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AGK CUTTING RULES AND PERTURBATIVE QCD

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ABSTRACT

The purpose of the present note is to describe a few features of semi-hard interactions, in high energy nuclear collisions, that are better understood with the help of the AGK cutting rules and of the probabilistic picture of the interaction which follows. In the first part of the article the cutting rules are discussed for a simplest component of the forward three-body-parton amplitude in the large- s fixed- t limit. The case considered corresponds to the term, at the lowest order in the coupling constant and with vacuum quantum number exchange in both t -channels, of the amplitude which describes the interaction of a high energy quark with the two target quarks. The different leading cuts of the amplitude are shown to be proportional to one another with the same weights of the cutting rules discussed in the context of multi-Pomeron exchange. The probabilistic picture of the multiple interactions, which is originated from the cutting rules, and the self shadowing cross sections are then discussed. The second part of the article is dedicated to the semi-hard interactions. The semi-hard cross section in high energy nucleus-nucleus collisions is represented as a self-shadowing cross section, and a feature which is pointed out is that the single scattering factorized expression of the perturbative-QCD-parton model holds at any order in the multi-parton correlations, the relation being the analogous of the AGK cancellation for the average number of soft interactions in high energy hadron-nucleus collisions. An infrared problem which finds a solution within the self shadowing representation of the semi-hard cross section is finally discussed.

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I. Introduction

All intermediate states, which are allowed by energy-momentum and quantum number conservation, contribute to the forward amplitude in high energy hadronic and nuclear reactions. A definite contribution to the forward amplitude, with a given number of particles produced, may be expressed by drawing a loop diagram with the corresponding particles in the s -channel. If one makes a s -channel cut, namely if one draws a line which goes thru the s -channel particles, one notices that, by multiplying by two the imaginary part of each cut propagator, the four-momentum loop integrals, together with the δ functions of the imaginary parts of the cut propagators, give precisely the invariant phase space integral with the particles in the cut as asymptotic outgoing states. The left and the right hand side of the cut represent, moreover, the probability amplitudes for the transition and its conjugate, in such a way that the cut in the forward amplitude is proportional to the cross section for producing the particles in the s -channel. By considering all possible diagrams and by drawing all possible cuts, one obtains explicitly, from the forward amplitude, all different contributions to the total cross sections. Increasing the c.m. energy the number of possible inelastic channels grows rapidly and it seems reasonable to expect that the relations between the forward amplitude and the different contributions to the inelastic cross section become of little practical use. Indeed an important observation has been done in this respect long time ago by Abramovskii, Gribov and Kancheli[1]. The observation was that, although at high energy the number of possible cuts of a forward amplitude becomes increasingly large with energy, still only few cut amplitudes are leading in the high- s fixed- t limit. Moreover the leading cut amplitudes are all proportional to one another with given proportionality factors, the AGK weights.

A consequence of this property is that also in a very complicated case, like in high energy hadron-nucleus interaction, where the forward amplitude is represented, at high energy, by a series of terms with many exchanged Pomerons, one can obtain the whole inelastic hadron-nucleus cross section by summing explicitly all contributions from the leading cuts of the forward hadron-nucleus amplitude[2]. Remarkably the result which is obtained, after summing all different cut diagrams, is that the cross section acquires the simple probabilistic meaning of sum of multiple interaction probabilities[3], the probabilistic picture of the interaction being the quantum mechanical result of summing all the leading cut amplitudes.

The aim of the present note is to describe the mechanism which gives rise to the cutting rules and to point out a few interesting features of the interaction which are most conveniently understood in this context. In a simplest case, which can be discussed in perturbative QCD at the lowest orders in the coupling constant, the derivation of the cutting rules is particularly transparent. For this reason a preliminary description of the main features of the quark-quark scattering amplitude is outlined in the next paragraph: The large- s fixed- t limit of the lowest order terms in the coupling constant is worked out and the Reggeization of the t -channel

gluons is described. The cutting rules are then discussed in the following section where the derivation of all the dominant terms at high energy, of all the leading cuts of the amplitude, is sketched in a definite case of three-body quark interaction. The probabilistic picture of the many body interaction, which is originated from the cutting rules and the self-shadowing cross sections[4] are the argument of the fourth paragraph. The fifth part is devoted to the semi-hard cross section which is expressed as a self shadowing cross section, while the input multi-parton distributions are approached with a generating functional formalism. A general result which is obtained is that, after keeping into account of all unitarity corrections, the single scattering expression of the perturbative-QCD parton model is shown to be still a meaningful one, also in semi-hard nuclear interactions at very high energy: At any order in the multi-parton correlations the single scattering expression gives the average number of semi-hard interactions at a fixed value of the impact parameter of the nuclear collision. In the last part of the note the self-shadowing picture of semi-hard interactions is compared with the most conventional eikonal approach. It is pointed out how the eikonal scheme gives rise to a whole class of infrared divergent quantities and, by making a definite example, it is shown how this infrared problem is avoided within the self-shadowing approach.

II. Lowest order terms and gluon Reggeization

In this paragraph a few properties of the quark-quark amplitude, at the lowest order in the coupling constant and with vacuum quantum number exchange, are discussed in the large- s fixed- t limit. The lowest order Born term, the box and crossed box diagrams of the quark-quark scattering amplitude are drawn in fig.1.

Fig.1: Lowest order Feynman diagrams for quark-quark scattering

Each different diagram is characterized by a different color factor. Introducing the color matrices $T_a \equiv 1/2\lambda_a$, with λ_a the SU(3) Gell Mann matrices, the color factors are

$$\begin{array}{ll}
 T_a^{(1)} T_a^{(2)} & \text{Born - term} \\
 T_a^{(1)} T_b^{(1)} T_a^{(2)} T_b^{(2)} & \text{box - diagram} \\
 T_b^{(1)} T_a^{(1)} T_a^{(2)} T_b^{(2)} & \text{crossed - box - diagram}
 \end{array} \tag{1}$$

By introducing the commutators, $[T_a, T_b]$, and the anti-commutators, $\{T_a, T_b\}$, of the color matrices, one may write:

$$\begin{aligned}
T_a^{(1)}T_b^{(1)}T_a^{(2)}T_b^{(2)} &= \frac{1}{4}\{T_a^{(1)}, T_b^{(1)}\}\{T_a^{(2)}, T_b^{(2)}\} + \frac{1}{4}[T_a^{(1)}, T_b^{(1)}][T_a^{(2)}, T_b^{(2)}] \\
T_b^{(1)}T_a^{(1)}T_a^{(2)}T_b^{(2)} &= \frac{1}{4}\{T_b^{(1)}, T_a^{(1)}\}\{T_a^{(2)}, T_b^{(2)}\} + \frac{1}{4}[T_b^{(1)}, T_a^{(1)}][T_a^{(2)}, T_b^{(2)}]
\end{aligned} \tag{2}$$

The color factors for the box and for the crossed box diagrams are therefore split into a symmetric component, which is the same for the two terms, and into a anti-symmetric component, with a different sign for the box and for the crossed box diagrams. By using the commutator rules of the color algebra

$$[T_a, T_b] = if_{abc}T_c$$

and the relation for the SU(3) structure constants

$$f_{abc}f_{abc'} = 3\delta_{cc'}$$

one may express the anti-symmetric color term as

$$[T_a^{(1)}, T_b^{(1)}][T_a^{(2)}, T_b^{(2)}] = -3T_c^{(1)}T_c^{(2)} \tag{3}$$

which shows that the anti-symmetric color component has the same color factor as the lowest order Born diagram. In the color factors of the diagrams with two gluon exchange there is a component with the same color of the diagram with one gluon exchange, namely which corresponds to exchange of a color octet, and a component which cannot be simply added to the lowest order term, since it is characterized by the exchange of different quantum numbers, actually it contains also a color singlet exchange.

To perform the analysis of the box and of the crossed box diagrams one needs to work out the leading terms of the space-time factors. In fig.2 the kinematical variables characterizing the space-time factor $\mathcal{M}$ of the box diagram are explicitly indicated.

Fig.2: The box diagram

The light-cone components of the momenta shown in fig.2, which are needed to estimate the leading behaviour of $\mathcal{M}$ at high energy, are written below:

$$\begin{aligned}
p_1 &\equiv \sqrt{s}(1, 0, 0) \\
p_2 &\equiv \sqrt{s}(0, 1, 0) \\
\Delta &\equiv \sqrt{s}(0, 0, \Delta_t/\sqrt{s}) \\
q &\equiv \sqrt{s}(\alpha, \beta, \mathbf{q}_t/\sqrt{s})
\end{aligned} \tag{4}$$

Δ is the overall momentum transfer and, being the final state on shell, in the limit $s \rightarrow \infty$, $t/s \rightarrow 0$, only Δ_t needs to be taken into account. q is the loop variable and α, β are the fractional components of q in the direction of the incoming momenta p_1 and p_2 . If one writes the diagram in the Feynman gauge, at the leading order the numerators in the propagators give a factor $\sqrt{s}$ at each vertex. To work out the diagram one needs to look at the position of the poles which correspond to the mass shell condition for the four propagators:

$$\begin{aligned}
q^2 &= \alpha\beta s - q_t^2 \\
(p_1 + q)^2 &= (1 + \alpha)\beta s - q_t^2 \\
(q - \Delta)^2 &= \alpha\beta s - (\mathbf{q}_t - \Delta_t)^2 \\
(p_2 - q)^2 &= -(1 - \beta)\alpha s - q_t^2
\end{aligned} \tag{5}$$

