EFFECT OF RADIO FREQUENCY PROGRAMME NOISE ON THE STACKING PROCEDURES IN STORAGE RINGS [WITH PARTICULAR APPLICATION TO THE CERN ELECTRON STORAGE AND ACCUMULATING RING (CESAR)]

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## CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2. DERIVATION OF THE FOKKER–PLANCK EQUATION (^1) IN PHASE SPACE ((y, \varphi, \tau))</td>
<td>1</td>
</tr>
<tr>
<td>3. EFFECT OF NOISE ON THE DISTRIBUTION OF STACKED PARTICLES</td>
<td>7</td>
</tr>
<tr>
<td>4. AN EXPERIMENT WITH &quot;NOISY BUCKETS&quot;</td>
<td>9</td>
</tr>
<tr>
<td>4.1 The diffusion model</td>
<td>9</td>
</tr>
<tr>
<td>4.2 Definition of (p_0)</td>
<td>10</td>
</tr>
<tr>
<td>4.3 Integration of (n(t))</td>
<td>11</td>
</tr>
<tr>
<td>4.4 Estimation of two-thirds filling time (T(\frac{2}{3}))</td>
<td>11</td>
</tr>
<tr>
<td>5. CONCLUSIONS</td>
<td>12</td>
</tr>
<tr>
<td>APPENDIX 1 Derivation of Equations of Motion for Noisy Buckets</td>
<td>13</td>
</tr>
<tr>
<td>APPENDIX 2 Calculation of (\Delta \varphi^T)</td>
<td>19</td>
</tr>
<tr>
<td>APPENDIX 3 Calculation of (\Delta y^T) and (\Delta y \Delta \varphi)</td>
<td>21</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>23</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

The theoretical description of the phenomena associated with radio-frequency acceleration in a synchronous particle accelerator is often given in terms of the so-called "bucket" theory, a bucket being a region of phase space containing the closed, almost circular, trajectories resulting from the Hamiltonian description of the motion of particles subject to the acceleration. When this motion is also subjected to small random perturbations, it is shown below that, under certain limiting conditions, a process similar to diffusion may occur tending to produce an average flow of particles across and between the individual trajectories of the unperturbed phase space. The limiting conditions are, of course, the normal ones associated with any diffusion process, i.e. the mean free time between applications of the perturbations should be small compared with the over-all times during which the phenomena are studied. The source of the perturbations is the ever-present electronic noise in the frequency programme generator of the acceleration system. Hence the term "noisy" bucket used in the following discussion.

The symbols for the variables used below are those adopted by Vogt-Nielsen, and are in such common usage that they should need no explanation to those people who are likely to read more than this introduction.

In the next section the Fokker-Planck equation obtaining to the relevant phase space is derived. Later sections making use of this equation deal with the effect of noisy buckets on the distribution of particles stacked in a storage ring such as the CERN Electron Storage and Accumulating Ring (CESAR). Some of the anomalous stacking efficiencies observed in CESAR are examined, and finally the results of an experiment with noisy buckets are shown to compare well with the diffusion model.

For the sake of clarity, at least in the narration, most of the tedious details of the derivation of some of the equations are given in the appendices.

2. DERIVATION OF THE FOKKER-PLANCK EQUATION² IN PHASE SPACE \((y,\dot{y},\tau)\)

The equation of motion of the particles may be represented by the Hamiltonian (see Appendix 1)

\[
H(y,\dot{y},\tau) = \Gamma \phi - \cos \phi - \frac{1}{2}y^2 - yB(\tau)
\]  

(1)

where \(B(\tau)\) is a random function of the scaled time \(\tau\), and in fact in the usual nomenclature \(B(\tau) = K\Delta \omega(\tau)\) and \(K\) is a scaling factor; \(\Delta \omega\) is the random error in the frequency of the RF voltage.

The variables \((y,\phi)\) are canonical conjugate so that we can represent a particle distribution in phase space in terms of a Liouvillian density \(P(y,\phi,\tau)\).

If during a time interval \(\Delta \tau\), long enough for the system to experience many random fluctuations of \(B(\tau)\), \(y\) changes by an amount \(\Delta y\) and \(\phi\) by \(\Delta \phi\), we can define a transition probability \(\Psi(y,\phi,\Delta y,\Delta \phi,\tau)\) such that

\[
P(y',\phi',\tau+\Delta \tau) = \iint P(y,\phi,\tau)\Psi(\Delta y)\Psi(\Delta \phi)
\]

(2)
where the integration is taken over all possible values of $\Delta y$ and $\Delta \phi$. It is apparent from the form of this integral equation that we have assumed that the course taken by a particle is dependent only upon the instantaneous values of $\phi$, $y$, and $\tau$, i.e., its course is entirely independent of its previous history. (This type of stochastic process is usually termed a Markoff process, and is assumed to be applicable to the case of a particle subjected to the forces produced by RF acceleration and noise.)

Substitute $y' = \Delta y$ for $y$ and $\phi' = \Delta \phi$ for $\phi$ in the integral and develop in terms of $\Delta \phi$ and $\Delta y$, while at the same time developing the left-hand side of Eq. (1) in terms of $\Delta \tau$. This method of deriving the Fokker-Planck equation is described in detail on page 33 of Ref. 3). Only an outline is given here.

