A SOVIET FORMALISM IN BETATRON OSCILLATION THEORY
AND ITS COMPARISON WITH OTHER CONCEPTS

by

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The theory of betatron oscillations in a synchrotron is presented according to a logical development using first the classical COURANT-SNYDER formalism and second a Soviet formalism. It is then possible to compare the concepts and the terminology used in both cases, and this could help people who will have to work on the Serpukhov synchrotron. The Soviet formalism is based on the concept of complex "Floquet function $\varphi(s)$" and of its derivative $\varphi'(s)$, as a solution of the equation of motion. The phase $\varphi(s)$ of the Floquet function and the phase $\chi(s)$ of its derivative are used for the representation of motion and for the computation of the transfer matrices. A table gives the correspondence between the usual $\beta$, $\gamma$, $\alpha$, $\mu$, $\varphi$ parameters and the soviet terms $|\varphi||\varphi'|$, $|\varphi'|$, $\varphi$, $\chi$ and $\mu$. A similar correspondence is done for the computation of the perturbed closed orbit.

On présente la théorie des oscillations bétatrontiques dans un synchrotron suivant un développement logique utilisant d'abord le formalisme classique de COURANT-SNYDER et ensuite un formalisme soviétique. Il est alors possible de comparer la terminologie et les concepts dans chacun des cas, et cela devrait aider les personnes devant travailler autour du synchrotron de Serpukhov. Le formalisme soviétique est fondé sur le concept de "fonction de Floquet complexe $\varphi(s)$" et de sa dérivée $\varphi'(s)$, considérée comme solution de l'équation du mouvement. La phase $\varphi(s)$ de cette fonction de Floquet et la phase $\chi(s)$ de sa dérivée sont utilisées pour la représentation du mouvement et des matrices de transfert. Un tableau indique la correspondance entre les paramètres habituels $\beta$, $\gamma$, $\alpha$, $\mu$, $\varphi$ et les termes soviétiques $|\varphi||\varphi'|$, $|\varphi'|$, $\varphi$, $\chi$ et $\mu$. On présente une correspondance analogue pour le calcul de l'orbite fermée perturbée.
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REFERENCES
1. INTRODUCTION

Those who will have to work on the 80 GeV proton synchrotron at Serpukhov, as well as other interested persons, should know the Soviet formalism used there for particle motion in order to link it up to our classical concepts. We take advantage of this presentation to introduce some generalities which should clarify the comparison, such as, for example, the rotations in normalized phase spaces.

We make the following assumptions:

a) The machine is ideal; there is no high-order multipole field above the quadrupole field.

b) The motion is conservative; no energy is radiated from particles.

c) The beam is mono-energetic; a momentum dispersion appears only in the appendix about the perturbed closed orbits.

d) The infinite motion of particles around the machine is stable; the transformation is said to be "elliptic" and not "hyperbolic".

e) There is no coupling between the so-called "horizontal plane" (or median plane) and the "vertical plane" motions.

f) The following formalism describes the motion in any of these planes.

2. GENERALITIES

2.1 Definition of a periodic system

We postulate the existence, in the horizontal plane, of a closed circuit \(C\) called the "unperturbed equilibrium orbit". We characterize the position of a point \(P\) near to \(C\) by the following coordinates:

- \(s\) = the distance along \(C\) measured from some origin to that point on \(C\) closest to \(P\);
- \(y\) = the horizontal component of the displacement of \(P\) from \(C\);
- \(z\) = the vertical component of this displacement.

We form the derivative \(\dot{y} = \frac{dy}{ds}\) so that \(y\) and \(\dot{y}\) will be the coordinates in a degenerated form of phase space.

The motion along a closed circuit such as \(C\) is said to be periodic because at each turn the particles meet the same fields at the same points. The condition for periodicity remains if \(C\) can be divided into an integer number \(N\) of identical periods of length \(L\), such that the field \(H\) satisfies the relation \(H(y, z, s) = H(y, z, s + L)\). (In fact, the existence of identical periods is sufficient for periodicity though not necessary, but for the sake of clarity we shall keep the following criterion: \(C\) can be divided into \(N\) identical cells, \(N\) being at least equal to 1.)

2.2 The Hill equation

We suppose that the particles are submitted only to restoring forces proportional to \(y\), and such that the equation of motion can be written

\[
\frac{d^2y}{ds^2} + K(s)y = 0 ,
\]

(1)
and from periodicity we know (see Section 2.1) that
\[ K(s + L) = K(s). \]  
(2)

One calls a "Hill equation" any homogeneous linear differential equation of second order with real periodic coefficients, and Eq. (1) belongs to this type\(^1\). Thus Eq. (1) has four main properties:

i) it is a differential equation with periodic coefficients;

ii) the coefficients are real;

iii) this equation is linear and of second order;

iv) it has no term in \( y' \) (this is due to assumption 1.b).

We shall also use the fact that any complex function can be represented as
\[ \varphi(s) = |\varphi(s)| e^{i \varphi(s)} \]  
(3)
where \(|\varphi(s)|\) is its modulus and \(\varphi(s)\) its argument (or its phase).

From property (i) essentially (but also from others) the Floquet theorem\(^1\) states that Eq. (1) has two linearly independent solutions which are respectively complex conjugates and whose modulus has the period of \(K(s)\). These solutions can be written
\[ \varphi(s) = |\varphi(s)| e^{i \varphi(s)} \]  
(4)
with
\[ |\varphi(s)| = |\varphi(s + L)| \quad \text{or} \quad \varphi(s + L) = \varphi(s)e^{i \mu}. \]  
(5)

From property (iii) we deduce that any solution \(y(s), y'(s)\) can be obtained from its initial conditions \(y_0, y'_0\) by means of a linear transformation such as
\[ \begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = M \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \]  
(6)
where \(M\) is a \(2 \times 2\) matrix.

From property (iii) again, we know that the general solution \(y(s)\) is a linear combination of two particular solutions linearly independent as described by Eq. (4).

