(3)_{\mathcal{F}} \rightarrow \text{Aut}.\mathcal{D}(m), v \rightarrow I_m^{\mathcal{F}}(v)$ on $\mathcal{D}(m)$ with $I_m^{\mathcal{F}}(v) = I_m(1, v)$ explicitly given in the equations (53), (54) and satisfying the condition (95) [Def.[42], Theorem 3.1 and 5.1].

Summing up we have arrived at the result:

Theorem (Flavor Symmetry): The four conserved vector currents $j_{a,V}^{8,\mu}(f)_{m,\mathcal{F}}, a \in \{(1,0,0), (1,\pm 1,0), (0,0,0)\}$ generate a η_m -isometric representation on $\mathcal{D}(m)$ [unitary one on $(\overline{H(m)}/\overline{H_0(m)}, \langle \cdot, \cdot \rangle)$ respectively $\mathcal{H}_{phys}$] of the algebraic flavor group $G(3)_{\mathcal{F}} \sim SU(2)_{\mathcal{F}} \times U(1)_{\mathcal{F}}$ which commutes with the translation group, whereas the algebraic chiral flavor symmetry is spontaneously broken due to the presence of the pseudo-scalar mesons (with finite masses).

In order to extend the investigation above to the color group the spectrum condition (64) respectively (65) has to be reinvestigated, that is its domain of validity enlarged from the physical section of $H(m)$ into unphysical parts of $\mathcal{H}(m)$. In the standard case of quantum gauge theories [13]-[22] the spectrum condition (64)/(65) is postulated to hold on the whole domain $\mathcal{D}(m)$ instead only on the subdomain $\mathcal{D}_H(m) = \mathcal{D}(m) \cap H(m)$ as required by the physical interpretation of the theory. In the case at hand this would be deadly bold at least for all colored states $\Psi(m) \in \mathcal{D}(m)$ with negative η_m -norm. Since the application of the translation operator $\mathcal{T}_m(y)$ to them produces an exponential factor $\exp.[Q(c)_c(\delta \cdot y)]$, the left hand side of equation (65) would not exist for testfunctions $h(y)$ whose Fourier transform has the required support properties in momentum space, for instance compact support. However due to part b.) of the Theorem on Color-Confinement in Section IV all colored states produced by a single basic field or conserved current from $\Psi_0(m)$ have zero η_m -norm. Hence we may extend the spectrum condition (64) respectively (65) at least to these states and assume in addition to (64)/(65):

$$\begin{aligned} \int d^4y \exp.[-i(q \cdot y)] < \Phi(m), \mathcal{T}_m(y) j_c^{8,\mu}(f)_{m,c} \Psi_0(m) >_m = 0 \\ \text{for all } q \notin \overline{V^0}_+, \text{ all } \Phi(m) \in \mathcal{H}(m) \text{ and almost all } m \in \mathcal{O}_{g,M^2}^\theta \end{aligned} \quad (99)$$

Now the generalized Goldstone Theorem and its consequences for color transformations are obtained in literally the same way as above just by replacing everywhere the flavor current by the color current and $p(a) = p_{\mathcal{M}}(8, a)$ by $(-iQ(a)_c\delta)$.

(Generalized Color-)Goldstone-Theorem: The color symmetry generated by the conserved current $j_a^{8,\mu}(h)_{m,c}$ is not spontaneously broken, that is:

$$\langle \Psi_0(m), \int d^3\mathbf{x} f_R(|\mathbf{x}|) [j_a^{8,0}(x)_{m,c}, \phi(f)]_- \Psi_0(m) \rangle_m = 0 \quad (100)$$

for all $|x^0| < \epsilon$, $\phi(f) \in \pi(\varphi_m, j_m)_{SL}$ and all $R > R(\epsilon, f)$, unless there exists a state $\Psi_{\kappa(a)}(m) \in \mathcal{H}(m)$ with mass $\kappa(a)^2 = -Q(a)_c^2 \delta^2 = -Q(a)_c^2 g \geq 0$ and $(\Psi_{\kappa(a)}(m), j_a^{8,0}(f_R)_{m,c} \Psi_0(m))_m \neq 0$.

In other words: In the representations with $g = \delta^2 > 0$ at most uncolored Goldstone states with zero mass can exist but no colored ones. If uncolored ones occur at all, they may appear in any part of $\mathcal{H}(m)$. - For $g = \delta^2 = 0$ both colored and uncolored Goldstone states of mass zero are admitted. If they occur at all, the former are confined to $\mathcal{H}_0(m)$ whereas the latter ones may appear in any part of $\mathcal{H}(m)$. - For $g = \delta^2 < 0$ again colored and uncolored Goldstone states can occur, the former with mass larger than zero confined in $\mathcal{H}_0(m)$ and the latter with mass zero in any part of $\mathcal{H}(m)$. - If in addition a local gauge condition of the type (26) holds in $\mathcal{H}(m)$ then also the uncolored Goldstone states are confined in the unphysical section $(\mathcal{H}(m) \setminus H(m)) \cup H_0(m)$.

Similarly to Wightman theories there may be further restrictions concerning the spin - or helicity - quantum numbers of the Goldstone states [28], [43], [44]. However in view of the indefinite metric and the direct integral structure of the representation of the Lorentz group this is a problem of its own beyond the present note.

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