The integrals over $d(\Delta y)d(\Delta \phi)$ may be considered to have the following meanings:

\[
\iint \psi(y', \phi', \Delta y, \Delta \phi, \tau) \, d(\Delta y)d(\Delta \phi) = 1
\]
\[
\int \Delta y \, \psi' \, d(\Delta y)d(\Delta \phi) = \Delta y
\]

where

\[
\psi' = \psi(y', \phi', \Delta y, \Delta \phi, \tau).
\]

Further,

\[
\iint \Delta y^2 \, \psi' \, d(\Delta y)d(\Delta \phi) = \Delta y^2
\]
\[
\iiint \Delta \phi \, \psi' \, d(\Delta y)d(\Delta \phi) = \Delta \phi
\]
\[
\iiint \Delta y \, \Delta \phi \, \psi' \, d(\Delta y)d(\Delta \phi) = \Delta y \Delta \phi
\]

After dividing through by $\Delta \tau$ and taking the limit as $\Delta \tau \to 0$, Eq. (2) leads to the Fokker-Planck equation [see Ref. 3)]

\[
\frac{\partial \mathcal{F}}{\partial \tau} = - \frac{\partial}{\partial y} + f_1 \frac{\partial \mathcal{F}}{\partial y} - f_2 \frac{\partial \mathcal{F}}{\partial \phi} + f_3 \frac{\partial^2 \mathcal{F}}{\partial \phi^2} + 2 \frac{\partial \mathcal{F}}{\partial y} \frac{\partial \mathcal{F}}{\partial \phi} + 2 \frac{\partial^2 \mathcal{F}}{\partial y \partial \phi} + 2 \frac{\partial^2 \mathcal{F}}{\partial \phi^2}
\]

\[+ \frac{\partial}{\partial y} \frac{\partial \mathcal{F}}{\partial \phi} + 2 \frac{\partial \mathcal{F}}{\partial \phi} \frac{\partial^2 \mathcal{F}}{\partial \phi^2} + \frac{\partial^3 \mathcal{F}}{\partial \phi^3} + 2 \frac{\partial \mathcal{F}}{\partial y} \frac{\partial \mathcal{F}}{\partial \phi} + 2 \frac{\partial^2 \mathcal{F}}{\partial \phi^2} + 2 \frac{\partial \mathcal{F}}{\partial y} \frac{\partial \mathcal{F}}{\partial \phi} + 2 \frac{\partial^2 \mathcal{F}}{\partial \phi^2} + \text{higher order terms} \ldots (3)\]
where

\[ f_1 = \lim_{\Delta r \to 0} \frac{\Delta y}{\Delta r} ; \quad g_1 = \lim_{\Delta r \to 0} \frac{\Delta y}{\Delta r} \]

\[ f_2 = \lim_{\Delta r \to 0} \frac{\Delta y}{\Delta r} ; \quad g_2 = \lim_{\Delta r \to 0} \frac{\Delta y}{\Delta r} \]

\[ h_1 = \lim_{\Delta r \to 0} \frac{\Delta y}{\Delta r} . \]

It should now be possible to find \( f, g, \) and \( h \) from the solution of the equations of motion given by the Hamiltonian equation (1). These equations are, from Appendix 1,

\[ \frac{\Delta y}{\Delta r} = -y - B(r) \]

\[ \frac{\Delta y}{\Delta r} = -r - \sin \phi \]

where \( r = -\sin \phi \).

Unfortunately these equations are non-linear so that \( f, g \) and \( h \) cannot be obtained easily without first linearizing by considering small amplitude oscillations about the phase stationary points \( \phi_s \) and \( \phi_i \).

Near \( \phi_s \), where \( \phi = \phi_s + \phi^* \)

\[ \phi^* + \omega_s^2 \phi = A(r), \quad \text{for all } r . \]  \tag{4}

Near \( \phi_i \), where \( \phi = \phi_i + \phi^* \)

\[ \phi^* - \omega_s^2 \phi = A(r), \quad \text{for } r \leq \omega_s^{-1} \]  \tag{5}

where

\[ \omega_s^2 = \sqrt{1 - r^2} \]

and

\[ A(r) = -\frac{\Delta y}{\Delta r} . \]

The solution of Eq. (4) may be written as

\[ \phi^* = \int_0^r B(T) \cos \omega_s (r - T) dT + \phi^* \cos \omega_s (r + \tau_0) \]

as long as displacements due to \( B(r) \) are small compared with the \( \Delta \phi's \) considered in the derivation of the Fokker-Planck equation.
Then
\[ \phi^*(\tau + \Delta \tau) - \phi^*(\tau) = \Delta \psi \]
where, for \( \omega_s \Delta \tau \ll 1, \)
\[
\Delta \psi = \int_\tau^{\tau + \Delta \tau} B(T) \cos \omega_s (\tau - T) \, dT - \phi_0^* \omega_s \Delta \tau \sin \omega_s (\tau + \tau_0) \]
\[= \int_\tau^{\tau + \Delta \tau} B(T) \cos \omega_s (\tau - T) \, dT - \Delta \tau \left[ y - \omega_s \int_0^{\tau + \Delta \tau} B(T) \cos \omega_s (\tau - T) \, dT \, d\tau \right]. \quad (6) \]

In the same manner, an expression for \( \Delta y \) is obtained:
\[
\Delta y = \omega_s^2 \int_\tau^{\tau + \Delta \tau} \varepsilon \int_0^T B(T) \cos \omega_s (\varepsilon - T) \, dT \, d\varepsilon + \phi_0^* \omega_s^2 \Delta \tau \cos \omega_s (\tau + \tau_0) \]
\[= \omega_s^2 \int_\tau^{\tau + \Delta \tau} \varepsilon \int_0^T B(T) \cos \omega_s (\varepsilon - T) \, dT \, d\varepsilon + \omega_s^2 \Delta \tau \left[ \phi^* - \int_0^{\tau + \Delta \tau} B(T) \cos \omega_s (\tau - T) \, dT \right]. \quad (7) \]

The solution of Eq. (5) yields integrals for \( \Delta \psi \) and \( \Delta y \) near \( \phi_1 \) and \( \phi_2 \) which are similar to Eqs. (6) and (7) except that \( \cosh \) should be inserted for \( \cos \) and \( \sinh \) for \( \sin \). In the following, we consider in detail only the effects of noise near \( \phi_2 \).