The integration on α is done by taking the residuum of the pole of $1/(p_2 - q)^2$, giving as a result $\alpha = -q_t^2/(1 - \beta)s$ together with the condition $0 < \beta < 1$, which guarantees that, in the complex α -plane, the singularities of the propagators are not all on the same side of the integration contour. After substituting the value of α in the loop integral, the leading behaviour is obtained from the integration region with $\mathbf{q}_t$ fixed and β small. q and $q - \Delta$ have therefore only transverse components and the dominant contribution comes from the behaviour of the quark propagator with momentum $p_1 + q$ for $\beta \rightarrow 0$:

$$\mathcal{M} \propto \frac{1}{q_t^2(\mathbf{q}_t - \Delta_t)^2 s} \int_0^\Lambda \frac{d\beta}{\beta s - q_t^2 + i\epsilon} \tag{6}$$

where Λ is a small number, constant with s . The integral on β gives as a result:

$$\mathcal{M} \propto \frac{1}{q_t^2(\mathbf{q}_t - \Delta_t)^2 s} \frac{1}{s} [\ln(s\Lambda) - i\pi] \tag{7}$$

The space-time factor in the box diagram, $\mathcal{M}$, is therefore characterized by a large real term, which grows as $\ln s$, and by an imaginary term which arises because of the position of the singularity of the propagator of the quark line, with momentum $p_1 + q$, with respect to the integral on β and corresponding to the contribution from the quark line on shell.

The space-time factor of the crossed box diagram differs, with respect to the case of the box diagram, in the intermediate quark propagator with momentum $p_1 + \Delta - q$, which substitutes the quark propagator with momentum $p_1 + q$ as indicated in fig.3.

Fig.3: Crossed-box diagram

The dominant term at large s is given by the same integration region as in the former case, namely by the behaviour of the propagator with momentum $p_1 + \Delta - q$ for $\beta \rightarrow 0$:

$$\mathcal{M}' \propto \frac{1}{q_t^2(\mathbf{q}_t - \Delta_t)^2 s} \int_0^\Lambda \frac{d\beta}{-\beta s - (\mathbf{q}_t - \Delta_t)^2 + i\epsilon} \quad (8)$$

The leading term from the integral on β gives:

$$\mathcal{M}' \propto \frac{1}{q_t^2(\mathbf{q}_t - \Delta_t)^2 s} \frac{-1}{s} \ln(s\Lambda) \quad (9)$$

showing that the differences with the leading term of the box diagram are the opposite sign in the real part and the absence of an imaginary part.

By taking into account the color factors, the dominant contribution to the forward amplitude, from the box and from the crossed box diagrams, is therefore expressed as:

$$\begin{aligned} & \frac{1}{4} \{T_a^{(1)}, T_b^{(1)}\} \{T_a^{(2)}, T_b^{(2)}\} \times (\mathcal{M} + \mathcal{M}') \\ & - \frac{3}{4} T_c^{(1)} T_c^{(2)} \times (\mathcal{M} - \mathcal{M}') \end{aligned} \quad (10)$$

One may notice that the correction to the lowest order Born diagram is represented by the second line in Eq.(10) which, being characterized by the same color factor $T_c^{(1)} T_c^{(2)}$ of the Born term, represents a color octet exchange. This correction term is dominantly real and grows as a $\ln s$, since it is proportional to $(\mathcal{M} - \mathcal{M}')$. The symmetrized color factor in the first line in Eq.(10) contains the exchange of a color singlet in the t -channel and, for this reason, it cannot be simply added to the lowest order term. Differently with respect to the previous case of color octet exchange, this term is mainly imaginary, being proportional to $(\mathcal{M} + \mathcal{M}')$. Actually:

$$\begin{aligned} \mathcal{M} + \mathcal{M}' & \propto \frac{i\pi}{q_t^2(\mathbf{q}_t - \Delta_t)^2 s} \\ \mathcal{M} - \mathcal{M}' & \propto \frac{-\ln s}{q_t^2(\mathbf{q}_t - \Delta_t)^2 s} \end{aligned} \quad (11)$$

The analysis just described can be pushed to higher orders in the coupling constant g [5], a rather detailed discussion of the order g^6 is presented in ref.[6].

At each order in g one has a different set of Feynman diagrams each characterized by a different topology and, therefore, by a different color factor. The way to proceed is to introduce, at each order in g , a set of color factors which may act as a base for the whole set of color factor of the Feynman diagrams at that order and which contains all base-color factors of the lower orders in g . The example of the order g^6 is shown in fig.4, where all Feynman diagrams, which contribute, in the Feynman gauge, at order g^6 to the leading term of the amplitude at large s , are shown.

Diagrams	Color Group Factors	G_1	G_2	G_3	G_4
		1		0	0	1	0
2		$\frac{9}{4}$	0	1	0
3	4	0	$-\frac{3}{2}$	1	0
5	6
7	8	0	$-\frac{3}{2}$	0	0
9	10
11	12	$-\frac{9}{4}$	$-\frac{3}{2}$	0	0
13	14
15		0	0	0	1
16		$\frac{9}{4}$	$\frac{9}{2}$	-1	1
17	18	0	$\frac{3}{2}$	0	1
19	20	0	3	-1	1
21					

Fig.4: Color-group factors for the diagrams at order g^6 in quark-quark scattering

The color factor of each diagram is identified by its topology. Each different color factor can be expressed as a superposition of the four color terms $G_1 \dots G_4$

identified by the topologies of the graphs in the upper part of the figure. Fig.4 shows the coefficients of the expansion. One notices that G_1 is the color factor of the lowest order g^2 Born diagram and G_1 and G_2 allow one to express the color factors of the Feynman diagrams at order g^4 , actually the box and the crossed box diagrams discussed above. Each base-color factor multiplies a space-time factor which is obtained by projecting all Feynman diagrams into the given base-color factor. A regularity which one may notice is that the lowest order term in g of the space-time factor, which multiplies the color-base factors G_1 , G_2 and G_4 , namely those with no interaction between the exchanged lines, is precisely the space-time factor of an Abelian theory at the same order in the coupling constant. For example the lowest order term of the space-time factor of G_2 is given by the sum of the box and the crossed box diagram, the same as in an Abelian theory at order g^4 . At order g^6 , by looking at fig.4, one obtains that G_4 multiplies a space-time factor which is obtained by summing, with equal weight, all diagrams from diagram number 15 to diagram number 20 included, corresponding to the six diagrams at order g^6 in the Abelian theory.

A further regularity is observed by looking at the expansion in powers of the coupling constant of the space-time factor of each color-base factor. The simplest case is for G_1 . At the order g^6 one obtains:

$$G_1 \frac{g^2}{\Delta_t^2} \left[1 - \bar{\alpha}(\Delta_t) \ln s + \frac{1}{2} \bar{\alpha}^2(\Delta_t) \ln^2 s \right] \quad (12)$$

where

$$\begin{aligned} \bar{\alpha}(\Delta) &= \frac{3g^2}{4\pi} \Delta^2 I(\Delta) \\ I(\Delta) &= \int \frac{d^2 q_t}{(2\pi)^2} \frac{1}{q_t^2 (\Delta_t - q_t)^2} \end{aligned} \quad (13)$$

The first term in the square parenthesis in Eq.(12) is the Born term, the second is the result of the sum of the box and of the crossed box diagrams, the third term is obtained by summing all leading contributions at order g^6 of the 21 diagrams in fig.4 with the weights indicated there. The relevant observation is that the term in the square parenthesis in Eq.(12) may be obtained by expanding $s^{-\bar{\alpha}(\Delta_t)}$. One can therefore keep into account of the leading terms, at large s , of the higher order corrections in the coupling constant, with a simple modification of the lowest order Born term diagram. Namely it is enough to multiply the propagator of the exchanged gluon by $s^{-\bar{\alpha}(\Delta_t)}$. It is remarkable that this simple modification is able to take into account of the leading terms also when considering higher orders, actually the statement has been verified by pushing the perturbative analysis till the tenth perturbative order[7]. Moreover the statement does not hold only for G_1 , it holds also for all other color-base factors. The leading terms in s of the higher order corrections in g are taken into account simply by multiplying each t -channel-gluon, in each color-base diagram, by $s^{-\bar{\alpha}(q)}$, where q is the momentum of the exchanged

gluon. This feature of the interaction, which results from the perturbative analysis of the lower order diagrams, has been called *gluon Reggeization*[6,8].

One may notice that while the lowest order Born diagram gives a purely real amplitude and the terms at order g^4 give a dominantly real contribution (actually the real parts of the box and of the crossed box diagrams grow as $\ln s$) the sum of all these large real parts results in the factor $s^{-\bar{\alpha}(q)}$ which is a decreasing function of s . Gluon Reggeization is therefore able to show how an amplitude, which is real at the lowest perturbative orders, may eventually become dominantly imaginary after resummation. A consequence is that, when analyzing an amplitude perturbatively, one should not select the dominant component with the only criterion of the leading behaviour at high energy. Rather one should focus on quantum number exchange. In this respect one may notice that the term in the quark-quark amplitude, at the lowest order in the coupling constant and with vacuum quantum number exchange, is represented by the lowest order term in the expansion of the space-time factor of the color-base factor G_2 , namely it is given by the sum of the box and of the crossed-box diagrams.