From property (iv) we can prove that the Jacobian of the system, that is to say the determinant of \(M\), is equal to 1:
\[ \det M = 1. \]  
(7)

From property (ii) essentially [but also from (iii) and (iv)] we deduce the important relation
\[ \varphi' = k \frac{\varphi}{|\varphi(s)|^2} \quad \text{or} \quad \varphi' = k \int_0^s \frac{ds}{|\varphi(s)|^2}, \]  
(8)
k being a constant. This constant is currently taken to be 1 [see demonstration from Eqs. (79) and (80) through the normalization relation (43), and also remark (b) in Section 5].

To prove Eq. (8), we derive Eq. (3) and put it in Eq. (1). Separating imaginary and real parts, we get

\[ 2|\varphi|^2 \varphi' + |\varphi| \varphi' = 0, \]

which is nothing else than

\[ \frac{d}{ds} [ |\varphi|^2 \varphi'] = 0. \]

3. THE COURANT-SNYDER FORMALISM

We summarize very briefly the pioneer work of E. Courant and H. Snyder described by them\(^2\), and which will be referred to as CS.

3.1 Solution of the Hill equation

As seen above, the general solution of Eq. (1) is a linear combination of the two forms

\[ y(s) = k^{1/2} \omega(s)e^{i\varphi(s)}. \]

We introduce the unusual constant \( k \) which is not made explicit in either of the formalisms (CS or Russian) because it is equal to 1. But in the present report, when comparing both formalisms we shall meet homogeneity problems which will only be solved by introducing an "homogeneity constant" \( k \), equal to one but having the dimension of a length.

\[ \boxed{\text{k is the unit of length}}. \]

The transfer matrix \( M \) of Eq. (6) is represented through one period as

\[ M_k(s \rightarrow s + L) = \begin{pmatrix} \cos \mu + \alpha(s) \sin \mu & \mu(s) \sin \mu \\ - \gamma(s) \sin \mu & \cos \mu - \alpha(s) \sin \mu \end{pmatrix}, \]

where

\[ \cos \mu = \frac{1}{2} \text{Tr of } M_k. \]

The assumption \( s \) imposes the criterion for stability: \( \mu \) real or \( \cos \mu \leq 1 \). It is also shown that \( \mu \) is independent of \( s \). One then makes the following identifications:\(^3\):

\[ \boxed{\text{.}} \]
\[ w(s) = \sqrt{\beta(s)} \]  
\[ \frac{d\phi}{ds} = -2\alpha(s) \]  
\[ \psi(s) = \int_0^s \frac{ds}{\beta(s)} . \] 

So

\[ \mu = \int_s^{s+L} \frac{ds}{\beta(s)} . \]  

\[ \beta(s) \] is the "amplitude function" and \( \psi(s) \) the "phase function".

Because of the periodicity one has

\[ \beta(s+L) = \beta(s), \quad \alpha(s+L) = \alpha(s), \quad \gamma(s+L) = \gamma(s), \quad \text{and} \quad \phi(s+L) = \phi(s) + \mu . \]  

The determinant of \( \mathbf{M} \) being equal to one, one gets

\[ \beta(s)\gamma(s) = 1 + \alpha^2(s) . \]  

One then writes the real solution of Eq. (1) under the form

\[ y(s) = a k^{1/2} \beta^{1/2}(s) \cos \left[ \psi(s) + \delta \right] \]  

where \( a \) and \( \delta \) are non-dimensional constants from integration; \( k \) is the unit length, as explained above. The dimensions are the following: \( y \) and \( s \) in \( \text{m} \), \( \beta \) in \( \text{m} \), \( \gamma \) in \( \text{m}^{-1} \), and \( \alpha, \psi, \delta \) without dimension.

Deriving Eq. (20) one gets

\[ y'(s) = a k^{1/2} \beta^{-1/2}(s) \cos \left[ \psi(s) + \delta \right] - \frac{d\phi}{ds} \beta^{1/2}(s) \sin \left[ \psi(s) + \delta \right] . \]  

Using Eqs. (15), (16), and (19), one then can write

\[ y'(s) = a k^{1/2} \gamma^{1/2}(s) \cos \left[ \chi(s) + \delta \right] \]  

in which we have anticipated by already introducing the Soviet function \( \chi(s) \) given by

\[ \text{tg} \left[ \psi(s) - \chi(s) \right] = \frac{1}{\alpha(s)} . \]
So $\phi(s)$ is the phase of $y$ and $\chi(s)$ the phase of $y'$. If we could get a complete table giving $\beta(s), \gamma(s), \alpha(s),$ and $\phi(s)$, all along the closed orbit $C$, we should have enough information to compute the particle motion and also to deduce the transfer matrix of any portion of the machine, as we shall now see.

### 3.2 The transfer matrices

The transfer matrix from $s_1$ to $s_2$ can be deduced by the following formula\(^{2,3}\)

\[
\mathbf{M}(s_1 \rightarrow s_2) = \begin{pmatrix}
\sqrt{\frac{\beta_1}{\beta_1}} (\cos \Delta \phi + \alpha_1 \sin \Delta \phi) & \sqrt{\frac{\beta_1}{\beta_2}} \sin \Delta \phi \\
- \frac{1}{\sqrt{\beta_1 \beta_2}} [(1 + \alpha_1 \alpha_2) \sin \Delta \phi + (\alpha_2 - \alpha_1) \cos \Delta \phi] & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \Delta \phi - \alpha_2 \sin \Delta \phi)
\end{pmatrix}
\]

where $\beta_1, \alpha_1$ stands for $\beta(s_1), \alpha(s_1)$ and

\[
\Delta \phi = \phi(s_2) - \phi(s_1).
\]

This classical formula, as well as the following, may be presented as resulting from a geometrical transformation of the invariant ellipse, as will be explained in Section 6.