If averages are now taken over a large number of ensembles of phase spaces, and it is assumed that the random perturbations are such that
\[ \bar{B}(T) = 0, \]
then it follows that
\[ g_1 = \lim_{\Delta \tau \to 0} \frac{\Delta \psi}{\Delta \tau} = -\bar{y} = 0 \]
\[ f_1 = \lim_{\Delta \tau \to 0} \frac{\Delta y}{\Delta \tau} = \omega_s^2 \bar{\psi} = 0. \quad (8) \]

Further, it may be shown that
\[ g_2 = \lim_{\Delta \tau \to 0} \frac{\Delta \psi}{\Delta \tau} \]
\[= \lim_{\Delta \tau \to 0} \int_\tau^{\tau + \Delta \tau} \gamma(T_1, T_2) \cos \omega_s (\tau - T_1) \cos \omega_s (\tau - T_2) \, dT_1 \, dT_2 \quad (9) \]
where \( \gamma(T_1, T_2) \) is the autocorrelation function of \( B(T) \).
If the theory is to be developed any further, some assumptions concerning the type of noise must be made. The simplest assumption consistent with a minimum loss of generality appears to be that of assuming bandwidth-limited stationary noise. In this case $\gamma$ may be expressed in terms of the spectral density function $\phi(a)$ as

$$\gamma(T_1, T_2) = \gamma(T_1 - T_2) = 2 \int_{a_1}^{a_2} \phi(a) e^{i\alpha(T_1 - T_2)} da .$$

The equations may then be integrated in the usual manner to yield some highly unpalatable algebra. If, however, a further assumption is made (careful analysis indicates that not much inaccuracy is introduced) the equations are easily resolved. This assumption is that all the noise is "white", i.e. the spectral density function $\phi(a)$ is assumed constant throughout the bandwidth considered, or

$$\gamma(T_1 - T_2) = 2\pi \phi \delta(T_1 - T_2)$$

In this case, it is easily shown that (see Appendices 2 and 3).

$$g_s = 2\pi \phi \quad \text{[also see Hereward and Johnsen\textsuperscript{3}]}$$

$$f_s = 0$$

and

$$h_s = \lim_{\Delta r \to 0} \frac{\Delta \gamma \Delta \phi}{\Delta r} = 2\pi \omega \phi \sin \omega r .$$

The Fokker–Planck equation then reduces to the diffusion-type equation for particles near the phase stationary point $\phi_s$,

$$\frac{\partial P}{\partial r} = g_s \frac{\partial^2 P}{\partial \phi^2} + 2h_s \frac{\partial^2 P}{\partial \gamma \phi} . \quad (10)$$

The general theory of noise\textsuperscript{6} indicates that for stationary random processes

$$\gamma(T) \leq \gamma(0) = B^\phi = K^\phi \omega^T ,$$

and further for white noise bandwidth limited we may write

$$K^\phi \omega^T = 2\Phi \Delta \alpha$$

where $\Delta \alpha$ is the bandwidth.

In the more usual nomenclature in terms of $\Delta F = \Delta \omega / 2\pi$ and unscaled variables (i.e. eliminating $K$ and using real time $t = Kr$ and $\omega_0 = K\nu_0$),

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial \phi^2} + \frac{2D
\nu_0}{\pi \left| \nu_0 \right|^2} \sin \nu_0 t \frac{\partial^2 P}{\partial \phi \partial W} \quad (11)$$
where $h$, $\Omega_s$, and $W^*$ are defined in Appendix 1. The diffusion coefficient $D$ is then given as

$$D = 2\pi \frac{\Delta f^2}{\Delta f}$$  \hspace{1cm} (12)

where $\Delta f^2$ is the mean square error in the frequency of the voltage applied to the RF cavity and $\Delta f$ is the bandwidth of the system.

It is often more convenient to consider the differential of $F$ with respect to $E$, the energy, rather than $W^*$ -- in this case for a storage ring where the revolution frequency $\Omega$ does not change much over all stacking orbits,

$$\frac{\partial F}{\partial t} = D \frac{\partial^2 F}{\partial \phi^2} + \nu_s H \sin \nu_s t \frac{\partial^2 F}{\partial \phi \partial E}$$  \hspace{1cm} (13)

where $H \approx \text{const} = 2D/h(\partial \Omega_s / \partial E)$; $\Omega_s$ is the revolution frequency of a particle at the phase stable point $\phi_s$.

Equation (13) is valid in the regions of phase space containing particles near the centre of a bucket centred at $\phi = \phi_s$, $E = E_s$ (energy of phase stable particle) taken to be the origin of the phase space $(E, \phi)$.

The solution of Eq. (13) is found to be of the form

$$F(E, \phi, t) = \frac{1}{2\pi D t} \exp \left[ \frac{-(E^2 + \phi^2 + 2\phi E)}{4(Dt - H(\cos \nu_s t - 1))} \right]$$

and at $t >> H/D = 2/h(\partial \Omega_s / \partial E)$, this reduces to

$$F = \frac{1}{2\pi D t} \exp \left[ \frac{-E^2}{4Dt} \right] = F(\theta, t)$$

where $\theta = E + \phi$, i.e. the solution of the one-dimensional diffusion equation,

$$\frac{\partial F}{\partial t} = D \frac{\partial^2 F}{\partial \theta^2}.$$  \hspace{1cm} (14)

The above equations are expected to be valid for particles near the centre of a bucket, i.e. near $\phi_s$, and the expression for $\Delta f^2$ is the same as that obtained by Hereward and Johnsen in an earlier analysis of the effects of noise in proton synchrotrons. If the analysis is repeated for particles near the edges of a bucket, i.e. near $\phi_1$ (by substituting $\cosh$ for $\cos$, etc., as stated above), Eq. (14) may again be obtained but now the diffusion coefficient is best taken to be

$$D = \frac{\cosh 2\pi \phi}{2} = \frac{2}{4} \cosh 2\pi \frac{\Delta f^2}{\Delta f}$$
3. EFFECT OF NOISE ON THE DISTRIBUTION OF STACKED PARTICLES IN CESAR

Stacking efficiencies have been measured with CESAR \(^2\) using both full and partially empty buckets. The full buckets yielded results which compared well with theory, while the partially empty buckets yielded anomalous efficiencies sometimes in excess of 120\%. The stacking efficiency \(\varepsilon_s\) was defined as the ratio of the theoretical energy width of the stack (calculated from a knowledge of the bucket area) to the measured width.

Now if the partially empty bucket had been noisy, then, by a diffusion process, it is possible for the bucket to pick up particles as it moves through the tail of the stack and deposit them at the top when the bucket is switched off.

