BOUNDARY CONDITIONS FOR THE SCALAR FIELD 
IN THE PRESENCE OF SIGNATURE CHANGE

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ABSTRACT

We show that, contrary to recent criticism, our previous work yields a reasonable class of solutions for the massless scalar field in the presence of signature change.

1. INTRODUCTION

In a recent letter [1], Hayward purports to show that our earlier work on signature change, especially [2] but also [3], is “mathematically inconsistent”, “entails a non-uniqueness which destroys predictability”, and does not “make sense of the relevant field equations”. He has made similar criticisms elsewhere [4]. Contrary to Hayward’s claims, our approach is completely consistent and makes sense of our field equation.

In many situations in physics one deals with equations that admit singular solutions. Sometimes one can simply excise those domains of the manifold where the singularity occurs and replace the effect of the singularity by a suitable set of parameters. But sometimes the components of tensor equations themselves are singular. In this case a strategy must be adopted to formulate the problem before attempting a solution. This

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is precisely the situation under discussion here, where a scalar field equation is sought on a manifold with a degenerate metric which changes signature at a hypersurface. Our formulation of the problem is different from Hayward’s, leading to different equations with different solution spaces. In the absence of experimental guidance, both approaches are viable.

Hayward’s approach [1, 5] is to give a global definition of the scalar field equation on such a manifold and to demand that, since the field equation is second order, its solutions be globally $C^2$. He concludes that the field momentum must vanish at the surface of signature change. Our approach [3] is to demand instead that the scalar field equation be defined in a piecewise fashion on the manifold and to admit piecewise smooth solutions. We then choose conditions that match the piecewise solutions across the degeneracy hypersurface. Our conditions are simply that the field and the unit normal derivative of the field (or equivalently the canonical momentum) exist and be continuous. These conditions can be derived by promoting the piecewise formulation of the field equation into a single global distributional equation [3]. Hayward [1] dismisses our approach claiming that it leads to non-unique solutions and hence “destroys predictability”. This claim is false. We show below how solutions may be completely determined up to normalization. Our procedure does however select a class of solutions that are not necessarily globally $C^2$.

We note with interest that recent work [6, 7] in the context of Kleinian signature change also argues in favor of a continuity condition on the unit normal derivative of the field, rather than requiring that it vanish.

We begin this rebuttal by giving the simple example, first introduced in [2], which Hayward uses in his attempt [1] to show that our solutions are non-unique. We demonstrate that our approach does indeed determine the arbitrary constants.

In Section 3, we examine several different generalizations of the standard action for the massless scalar field to a signature changing background. For each such generalization, a variational principle requires that the appropriate canonical momentum be continuous (and in one case, zero) at the boundary. In this way, we see how both our boundary conditions and Hayward’s can be derived in a parallel manner, but from different actions. From this point of view, it is not surprising that the theories obtained from these actions have different spaces of classical solutions. We also discuss the implications of each of the resulting theories.

Contrary to Hayward’s claims, our approach can be derived from an explicit field equation using standard techniques. We discuss our field equation in Section 4. The fact that our distributional field equation is not the one that Hayward uses seems to have been lost in his claims that we do not make sense of “the” field equations. To discuss a scalar field equation on a manifold with a singular metric, one must first formulate the problem in an unambiguous manner. It is simply incorrect to refer to “the” scalar field equation on such a manifold. In the Appendix, we further show that our approach yields the standard junction conditions for discontinuous Maxwell fields.

An important aspect of quantum cosmology concerns semiclassical approximations to path integrals, which exploit particular classical solutions — hence the desire of some to look at “real-tunneling” solutions. Our theory here is purely classical, and we wish to consider all classical solutions. We discuss these differing motivations in Section 5.
2. EXAMPLE

The differences in the two approaches can be best seen in terms of an example, first introduced in [2] and also discussed in [1]. Consider the singular differential equation

\[ 2t \frac{d\Phi}{dt} = \Phi \]  

(1)

This equation can be viewed as the massless scalar wave equation for the 1-dimensional signature-changing metric \( t \, dt^2 \).

The general solution to (1) is

\[ \Phi = \begin{cases} 
A(-t)^{3/2} + B & (t < 0) \\
Ct^{3/2} + D & (t > 0)
\end{cases} \]

(2)

where \( A, B, C, D \) are constants. The requirement that \( \Phi \) be continuous at \( t = 0 \) fixes \( D = B \).