Consequently, we anticipate by presenting a new form of the transfer matrix (the "$\gamma$ type"), though the demonstration will be given only in Section 6 [see also remark $(c)$ in Section 5]

\[
\mathbf{M}(s_1 \rightarrow s_2) = \begin{pmatrix}
\frac{\chi_1}{\gamma_1} (\cos \Delta \chi + \alpha_2 \sin \Delta \chi) & \frac{1}{\sqrt{\gamma_2 \gamma_1}} [(\alpha_1 - \alpha_2) \cos \Delta \chi + (1 + \alpha_1 \alpha_2) \sin \Delta \chi] \\
- \sqrt{\gamma_2 \gamma_1} \sin \Delta \chi & \frac{\chi_2}{\gamma_1} (\cos \Delta \chi - \alpha_1 \sin \Delta \chi)
\end{pmatrix}
\]

where, as above, $\gamma_1, \alpha_1$ stand for $\gamma(s_1), \alpha(s_1)$ and

\[
\Delta \chi = \chi(s_2) - \chi(s_1).
\]

$\chi(s)$ is the angle defined in Eq. (22) as the phase of $y'$, and one deduces

\[
\tan (\Delta \phi - \Delta \chi) = \frac{\alpha_1 - \alpha_2}{1 + \alpha_1 \alpha_2}.
\]

If $\mathbf{M}$ is the matrix of one period, then $\beta(s_2) = \beta(s_1), \alpha(s_2) = \alpha(s_1), \gamma(s_2) = \gamma(s_1), s_2 - s_1 = L$, and $\Delta \phi = \Delta \chi = \mu$, such that $\cos \mu = \frac{1}{2} \text{Tr } \mathbf{M}$; in this case Eqs. (24) and (26) reduce to Eq. (12).
4. A SOVIET FORMALISM

We must go now into more details in order to show the logical development. A part of this section can be found elsewhere, another part comes from private communications, and a third part is a personal interpolation. Unfortunately, the use of the symbols is not always consistent:

1) the phase of the "Floquet function \( \psi(s) \)" will be called here \( \varphi(s) \), which is consistent with Refs. 2, 6 and 7, but not with Ref. 4 where \( \psi(s) \) is written \( f(s)e^{i\chi(s)} \);
2) the phase of the derivative \( \psi'(s) \) of the Floquet function will be called here \( \chi(s) \);
3) in Ref. 5, a symbol \( \psi(s) \) is used for the amplitude of the perturbed closed orbit divided by \( \Delta p/p \), as we shall see in the Appendix.

4.1 The Floquet functions as solutions of the Hill equation

We still start from the Hill equation (1) with the condition of periodicity (2). In Ref. 5, Eq. (1) is written as

\[
\frac{d^2y}{ds^2} + \omega^2(s)y = 0,
\]

(29)

where \( \omega \) is a real or imaginary frequency. So \( \omega(s) \) can be positive or negative and is periodic [\( \omega(s) = K(s) \)].

From property (iii) (page 2), we know the existence of a \( 2 \times 2 \) matrix \( M \) such that any solution of Eq. (1) can be built from its initial conditions according to

\[
\begin{pmatrix} \psi(s) \\ \psi'(s) \end{pmatrix} = \begin{pmatrix} M(s_0 - s) \end{pmatrix} \begin{pmatrix} \psi(s_0) \\ \psi'(s_0) \end{pmatrix}
\]

(30)

and later we shall see how to choose a solution that is as simple as possible [see Eq. (46)]. Here we will anticipate, i.e., we will consider that this choice has been made.

We now decide to write this solution of Eq. (1), the so-called complex "Floquet function", as already given by Eq. (3):

\[
\psi(s) = |\psi(s)|e^{i\varphi(s)}.
\]

(3)

\( K \) being real, if \( \psi(s) \) is a solution then its complex conjugate \( \psi^*(s) \) is also a solution, because we may write:

\[
\varphi^* + K\varphi = 0
\]

(31)
\[
(\varphi^*)^* + K\varphi^* = 0
\]

(32)
\[
(\varphi^*)^* + K\varphi^* = 0
\]

(33)

Furthermore, \( \psi, \psi^* \) are linearly independent because the Wronskian, which is constant as we shall see later in Eq. (41), may be chosen different to zero.
Thus any complex solution of the Hill equation (1) can be expressed as
\[ y(s) = A\varphi(s)e^{i\psi(s)} + B\varphi(s)e^{-i\psi(s)} \]  
(34)
where \( A \) and \( B \) are complex constants. Because we are interested in the real solution which depends only on two real constants of integration, it is sufficient to consider the particular case
\[ y(s) = A\varphi(s) + A^*\varphi^*(s) \]  
(35)
and its derivative
\[ y'(s) = A\varphi'(s) + A^*\varphi'^*(s) \]  
(36)
We introduce the angle \( \chi(s) \) as the phase of the derivative of the Floquet function, and we now drop the \((s)\) for clarity, although \( \varphi, \varphi', \chi, \) and \( \chi \) are still functions of \( s \).

The usual representation of the Floquet function is therefore:
\[
\begin{align*}
\varphi &= |\varphi|e^{i\psi} \\
\varphi' &= |\varphi'|e^{i\chi}
\end{align*}
\]  
(37)
(38)
and the real solution of Eq. (1) can be written
\[
\begin{align*}
y(s) &= |\varphi|[Ae^{i\psi} + A^*e^{-i\psi}] = a|\varphi| \cos(\psi + \delta) \\
y'(s) &= |\varphi'|[Ae^{i\chi} + A^*e^{-i\chi}] = a|\varphi'| \cos(\chi + \delta) 
\end{align*}
\]  
(39)
(40)
\( \delta \) being any phase constant between 0 and \( 2\pi \) and \( a \) being a scalar constant. The \( \varphi \) and \( \chi \) of Eqs. (39) and (40) are the same as those already used in Eqs. (20) and (22).

The dimension is a length for \( y, s, \) and \( \varphi, \) whilst \( y', \varphi', a, \) and \( \delta \) have no dimensions.