This effect can then lead to efficiencies, as defined above, which exceed 100\% when the distribution of particles in the stack is such that the density (number per unit energy interval) \(\sigma(E)\) is given by the following expression,

\[
\sigma(E) = \frac{E^2}{E_m^2} \exp \left( -\frac{E^2}{E_m^2} \right).
\]

(E \(_m\) is the value of \(E\) at which \(\sigma\) is a maximum, and the stack width may be considered to be of the order of \(3 E_m\). Such density distributions are, in fact, observed in CESAR.)

A quantitative estimate of the amount of noise needed to account for such efficiencies can be made with the help of the diffusion equations (13) and (14) of the previous section.

The number of particles \(N\) in the bucket centred at energy \(E\) must be equal to the number lost from the stack during the bucket's traversal of the energy interval \(E\) [assuming that the stack starts at \(E = 0\) as implied in Eq. (15)]. Thus if \(\sigma(E)\) denotes the density in the stack after the passage of, on the average, one empty noise bucket, then

\[
\overline{\sigma} = \sigma - \frac{dN}{dE}.
\]

The calculation of \(N\) presents no difficulties, if the solution of the diffusion equation (14) can be found. If one imposes the boundary conditions:

i) diffusion occurs across a separatrix at \(\Theta = 0\);

ii) reflection occurs, due to the finite extent of the bucket such that

\[
\frac{\partial P}{\partial \Theta} = 0 \quad \text{at} \quad \Theta = \Theta_s
\]

iii) the probability density function \(P(\Theta,t)\) is the solution of

\[
\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial \Theta^2}, \quad P \to 1 \text{ as } D \to \infty,
\]

and is given by
\[ P(\Theta, t) = 1 - \frac{1}{\pi^2} \left[ \int_0^{\Theta_1/2} \exp \left( -x^2 \right) \, dx + \int_{\Theta_1/2}^{\infty} \exp \left( -x^2 \right) \, dx \right] \]  

(17)

where \[ \lim_{t \to 0} P = 0, \quad 0 < \Theta < \Theta_1; \quad \lim_{t \to 0} P = 1, \quad \Theta > 0. \]

The number of particles inside the bucket at time \( t \) is then

\[ N(t) = k \int_0^{\Theta_1} \int_0^t \frac{\Theta}{\Theta_1^2} P(\Theta, t-\tau) \, d\tau \, d\Theta \]  

(18)

where \( k \) is a normalization constant and is equal to the area of the bucket divided by \( 2\pi \Theta_1 \), when \( \Theta_1 \) is considered constant over the energy width of the bucket, i.e., bucket area small compared with stack. The energy is simply related to the time because the bucket is considered to be moving through the stack at velocity \( v = d\Theta/dt \) and it is arranged that the stack is encountered at \( \Sigma = 0 \) at \( t = 0 \).

Thus Eq. (18) yields an expression for \( N \), which, when substituted into Eq. (16), gives \( \bar{\Omega} (\Sigma) \) the density distribution after the passage of a noisy bucket. The energy width of the stack, before and after the passage of an empty noisy bucket, can be compared, and the stacking efficiency already defined at the beginning of this section can be calculated as a function of the amount of noise on the bucket, or (more specifically) as a function of the magnitude of the diffusion coefficient \( D \).

In order to apply this analysis to the anomalous results obtained with CESAR, a slightly more sophisticated calculation had to be carried out using Eq. (13) as a basis for the analysis. This was because results with CESAR were obtained under such conditions that \( t = H/D \). The results are given in Fig. 1. The energy width of the stack is typical of CESAR's, as are the energy widths of the bucket.

Examination of Fig. 1 shows that a diffusion coefficient of the order of \( 100 \) (rad)\(^2\) · (sec\(^{-1}\)) can explain the anomalous stacking efficiencies. In terms of the spectral density function \( \Phi \), and depending upon the relationship \( D = m \Phi \) (where \( m \) is \( 2\pi \) for particles near the centre of the bucket and may be as high as 55 for those near the edges), it is apparent that a \( \Phi \) of approximately \( 4 \) rad·sec\(^{-1}\) for \( m = 25 \) could explain the anomalous efficiencies. The reasons behind this choice of \( m \) will emerge from the experimental work described in the next section. The values of \( \Phi \) for various frequency programmes in CESAR have been measured by Brückner and in fact range from \( 3 \) to \( 0.03 \) rad·sec\(^{-1}\) in a noise bandwidth (not at all white) of \( 0 \) to \( 40 \) KHz.

The agreement between experiment and theory is tenuous. However, it must be remembered that many of the assumptions (in particular, that of white noise) made in the theory may not be valid for the particular conditions obtaining to CESAR during the anomalous efficiency measurements. Therefore, one hopes that a less ambiguous check on the theory will be obtained by setting up an experiment specifically to measure the effects of known amounts of noise artificially injected into the RF programme. Such an experiment is described in the next section, and has in fact been performed with CESAR. The results are also given below and allow an empirical estimate of \( m \) to be made. It was found that \( m \) was about 25.
4. An Experiment with "Noisy Buckets"

An experiment which is not subject to much ambiguity is one involving what one might call the "efficiency of depletion" of an existing stack.

More explicitly, the experiment would be conducted as follows.

A large stack is first built up in CESAR, and its average density in the relevant phase space would be measured using the scanning bucket technique\(^2\). The size of the "hole" produced at the centre of such a stack by an empty stationary scanning bucket is the particular parameter which would be measured.

Next, some noise is introduced into the RF programme of the scanning bucket system, and in principle, if particles move into the bucket as a result of the random displacements in phase space, then the "hole" will gradually fill up and the signal observed at the pick-up station will diminish. If, in the time available, observable differences in signal at the pick-up station are obtained, then the experiment is complete.

In order to compare the results of such an experiment with the theory of Section 2, an expression must be found which relates the number of particles one might expect to find inside such a bucket after a given lapse of time, with the diffusion coefficient or the spectral density function of the noise spectrum. The rest of this section discusses such a relation in terms of a model based on what is envisaged to be the effect of a noisy bucket on particles surrounding its separatrix.