Our approach [3] is to demand in addition that the (unit) normal derivative of \( \Phi \) at \( t = 0 \), defined in terms of 1-sided limits, be continuous, namely

\[ \lim_{t \to 0^+} \frac{\Phi}{\sqrt{t}} = \lim_{t \to 0^-} \frac{\Phi}{\sqrt{-t}} \]

(3)

This fixes \( C = -A \) in (2), and the solution becomes

\[ \Phi = \begin{cases} 
A(-t)^{3/2} + B & (t < 0) \\
-At^{3/2} + B & (t > 0)
\end{cases} \]

(4)

The integration constant \( B \) can be determined, e.g. by the choice of \( \Phi(0) \), and \( A \) can be fixed by appropriate normalization. If however, following Hayward [1], one instead demands that \( \Phi \) be globally \( C^2 \), then the only solution is \( \Phi = B \). Each choice leads to its own particular class of solutions.

Alternatively, in [1], Hayward examines the solutions (2) without imposing any further conditions such as (3). It is these solutions which Hayward claims exhibit non-uniqueness. We have never suggested using solutions of this form.

The controversy, therefore, apparently comes down to whether or not it is reasonable to allow solutions to a second order differential equation which are not globally \( C^2 \). This is commonly done in elementary physics across regions which contain physical discontinuities such as the electrostatic field across a hollow charged conductor; see the Appendix.

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2 As discussed in [3], an alternative boundary condition, corresponding to the freedom in choosing the relative orientation of the normal derivatives, is to insert a minus sign on one side of (3), which would lead to \( C = A \) in (2). Both of these (classes of) solutions were given in [2].
3. VARIATIONAL APPROACH

The standard action for a real massless scalar field on an $n$-dimensional (non-degenerate) Lorentzian background is:

$$S_L = \frac{1}{2} \int g^\mu\nu \Phi_{,\mu} \Phi_{,\nu} \sqrt{-g} \, d^{n-1}x \, dt$$  \hspace{1cm} (5)

There are several possible generalizations of this action which might be used to accommodate a background spacetime which changes signature from Lorentzian to Euclidean at a hypersurface $\Sigma = \{ t = \text{constant} \}$. It is illuminating to examine the consequences of varying these actions.

One possibility is to extend (5) to the Euclidean region by putting an absolute value in the square root. Such an action has the advantage that it naturally remains real even in the Euclidean region:

$$S_1 = \frac{1}{2} \int_{\text{Lorentzian}} g^\mu\nu \Phi_{,\mu} \Phi_{,\nu} \sqrt{|g|} \, d^{n-1}x \, dt + \frac{1}{2} \int_{\text{Euclidean}} g^\mu\nu \Phi_{,\mu} \Phi_{,\nu} \sqrt{|g|} \, d^{n-1}x \, dt$$  \hspace{1cm} (6)

We vary $S_1$ and demand that it be stationary for variations of $\Phi$ that do not necessarily vanish on $\Sigma$. The variation is

$$\delta S_1 = - \int_{\text{Lorentzian}} \left( g^\mu\nu \Phi_{,\mu} \sqrt{|g|} \right)_{,\nu} \delta \Phi \, d^{n-1}x \, dt + \int_{\Sigma} \left( g^{0\nu} \Phi_{,\nu} \sqrt{|g|} \right) \delta \Phi \, d^{n-1}x$$

$$- \int_{\text{Euclidean}} \left( g^\mu\nu \Phi_{,\mu} \sqrt{|g|} \right)_{,\nu} \delta \Phi \, d^{n-1}x \, dt - \int_{\Sigma} \left( g^{0\nu} \Phi_{,\nu} \sqrt{|g|} \right) \delta \Phi \, d^{n-1}x$$  \hspace{1cm} (7)

and we immediately obtain the scalar field equations in the Lorentzian and Euclidean regions separately from variations whose support does not intersect $\Sigma$. If we now seek solutions such that $\Phi$ is continuous, it is natural to assume that $\delta \Phi$ is continuous at $\Sigma$, and the remaining variations yield a boundary condition on the canonical momentum $^3$

$$\Pi_1 := \frac{\delta S_1}{\delta \Phi^0} = \sqrt{|g|} g^{0\nu} \Phi_{,\nu}$$  \hspace{1cm} (8)

namely that $\Pi_1$ be continuous at $\Sigma$. This is precisely the boundary condition which we have proposed $[2,3]$; it is equivalent to continuity of the normal derivative, as in (3). Note that this boundary condition is required by a variational principle. Solutions of the equations of motion for which $\Phi$ and $\Pi_1$ are continuous do not suffer from any "non-uniqueness" which

$^3$ For all of the actions in this section, we consider only fields for which the canonical momenta have bounded limits to $\Sigma$; this implies that the integrands of the various actions are well-behaved near $\Sigma$. The variation of each action then yields the usual scalar field equations in the Lorentzian and Euclidean regions separately; it is the boundary conditions which differ in each case.
“destroys predictability”. Quite the contrary, any real Euclidean solution is matched to precisely one Lorentzian solution by this prescription.