III Three-body-quark interaction and cutting rules

III.1 Forward amplitude

The analysis of the previous paragraph allows one to identify the lowest order term in the two-body amplitude with the right quantum number exchange to give a non-vanishing contribution at high energy. The leading behaviour of $\mathcal{M} + \mathcal{M}'$, which is the lowest order term in g of the space-time factor of G_2 , is obtained by taking the on shell contribution of the propagators of the quarks with momenta $p_1 + q$ and $p_2 - q$. The dominant configuration corresponds therefore to $(p_1 + q)^2 = 0$ which gives

$$q_0 - q_z \simeq \frac{q_t^2}{2\omega}$$

with ω the energy component of the incoming four-momentum p_1 . If q_t is kept fixed while $s \rightarrow \infty$, $q_0 - q_z \rightarrow 0$. The interaction region is therefore squeezed in the transverse direction to a transverse size of order $1/q_t < 1/q_t^{min}$, with q_t^{min} a lower cut-off of a few GeV in order to allow a perturbative calculation. The interaction region however *is not squeezed in the longitudinal direction conjugate to q_-* : even if the regime is perturbative, the + longitudinal component the interaction region grows as $\sqrt{s}$, when q_t^{min} , although large, does not grow with s .

A case which one may conceive is the one of a high energy parton interacting with a nucleus. All the target partons inside the interaction region are allowed to interact coherently with the projectile. The simplest possibility to be considered, other than the two-body interaction, is the case of a three-body interaction of a projectile with two target partons. The problem which will be discussed in the

present paragraph is the simplest one where the three body quark interaction is constructed by superimposing coherently two-body quark interactions, each represented by the exchange of a box or of a crossed box diagram. Although this is the only possibility, at the lowest order in g , with the correct quantum number exchange in a two-body interaction, it is not the only possibility for a three body interaction. Indeed in the three body interaction one has two t -channels. With the present choice one selects one of the color factors which contain color singlet exchange in both t -channels[9], and to have a finite contribution at high energy it is enough to have vacuum quantum number exchange with the whole target. Also this restricted case requires however some effort to be analyzed, in fact it corresponds to the three-body amplitude described by the sum on the 24 QED-like Feynman diagrams shown in fig.5.

Fig.5: QED-like diagrams in three-body quark interaction

The horizontal line in the middle of each diagram in fig.5 represents the projectile quark, the horizontal lines in the top and in the bottom are the target quarks and the vertical lines are the exchanged gluons. The sum of the space time parts of the diagrams of the set in fig.2 is gauge independent. The reason is that the set of diagrams in fig.2 is a complete sub-set of diagrams, in electrodynamics, with a given quantum number exchange and at a given order in the coupling constant. Actually since electrodynamics is gauge independent for any value of the electric charge, one obtains a gauge independent set also after restricting to the diagrams which do not include interactions between the target quarks.

All different contributions to the inelastic cross section are obtained from the forward amplitude by taking all the s -channel cuts. The aim of the analysis in the present paragraph is to obtain all the leading cuts at high energy of the three-

body quark amplitude given by the sum of the 24 diagrams in fig.5, and to show that they are all proportional to one another with the same proportionality factors which have been obtained by AGK.

Most of the considerations of this paragraph may find a justification in the following eikonal expression[7]: Suppose to have a high energy particle with momentum p and n quanta with momenta $q_1 \dots q_n$ attached to the high energy line. Let β_i be the scaled-light-cone components of q_i in the direction opposite to p . One may then write:

$$\begin{aligned} & \sum_{\{\text{permutations}\}} \delta\left(\sum_{i=1}^n \beta_i\right) \frac{1}{\beta_1 + i\epsilon} \cdot \frac{1}{\beta_1 + \beta_2 + i\epsilon} \cdots \frac{1}{\beta_1 + \beta_2 \dots + \beta_{n-1} + i\epsilon} \\ &= \frac{1}{(-2\pi i)^{n-1}} \prod_{i=1}^n \delta(\beta_i) \end{aligned} \quad (14)$$

When all permutations of the quanta attached to the projectile are taken, the leading term associated to the propagator of the eikonal line is obtained by putting on shell all intermediate propagators of the line with momentum p . The result of the sum of the box and of the crossed box diagrams, discussed in the previous paragraph, may be obtained by using this relation. With the help of this argument one may easily evaluate the sum of all the 24 diagrams in fig.5. A feature of the result is that, although several terms in the sum grow separately as $(\ln s)^2$, the sum does not contain any $\ln s$ factor[9].

III.2 Two-box cuts

In order to discuss the cuts of the amplitude described by the diagrams in fig.5, one may classify all possible cuts by counting the number of cut boxes. One may consider a vertical cut which involves both boxes. Although there are altogether 24 diagrams, only 4 of them, the cut diagrams in fig.6, contribute.

Fig.6: Diagrams for the two-box cut amplitude

In fact each exchanged gluon has a transverse momentum larger than q_t^{min} while all external lines are on shell and with negligible transverse momenta. These requirements allow one to reduce greatly the possible cuts in the set of diagrams

in fig.5. In fact if one takes, for example, the cut diagram in fig.7, one observes that, as shown in fig.7b, an external line, with negligible transverse momentum, radiates a real gluon with transverse momentum $\mathbf{q}_t$ acquiring a transverse momentum $-\mathbf{q}_t$ and remaining on shell. The process is possible only if the external line has a virtuality of order q_t . Actually the diagrams in fig.5 are to be understood as the perturbative part of a much more complicated amplitude. The external lines are therefore meant to be attached to a soft vertex, which does not allow large virtualities nor large transverse momenta, and the configuration of the cut diagram in fig.7 is therefore depressed.

Fig.7

- a) A two-box cut diagram which does not contribute to the leading term
- b) A component of the right hand side of the cut in fig.7a

The sum of the four diagrams in fig.6 may be split into the convolution of two amplitudes, with external lines on shell, each describing a three-body-quark interaction as represented in fig.8.

$$\begin{aligned}
 & \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} \right) \\
 &= \left(\text{diagram 5} + \text{diagram 6} \right) \otimes \left(\text{diagram 7} + \text{diagram 8} \right) = \text{diagram 9} \otimes \text{diagram 10}
 \end{aligned}$$

Fig.8: Leading term of the three-body cut amplitude

Each of the three-body-quark amplitudes, which appear in the left hand side of the convolution in the second line in fig.8, is expressed by the sum of two terms and these two terms differ only because of the ordering of the gluon lines attached to the projectile. By using the eikonal rule mentioned above the leading term in the three-body-quark amplitude is given by the configuration with the intermediate

projectile on shell. If β is the fractional light cone component of the gluon line, the leading term from the propagator of the projectile quark in each of these the three-body amplitudes is proportional to:

$$\frac{1}{\beta + i\epsilon} + \frac{1}{-\beta + i\epsilon} = \frac{P}{\beta} - i\pi\delta(\beta) - \frac{P}{\beta} - i\pi\delta(\beta) \quad (15)$$

which shows how the leading terms of the real parts of the two intermediate quark propagators interfere destructively while the imaginary parts add constructively. The consequence is that each of the three-body amplitudes in the second line in fig.8 reduces to a product of two-body amplitudes with external lines on shell. The relation is represented graphically in the last line of fig.8, where the cross means that one has to take twice the imaginary part of the intermediate quark propagator. As a summary the whole vertical cut of the three body amplitude, represented by the set of diagrams in fig.5, which is given by the sum of the cut diagrams in fig.6, reduces to the convolution of the much simpler three-body-parton amplitudes shown in fig.8. In the dominant configuration the three-body amplitudes in fig.8 factorize into a product of two-body amplitudes

$$A(3 \rightarrow 3) = A(2 \rightarrow 2) \otimes A(2 \rightarrow 2)$$

in such a way that the whole cut amplitude is expressed as

$$A(3 \rightarrow 3) \otimes A^*(3 \rightarrow 3) = |A(2 \rightarrow 2)|^2 \otimes |A(2 \rightarrow 2)|^2$$

and therefore it is proportional to the square of a two-body cross section.

A simple modification of the cut just discussed is shown in fig.9. In this context also a cut like the one in fig.9 is meaningful. Indeed the amplitude under consideration is the perturbative part of a much more complicated structure where the external lines are hadrons and nuclei. In fig.9 only the piece of the cut which involves the perturbative part of the interaction is drawn and an s -channel cut can be obtained by continuing the cut in the soft region.

Fig.9: Diagrams for a non leading cut amplitude

There are altogether eight diagrams which contribute to this cut. They can be grouped in pairs. One may consider the first two diagrams in the first line in fig.9. The two diagrams differ because of the left-hand side of the cut. The contribution of the two cut diagrams to the cut amplitude is obtained by convoluting the sum of the two terms in fig.10, which represent the left hand side of the cut, with the right hand side, which is the same for the two diagrams.

Fig.10: Destructively interfering terms in the cut-amplitude in fig.9

The two terms in fig.10 are summed coherently to construct the amplitude at the left hand side of the cut. One may notice that in the first term in fig.10 the high energy incoming projectile radiates a gluon to the final state, leaving the intermediate quark propagator with a negative invariant mass. In the second term in fig.10 the high energy projectile which radiates the gluon goes to the final state without interacting and the invariant mass of the intermediate quark is therefore positive. As a consequence the two terms interfere destructively and the cut amplitude corresponding to the case shown in fig.9 is subleading with respect to the case of the cut represented in fig.6. The difference in the two cases is due to the different result which is obtained when summing the two terms, representing the three body amplitude in each of the parenthesis of the second line in fig.8, and the two terms in fig.10. In the first case one has destructive interference for the real parts and constructive interference for the imaginary parts, as emphasized in Eq.(15). In the second case one has still destructive interference for the real parts, while there is no contribution from the imaginary part, since the configuration with the intermediate quark propagator on shell is not allowed with the kinematics shown in fig.10.