**Remark**

As there is no dissipative term in Eq. (1), the Wronskian \( W \) of the system [Eqs. (35) and (36)] should be constant:
\[
W = \begin{vmatrix} \varphi & \varphi^* \\ \varphi' & \varphi'^* \end{vmatrix} = \text{const, because } \frac{dW}{dt} = 0 \]  
(41)
This is one possible aspect of Liouville's theorem\(^a\). The Wronskian has the dimension of \( \varphi(s) \) and is imaginary (it is easy to verify that \( W^* = -W \)). \( \varphi \), the solution of Eq. (1), is first defined up to a multiplicative constant. It is then currently normalized by choosing the following value for the Wronskian [see remark (b) in Section 5].
\[ W = -21k \]  
(42)
Here again we introduce the homogeneity constant $k$ as in Eq. (11), though it is equal to 1 and is never explicitly written. From Eqs. (37), (38), (41), and (42) one gets a very important form of the normalisation condition (42)

$$|q| |q'| \sin (\phi - \chi) = - k$$

(43)

If we now introduce the "generalized Floquet function matrix" as

$$\psi = \begin{pmatrix} \text{Re } \psi & \text{Im } \psi \\ \text{Re } \psi' & \text{Im } \psi' \end{pmatrix}$$

(44)

one can verify that

$$\text{Det } \psi = k$$

(45)

4.2 The stability of betatron oscillations

Since we are free to choose arbitrarily the first particular solution of the Hill equation as described in Eq. (30), we decide to define it by initial conditions satisfying the relation:

$$M_p \begin{pmatrix} \psi(s_0) \\ \psi'(s_0) \end{pmatrix} = \lambda \begin{pmatrix} \psi(s_0) \\ \psi'(s_0) \end{pmatrix}$$

(46)

where $M_p$ is the transfer matrix through one period starting at the origin $s_0$. This well-known equation is already solved in Ref. 2 [and in other papers such as that of Schoch11*, for example]; it gives

$$\lambda = e^{i \mu} = \cos \mu + i \sin \mu$$

(47)

with

$$\cos \mu = \frac{1}{2} \text{ Tr } M_p - \frac{\sum_{i,j} M_{ij}}{2}$$

(48)

$m_{ij}$ being the elements of $M_p (s_0 \to s_0 + L)$.

The modulus of $\lambda$ remains equal to 1 if $\mu$ is real, that is to say if

$$|\sum_{i,j} + m_{12}| \leq 2$$

(49)

as already discussed in Eq. (13). Thus, using one solution of Eq. (47) and putting it in Eq. (46) for the period starting at $s_0$, one obtains:

$$\psi(s_0 + L) = \lambda \psi(s_0) = \psi(s_0) e^{i \mu}$$

(50)

$$\psi'(s_0 + L) = \lambda \psi'(s_0) = \psi'(s_0) e^{i \mu}$$

(51)

Then, from

$$\begin{pmatrix} \psi(s) \\ \psi'(s) \end{pmatrix} = M(s_0 \to s) \begin{pmatrix} \psi(s_0) \\ \psi'(s_0) \end{pmatrix}$$

(52)
one gets, using periodicity and Eqs. (50)-(51),
\[
\begin{pmatrix}
\psi(s + L) \\
\psi'(s + L)
\end{pmatrix}
= M(s_0 + s)
\begin{pmatrix}
\psi(s_0 + L) \\
\psi'(s_0 + L)
\end{pmatrix}
= M(s_0 + s)e^{i\mu}
\begin{pmatrix}
\psi(s_0) \\
\psi'(s_0)
\end{pmatrix}
\]  
(53)

which is, in complete generality (i.e. whatever \( s \)),
\[
\begin{pmatrix}
\psi(s + L) \\
\psi'(s + L)
\end{pmatrix}
= e^{i\mu}
\begin{pmatrix}
\psi(s) \\
\psi'(s)
\end{pmatrix}
\]  
(54)

This means that providing that Eq. (49) is satisfied, stability is assured [see Eqs. (39), (40), and (54)]. It may be useful to write Eq. (54) in detail. This gives:
\[
|\psi(s + L)| = |\psi(s)|
\]  
(55)
\[
\phi(s + L) = \phi(s) + \mu
\]  
(56)
\[
|\psi'(s + L)| = |\psi'(s)|
\]  
(57)
\[
\chi(s + L) = \chi'(s) + \mu
\]  
(58)

with [due to Eq. (8)],
\[
\mu = \chi(s + L) - \chi(s) = \phi(s + L) - \phi(s) = k \int_s^{s+L} \frac{ds}{|\psi(s)|^2}
\]  
(59)

4.3 The transfer matrices

Let us consider the solution of Eq. (1) in its general form (54) and let us calculate the constants \( A \) and \( B \) in terms of initial conditions. We solve the linear system
\[
\begin{pmatrix}
y_1 \\
y'_1
\end{pmatrix}
= A|\psi_1|e^{i\psi_1} + B|\psi_1|e^{-i\psi_1}
\]  
(60)
\[
\begin{pmatrix}
y_1' \\
y'_1'
\end{pmatrix}
= A|\psi_1'|e^{i\psi_1'} + B|\psi_1'|e^{-i\psi_1'}
\]  
(61)

(The low index 1 means these values at \( s = s_1 \), and the index 2 will mean these values at \( s = s_2 \).) One then gets
\[
A = \frac{1}{2} [y_1|\psi_1'|e^{-i\psi_1'} - y_1'|\psi_1|e^{-i\psi_1}]k^{-1}
\]  
(62)
\[
B = \frac{1}{2} [y'_1|\psi_1'|e^{i\psi_1'} - y_1'|\psi_1|e^{i\psi_1}]k^{-1}
\]  
(63)

The constant \( k \) equal to one comes from Eqs. (41) and (42).
Using these values for A and B and writing Eq. (34) for \( s = s_2 \), it becomes:

\[
\begin{align*}
  y_2 &= k^{-1} \left[ -y_1 |\psi_2| |\psi_1'| \sin (\varphi_2 - \chi_1) + y_1' |\psi_2| |\psi_1| \sin (\varphi_2 - \varphi_1) \right] \\
  y_2' &= k^{-1} \left[ -y_1 |\psi_2'| |\psi_1'| \sin (\chi_2 - \chi_1) + y_1' |\psi_2'| |\psi_1| \sin (\chi_2 - \varphi_1) \right].
\end{align*}
\]

(64) (65)

We can now write the transfer matrix from \( s_1 \) to \( s_2 \) as:

\[
M(s_1 \rightarrow s_2) = k^{-1} \begin{pmatrix}
-|\psi_2| |\psi_1'| \sin (\varphi_2 - \chi_1) & |\psi_2| |\psi_1| \sin (\varphi_2 - \varphi_1) \\
-|\psi_2'| |\psi_1| \sin (\chi_2 - \chi_1) & |\psi_2'| |\psi_1| \sin (\chi_2 - \varphi_1)
\end{pmatrix}.
\]