4.1 The diffusion model

\[
\begin{align*}
&\text{REFLECTING BARRIER} \\
&\begin{array}{c}
\text{\(p\)} \\
\text{\(t=0\)} \\
\text{\(p_0\)} \\
\text{\(\Theta=0\)} \\
\text{\(\Theta=\Theta_1\)} \\
\text{\(\Theta\)}
\end{array}
\end{align*}
\]

It is assumed that, at \(t = 0\), the situation near the separatrix of a noisy bucket is described by the diagram given above; the phase unstable point is at \(\Theta = 0\), and outside the bucket at \(\Theta < 0\) there is an infinite number of particles uniformly distributed in phase space having an average density \(p_0\) (to be defined later).

Define a distribution function \(P(\Theta, t)\) such that the probable number of particles to be found between \(\Theta\) and \(\Theta + \Delta\Theta\) at time \(t\), is given by \(p_0 P(\Theta, t) d\Theta\).

The conditions to be imposed upon \(P(\Theta, t)\) are that

\[P \geq 0; \frac{\partial P}{\partial \Theta} = 0 \text{ at } \Theta = \Theta_1,\]

i.e., there is a perfectly reflecting barrier at \(\Theta = \Theta_1\). Further

\[p_0 \int_{-\infty}^{\Theta_1} P(\Theta, t) d\Theta = \text{total number of particles in the stack}\]

and this expression may be put equal to infinity without introducing large errors, as long as the bucket area is small compared with the area of the stack.
Further conditions are (as in the previous section)

\[ \lim_{t \to 0} P = 0, \quad 0 < \Theta < \Theta_1 \]

and

\[ \lim_{t \to 0} P = 1, \quad \Theta < 0 \]

and finally \( P \) must satisfy the diffusion equation

\[ \frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial \Theta^2} D; \quad t \gg \frac{H}{D}. \]

A probability function which meets all of the above conditions is already given above in Eq. (17).

Thus if \( \rho_0 \) is properly defined, then the number of particles to be found inside the bucket between \( \Theta = 0 \) and \( \Theta = \Theta_1 \) at time \( t \) is simply

\[ n(t) = \rho_0 \int_{0}^{\Theta_1} P(\Theta, t) d\Theta. \]

\( \rho_0 \) is some average value of a density obtaining outside the bucket at \( t = 0 \) and as such is a constant. It is not absolutely necessary to know its actual value in so far as it concerns an experiment designed to measure the coefficient of diffusion, because, if we have a measure of \( n(t) \) at two different times we may use

\[ \frac{n(t_1)}{n(t_0)} = \frac{\int_{0}^{\Theta_1} P(\Theta, t_1) d\Theta}{\int_{0}^{\Theta_1} P(\Theta, t_0) d\Theta} = f(d). \]

However, to prove that the theoretical expression obtained for \( n(t) \) is correct, we need to know \( \rho_0 \) so that the theoretical and experimental results may be compared directly.

4.2 Definition of \( \rho_0 \)

The average value of \( \rho_0 \) may be defined quite simply by the following expression:

\[ \lim_{t \to \infty} \rho_0 \int_{0}^{\Theta_1} P(\Theta, t) d\Theta = \bar{\rho}. \]

\( B \) is the area inside a bucket contained between the trajectories which pass through \( \Theta = 0 \) and \( \Theta = \Theta_1 \); \( \bar{\rho} \) is the density of particles in the stack defined by

\[ \bar{\rho} = \frac{N}{S}. \]

\( N \) is the total number of particles stacked and \( S \) is the area of the stack measured in the same units of phase space as was the area \( B \).

This definition of \( \rho_0 \) is sufficiently accurate if the bucket area is small compared with the stack. This condition has already been invoked in setting up this diffusion model so that no greater loss of generality is implied in the definition.
Therefore since

\[
\lim_{{t \to \infty}} \int_0^{\Theta_t} P(\Theta, t) d\Theta = \Theta_t
\]

then

\[
\rho_0 = \frac{B}{\Theta_t}; \quad \rho = \frac{BN}{\Theta_t S}.
\]

4.3 Integration of \( n(t) \)

The expression \( n(t) \) may be found by numerical integration if the value of \( \Theta_t \) is known. Such an integration has been carried out and the results are given in Fig. 2.

The RF cavities in CESAR operate on the second harmonic of the revolution frequency so that there are two buckets. If these buckets are stationary, then a reasonable value for \( \Theta_t \) would appear to be \( \pi/2 \), i.e. the half-width of the bucket.

4.4 Estimation of two-thirds filling time \( T(\gamma) \)

The form of the curve \( n(t) \) versus \( t \) is given in Fig. 2. A quantitative comparison between theory and experiment may be obtained by estimating the two-thirds filling time for different values of \( D \). It may be shown from Fig. 2 that for \( \Theta_t = \pi/2 \) and

\[
\frac{4Dt}{\Theta_t^2} = 14
\]

then

\[
\frac{n(t)}{n(\infty)} = \frac{2}{3}.
\]

The area \( B \) obtaining to this value of \( \Theta_t \) is taken to be full at \( t = \infty \); thus at

\[
t = \frac{14 \Theta_t^2}{4D}
\]

the bucket is two-thirds full.

Thus we may call this value of \( t \) the two-thirds filling time \( T(\gamma) \). In an experiment it would be a simple matter to measure \( T(\gamma) \) since it is equal to that time which elapses between the appearance of the maximum signal on the pick-up electrode (produced by the initially empty stationary "noisy" bucket in the full stack) and a signal of one-third height. Therefore inserting \( \pi/2 \) for \( \Theta_t \) yields

\[
T(\gamma) = 8.6/D \text{ seconds}.
\]

Figure 3 shows the results of exactly this experiment with CESAR in which \( T(\gamma) \) was measured as a function of \( \Psi \). Since \( D = m \Phi \), these results when plotted as \( T(\gamma) \) versus \( \Phi^{-1} \) yield a linear curve, the slope of which is equal to \( 8.6/m \). Examination of the slope indicates that \( m = 24 \pm 5 \). The theory of Section 2 indicated that \( 2\pi < m < 55 \), giving very good agreement with experiment when it is remembered that it is a linear theory attempting to describe in effect a non-linear problem. Some more comments concerning this difficulty are given below in the conclusions.
5. CONCLUSIONS

The theory of Section 2 was linearized at an early stage, and this means that each particle, no matter where it is in the phase space (it will be performing elliptical trajectories but no separatrix is built into the linear theory), will be subject to the same random displacement. This means that if we are to apply the linear theory to the case of a single noisy bucket, it will tell us only the probability that the centre of density of the particles may be found on a particular trajectory, e.g.