For all other cuts of the amplitude, which involve both exchanged boxes and which are different with respect to the case represented in fig.6, similar considerations hold[9]. The cut in fig.6 is therefore the leading cut of the three body amplitude, given by the set of diagrams in fig.5, where both exchanged boxes are cut. The resulting leading term of this cut amplitude is proportional to the square of a two-body parton cross section.

III.3 Further leading cuts and AGK weights

The different cuts of the three body amplitude can be organized counting the number of cut boxes. The case which has been discussed in the previous paragraph is the one of two cut boxes. A leading cut which involves one box is shown in fig.11.

Fig.11: Diagrams for the leading one-box-cut amplitude

One may notice that the six diagrams in fig.11 correspond to all the possible orderings of the three gluons attached to the projectile on the left side of the cut. When making the sum of the cut diagrams in fig.11 one therefore sums over all permutations of the gluons attached to the projectile-quark. By using Eq.(14) the leading term in the sum is obtained by putting all intermediate quark propagators on shell. As a result the leading term of the cut amplitude gets factorized into a product of two-body amplitudes with external lines on shell. Analogously to the case of the leading cut amplitude discussed in the previous paragraph, one obtains, also in this case, a result proportional to a two body cross section squared. The cut shown in fig.11, together with the other three possibilities of cutting the amplitude in an analogous way, are the only leading cuts of this kind. As in the case of the two box cut amplitude, discussed previously, cuts like the one shown in fig.12 are suppressed by destructive interferences among the different diagrams which contribute to the cut amplitude.

Fig.12: Diagrams for a non-leading one-box-cut amplitude

The reason is essentially that the condition for the cut lines to become physical

does not allow all intermediate propagators of the projectile to reach the mass shell condition. The kinematical configuration which dominates in the eikonal approximation[10] is not approached in this case and, as a consequence, only a subleading contribution is obtained.

The case where no box is cut is shown in fig.13

Fig.13: Diagrams for the no-cut-box amplitude

As for the leading cuts of the amplitude already described one may notice that all orderings of the exchanged quanta are included in the diagrams in fig.13, both on the left and on the right side of the cut. As previously shown the amplitude corresponding to the sum of the box and the crossed box diagrams admits a dominant term which corresponds to the mass shell condition for the intermediate particles. The dominant contribution is therefore obtained by the configuration with the intermediate quark propagators on shell. As a consequence one may factorize the cut amplitude as a product of two-body amplitudes with on shell external lines. The leading term is therefore proportional to a two-body quark cross section squared, like in the previous cases. One may therefore conclude that *the dominant terms of the leading cut amplitudes are all proportional to one another*, which is the basic feature required by the cutting rules.

The different leading cut amplitudes are compared below, in order to show how the AGK weights are obtained.

1) *Two box cut.* The leading cut amplitude corresponds to the cut in fig.14

Fig.14: Diagrams for the leading two-box cut amplitude

- Sign is +, it is a modulus squared.
- Multiplicity is 1, the only contribution is from the diagrams in fig.14

- $(2\pi)^5$ factor, a factor 2π for each quark propagator: three quark propagators are cut (a factor 2π for each of them as from the definition of cut) and two are put on shell by a pinch. As one may see in Eq.(15) a pinch gives twice the factor $i\pi\delta$.

2) *One box cut.* The leading cut amplitudes correspond to the cuts in fig.15

Fig.15: The four different one-box-cut amplitudes

- Sign is -: By using the Landau-Cutkosky rules, as from Itzykson and Zuber[11], the propagator is $i/(p^2 - m^2 + i\epsilon)$. The principal value of the propagator is therefore imaginary, while the contribution of the pole is real. When going thru the cut one has to take the complex conjugate. As a consequence the contribution from the pole is not changed while the contribution from the principal value changes sign. In the dominant kinematical configuration the leading term from each horizontal propagator is the contribution of the pole, while the leading term from each vertical propagator is the principal value. When moving from the leading two-box cut amplitude to the leading one-box cut amplitude, one vertical line goes thru the cut. As a consequence the sign of the one-box cut amplitude is opposite with respect to the sign of the two-box cut amplitude. Namely it is negative.
- Multiplicity is 4, as shown in fig.15 there are four different ways to cut the amplitude
- $\pi \times (2\pi)^4$ factor. There is a factor 2 less with respect to the previous case because each diagram with an uncut box is accompanied by another one where the box is replaced by a crossed box. As discussed in the second paragraph of the present note, when performing the sum of the box and of the crossed box diagram both quark propagators in the diagram are put on shell and the overall factor obtained from the integrations on the longitudinal variables is $2\pi^2$.

3) *No cut box.* The cuts are shown in fig.16

Fig.16: The two different no-cut-box amplitudes

- Sign is +, it is a modulus squared.
- As shown in fig.16 the multiplicity is 2.
- $\pi^2 \times (2\pi)^3$ factor. There are two factors 2 less now because both boxes are uncut. In fact for each uncut box in each diagram there is another diagram which differs because the box is replaced by a crossed box.

The AGK weights are therefore obtained: if the no-box cut amplitude is normalized to +1, the one-box cut amplitude is normalized to -4 and the two-box cut amplitude is normalized to +2. The sum of all the leading cut amplitudes gives as a result the opposite of the two-box cut amplitude divided by two.

IV Cutting rules and probabilistic picture of the interaction

IV.1 Hadron-nucleus interaction in the optical limit

As an heuristic example of the use of the cutting rules in high energy hadron-nucleus collisions, the simplest case of the Glauber description of the interaction will be discussed in the present paragraph. The forward elastic scattering amplitude of a high energy hadron-nucleus collision may be expressed as:

$$f_A(q) = \frac{ik}{2\pi} \int d^2b e^{iqb} \left(1 - e^{-\frac{\sigma_T}{2} T(b)} \right) \quad (16)$$

where q is the momentum transfer. k is the energy of the incoming hadron in the nuclear c.m. system. b is the impact parameter. $\sigma_T/2$ is the forward hadron-nucleon scattering amplitude, expressed as a function of the total hadron-nucleon cross section by using the optical theorem and neglecting the real part of the amplitude. $T(b)$ is the nuclear thickness function, namely the integral of the nuclear density at fixed impact parameter and normalized to the atomic mass number A . The normalization of the scattering amplitudes is such that the modulus square of f_A gives $d\sigma/d\Omega$. The exponential in Eq.(16) is obtained by performing the limit $A \rightarrow \infty$ in the forward hadron-nucleus amplitude expressed according with Glauber and Gribov[12]. By using the optical theorem one may write the total hadron-nucleus cross section:

$$\begin{aligned} \sigma_T^A &= \frac{4\pi}{k} \text{Im} f_A(0) \\ &= 2 \int d^2b \left(1 - e^{-\frac{\sigma_T}{2} T(b)} \right) \end{aligned} \quad (17)$$

and, by squaring the forward amplitude and integrating on the solid angle, one obtains the elastic cross section:

$$\sigma_{el}^A = \int d^2b \left(1 - e^{-\frac{\sigma_T}{2} T(b)} \right)^2 \quad (18)$$

One may construct $\sigma_T^A - \sigma_{el}^A$, which is therefore expressed as:

$$\sigma_T^A - \sigma_{el}^A = \int d^2b \left(1 - e^{-\sigma_T T(b)} \right) \quad (19)$$

In the black disk limit Equations (17) and (18) give for the total cross section a value that is twice the value of the elastic one. The factor which relates the integral on the impact parameter and the cross section is therefore different for σ_T^A and for $\sigma_T^A - \sigma_{el}^A$. The argument of the integral on b is, in both cases, a dimensionless number smaller than one. One may write:

$$\begin{aligned} \sigma_T^A &= 2 \int d^2b \left(1 - e^{-\frac{\sigma_T}{2} T(b)} \right) \\ &= 2 \int d^2b \sum_{n=1}^{\infty} \frac{\left(\frac{\sigma_T}{2} T(b) \right)^n}{n!} e^{-\frac{\sigma_T}{2} T(b)} \end{aligned} \quad (20)$$

and

$$\begin{aligned} \sigma_T^A - \sigma_{el}^A &= \int d^2b \left(1 - e^{-\sigma_T T(b)} \right) \\ &= \int d^2b \sum_{n=1}^{\infty} \frac{(\sigma_T T(b))^n}{n!} e^{-\sigma_T T(b)} \end{aligned} \quad (21)$$

In Eq(20) and in Eq(21) σ_T^A and $\sigma_T^A - \sigma_{el}^A$ are both expressed as a sum of positive terms whose sum is explicitly bounded by one. The question which arises is whether the terms may be understood as probabilities of multiple interactions. This is indeed the case as one may realize with the help of the cutting rules. One may check, in fact, that the probabilistic picture is consistent, at the level of double interactions, by comparing the expansions in Eqs.(20, 21) with the AGK weights discussed in the previous paragraph. In the expression of the total cross section, represented as a Poissonian probability distribution of interactions, $i\frac{\sigma_T}{2}$ is the forward amplitude of the hadron-nucleon interaction and $\frac{\sigma_T}{2} T(b)$ is the average number of interactions. If the hadron-nucleon interaction is described by the exchange of a Pomeron, the probability distribution which constructs the nuclear total cross section, as expressed in Eq.(20), is therefore a probability distribution in multiple uncut Pomeron exchanges. The term which corresponds to the exchange of *two uncut Pomerons* is:

$$2 \times \left(\frac{\sigma_T}{2} T(b) \right)^2 \frac{1}{2!} \times \left(1 + \mathcal{O}\left(\frac{\sigma_T}{2} T(b)\right) \right) = +1 \times \left(\frac{\sigma_T}{2} T(b) \right)^2 + \mathcal{O}(\sigma_T T(b))^3 \quad (22)$$

In Eq.(21) $\sigma_T^A - \sigma_{el}^A$ is represented as a Poissonian probability distribution of cut exchanged Pomerons: If $i\sigma_T/2 = f(0)$ is the forward amplitude of the hadron-nucleon interaction, $\sigma_T = 2\text{Im}f(0)$ corresponds to the cut-Pomeron exchange.