(66)

Using the normalization (43) which cancels out the constants \( k \), it is easy to verify that the determinant of \( M \) is 1 without dimension, which is necessary, since \( M \) is the Jacobian

\[
\frac{D(y_2, y_2')}{D(y_1, y_1')}
\]

Through one period we have Eqs. (55) to (58). Using Eq. (43) and introducing

\[
\nu(s) = \varphi(s) - \chi(s) = \nu(s + L),
\]

(67)

the matrix through one period becomes

\[
M_p(s \rightarrow s + L) = \begin{pmatrix}
\sin \left[ \frac{\nu(s) - \mu}{\nu(s)} \right] & k^{-1} |\psi(s)|^2 \sin \mu \\
- k^{-1} |\psi'(s)|^2 \sin \mu & \frac{\sin \left[ \frac{\nu(s) - \mu}{\nu(s)} \right]}{\sin \left[ \frac{\nu(s)}{\nu(s)} \right]}
\end{pmatrix}.
\]

(68)

4.4 The numerical calculation of the Floquet functions

Equation (46) is

\[
\begin{align*}
(m_{11} - \lambda) \varphi(s_0) + m_{12} \varphi'(s_0) &= 0 \\
m_{21} \varphi(s_0) + (m_{22} - \lambda) \varphi'(s_0) &= 0,
\end{align*}
\]

(69) (70)

\( m_{ij} \) being the elements of the transfer matrix through one period starting at \( s_0 \). In order to avoid the trivial solution \( \varphi = \varphi' = 0 \), we know that \( \lambda \) should satisfy Eq. (47); the system is then undetermined and we can choose \( \varphi(s_0) \) arbitrarily. But we know from Eq. (66) that \( |\varphi(s_0)| \) at the entrance of one period is \( k^{1/2} \sqrt{m_{12}/\sin \mu} \). The possibility of choosing the original phase remains, and we write

\[
\varphi(s_0) = \varphi_0 = \text{arbitrary constant}.
\]

(71)
Moduli and phases of the Floquet functions through a superperiod of a synchrotron (vertical plane).
Then we can solve the system (69), (70) and, using the same convention for the index \( \varphi_0 = \varphi(s_0), \text{ etc.} \), one gets:

\[
\begin{align*}
\text{Re } \varphi_0 &= k^{1/2} \sqrt{\frac{m_{12}}{\sin \mu}} \cos \varphi_0 \\
\text{Im } \varphi_0 &= k^{1/2} \sqrt{\frac{m_{12}}{\sin \mu}} \sin \varphi_0 \\
\text{Re } \varphi' &= k^{1/2} \sqrt{\frac{\sin \mu}{m_{12}}} \left( \frac{\cos \mu - m_{11}}{\sin \mu} \cos \varphi_0 - \sin \varphi_0 \right) \\
\text{Im } \varphi' &= k^{1/2} \sqrt{\frac{\sin \mu}{m_{12}}} \left( \cos \varphi_0 + \frac{\cos \mu - m_{11}}{\sin \mu} \sin \varphi_0 \right).
\end{align*}
\]  

(Note: One generally chooses \( \varphi_0 = 0 \) at the entrance to the first period.)

We now want to know the Floquet functions at each abscissa \( s \) along the accelerator. As the Hill equation is linear, Eq. (30) gives \( \varphi(s) \) and \( \varphi'(s) \), using the usual transfer matrix \( \mathcal{M} \)

\[
\begin{pmatrix}
\varphi(s) \\
\varphi'(s)
\end{pmatrix} = \mathcal{M}(s_0 \to s)
\begin{pmatrix}
\varphi(s_0) \\
\varphi'(s_0)
\end{pmatrix}
\]

which can also be written, by using the generalized matrix \( (\mathcal{M}) \) of Eq. (44), as

\[
[\varphi(s)] = \mathcal{M}(s_0 \to s)[\varphi(s_0)].
\]

So the Floquet functions are computed from point to point starting at the origin with the function [\( \varphi(0) \)] given by Eqs. (72) to (75) where usually \( \varphi(0) = 0 \). The results are presented in a complete table giving [\( \varphi(s) \)], [\( \varphi'(s) \)], \( \varphi(s) \), and \( \chi(s) \) all along the machine, and such a table appears in detail in Ref. 5 for the Serpukhov synchrotron. It allows a direct computation of the particle motion and a rapid building up of the transfer matrix of any portion between \( s_1 \) and \( s_2 \) according to Eqs. (39), (40), and (66). Figure 1 gives the curves of these functions in the vertical plane for one superperiod. To end with the presentation of the formalism we must compute \( \varphi'(s) \) and \( |\varphi(s)|' \), which are the derivative of the phase and the derivative of the modulus \(^*\) of the Floquet function, respectively. For any complex function \( \varphi(s) = |\varphi(s)| e^{i \varphi(s)} \) we have the relations

\[
\begin{align*}
|\varphi'| &= |\varphi'\!| \cos (\chi - \varphi) \\
\varphi' &= \frac{|\varphi'|}{|\varphi|} \sin (\chi - \varphi),
\end{align*}
\]

where \( \chi \) is the phase of the derivative \( \varphi'(s) \).

\(^*\) [\( \varphi'(s) \)] and [\( |\varphi(s)|' \)] should not be confused. They are the modulus of the derivative and the derivative of the modulus, respectively.
Introducing now the properties of the Hill equation through the normalization formula (43) we get

\[ \psi'(s) = \frac{k}{|\psi(s)|^2} \quad \text{or} \quad \psi(s) = k \int_0^s \frac{ds}{|\psi(s)|^2} \]

(80)

and

\[ \tan [\psi(s) - \chi(s)] = -\frac{k}{|\psi'|/|\psi|} \]

(81)

5. CORRESPONDENCE BETWEEN THE TWO FORMALISMS

From the preceding chapters, one concludes that it is possible to make the following identifications:

- **Amplitude function**
  \[ \beta(s) = k^{-1}|\psi(s)|^2 \]
  \[ \gamma(s) = k^{-1}|\psi(s)|^2 \]
  \[ \alpha(s) = \frac{1}{\tan [\psi(s) - \chi(s)]]} \]