We showed in [3] that the Klein-Gordon product of such solutions is conserved even across the hypersurface of signature change. This is a necessary condition for a corresponding quantum theory to be unitary and is thus an attractive feature of this approach.

The action $S_1$ can be extended to the complex scalar field in the usual way, yielding the action

$$S_2 = \int_{\text{Lorentzian}} g^{\mu \nu} \Phi_{,\mu} \Phi^{*, \nu} \sqrt{|g|} d^{n-1}x \, dt + \int_{\text{Euclidean}} g^{\mu \nu} \Phi_{,\mu} \Phi^{*, \nu} \sqrt{|g|} d^{n-1}x \, dt \quad (9)$$

Because $S_2$ is real, it is essentially two copies of the action for the real scalar field, as usual. The boundary condition obtained from varying $S_2$ is just the continuity of the (now complex) canonical momentum (8). Such an action is a natural candidate for extension, via minimal coupling to an electromagnetic field in a $U(1)$ invariant way.

A third possibility involves extending (5) to the Euclidean region via analytic continuation of $\sqrt{-g}$ in the variable $t$, yielding

$$S_3 = \int_{\text{Lorentzian}} g^{\mu \nu} \Phi_{,\mu} \Phi^{*, \nu} \sqrt{-g} d^{n-1}x \, dt + \int_{\text{Euclidean}} g^{\mu \nu} \Phi_{,\mu} \Phi^{*, \nu} \sqrt{-g} d^{n-1}x \, dt \quad (10)$$

where we have implicitly chosen a branch. We allow $\Phi$ to take complex values, even in the Lorentzian region, and discuss later the consequences of restricting to real $\Phi$.

Varying the real part of $S_3$ yields the boundary condition that the appropriate canonical momentum, given by

$$\Pi_3 := \frac{\delta S_3}{\delta \Phi_{,0}} = \sqrt{-g} g^{0\nu} \Phi_{,\nu}$$

should be continuous at $\Sigma$. Notice that this canonical momentum differs from the previous one (8) by a factor of $i$ in the Euclidean region. The imaginary part of $S_3$ is redundant, yielding the same field equations and the same boundary condition, (11), so that the real part of $S_3$ can be used alone as a real action for this theory.

Unlike $S_1$ and $S_2$, the Klein-Gordon products of some solutions of the equations of motion matched by (11) are not conserved across the hypersurface of signature change, so that it would be difficult to build a unitary quantum theory from such an action. Also, it is not obvious how to couple the scalar field in (10) to an electromagnetic field in a $U(1)$ invariant way, even in the Lorentzian region.

If one makes the added restriction that solutions $\Phi$ must be real in both the Lorentzian and Euclidean regions, it is impossible to satisfy continuity of $\Pi_3$ unless the canonical momentum is zero at the boundary. This is the junction condition which Hayward prefers. Solutions of this type do have conserved Klein-Gordon product, but the class of allowable solutions is restricted by this extra reality condition. We discuss in Section 5 why such particular solutions are of relevance in the context of a Euclidean approach to the quantum theory.
A fourth possibility involves starting with the conventional action for the complex massless scalar field in the Lorentzian region and using the analytic extension of $\sqrt{-g}$ into the Euclidean region. Then the action becomes

$$S_4 = \int_{\text{Lorentzian}} g^\mu\nu \Phi_{,\mu} \Phi^*_{,\nu} \sqrt{-g} \, d^{n-1}x \, dt + \int_{\text{Euclidean}} g^\mu\nu \Phi_{,\mu} \Phi^*_{,\nu} \sqrt{-g} \, d^{n-1}x \, dt \quad (12)$$