$\sigma_T T(b)$ is therefore the average number of cut-Pomeron exchanged with the nucleus by the incoming hadron at impact parameter b . in Eq.(20) the term which gives the probability of one cut-Pomeron exchange is:

$$(\sigma_T T(b)) e^{-\sigma_T T(b)} = \sigma_T T(b) (1 - \sigma_T T(b)) + \mathcal{O}(\sigma_T T(b))^3$$

and the term from the *one cut-Pomeron* exchange at order $(\sigma_T T(b))^2$ is:

$$-(\sigma_T T(b))^2 = -4 \times \left(\frac{\sigma_T}{2} T(b) \right)^2 \quad (23)$$

In Eq.(21) the term which represents the *two cut-Pomeron* exchange is given, at order $(\sigma_T T(b))^2$ by:

$$\frac{1}{2!} (\sigma_T T(b))^2 = +2 \times \left(\frac{\sigma_T}{2} T(b) \right)^2 \quad (24)$$

One may recognize in the coefficients in Eq.s (22, 23, 24) the AGK weights for the double interaction, $+1$, -4 and $+2$. The sum of all terms at order $(\sigma_T T(b))^2$ gives the corresponding contribution to the total cross section:

$$(1 - 4 + 2) \times \left(\frac{\sigma_T}{2} T(b) \right)^2 = - \left(\frac{\sigma_T}{2} T(b) \right)^2 \quad (25)$$

as one may verify by expanding directly the total cross section as expressed in Eq.(17):

$$2 \left(1 - e^{-\frac{\sigma_T}{2} T(b)} \right) \rightarrow 2(-) \frac{1}{2!} \left(-\frac{\sigma_T}{2} T(b) \right)^2 = - \left(\frac{\sigma_T}{2} T(b) \right)^2 \quad (26)$$

IV.2 Self shadowing cross sections

The probabilistic picture of the interaction allows one to understand, at high energy, $\sigma_T^A - \sigma_{el}^A$ as the simple superposition of hadron-nucleon events. It is consequently possible to recognize that there is a whole class of hadronic events which give rise to a rather peculiar expression for the corresponding cross section on a nucleus. In fact one may classify all hadron-nucleon events according with some criterion $\mathcal{C}$. All events of kind $\mathcal{C}$ contribute to $\sigma_{\mathcal{C}}$, all other events contribute to $\sigma_{\mathcal{N}}$ and the total hadron-nucleon cross section is given by

$$\sigma_T = \sigma_{\mathcal{C}} + \sigma_{\mathcal{N}} \quad (27)$$

The problem is to express $\sigma_{\mathcal{C}}^A$, namely the cross section for events of kind $\mathcal{C}$ on a nuclear target. One may suppose that $\mathcal{C}$ is such that any superposition of an event of kind $\mathcal{C}$ both with events of kind $\mathcal{C}$ and with events of kind $\mathcal{N}$ gives always rise to an event of kind $\mathcal{C}$. If this property holds then the expression for $\sigma_{\mathcal{C}}^A$ is particularly

simple. $\sigma_T^A - \sigma_{el}^A$ is given by a Poissonian superposition of inelastic hadron-nucleon events, one may analyze the term in the sum where n hadron-nucleon events are superposed:

$$\sigma_T^n = (\sigma_C + \sigma_N)^n = \sum_{k=0}^n \binom{n}{k} \sigma_C^k \sigma_N^{n-k} \quad (28)$$

All terms in the sum in Eq(28), with the only exception of the term $k = 0$, contribute to σ_C^A . One can obtain σ_C^A by making the superposition of all events which contribute:

$$\sigma_C^A = \int d^2b \sum_{n=1}^{\infty} \left[\sum_{k=1}^n \binom{n}{k} \sigma_C^k \sigma_N^{n-k} \right] \frac{(T(b))^n}{n!} e^{-\sigma_T T(b)} \quad (29)$$

the sum on k is easily done:

$$\sum_{k=1}^n \binom{n}{k} \sigma_C^k \sigma_N^{n-k} = \sigma_T^n - \sigma_N^n$$

and the sum on n gives as a result:

$$\sigma_C^A = \int d^2b \left(1 - e^{-(\sigma_T - \sigma_N)T(b)} \right) \quad (30)$$

The expression for σ_C^A is therefore:

$$\begin{aligned} \sigma_C^A &= \int d^2b \left(1 - e^{-\sigma_C T(b)} \right) \\ &= \int d^2b \sum_{n=1}^{\infty} \frac{(\sigma_C T(b))^n}{n!} e^{-\sigma_C T(b)} \end{aligned} \quad (31)$$

which shows how σ_C^A is given by a Poissonian distribution of multiple hadron-nucleon events of kind C , precisely as $\sigma_T^A - \sigma_{el}^A$ is expressed by a Poissonian distribution of multiple hadron-nucleon events. This property of the events of kind C is called *self shadowing*[4]. A simplest example of σ_C is given by σ_{inel} , after neglecting the possibility for an inelastic state, produced in an interaction on a nucleon, to make transitions to the elastic channel in further interactions. As a consequence one may write for the inelastic hadron-nucleus cross section σ_{inel}^A the relation:

$$\begin{aligned} \sigma_{inel}^A &= \int d^2b \left(1 - e^{-\sigma_{inel} T(b)} \right) \\ &= \int d^2b \sum_{n=1}^{\infty} \frac{(\sigma_{inel} T(b))^n}{n!} e^{-\sigma_{inel} T(b)} \end{aligned} \quad (32)$$

A further property which holds in this context is the *AGK cancellation*[1,13]: Rather than computing σ_C^A one may compute $\langle n \rangle \sigma_C^A$, where $\langle n \rangle$ is the average number of interactions of kind C on the nucleus:

$$\begin{aligned} \langle n \rangle \sigma_C^A &= \int d^2b \sum_{n=1}^{\infty} n \frac{(\sigma_C T(b))^n}{n!} e^{-\sigma_C T(b)} = \int d^2b \sigma_C T(b) \\ &= A \sigma_C \end{aligned} \quad (33)$$

Because of the Poisson distribution in multiple interactions the average number is given by the single scattering term. To construct σ_C^A from σ_C one cannot simply multiply σ_C by the atomic mass number A , one needs to take into account unitarity corrections, which give rise to the expression for σ_C^A as given in Eq.(31). The remarkable property is that when computing $\langle n \rangle \sigma_C^A$, rather than σ_C^A , all unitarity corrections cancel and the naive result holds. A quantity which is directly related to $\langle n \rangle$ has therefore a simple expression also in the very complicated context of the multiple interactions of a high energy hadron with a nucleus. A quantity of this sort is for example the inclusive cross section to produce a high energy particle in the central rapidity region. Because of the inclusive nature of the cross section, produced particles are counted with their own multiplicity and a term, corresponding to a given number of hadron-nucleon interactions in the inelastic cross section, is therefore multiplied by the multiplicity of interactions. Hence the inclusive cross section on a nucleus is given directly by the inclusive cross section on a nucleon multiplied by the atomic mass number A [14]. As one can consider inclusive cross sections which, being proportional to $\langle n \rangle$, are given directly by the single scattering term, one may analogously consider double inclusive cross sections proportional to $\langle n(n-1) \rangle$ and therefore proportional to the double scattering term, without any shadowing correction. One has hence the possibility of exploiting the complexity of the interaction by moving from the simplest quantities, those which are sensible only to the average number of interactions, to quantities which are sensible to increasingly detailed aspects of the interaction, being related to higher moments.

V. Semi-hard interactions

An event is called semi-hard[15] if, in the overall hadronic or nuclear intercourse, there is at least one elementary partonic interaction with momentum exchange larger than a lower threshold $q_t^{min} \approx \text{few GeV}$, such as to allow the use of perturbative QCD. The rule to identify semi-hard events is an example of the criterion $\mathcal{C}$ discussed in the previous paragraph. To claim that semi-hard interactions self-shadow, one needs however to show that the requirement, which allows the probabilistic picture of the interaction, namely the cutting rules, relating different cut diagrams, hold also for semi-hard interactions. The proof is missing in the general case. The discussion on the cutting rules in pQCD, in the initial

part of this note, shows however that one of the components of the semi-hard cross section which is leading in the high energy limit satisfies the cutting rules. One may therefore argue that it is a good working hypothesis to assume that, in high energy nuclear collisions, semi-hard interactions self shadow, in such a way that the semi-hard nuclear interaction can be expressed in a probabilistic way as a function of "elementary" semi-hard parton-parton interaction probabilities.