(82)

(83)

(84)

- **Phase function**
  \[ \psi(s) = \int_0^s \frac{ds}{F(s)} = k \int_0^s \frac{ds}{|\psi(s)|^2} \]

(85)

We have also

\[ \alpha(s) = -\frac{1}{2} \frac{d\psi(s)}{ds} = -k^{-1}|\psi(s)||\psi'(s)|' \]

(86)

\[ \beta(s)\gamma(s) = 1 + \alpha^2(s) \]

(87)

\[ \sin [\psi(s) - \chi(s)] = -k/|\psi(s)||\psi'(s)| \]

(88)

\[ \tan [\psi(s) - \chi(s)] = -k/|\psi(s)||\psi'(s)|' = 1/\alpha(s) \]

(89)

When \( \alpha(s) = 0 \), \( \beta(s) \) is at an extremum, the transfer matrix for one period is symmetric, and the beam envelope is at an extremum; at these points \( \psi - \chi = \pi/2 \).

**Remarks**

a) Up to here we have carried the "invisible" constant \( k \) which is equal to one and which allows the homogeneity of the identifications, though it is never made explicit in the literature. Let us repeat that it has the same dimension as \( \psi(s) \) which is generally given in metres as \( \beta(s) \) and \( s \), while \( y', \phi \) and \( \chi \) are in radians.
b) The normalization chosen by Eq. (42) is perfectly arbitrary. It would have been possible to normalize \( \psi \) by:

\[
\psi = -2i\rho \text{ } \text{ } \text{ (42')} \]

which is a multiplication of \( \psi \) by \( \rho^{1/2} \) (with respect to our case). Other key-formulae would have been:

\[
|\psi| |\psi'| \sin(\psi - \chi) = -\rho \text{ } \text{ (43')} \]

\[
\psi = \rho \kappa \int_0^s \frac{ds}{|\psi(s)|^2} \text{ (80')} \]

(no change in the value of \( \psi \))

\[
M = k^{-1} \rho^{-1} \begin{pmatrix}
- |\psi| |\psi'| \sin(\psi - \chi) & |\psi| |\psi'| \sin(\psi - \psi') \\
- |\psi'| |\psi'| \sin(\chi - \chi) & |\psi'| |\psi'| \sin(\chi - \psi')
\end{pmatrix} \text{ (66')} \]

(no change in the value of the matrix terms).

c) It is easy to verify, using Eqs. (82), (83), (84), and (88), that Eqs. (24) and (26) are identical to Eq. (66).

6. INARIANT ELLIPSES AND NORMALIZATION

Most of the following chapters are not directly related to the comparison of formalisms, but they should help in the understanding of the geometrical content of betatron oscillation theory common to both systems.

6.1 The invariant ellipses of a periodic system

Let us consider a periodic system whose transfer matrix through one period is described by \( \beta(s) \), \( \alpha(s) \), \( \gamma(s) \), and \( \mu \) according to Eq. (12).

It is known\(^2\)\(^,\)\(^9\) that there exists, in the phase space \( y, y' \), a family of elliptical contours which are invariant through an integer number of periods, though individual states of particles seem to move along this contour from period to period. They are the so-called "invariant ellipses" generally described as\(^9\)\(^,\)\(^10\)

\[
E(s) = \begin{pmatrix}
\beta(s) \\
\alpha(s) \\
\gamma(s)
\end{pmatrix}
\]

and whose phase space area is \( A \) (in metres, in our units). The one having the largest possible area is the "acceptance ellipse" for an infinite number of identical periods.

The invariant ellipses form a family whose equation is

\[
\gamma(s)y^2 + 2\alpha(s)yy' + \beta(s)y'^2 = \epsilon \text{ (90)}
\]

or

\[
|\psi'(s)|^2y^2 + \frac{2kyy'}{t_k [\psi(s) - \chi(s)]} + |\psi(s)|^2y'^2 = k\epsilon \text{ (91)}
\]

\( \epsilon \) being the area \( A \) divided by \( \pi \).
These ellipses may also be described by the parametric equations (33) and (40) which become

\[
\begin{align*}
    y(s) &= \left(\frac{\epsilon}{k}\right)^{1/4} |\varphi(s)| \cos[\varphi(s) + \delta] \\
    y'(s) &= \left(\frac{\epsilon}{k}\right)^{1/4} |\varphi'(s)| \cos[\chi(s) + \delta]
\end{align*}
\] (92) (93)

the parameter \( \delta \) being any phase between 0 and 2\pi, \( \varphi(s) \) and \( \chi(s) \) still being linked by Eq. (88).

6.2 The principle of normalization

The phase space \( y, y' \) is transformed by means of a normalization matrix \( N \) in a normalized phase space \( \bar{Y}, \bar{Y}' \) where:

1) the coordinate axis bear the same dimension (metre \( 1/4 \));
2) the invariant ellipses become concentric circles along which rotations can be defined, but still depend on the type of normalization to be chosen. We try now to generalize these techniques \(^3, ^9, ^{11}, ^{18} \).

6.2.1 The \( x \)-type normalization

Any ellipse

\[
\begin{pmatrix}
    \beta \\
    \alpha \\
    Y
\end{pmatrix}
\]

whose equation in original phase space \( y, y' \), is

\[
y y^2 + 2\alpha y y' + \beta y'^2 = \epsilon
\] (94)

can be transformed in a circle of same area (\( \pi \epsilon \)) and same centre \( i' \), the phase-space coordinates undergo the following \( 2 \times 2 \) matrix transformation \( N(\alpha) \) \(^9 \)

\[
N(\alpha) = k(\alpha) \bar{N}^Y
\] (95)

where

\[
\bar{N}^Y = \begin{pmatrix}
    \sqrt{\gamma} & \alpha \\
    0 & \sqrt{\gamma}
\end{pmatrix}
\] (96)
and where \( R(\omega) \) is the rotation operator of an angle \( \omega \) counted positive clockwise (as \( \mu, \phi, \) and \( \chi \)) so that

\[
R(\omega) = \begin{pmatrix}
\cos \omega & \sin \omega \\
-\sin \omega & \cos \omega
\end{pmatrix}.
\]  

(97)

The parametric representation of the ellipse \( E \) according to Eqs. (92) and (93) gives a fast demonstration of that, because after transformation it becomes

\[
\begin{align*}
Y &= \epsilon \frac{1}{2} \sin (\chi + \delta + \omega) \\
Y' &= \epsilon \frac{1}{2} \cos (\chi + \delta + \omega).
\end{align*}
\]  

(98)  

(99)

It can be seen from these formulae that the angle \( \omega \) associated with a \( \gamma \)-type transformation is of \( \chi \)-type. If \( \omega = 0 \), then \( R(\omega) \) reduces to \( N^\gamma \) which we call the normalization matrix of the \( \gamma \)-type.