![Diagram showing two sets of trajectories at t=0 and t>0](image)

The linear theory only tells us that the mean square of the radius R after time t is proportional to t. The diffusion coefficient D defined above refers to the growth of this radius.

This is not to say that for one isolated noisy bucket in a stack, a similar sort of diffusion coefficient is not applicable. In practice, we have a particularly non-linear problem especially in the regions of the separatrix. In fact, one can see quite easily that near the separatrix a random perturbation will have different effects on different particles. The effect will depend upon the particle's particular trajectory as shown below

![Diagram showing trajectories and jumps](image)

Bucket jumps to A' and stops there until particle has moved to B'.

Bucket jumps back to A when particle reaches B' and is now "captured" by bucket so that number of particles inside is altered.

This type of "diffusion" (because if such jumps are randomly distributed then one can expect from physical reasons that the Fokker-Planck equation in its diffusion form will apply), depends upon the fact that not all particles feel the same about noise effects. Even without a separatrix this last statement is also true, because the phase oscillation frequency of a particle inside the bucket is a function of the area enclosed by a particular trajectory. This, of course, is the reason why the diffusion coefficients quoted above differ at the centre and edges of a bucket.

It is apparent from the experimental results outlined in the previous section that the problem of one noisy bucket isolated in phase space yields to the diffusion model approach. Further, theoretical predictions based on this model will give good agreement with experiment if the value of \( m \), relating the diffusion coefficient to the spectral density function, is taken to be about 24.
Derivation of Equations of Motion for Noisy Buckets

The theory of RF buckets as given by Vogt-Nielsen (CERN PS/NNNI 1958) is modified to include the effect of errors due to noise in the frequency of the RF voltage.

In this theory it is assumed that the frequency of revolution of a particle \( \Omega \) (measured in radians per unit time), is dependent solely upon the particle energy \( E \)

\[
\Omega = \Omega(E) . \tag{A1.1}
\]

Assume that there is present in the machine a single narrow-gap cavity carrying the voltage

\[
U = U_0 \sin \int_{t}^{t_0} (\omega + \Delta \omega) \, dt , \tag{A1.2}
\]

where \( \Delta \omega \) is some error in the mean frequency \( \omega \) of the RF voltage.

Then, following Vogt-Nielsen, the energy gained in one transit of the cavity at \( t = t_0 \) is

\[
\Delta E = - qU_0 \sin \int_{t_0}^{t} (\omega + \Delta \omega) \, dt . \tag{A1.3}
\]

Defining transit phase as \( \Theta_0 \) at \( t = t_0 \) where

\[
\Theta_0 = \int_{t_0}^{t} (\omega + \Delta \omega) \, dt , \tag{A1.4}
\]

between two consecutive transits

\[
\Delta \Theta_0 = \int_{t_0}^{t_0 + 2\pi/\Omega} (\omega + \Delta \omega) \, dt . \tag{A1.5}
\]

We are interested in studying the motions of those particles which gain relatively little energy per transit; then we may redefine the phase change such that

\[
\Delta \Theta_0 = \int_{t_0}^{t_0 + 2\pi/\Omega} (\omega + \Delta \omega) \, dt - 2\pi h
\]

where \( h \) is any integer greater than or equal to 1.

We may also define a phase angle \( \Theta \) which assumes the values \( \Theta_0 \) at each transit by

\[
\frac{d\Theta}{dt} = \omega + \Delta \omega - h\Omega \tag{A1.6}
\]

where \( \Omega \) is assumed constant between transits. Thus

\[
\Delta \Theta = \int_{t_0}^{t_0 + 2\pi/\Omega} (\omega + \Delta \omega - h\Omega) \, dt = \int_{t_0}^{t_0 + 2\pi/\Omega} (\omega + \Delta \omega) \, dt - 2\pi h = \Delta \Theta_0 .
\]
Equation (A1.3) indicates that the energy gain per revolution (where $\Delta t = \text{time per one revolution} = 2\pi/\Omega$) may be written

$$\frac{\Delta E}{\Delta t} = -\frac{qu_0}{2\pi} \Omega \sin \Theta_b$$

or in a smoothed approximation (where times are very large compared with $\Delta t$)

$$\frac{dE}{dt} = -\frac{qu_0}{2\pi} \Omega \sin \Theta,$$  \hspace{1cm} (A1.7)

where $\Theta$ is given by equation (A1.6) and $\Omega$ is now considered a function of $\Theta$ over the "long" times involved.

The above equations are approximate but may be considered valid as long as the change in $\Theta$ per revolution is a small fraction of $2\pi$.