The purpose of the present paragraph is to exploit a few physical features which characterize the interaction in the framework of the self shadowing hypothesis for the semi-hard interactions[16]. Analogously to the case described in the previous paragraph the dimensionality to the cross section is provided by the integration on the impact parameter of the collision, β . The probability of interaction is then given by the product of the probabilities of the parton configurations of the two nuclei A and B , $W(\eta_A)$ and $W(\eta_B)$ (where η a label characterizing the parton configuration) with the probability of interaction of the two configurations $\mathcal{P}(\eta_A, \eta_B)$. One may therefore write:

$$\sigma_H = \int d\beta \sum_{\eta_A} W(\eta_A) \sum_{\eta_B} W(\eta_B) \mathcal{P}(\eta_A, \eta_B) \quad (34)$$

where the sums extend to all possible parton configurations of the two nuclei. Because of the self-shadowing hypothesis $\mathcal{P}(\eta_A, \eta_B)$ can be expressed explicitly as a function of elementary two body parton interaction probabilities. Actually:

$$\mathcal{P}(n, m) = \left\{ 1 - \prod_{i=1}^n \prod_{j=1}^m [1 - \hat{\sigma}(u_i, u'_j)] \right\} \quad (35)$$

where, for simplicity, the label to identify the parton configuration is limited to the number of partons in each configuration only and $\hat{\sigma}(u_i, u'_j)$ is the *probability* for the parton i of the A -nucleus to have an hard interaction with the parton j of the B -nucleus. $u_i \equiv x_i, \mathbf{b}_i$ represents the fractional momentum and transverse coordinate of the parton labeled with i . The probability of interaction of the two configurations is expressed as one minus the probability of not having any semi-hard interaction. The probability of not having any interaction between the two configurations is obtained by making the product of all the probabilities of no interaction of all pairs of partons from the two configurations:

$$1 - \hat{\sigma}(u_i, u'_j)$$

is the probability of no interaction between the parton of nucleus A , labeled with i , and the parton of nucleus B , labeled with j ,

$$\prod_{i=1}^n \prod_{j=1}^m [1 - \hat{\sigma}(u_i, u'_j)]$$

is the probability of no interaction between the two configurations with n A -partons and m B -partons,

$$1 - \prod_{i=1}^n \prod_{j=1}^m [1 - \hat{\sigma}(u_i, u'_j)]$$

is the probability of having at least one semi-hard interaction, namely $\mathcal{P}(n, m)$, which is the same as the sum of all possible semi-hard interactions between the two configurations.

The self-shadowing hypothesis and the probabilistic picture which follows are not sufficient to write the semi-hard nuclear cross section. In fact, differently with respect to soft interactions, where the number of "elementary" interacting objects, which in that case are the nucleons, is fixed, when considering semi-hard interactions and partons as elementary interacting objects, the number is, on the contrary, varying. Actually the configurations with many interacting partons involve the multi-parton distributions[17] that are non perturbative quantities, independent of the single-parton distributions of large p_t physics. The complete description of the nuclear semi-hard interaction needs therefore, as a input, the infinite set of non perturbative quantities, represented by the multi-parton distributions[18]. Nevertheless, for the more limited program to evaluate a few average quantities, much less information is needed as a input. In this case it is very convenient to approach the problem by means of a generating functional formalism[19, 20] which allows one to avoid explicit expressions for the multiparton distributions.

To construct the generating functional for the multi-parton distributions one may start with the exclusive multi-parton distributions, namely, at a given scale provided by the cut off p_t^{min} , the probabilities for each of the configurations with a given number of partons. The $W^{(n)}(u_1 \dots u_n)$ are then the exclusive n -body parton distributions. The generating functional $\mathcal{Z}[J]$ is defined as:

$$\mathcal{Z}[J] = \sum_n \frac{1}{n!} \int J(u_1) \dots J(u_n) W^{(n)}(u_1 \dots u_n) du_1 \dots du_n, \quad (36)$$

where $J(u)$ are auxiliary functions. The normalization condition for the exclusive probabilities gives the value of $\mathcal{Z}[J]$ at $J = 1$:

$$\mathcal{Z}[1] = 1. \quad (37)$$

The exclusive distributions are obtained from the expansion of the generating functional around $J = 0$. The inclusive distributions are obtained by expanding $\mathcal{Z}[J]$ around $J = 1$. Explicitly the one-body and two-body inclusive distributions $D^{(1)}$ and $D^{(2)}$ are given by:

$$\begin{aligned}
D^{(1)}(u) &= W^{(1)}(u) + \int W^{(2)}(u, u') du' + \frac{1}{2} \int W^{(3)}(u, u', u'') du' du'' + \dots \\
&= \left. \frac{\delta \mathcal{Z}}{\delta J(u)} \right|_{J=1}, \\
D^{(2)}(u_1, u_2) &= W^{(2)}(u_1, u_2) + \int W^{(3)}(u_1, u_2, u') du' \\
&\quad + \frac{1}{2} \int W^{(4)}(u_1, u_2, u', u'') du' du'' \dots \\
&= \left. \frac{\delta^2 \mathcal{Z}}{\delta J(u_1) \delta J(u_2)} \right|_{J=1} \equiv D^{(1)}(u_1) D^{(1)}(u_2) + \frac{1}{2} C^{(2)}(u_1, u_2)
\end{aligned} \tag{38}$$

where the two body correlation $C^{(2)}(u_1, u_2)$, which measures the deviation from the Poisson distribution[19, 20], has been introduced.

In fact, to gain a better feeling on the multi-parton distributions, one may consider the simplest case where all correlations are zero. The n -body inclusive parton distributions are expressed in this case as:

$$D^{(n)}(u_1 \dots u_n) = D^{(1)}(u_1) \dots D^{(1)}(u_n) = \left. \frac{\delta^{(n)} \mathcal{Z}}{\delta J(u_1) \dots \delta J(u_n)} \right|_{J=1} \tag{39}$$

The simple factorized expression for the inclusive distributions allows one to perform the sum of the expansion of $\mathcal{Z}[J]$ around $J = 1$:

$$\mathcal{Z}[J] = \exp \left[\int D^{(1)}(u) [J(u) - 1] du \right] \tag{40}$$

and the expansion of $\mathcal{Z}[J]$ around $J = 0$ gives the explicit expression for the exclusive multiparton distributions:

$$W^{(n)}(u_1 \dots u_n) = \frac{D^{(1)}(u_1) \dots D^{(1)}(u_n)}{n!} e^{-\int D^{(1)}(u) du} \tag{41}$$

which, as one expects in absence of correlations, is Poisson-like.

By representing the multiparton distributions as functional derivatives of $\mathcal{Z}[J]$ around $J = 0$ and by using the expression in Eq.(35) for the interaction probability, the semi-hard cross section σ_H assumes the explicit form written below:

$$\sigma_H = \int d\beta \sigma_H(\beta)$$

$$\begin{aligned}
\sigma_H(\beta) = & \int \sum_n \frac{1}{n!} \frac{\delta}{\delta J(u_1)} \cdots \frac{\delta}{\delta J(u_n)} \mathcal{Z}_A[J] \Big|_{J=0} \\
& \times \sum_m \frac{1}{m!} \frac{\delta}{\delta J'(u'_1 - \beta)} \cdots \frac{\delta}{\delta J'(u'_m - \beta)} \mathcal{Z}_B[J'] \Big|_{J'=0} \\
& \times \left\{ 1 - \prod_{i=1}^n \prod_{j=1}^m [1 - \hat{\sigma}(u_i, u'_j)] \right\} \prod du du'
\end{aligned} \tag{42}$$

The expression of σ_H in Eq.(42) is completely general within the probabilistic description of the semi-hard interaction. The only limitation is in the restriction to the case of two-body parton interaction, more precisely the interaction between two parton configurations is assumed to be factorizable as a sum of products of two body parton interactions. In particular Eq.(42) is completely general as far as the multiparton distributions are concerned. Indeed some features of the interaction may be obtained from the expression in Eq.(42) without making any specific assumption. In fact one may evaluate the *average number of semi-hard interactions*.

To obtain the average number of semi-hard partonic collisions one expands the interaction probability as a sum of multiple interactions. Although one may be completely general[20], for simplicity, one may consider the heuristic case where all elementary interaction probabilities are equal to one another: $\hat{\sigma}_{i,j} \equiv \hat{\sigma}$. The interaction probability may be expanded in multiple interactions in the following way:

$$1 - \prod_{N=1}^Q (1 - \hat{\sigma}) = \sum_{N=1}^Q \binom{Q}{N} \hat{\sigma}^N (1 - \hat{\sigma})^{(Q-N)} \tag{43}$$

where the index N counts the possible interactions, in such a way that $Q = nm$. The average number of partonic collisions at a given value of Q , $\langle N \rangle_Q$, is given by:

$$\langle N \rangle_Q = \sum_{N=1}^Q N \binom{Q}{N} \hat{\sigma}^N (1 - \hat{\sigma})^{(Q-N)} = Q \hat{\sigma} \tag{44}$$