6.2.2 The \( \beta \)-type normalization

The same ellipse \( E \) as in Eq. (94) above can also be transformed in the same circle by a transformation \( N(\xi) \) defined by

\[
N(\xi) = R(\xi)N^\beta
\]  

(100)

where

\[
N^\beta = \begin{pmatrix}
\frac{a}{\sqrt{\beta}} & \sqrt{\beta} \\
-\frac{1}{\sqrt{\beta}} & 0
\end{pmatrix}
\]  

(101)

and is called the \( \beta \)-type normalization matrix.

\( R(\xi) \) is the rotation operator as described in Eq. (97). The parametric representation of the ellipse \( E \) becomes in this new phase space

\[
\begin{align*}
Y &= -\epsilon \frac{1}{2} \sin (\phi + \delta + \xi) \\
Y' &= -\epsilon \frac{1}{2} \cos (\phi + \delta + \xi).
\end{align*}
\]  

(102)  

(103)

We see that the \( \xi \)-angle associated with a \( \beta \)-type transformation is of \( \phi \)-type. Furthermore, starting from the same point of an ellipse

\[
\begin{pmatrix}
\beta \\
E \\
\alpha \\
Y
\end{pmatrix}
\]

one reaches the same point of the normalized circle either by \( N^\gamma \) or by \( N^\beta \) if we have the relation

\[
\phi - \chi = \Theta - \xi + (2k - 1)\pi.
\]  

(104)
6.2.3 Matrix through one period

If we apply the above results to the invariant ellipse, we go into a normalized phase space where particles seem to advance by an angle \( \mu \) at each period, \( \mu \) being the increase of \( \phi(s) \) as well as the one of \( \chi(s) \). We now understand the meaning of \( \phi(s) - \chi(s) \) as a difference of rotations in different types of normalized phase spaces. The transformation matrix through a period is

\[
M_P(s + s + L) = N^{-1} R(\mu)N
\]

(105)

where \( N \) is either Eq. (96) or Eq. (101). Equation (105) gives back Eq. (12).

7. ELLIPSE TRANSFORMATION

In order to find by geometrical considerations the transfer matrices (24) and (26) [which are also given by Eq. (66), cf. remark (c) in Section 5], we shall make explicit the matrices \( N(E_1 \to E_2) \) which transform an ellipse

\[
E_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
\]

into an ellipse

\[
E_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
\]

of same area and same centre [the equations of \( E_1 \) being still Eq. (34) with the appropriate indices].

To do so we first transform \( E_1 \) into a circle by means of a normalization, then we rotate on the circle, which is then transformed back into \( E_2 \) by an inverse normalization.

7.1 The \( \gamma \)-type transformation

Using the normalization matrices

\[
N_1 = \begin{pmatrix} \sqrt{y_1} & 0 \\ \frac{1}{\sqrt{y_1}} & 0 \\ 0 & 1 \end{pmatrix}
\]

(106)

the transformation we look for is

\[
M(E_1 \to E_2) = (N_2^T)^{-1} R(\Delta \chi) N_1
\]

(107)
For any rotation \( R(\Delta \chi) \) on the common circle [for the rotation operator \( R \), see the definition (97)]. It is easy to verify that Eq. (107) is strictly the matrix of Eq. (26) using the same notations. There is an infinity of such matrices transforming \( E_1 \) into \( E_2 \), but only one, with \( \Delta \chi = \chi_2 - \chi_1 \), corresponds to the "mechanical" transformation defined by the Hill equation. This one is the transfer matrix from \( s_1 \) to \( s_2 \).

7.2 The \( \beta \)-type transformation

Using the normalization matrices

\[
N^\beta_1 = \begin{pmatrix} a_1 & \sqrt{b_1} \\ \frac{1}{\sqrt{b_1}} & 0 \end{pmatrix}
\]

(108)

the transformation is

\[
M(E_1 \to E_2) = (N^\beta_2)^{-1} R(\Delta \varphi) N^\beta_1
\]

(109)

for any rotation \( R(\Delta \varphi) \) on the common circle as defined by Eq. (97). We can again verify that Eq. (109) is strictly Eq. (24) with the same notations. There is also an infinity of such matrices transforming \( E_1 \) into \( E_2 \), but only one, with \( \Delta \varphi = \varphi_2 - \varphi_1 \), corresponds to the real "mechanical" transformation. This one is the transfer matrix from \( s_1 \) to \( s_2 \).

Our purpose was to show that any transfer matrix from \( s_1 \) to \( s_2 \) is a sort of geometrical transformation of the invariant ellipse, and that the angles \( \psi(s) \) and \( \chi(s) \) are some sort of rotation in some sort of normalized phase space.

We should like to thank Dr. H. Hereward for his encouraging support.
1. A POSSIBLE GENERALIZATION OF THE CS METHOD

The perturbed transfer matrix through one period is generally defined as follows:

\[
\begin{pmatrix}
    y(s + L) \\
    y'(s + L)
\end{pmatrix} = \begin{pmatrix}
    m_{11}^P & m_{12}^P & m_{13}^P \\
    m_{21}^P & m_{22}^P & m_{23}^P \\
    0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    y(s) \\
    y'(s)
\end{pmatrix},
\]

where \( m_{ij}^P \) are the usual elements of the \( 2 \times 2 \) transfer matrix through one period as given by Eq. (12) if \( i, j = 1, 2 \), and are additional perturbation terms given for one period if \( j = 3 \). If the perturbation is caused by a change \( \Delta p \) in the momentum \( p \), the elements \( m_{ij}^P \) and \( m_{i3}^P \) should already contain the term \( \Delta p^2 / p \). A period in this case is defined by the periodicity of the perturbations; if they are not evenly distributed, one has one period per turn.