This results in a non-analytic integrand and a complex action. By varying $S_4$ with respect to the real and imaginary parts of $\Phi$ we obtain the equations of motion together with the following boundary conditions:

$$\lim_{t \to \Sigma^-} \sqrt{-g} \, g^0\nu \Phi_{,\nu} = \lim_{t \to \Sigma^+} \sqrt{-g} \, g^0\nu \Phi_{,\nu}$$
$$\lim_{t \to \Sigma^-} \sqrt{-g} \, g^0\nu \Phi^*_{,\nu} = \lim_{t \to \Sigma^+} \sqrt{-g} \, g^0\nu \Phi^*_{,\nu} \quad (13)$$

This action is complex, so that the boundary condition on $\Phi^*$ is not the conjugate of the condition on $\Phi$, yielding an additional constraint on the allowed solutions. We find this property of complex actions like $S_4$ rather disturbing, as it restricts the allowed solutions away from the boundary in a way which the other actions considered here do not. Solving both equations in (13) simultaneously requires both sides of each equation to be zero; this is again the junction condition which Hayward prefers. However, as the resulting solution space turns out to be a proper subset of that determined by $S_1$, it is clear that Klein-Gordon products are preserved. Furthermore, and despite the action being complex, this theory is a candidate for extension, via minimal coupling to an electromagnetic field in a $U(1)$ invariant way.

Of the two real actions considered here, $S_1$ and the real part of $S_3$, only the boundary condition obtained from a variational principle for $S_1$ matches real solutions to real solutions. Furthermore, only $S_1$ conserves Klein-Gordon products of solutions across the signature change unless the solution space for $S_3$ is additionally restricted by a reality condition. Of the actions for the complex field considered here, both $S_2$ and $S_4$ conserve Klein-Gordon products of solutions and both appear to provide candidates for minimal coupling to an electromagnetic field. However, in the case of $S_4$ our procedure yields two boundary conditions, instead of the expected one, since the boundary condition on $\Phi^*$ is not the complex conjugate of the boundary condition on $\Phi$.

The set of actions given here is by no means exhaustive. There are clearly many other possibilities, such as other relative factors between the Lorentzian and Euclidean actions, the addition of surface terms on $\Sigma$, leading to discontinuous canonical momenta. Nevertheless, we feel that this language is a good one for illustrating how a variety of boundary conditions can be obtained. It also provides a framework for discussing some of the considerations which might arise when one attempts to find a physical interpretation of the extension of the theory of the massless scalar field to a signature changing background. A recent paper by Embacher [8] considers similar generalizations of the Einstein-Hilbert action to accommodate signature change.
4. DISTRIBUTIONAL WAVE EQUATION

An alternate approach is to use the language of tensor distributions. We summarize here our approach to the massless scalar field in this language, as presented in detail in [3], and give a 1-dimensional example.

Suppose one is given two regions $U^\pm$ of a manifold $M$ sharing a common boundary $\Sigma$ given by $\{\chi = 0\}$. Suppose further that the field equations $dF^\pm = 0$ hold separately on the 2 regions, where $F^\pm$ are differential forms on $U^\pm$. Introduce the distributional field

$$ F = \Theta^+ F^+ + \Theta^- F^- $$

in terms of the Heaviside distributions $\Theta^\pm$ with support in $U^\pm$ and such that

$$ d\Theta^\pm = \pm \delta $$

where $\delta = \delta(\chi) d\chi$ in terms of the Dirac delta “function” $\delta(\chi)$. We shall call a differential form $F$ on $U^+ \cup U^-$ regularly discontinuous if the restrictions $F^\pm = F|_{\Sigma^\pm}$ are smooth and the (1-sided) limits $F^\pm|_{\Sigma} = \lim_{t \to \Sigma^\pm} F$ exist. It follows that

$$ dF = \Theta^+ dF^+ + \Theta^- dF^- + \delta \wedge [F] $$

where $[F] := F^+|_{\Sigma} - F^-|_{\Sigma}$ is the discontinuity in $F$. If we now postulate the distributional field equation

$$ dF = 0 $$

then we obtain both the original field equations

$$ dF^\pm = 0 $$

and the boundary condition

$$ \delta \wedge [F] = 0 $$

This formalism is valid irrespective of whether the metric signature changes at $\Sigma$. An example with constant signature is given in the Appendix.