The average number of semi hard interactions, at a given value of the impact parameter β , is obtained by replacing the interaction probability $\mathcal{P}(n, m)$ with $\langle N \rangle_Q = Q \hat{\sigma} = nm \hat{\sigma}(u_1, u'_1)$ in the expression of $\sigma_H(\beta)$, as given in Eq.(42):

$$\begin{aligned}
\langle N(\beta) \rangle &= \int \sum_n \frac{1}{n!} \frac{\delta}{\delta J(u_1)} \cdots \frac{\delta}{\delta J(u_n)} \mathcal{Z}_A[J] \Big|_{J=0} \times n \\
&\times \sum_m \frac{1}{m!} \frac{\delta}{\delta J'(u'_1 - \beta)} \cdots \frac{\delta}{\delta J'(u'_m - \beta)} \mathcal{Z}_B[J'] \Big|_{J'=0} \times m \\
&\times \hat{\sigma}(u_1, u'_1) \prod du du'
\end{aligned} \tag{45}$$

The sums on m and n can be performed explicitly. As a result both arguments of $\mathcal{Z}_A$ and of $\mathcal{Z}_B$ are shifted by one unit. One can then write:

$$\langle N(\beta) \rangle = \int \frac{\delta}{\delta J(u_1)} \mathcal{Z}_A[J+1] \Big|_{J=0} \frac{\delta}{\delta J'(u'_1 - \beta)} \mathcal{Z}_B[J'+1] \Big|_{J'=0} \hat{\sigma}(u_1, u'_1) du_1 du'_1 \tag{46}$$

Keeping in mind that the one-body inclusive parton distribution $D^{(1)}$ is given by:

$$D^{(1)}(u) = \frac{\delta \mathcal{Z}}{\delta J(u)} \Big|_{J=1}$$

one may express $\langle N(\beta) \rangle$ as:

$$\langle N(\beta) \rangle = \int D_A^{(1)}(u) D_B^{(1)}(u') \hat{\sigma}(u, u') du du' \tag{47}$$

which is the *factorized single scattering expression of the pQCD parton model*. Indeed by integrating Eq.(47) on β and by performing the integrations on the transverse parton coordinates one obtains the average number of semi-hard partonic collisions multiplied by the semi-hard nuclear cross section:

$$\langle N \rangle \sigma_H = \int d\beta \langle N(\beta) \rangle = \int_{q_t > q_t^{min}} G_A(x) G_B(x') \sigma(xx') dx dx' \tag{48}$$

where

$$G(x) \equiv \int D^{(1)}(x, b) d^2 b \tag{49}$$

is the average number of partons with fractional momentum x and

$$\sigma(xx') \equiv \int_{q_t > q_t^{min}} \hat{\sigma}(xx', \mathbf{b} - \mathbf{b}') d^2(\mathbf{b} - \mathbf{b}') \tag{50}$$

is the integrated parton-parton cross section with the lower cut-off in the momentum transfer q_t^{min} .

The factorized expression of the perturbative QCD parton model is therefore shown to be meaningful in a much wider context than the regime of large p_t physics

where it is usually employed. It holds also in the regime of semi-hard nuclear collisions, where the number of semi-hard partonic interactions in a single event is very large and also the number of semi-hard interactions of every interacting parton is large. The quantities which enter in the single scattering factorized expression acquire in this context the meaning of average quantities and also the integral in Eq.(48) gives an average number which multiplies the cross section rather than the cross section itself, as in the large p_t regime. One needs to emphasize that the factorized expression Eq.(47), as it was obtained with a functional method, *holds at any order in the multi-parton correlations*, therefore it is completely general in the probabilistic approach discussed here. A further observation is that the single scattering term representing the average number of collisions, namely Eq.(47), is precisely the statement of the AGK cancellation discussed in the previous paragraph.

An analogously general result can be obtained for the average number of collisions of each parton. The sum over m in Eq.(42) can be performed explicitly and $\sigma_H(\beta)$ is expressed as:

$$\begin{aligned} \sigma_H(\beta) = & \int \sum_n \frac{1}{n!} \frac{\delta}{\delta J(u_1)} \cdots \frac{\delta}{\delta J(u_n)} \mathcal{Z}_A[J] \Big|_{J=0} \\ & \times \left\{ 1 - \prod_{i=1}^n \mathcal{Z}_B[1 - \hat{\sigma}(u_i, \cdot)] \right\} \prod_{i=1}^n du_i \end{aligned} \quad (51)$$

In Eq.(51) every configuration with a given number of partons from the A -nucleus is required to interact at least once with nucleus B . The probability for one of the partons from A , which we label with i , not to interact with the whole nucleus B is $\mathcal{Z}_B[1 - \hat{\sigma}(u_i, \cdot)]$. The probability for the A -parton with label i to have at least one semi-hard interaction with nucleus B is therefore given by:

$$1 - \mathcal{Z}_B[1 - \hat{\sigma}] \quad (52)$$

where, for simplicity, although one might consider the most general case, the elementary interaction probability $\hat{\sigma}$ has been taken as a constant. After writing $1 = \mathcal{Z}_B[1]$, the expression in Eq.(52) can be written as an expansion of $\mathcal{Z}_B[J']$ around $J' = 0$:

$$1 - \mathcal{Z}_B[1 - \hat{\sigma}] = \int \sum_m \frac{1}{m!} \frac{\delta}{\delta J'_1} \cdots \frac{\delta}{\delta J'_m} \mathcal{Z}_B[J'] \Big|_{J'=0} [1 - (1 - \hat{\sigma})^m] \prod du' \quad (53)$$

The interaction probability in Eq.(53), namely $[1 - (1 - \hat{\sigma})^m]$, can be expanded in multiple interactions:

$$[1 - (1 - \hat{\sigma})^m] = \sum_{k=1}^m \binom{m}{k} \hat{\sigma}^k (1 - \hat{\sigma})^{(m-k)} \quad (54)$$

To determine the average number of interactions one needs to multiply by k each term in the sum on k :

$$\langle k \rangle_m = \sum_{k=1}^m k \binom{m}{k} \hat{\sigma}^k (1 - \hat{\sigma})^{(m-k)} = m \hat{\sigma} \quad (55)$$

By replacing the interaction probability in Eq.(53) with $\langle k \rangle_m$ the average number of interactions of the parton under consideration with the target nucleus B is obtained:

$$\begin{aligned} \langle k(u) \rangle_B &= \int \sum_m \frac{1}{m!} \frac{\delta}{\delta J'_1} \cdots \frac{\delta}{\delta J'_m} \mathcal{Z}_B[J'] \Big|_{J'=0} m \hat{\sigma}(u, u'_1) \prod du' \\ &= \int \frac{\delta \mathcal{Z}_B[J']}{\delta J'(u' - \beta)} \Big|_{J'=1} \hat{\sigma}(u, u') du' \\ &= \int D_B^{(1)}(u' - \beta) \hat{\sigma}(u, u') du' \end{aligned} \quad (56)$$

After integrating on the transverse coordinate b' the average number of interactions is finally expressed as:

$$\langle k(x, b) \rangle_B = \int_{xx's > 4q_t^{min}} D_B^{(1)}(x', \mathbf{b} - \beta) \sigma(xx') dx' \quad (57)$$

with $\sigma(xx')$ the elementary parton-parton cross section.

One may notice that the expression for $\langle k(x, b) \rangle_B$ in Eq.(57) is completely general and, analogously to Eq.(47) for $\langle N(\beta) \rangle$, it represents the AGK cancellation for the parton-nucleus case. One may therefore write:

$$\begin{aligned} \langle N(\beta) \rangle &= \int D_A^{(1)}(u) D_B^{(1)}(u') \hat{\sigma}(u, u') du du' \\ &= \int D_A^{(1)}(u) \langle k(u) \rangle_B du \\ &= \int \langle k(u') \rangle_A D_B^{(1)}(u') du' \end{aligned} \quad (58)$$

which expresses the average number of semi-hard interactions, in a nuclear collision at impact parameter β , as the product of the average number of partons in one nucleus multiplied by the average number of semi-hard interactions that each parton undergoes with the other nucleus. A further observation is that, in the simplest instance of uncorrelated multi-parton distributions, the parton-nucleus cross section can be expressed in closed form. Actually, in absence of correlations, one may write:

$$\mathcal{Z}[J] = \exp \left[\int D^{(1)}(u) [J(u) - 1] du \right]$$

and therefore

$$1 - \mathcal{Z}_B[1 - \hat{\sigma}(u, \cdot)] = 1 - \exp\left[\int -D^{(1)}(u')\hat{\sigma}(u, u')du'\right] \quad (59)$$

By looking at the expression for $\sigma_H(\beta)$ in Eq.(42) one realizes that, on the contrary, $\sigma_H(\beta)$ cannot be expressed in a closed form even in the simplest case of uncorrelated multi-parton distributions.