It is easy to verify that such a relation (A.1) may always be written for an elliptic periodic system (that is to say, a system where \( \mu \) is real) under the following form which generalises Eq. (12):

\[
\begin{pmatrix}
    y(s + L) - e(s) \\
    y'(s + L) - e'(s)
\end{pmatrix} = \begin{pmatrix}
    \cos \mu + a(s) \sin \mu & \beta(s) \sin \mu \\
    -\gamma(s) \sin \mu & \cos \mu + a(s) \sin \mu
\end{pmatrix} \begin{pmatrix}
    y(s) - e(s) \\
    y'(s) - e'(s)
\end{pmatrix}.
\]

Such an identification defines not only \( \beta(s) \), \( a(s) \), and \( \gamma(s) \) as we have done in Eqs. (12) to (19), but also two new functions \( e(s) \) and \( e'(s) \) given by

\[
\begin{pmatrix}
    e(s) \\
    e'(s)
\end{pmatrix} = \frac{1}{2(1 - \cos \mu)} \begin{pmatrix}
    1 - m_{22}^P & m_{12}^P & m_{13}^P \\
    m_{21}^P & m_{22}^P & m_{23}^P \\
    m_{31}^P & m_{32}^P & m_{33}^P
\end{pmatrix}.
\]

The relation (A.2) means that the \( 2 \times 2 \) transfer matrix Eq. (12) transforms the phase-space coordinates counted from a new axis, and that this new axis is a closed curve after an integer number of periods. So \( e(s) \), \( e'(s) \), defined for any value of \( \Delta p / p \), are the corresponding "perturbed closed orbit" [our notation is a generalization of a current use at CERN\(^{13,14}\)].

Now the values of \( e(s) \), \( e'(s) \) all around the machine may be computed numerically by applying the "closing condition" (A.3) point by point, i.e. for each value of \( s \).
2. THE SOVIET METHOD DESCRIBED IN REF. 5

In Ref. 5 the perturbed closed orbit is defined by the functions \( \psi(s) = \Delta \Phi(s)/(\Delta p/p) \) and \( \psi'(s) = d\psi/ds \). But we shall keep the notations of Eq. (A.2), considering \( e(s) \) as a solution of the non-homogeneous equation\(^4,15\):

\[
\frac{d^2 \psi}{ds^2} + K(s) \psi = \frac{1}{\rho(s)} \frac{\Delta p}{p}. \tag{A.4}
\]

So \( e(s) \) can be identified as \( \psi(s)/(\Delta p/p) \). This solution, as for any non-homogeneous linear differential equation, is the sum of the general solution of the homogeneous equation that we know already from Eq. (6), plus a special solution of the non-homogeneous equation:

\[
\begin{pmatrix} \psi(s) \\ \psi'(s) \end{pmatrix} = M(s_0 \to s) \begin{pmatrix} \psi(s_0) \\ \psi'(s_0) \end{pmatrix} + \begin{pmatrix} q(s) \\ q'(s) \end{pmatrix}. \tag{A.5}
\]

The numerical procedure will be as follows:

a) The perturbed orbit is first defined at the origin \( s_0 \) by the "closing condition" (A.3), \( e_{ij}^F \) being the elements of the transfer matrix for the period starting at \( s_0 \). This gives \( e(s_0), e'(s_0) \), and consequently the corresponding solution of the homogeneous equation:

\[
\begin{pmatrix} e_{H}(s_1) \\ e_{H}'(s_1) \end{pmatrix} = M(s_0 \to s_1) \begin{pmatrix} e(s_0) \\ e'(s_0) \end{pmatrix}. \tag{A.6}
\]

b) The period is now divided into \( n \) sections for which we know their individual \( 2 \times 2 \) transfer matrix \( M(s_{i-1} \to s_i) \) and the two individual dispersive terms (that is to say, the elements (1,3) and (2,3) of the individual \( 3 \times 3 \) matrix)

\[
\begin{pmatrix} q(s_{i-1} \to s_i) \\ q'(s_{i-1} \to s_i) \end{pmatrix} \quad \text{[we use the notation of Ref. 5]}. \tag{A.7}
\]

Then one applies the following recurrence formulae (easy to establish)

\[
\begin{pmatrix} Q(s_0 \to s_1) \\ Q'(s_0 \to s_1) \end{pmatrix} = M(s_{i-1} \to s_i) \begin{pmatrix} Q(s_0 \to s_{i-1}) \\ Q'(s_0 \to s_{i-1}) \end{pmatrix} + \begin{pmatrix} q(s_{i-1} \to s_i) \\ q'(s_{i-1} \to s_i) \end{pmatrix} \tag{A.7}
\]

for \( i = 1, \ldots, n \), starting with \( Q(s_0 \to s_0) = Q'(s_0 \to s_0) = 0 \).

c) Finally the perturbed closed orbit is obtained for any value of \( s_i \) by
\[
\begin{pmatrix}
    e(s_1) \\
    \varphi'(s_1)
\end{pmatrix} = M(s_0 \rightarrow s_1) \begin{pmatrix}
    e(s_0) \\
    \varphi'(s_0)
\end{pmatrix} + \begin{pmatrix}
    \eta(s_0 \rightarrow s_1) \\
    \eta'(s_0 \rightarrow s_1)
\end{pmatrix}.
\]

(A.5)

Numerical values are given in Ref. 5 for 60 points of the Serpukhov synchrotron, in the form of a table indicating \( \psi(s) = e(s)/(\dot{\omega}p/p) \), \( \psi'(s) \), \( q \) and \( q' \), these two last figures being given in fact as \( q/(\dot{\omega}p/p) \) and \( q'/(\dot{\omega}p/p) \).
REFERENCES


5) E.K. Tarasov, V.V. Vladimirskij and D.G. Kosharev, "Forecast characteristics of the 60-70 GeV proton synchrotron" (in Russian), ITEF/232-Moscow.


14) C. Bovet, "Variation des paramètres de l'éjection lente", CERN MPS/DL Int. 65-6 (1965).