Consider now the manifold $M = \mathbb{R}$ with signature-changing metric

$$ ds^2 = t \, dt^2 $$

Away from $\{t = 0\}$, the Hodge dual operator associated with this metric is given by

$$ *1 = \sqrt{|t|} \, dt $$

$$ *\sqrt{|t|} \, dt = \text{sgn}(t) $$

The massless wave equation for a 0-form $\Phi$ on a region of $M$ where the metric is non-degenerate may be written

$$ dF = 0 $$
where $F = *d\Phi$ in terms of the Hodge map $*$ defined by the metric. Setting $F^\pm = *d\Phi^\pm$ on $U^\pm := \{\pm t > 0\}$, where $\Phi^\pm = \Phi|_{u^\pm}$, we thus require that (22) be satisfied on $U^\pm$, i.e. that $dF^\pm = 0$. We now seek distributional solutions to (17) where $F$ is defined as above. In order for this to make sense, we admit only solutions such that the tensor $F$, defined for $t \neq 0$ by $F|_{u^\pm} = F^\pm = *d\Phi^\pm$, is regularly discontinuous at $\{t = 0\}$ so that $[F] = [*d\Phi]$ is well-defined. 4 Using (21) we see that

$$*d\Phi^\pm = \text{sgn}(t) \frac{\partial \Phi^\pm}{\sqrt{|t|}} \tag{23}$$

If $\partial \Phi^\pm/\sqrt{|t|}$ is bounded as $t \to 0^\pm$ then $*d\Phi$ is regularly discontinuous at $\{t = 0\}$ and

$$[*d\Phi] = \lim_{t \to 0^+} \frac{\partial \Phi^+}{\sqrt{|t|}} + \lim_{t \to 0^-} \frac{\partial \Phi^-}{\sqrt{|t|}} \tag{24}$$

Then (19) implies

$$dt \wedge [*d\Phi] = 0 \tag{25}$$

which for this simple example is equivalent to

$$[*d\Phi] = 0 \tag{26}$$

since $M$ is 1-dimensional.

In summary, the distributional field equation (17) requires satisfaction not only of the wave equations on each side, namely

$$d*\Phi|_{u^\pm} = 0 \tag{27}$$

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4 In [4], Hayward argues that, in the presence of signature change, $*$ must be interpreted as a distribution. He further argues that $d\Phi$ must be $C^\infty$ so that $*$ can act on it. There is no need for such a requirement; it is only necessary that $[*d\Phi]$ be well-defined. Furthermore, if $T^\pm$ are tensors on $U^\pm$ such that the tensor $W$, defined by $W|_{u^\pm} = *T^\pm$, is regularly discontinuous at $\Sigma$, then a natural Hodge dual operator $*$ can be defined via

\[ *((\Theta^+ T^+ + \Theta^- T^-)) = \Theta^* T^+ + \Theta^-* T^- \]

where $*$ refers to the Hodge dual operators on $U^\pm$ as appropriate. If we now impose the natural condition that $\Phi$ be continuous, then

$$F = \Theta^+ *d\Phi^+ + \Theta^- *d\Phi^- \equiv \hat{\Phi}$$

where $\Phi = \Theta^+ \Phi^+ + \Theta^- \Phi^-$, so that in this case (17) takes the usual form of the wave equation.
but also of the boundary conditions (26). Introducing new “time” parameters \( \tau \) for \( t < 0 \) and \( \sigma \) for \( t > 0 \) by

\[
\begin{align*}
\tau &= \int_0^t \sqrt{-t} \, dt \\
\sigma &= \int_0^t \sqrt{t} \, dt
\end{align*}
\]

(28)

allows us to rewrite (26) as

\[
\Phi_{\tau, \sigma} |_x = -\Phi_{\tau} |_x
\]

(29)

(Changing the relative orientations of \( U^\pm \) amounts to inserting a factor of \( \text{sgn}(t) \) into both of equations (21), resulting in (29) without the minus sign; this gives (3) for the 1-dimensional example discussed previously.)

5. GENESIS OF EUCLIDEAN METHODS IN QUANTUM THEORY

It is of interest to recall the genesis of the use of Euclidean methods in quantum theory. A convenient way to determine the spectrum \( \{ E_n \} \) of the quantum Hamiltonian \( H \) and its associated energy eigenfunctions \( | n \rangle \) in the position representation \( | x \rangle \) is to use the generating function

\[
\langle x_f | e^{-HT} | x_i \rangle = \sum_n e^{-E_n T} \langle x_f | n \rangle \langle n | x_i \rangle
\]

(30)

where \( T \) is a real parameter. The leading term in this expression for large \( T \) gives the energy and lowest energy wavefunction. The left hand side of this equation may be represented as a path integral over all trajectories \( x(\tau) \) satisfying \( x(0) = x_i \) and \( x(T) = x_f \)

\[
\langle x_f | e^{-HT} | x_i \rangle = \mathcal{A}_0 \int e^{S_T} [dx]
\]

(31)

where in terms of the classical potential \( V \)

\[
S_T = \int_0^T \left( (dx/d\tau)^2 / 2 + V(x(\tau)) \right) \, d\tau
\]

(32)

This representation has the advantage that it can be approximated in the semi-classical limit where the integral is dominated by the stationary paths \( x \) of \( S_T \).