The equations are more easily handled by the introduction of a new variable defined as

$$W(x) = \int dx/\Omega(x)$$  \hspace{1cm} (A1.8)

so that

$$\frac{dW}{dt} = -\frac{qu_0}{2\pi} \sin \Theta$$

and

$$\frac{d\Theta}{dt} = \omega + \Delta \omega - \hbar \Theta.$$  \hspace{1cm} (A1.9)

These may be considered canonical equations of motion derivable from the Hamiltonian

$$H(\Theta, W, t) = (\omega + \Delta \omega)W - hE(W) - \frac{qu_0}{2\pi} \cos \Theta.$$  \hspace{1cm} (A1.10)

Changing the variables to $\Theta = \Theta^*$, $W = W^* + W_s$ by means of the generating function

$$S = \Theta^* (W - W_s)$$  \hspace{1cm} (A1.11)

yields a new Hamiltonian

$$H^*(\Theta^*, W^*, t) = \left[ H - \frac{dS}{dt} \right]_{W = W^* + W_s}$$

$$= (\omega + \Delta \omega) \left[ W^* + W_s \right] - hE(W^* + W_s)$$

$$- \frac{qu_0}{2\pi} \cos \Theta^* + \Theta^* \frac{dW_s}{dt}.$$
Expanding $E$ in a Taylor series yields

$$E(W^* + W_s) = E(W_s) + W^* \left( \frac{\partial E}{\partial W} \right)_{W_s} + \frac{W^*^2}{2} \left( \frac{\partial^2 E}{\partial W^2} \right)_{W_s} + \ldots$$

Now

$$\left( \frac{\partial E}{\partial W} \right)_{W_s} = \Omega(E_s) = \Omega_s ; \quad W_s = \int \frac{dE}{\Omega(E)}$$

$$\left( \frac{\partial^2 E}{\partial W^2} \right)_{W_s} = \left( \frac{\partial^2 E}{\partial W} \right)_{W_s} = \Omega_s'$$

also

$$\frac{dW_s}{dt} = \left( \frac{dW}{dt} \right)_{W_s} = \frac{1}{\Omega_s'} \frac{d\Omega_s}{dt} .$$

The Hamiltonian then becomes

$$H^* = (\omega + \Delta\omega)W^* + (\omega + \Delta\omega)W_s - \hbar E(W_s) - \hbar \Omega_s W^* - \frac{\hbar}{2} \Omega_s' W^*^2 \ldots\ldots$$

$$= - \frac{\alpha_0}{2\omega} \cos \Theta^* + \frac{\alpha}{\Omega_s'} \frac{d\Omega_s}{dt} . \quad (A1.12)$$

The equations of motion in terms of these new variables are then

$$\frac{d\Theta^*}{dt} = \frac{\partial H^*}{\partial W^*} = \omega + \Delta\omega - \hbar \Omega_s - \hbar \Omega_s' W^* \ldots\ldots \quad (A1.13)$$

$$\frac{dW^*}{dt} = - \frac{\partial H^*}{\partial \Theta^*} = - \frac{1}{\Omega_s'} \frac{d\Omega_s}{dt} - \frac{\alpha_0}{2\omega} \sin \Theta^* . \quad (A1.14)$$

Now let $\omega$ be the average value of the frequency of the RF voltage and $\Delta\omega$ be the random error in this frequency — both $\omega$ and $\Delta\omega$ are functions of time.

If, then, we define the energy $E_s$ by

$$\Omega_s = \Omega(E_s) = \frac{\omega}{\hbar} \quad \frac{d\Omega_s}{dt} = \frac{\dot{\omega}}{\hbar}$$

then the equations of motion become

$$\frac{d\Theta^*}{dt} = \Delta\omega(t) - \hbar \Omega_s' W^* - \left( \text{terms of higher order in } W^* \right) \quad (A1.15)$$

$$\frac{dW^*}{dt} = - \frac{\dot{\omega}}{\Omega_s'} - \frac{\alpha_0}{2\omega} \sin \Theta^* .$$
In CESAR, the parameter $\Omega_s$ is such a function of energy that $\Omega'_s = \text{constant}$ and therefore all terms of higher order than $W^*$ in Eq. (A1.15) may be dropped out. This property is common to most synchrotrons and storage rings operating far from transition.

Furthermore, using the usual scaling factors

$$K = \sqrt{\frac{2\pi}{\hbar |\Omega'_s| q|u_0|}} \quad L = \sqrt{\frac{q|u_0|}{2\pi \hbar |\Omega'_s|}}$$

and putting $t = Kr$; $W^* = Ly$; $\phi = \pi - \Theta$; and finally

$$\Gamma = \frac{2\pi |\Omega'_s|}{q|u_0| \hbar |\Omega'_s|}$$,

the equations of motion may be written conveniently as

$$\frac{d\phi}{dt} = -y - B(r) \quad B(r) = K\omega(r)$$

(A1.16)

$$\frac{dy}{dt} = -\Gamma - \sin \phi$$

and are derivable from a Hamiltonian

$$H(y, \phi, r) = \Gamma\phi - \cos \phi - \frac{1}{2} y^2 - y B(r)$$.

The equations of motion (A1.16) may be linearized by considering small displacements $\phi^*$ about $\phi_s$ defined by $\Gamma = -\sin \phi_s$; in which case they reduce to (for $\phi^* < 1$)

$$\phi^* = \phi_s + \phi^* \sqrt{1 - \Gamma^2} = A(r) = -\frac{d}{dt} \left( B(r) \right)$$

$$y = \phi^* \sqrt{1 - \Gamma^2} \left[ 1 - \frac{\Gamma}{\sqrt{1 - \Gamma^2}} \phi_s \right] \approx \phi^* \sqrt{1 - \Gamma^2}$$

or

$$y = \phi^* \sqrt{1 - \Gamma^2}$$.

Likewise, linearization of the Hamiltonian leads to an expression for the "energy" of the system

$$H = \Gamma\phi_s - \cos \phi_s + \frac{1}{2} \phi^* \cos \phi_s - \frac{1}{2} y^2 - y B(r)$$

where $\cos \phi_s = -\sqrt{1 - \Gamma^2}$.
In order to examine the effect of a perturbation, let $B(\tau)$ be a small change in $\phi^*$, occurring at some time $\tau_1$, and for simplicity let us examine the case of $\Gamma = 0$; thus

$$H = 1 - \frac{1}{2} \phi^* z - \frac{1}{2} y^2; \quad \tau < \tau_1$$

for $B(\tau) = \epsilon U(\tau - \tau_1)$

and

$$H = 1 - \frac{1}{2} \phi^* - \frac{1}{2} y - y \epsilon; \quad \tau > \tau_1.$$