For potentials with a double-well shape the stationary paths \( x \) of \( S_T \) are the instanton solutions of the classical Euclidean equations of motion. By considering the fluctuations about the stationary points one can derive the standard transmission amplitude for a particle to tunnel through a potential barrier. In this case the stationary path \( x \) is known as the “bounce”. For such a solution the modulus of the transmission amplitude through a potential barrier of width \( x_2 - x_1 \) is given approximately by

\[
\exp \left( -S_\infty(x) \right) \equiv \exp \left( -\int_{x_1}^{x_2} \sqrt{2V(x)} \, dx \right)
\]

(33)

in full accord with the WKB approximation. It should be noted that the bounce is the zero “energy” Euclidean solution corresponding to vanishing “kinetic energy” at \( T = \infty \)
and the width of the Euclidean domain is determined by the potential. One might proceed in the semi-classical mode of thought and pretend that the classical particle materialises after barrier penetration from a Euclidean domain and evolves according to Newton’s laws of motion starting with zero momentum. However no-one would demand that quantum mechanics requires that all classical solutions have such a restrictive initial condition.

The above methodology exploits the analyticity of the standard Lagrangian for Newtonian mechanics in the evolution parameter to Wick rotate to imaginary time and dominate the continued action with particular real solutions to the continued equations of motion. With obvious caveats this philosophy has been generalised to field systems that possess the required analyticity.

When one deals with curved space-time metrics the above Wick rotation is not available and the lack of analyticity of the Riemannian volume element requires somewhat ad hoc continuation methods. For problems in quantum cosmology where the trajectories are 3-geometries, the insistence on real tunneling solutions across degenerate geometries that dominate the action gives rise to a behaviour analogous to that of the zero energy bounce solutions in particle mechanics.

The important point to stress is that in all these cases it is quantum amplitudes that are being defined by finite action contributions to the path integral. The properties of particular Euclidean configurations that enter into such a description are not shared by the general classical configurations that can be patched across degeneracy hypersurfaces to Lorentzian signature domains. It is our purpose to consider the so called theory of “classical signature change”, i.e. to treat field theories on a manifold with a degenerate metric tensor field in a coherent manner without recourse to any path integral technology. The quantisation of such configurations is a separate issue and deserves independent attention.

6. CONCLUSION
We have shown that Hayward’s criticisms of our work are untenable. Our field equations make sense and our solutions do not suffer from a lack of “predictability”. The question of what conditions are the most appropriate to describe the physics arising from the propagation of classical and quantum fields in the presence of a background which exhibits signature change can ultimately only be resolved by observation. Until then, the implications of all reasonable, internally consistent approaches should be investigated.

APPENDIX: Discontinuities in electromagnetic fields
The first two of Maxwell’s equations for a stationary electromagnetic field are

\[ dE = 0 = d\ast B \]

where \( \ast \) is now the Hodge dual operator on Euclidean \( \mathbb{R}^3 \). At a boundary \( \{ \chi = 0 \} \) in \( \mathbb{R}^3 \) we write each of the fields \( F = \ast B, E \) as in (14). Postulating the distributional versions (17) of Maxwell’s equations leads to Maxwell’s equations on each side of the boundary together with the boundary conditions

\[ d\chi \wedge [F] = 0 \]
This gives the standard boundary conditions:

\[ d\chi \wedge [E] = 0 \]

\[ d\chi \wedge [*B] = 0 \]

The remaining two equations are

\[ d*D = *\rho \quad dH = *J \]

where \( \rho \) is the electric charge density and \( J \) the electric current. A similar argument including sources shows that if \( J \) and \( \rho \) contain distributional components \( J^\Delta \delta \) and \( \rho^\Delta \delta \), respectively, then they must be related to the discontinuities of \( H \) and \( D \) by the standard boundary conditions:

\[ d\chi \wedge [H] = *J^\Delta \wedge d\chi \]

\[ d\chi \wedge [*D] = *\rho^\Delta \wedge d\chi \]

in terms of surface distributions of charge and current.

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