Let $2(1 - \epsilon) = R^2$ and we can then write

$$\phi^* z + y^2 = R^2 \quad \text{for} \quad \tau < \tau_1,$$

and

$$\phi^* z + y^2 + 2\epsilon y = R^2 \quad \text{for} \quad \tau > \tau_1,$$

and for $y = Y - \epsilon$, this last becomes

$$\phi^* z + y^2 = R^2 + \epsilon^2 \quad \text{for} \quad \tau > \tau_1.$$

Thus, it is easily seen that the phase trajectories are circles of radius $R$ before the perturbation, whereas afterwards, they are still circles but now the origin has shifted. It is also seen that in this approximation the increase in area is independent of the area enclosed by the initial trajectory. This is also illustrated by the form of the Fokker-Planck equation given in Section 2, and means that each particle is randomly displaced throughout the phase plane by the same amount. In effect, we have a new Hamiltonian after such a perturbation which is of the same form as the old; the "initial" conditions for particles described by this new Hamiltonian are to be found from the "final" conditions described at the instant of the occurrence of the perturbation by the old Hamiltonian. Thus if the particles start off by being enclosed by an ellipse surrounding the phase stable point (still defined as being a constant $\phi^*$), then at a later time they will still be enclosed by an ellipse (only for the linear approximation described above) of the same area, but the "centre" of this ellipse may now be elsewhere in the phase plane. Furthermore, this "centre" may be following an elliptical trajectory but is diffusing away from its initial position at the phase stable point. The rate of drift away from the initial position is given by the Fokker-Planck equation of Section 2. This point is discussed further in Section 5.
Calculation of $\overline{\Delta \Phi^2}$

If the noise is assumed to be stationary and bandwidth limited, $\gamma$ may be expressed in terms of the spectral density function $\Phi(a)$ as

$$\gamma(T_1, T_2) = 2 \int_{\alpha_1}^{\alpha_2} \Phi(a) e^{i a (T_1 - T_2)} da$$

and Eq. (9) of Section 2 then becomes

$$\overline{\Delta \Phi^2} = 2 \int_{\alpha_1}^{\alpha_2} \Phi(a) da \int_{T}^{T + \Delta T} e^{i a (T_1 - T_2)} \cos \omega_3 (r - T_1) dT \times \cos \omega_3 (r - T_2) dT_2.$$

Using $\cos x \cos y = \frac{1}{2} [\cos (x + y) + \cos (x - y)]$ and changing the variable of the integration to $X = r - T$, leads to

$$\overline{\Delta \Phi^2} = \int_{\alpha_1}^{\alpha_2} \Phi(a) da \left\{ \int_{0}^{\Delta T} e^{i a (X_2 - X_1)} \cos \omega_3 (X_1 + X_2) dX_1 dX_2 + \int_{0}^{\Delta T} e^{i a (X_2 - X_1)} \right\}$$

$$\times \cos \omega_3 (X_1 - X_2) dX_1 dX_2$$

which may be integrated to yield

$$\overline{\Delta \Phi^2} = \int_{\alpha_1}^{\alpha_2} \Phi(a) da \left\{ -\frac{4 \sin^2 \left( \frac{a - \omega_3}{2} \right) \Delta T}{(a^2 - \omega_3^2)} - \frac{2 \sin^2 \omega_3 \Delta T}{(a^2 - \omega_3^2)} + \frac{4 \sin^2 \left( \frac{a - \omega_3}{2} \right) \Delta T}{(a - \omega_3)^2} \right\}.$$

However, if the noise is also assumed to be white, then

$$\gamma(T_1 - T_2) = 2\pi \Phi(a) \delta(T_1 - T_2)$$

leading directly to

$$\overline{\Delta \Phi^2} = 2\pi \Phi \int_{T}^{T + \Delta T} \delta(T_1 - T_2) \cos \omega_3 (r - T_1) dT_1 \cos \omega_3 (r - T_2) dT_2$$

$$= 2\pi \Phi \int_{T}^{T + \Delta T} \cos^2 \omega_3 (r - T_2) dT_2$$

$$= \frac{2\pi \Phi}{\omega_3} \left[ \frac{\omega_3 \Delta T}{2} + \frac{1}{4} \sin 2\omega_3 \Delta T \right]$$
Calculation of $\Delta y$ and $\Delta y\Delta\phi$

Using Eq. (7) of Section 2

$$\Delta y^2 = \omega^4 \int d\xi d\xi' \frac{\xi}{(T_1 - T_2)} \cos \omega_s (\xi - T_1) \cos \omega_s (\xi - T_2) d\xi, d\xi'$$

where $\xi$ has replaced $r$ in the integral for reasons of clarity.

Again for white noise $\gamma(T_1 - T_2) = 2\pi \delta(T_1 - T_2)$

and therefore

$$\Delta y^2 = \omega^4 \int d\xi d\xi' \frac{\xi}{(T_1 - T_2)} \left[ \frac{\omega \xi^2}{4} + \frac{\omega \xi^2 (\xi + \Delta \xi)}{4} - \frac{1}{8 \omega_s} \cos 2\omega_s (\xi + \xi') + \frac{1}{8 \omega_s} \cos 2\omega_s \xi \right]$$

and remembering that $\xi$ is really $r$, i.e. $d\xi = dr$, we have as $\Delta r \to 0$

$$\Delta y^2 = \omega^4 \int d\xi \left[ \frac{\omega \xi^2}{4} + \frac{\omega \xi^2 (\xi + \Delta r)}{4} - \frac{\omega \xi^2 \Delta r^2}{4} \right].$$

Thus

$$\lim_{\Delta r \to 0} \frac{\Delta y^2}{\Delta r} = \omega^4 \int d\xi \left[ \frac{\Delta r^2}{2} + \frac{\omega \xi^2 (\xi + \Delta r)}{2} \right] = 0 = f_2.$$
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Anomalous stacking efficiencies as a function of the magnitude of the diffusion coefficient $D$.  

**Fig. 1**

Stack width 36 keV

Bucket width

- $10 \text{ keV}$
- $4 \text{ keV}$
- $2 \text{ keV}$
Fig. 2

The number of particles diffusing into an initially empty noisy bucket as a function of bucket size.
Fig. 3

The "filling" time as a function of the spectral density.