VI. Eikonal unitarization and infrared behaviour

The pQCD parton model single scattering expression which has been discussed at length in the previous paragraph was introduced to describe large p_t phenomena[21]:

$$\langle N \rangle \sigma_H = \int_{q_t > q_t^{min}} G_A(x) G_B(x') \sigma(xx') dx dx' \quad (60)$$

When in Eq.(60) q_t^{min} grows as $\sqrt{s}$, $\langle N \rangle \sigma_H$ behaves as $1/(q_t^{min})^2$. In each inelastic event there is at most one hard parton interaction, in such a way that $\langle N \rangle$ is equal to one and Eq.(60) gives directly σ_H . When the cut off is moved towards smaller values, on the contrary, the expression in Eq.(60) gives larger and larger values, which soon exceed the value of the inelastic cross section σ_{inel} . In this regime $\langle N \rangle$ becomes larger than one and a direct relation between the single scattering expression and the semi-hard cross section σ_H is lost[22]. A way to pose the problem is to say that, for small values of q_t^{min} , the expression for σ_H , given by Eq.(60) with $\langle N \rangle = 1$, becomes too large and the cross section needs to be unitarized. To proceed further the relevant observation is that

$$\langle N \rangle \sigma_H = \int_{q_t > q_t^{min}} G_A(x) G_B(x') \sigma(xx') dx dx' = \int d\beta \langle N(\beta) \rangle \quad (61)$$

To obtain the cross section σ_H , rather than $\langle N \rangle \sigma_H$, one needs to replace in Eq.(61) the average number of semi-hard interactions $\langle N(\beta) \rangle$ with the interaction probability. The simplest possibility is the following: For each value of the impact parameter of the collision β there are $\langle N(\beta) \rangle$ elementary semi-hard interactions. If all these elementary interactions are independent one to another the probability distribution of multiple interactions is a Poissonian. If this is the case the probability of interaction is $1 - \exp[-\langle N(\beta) \rangle]$ and one may simply write:

$$\sigma_H = \int d^2\beta \left[1 - \exp(-\langle N(\beta) \rangle) \right] \quad (62)$$

Even if $\langle N(\beta) \rangle$ becomes infinitely large σ_H , as expressed in Eq.(62), remains finite. In the limiting case of large $\langle N(\beta) \rangle$ the size to σ_H is given by geometry: in fact

if $\langle N(\beta) \rangle \rightarrow \infty$ the exponential in Eq.(62) goes to zero and the argument of the integral on β becomes one. However when the impact parameter becomes too large there is no overlap in the transverse plane between the two interacting nuclei. As a consequence when β becomes larger than some critical value, which is roughly the sum of the two nuclear radii, $\langle N(\beta) \rangle$ drops to zero also if the lower cut off q_t^{min} is very small. For those values of β the exponential gives one and there are no more contributions to the integral on β . The resulting value for σ_H is therefore roughly $\pi(R_A + R_B)^2$, with R_A and R_B the radii of the two interacting nuclei.

The approach just described is equivalent to the one followed to include semi-hard interactions in the eikonal models for high energy hadronic and nuclear interactions[23]. The expression for σ_H as given in Eq.(62), is obtained in the eikonal models when writing the cross section to have at least one semi-hard elementary parton interaction while allowing any number of soft interactions. Differently with respect to the case of eikonal models, in the self-shadowing approach to semi-hard interactions, which has been described in the previous paragraph, one does not obtain the relation in Eq.(62) for σ_H . Indeed one does not find any closed analytic expression for σ_H , even if one makes the drastic simplifying hypothesis that all multi-parton correlations are equal to zero. To obtain the expression for σ_H given in Eq.(62) within the self-shadowing approach, one needs to neglect both all multi-parton correlations and also all parton rescatterings: After making the hypothesis of no multi-parton correlations, one needs in fact to identify the parton-nucleus semi-hard interaction probability, Eq.(52), with the average number of parton-nucleus elementary semi-hard interactions, Eq.(56). Indeed the literal meaning of the expression for σ_H , as given in Eq.(62), is the one of a Poissonian probability distribution of elementary semi-hard parton interactions:

$$\sigma_H = \int d^2\beta \left[1 - \exp(-\langle N(\beta) \rangle) \right] = \int d^2\beta \sum_{N=1}^{\infty} \frac{\langle N(\beta) \rangle^N}{N!} \exp(-\langle N(\beta) \rangle) \quad (63)$$

One may easily realize which are the multiple interactions taken into account by the eikonal expression if one keeps in mind that

$$\langle N(\beta) \rangle = \int D_A^{(1)}(u) D_B^{(1)}(u') \hat{\sigma}(u, u') du du' \quad (64)$$

In Eq.(63) therefore only those nuclear collisions are included where the number of interacting partons from the two nuclei is the same and it is also equal to the number of elementary semi-hard interactions. As a consequence in Eq.(63) each parton interacts only once.

Although in Eq.(63) σ_H remains bounded also for small values of q_t^{min} , the physical picture implied by Eq.(63) is going to produce plenty of infrared problems. As an example one may look to the average value of semi-hard energy involved in the interaction:

$$\begin{aligned}
\langle E \rangle \sigma_H &= \int d^2\beta \sum_{N=1}^{\infty} \frac{E_1 + \dots + E_N}{N!} \frac{d\langle N(\beta) \rangle}{dE_1} \dots \frac{d\langle N(\beta) \rangle}{dE_N} e^{-\langle N(\beta) \rangle} dE_1 \dots dE_N \\
&= \int d^2\beta \sum_{N=1}^{\infty} \frac{1}{(N-1)!} \int E_1 \frac{d\langle N(\beta) \rangle}{dE_1} dE_1 \langle N(\beta) \rangle^{N-1} e^{-\langle N(\beta) \rangle}
\end{aligned} \tag{65}$$

which gives

$$\langle E \rangle \sigma_H = \int E \frac{d\sigma_s}{dE} dE \tag{66}$$

where $d\sigma_s/dE$ is the pQCD parton model single scattering expression. The point is that the Poisson distribution of multiple interactions allows one to write an infrared safe expression for σ_H , namely Eq.(62) which is obtained from Eq.(61) by replacing the average number of interactions $\langle N(\beta) \rangle$ with the interaction probability $1 - \exp(-\langle N(\beta) \rangle)$. However when any average quantity is evaluated from the expression of the semi-hard cross section, because of the Poissonian probability distribution of multiple interactions, the result which is obtained is precisely the average done by taking only the single scattering term. All average quantities are therefore affected by the same infrared problem which affects the pQCD parton model single scattering expression, as if no unitarization of the semi-hard cross section were performed. Since every differential cross section may be constructed from averages and higher moments, one may conclude that the eikonalization of the semi-hard cross section is able to solve the infrared problem of the integrated semi-hard cross section σ_H only.

In this respect the situation is quite different in the self-shadowing approach to semi-hard interactions. In fact a remarkable property of the self-shadowing approach is that it is able to screen the infinite quantities which are obtained in the infrared region in the eikonal approach. As an explicit example one may take the case of the average energy which goes into semi-hard interactions. There are two symmetric contributions, from nucleus A and from nucleus B . The contribution from nucleus B is given by:

$$\begin{aligned}
\langle E \rangle_{B\sigma_H} &= \frac{\sqrt{s}}{2} \int d\beta \sum_n W_A^{(n)}(u_1 \dots u_n) \times \sum_m W_B^{(m)}(u'_1 - \beta \dots u'_m - \beta) \\
&\quad \times (x'_1 \mathcal{P}_A(n, 1') + \dots + x'_m \mathcal{P}_A(n, m')) \prod du du'
\end{aligned} \tag{67}$$

where W represent the probability for the nuclear partonic configurations and $\mathcal{P}_A(n, j)$ is the probability for the parton with label j from nucleus B to have at least one semi-hard interaction with the configuration of n partons of nucleus A which is realized with probability $W_A^{(n)}(u_1 \dots u_n)$. More explicitly:

$$\begin{aligned}
\langle E \rangle_{B\sigma_H} &= \frac{\sqrt{s}}{2} \int d\beta \sum_n \frac{1}{n!} \frac{\delta}{\delta J(u_1)} \cdots \frac{\delta}{\delta J(u_n)} \mathcal{Z}_A[J] \Big|_{J=0} \\
&\quad \times \sum_m \frac{1}{m!} \frac{\delta}{\delta J'(u'_1 - \beta)} \cdots \frac{\delta}{\delta J'(u'_m - \beta)} \mathcal{Z}_B[J'] \Big|_{J'=0} \\
&\quad \times \sum_{j=1}^m x'_j \left\{ 1 - \prod_{r=1}^n [1 - \hat{\sigma}_{r,j}(u, u')] \right\} \prod du du' \\
&= \frac{\sqrt{s}}{2} \int d^2\beta \left\{ 1 - \mathcal{Z}_A[1 - \hat{\sigma}(\cdot, u')] \right\} x' D_B^{(1)}(x', b' - \beta) dx' db'
\end{aligned} \tag{68}$$

The result, last line in Eq.(68), is that the average fraction of energy of the B -nucleus, which goes into semi-hard interactions, is obtained by integrating on x' the fraction of energy carried by the B -partons multiplied by the probability for one B -parton, with given fractional momentum x' , to interact with the target nucleus A . In the simplest case of no multiparton correlations one may write a closed analytic expression:

$$\begin{aligned}
\langle E \rangle_{B\sigma_H} &= \frac{\sqrt{s}}{2} \int_{xx's > 4(q_t^{min})^2} dx' d^2\beta d^2b' \quad x' D_B^{(1)}(x', b' - \beta) \\
&\quad \times \left[1 - e^{-\int D_A^{(1)}(x, b') \hat{\sigma}(x, x') dx} \right]
\end{aligned} \tag{69}$$

which is finite in the limit of small q_t^{min} . When computing the same quantity with the eikonal model, the square parenthesis in Eq.(69) is replaced by $\int D_A^{(1)}(x, b') \hat{\sigma}(x, x') dx$, which represents the average number of semi-hard interactions of the B -partons with the nucleus A and the overall result is divergent because of the behaviour of the integrand for $x \rightarrow 0$. The self shadowing approach allows one to substitute this average number, which becomes divergent in the infrared limit, with the interaction probability, the square parenthesis in Eq.(69), which is a finite quantity. The feature is analogous to the eikonal unitarization of σ_H discussed at the beginning of this paragraph. In the eikonal unitarization one replaces the average number of interactions $\langle N(\beta) \rangle$ with the interaction probability, expressed as in a Poissonian probability distribution, and, as a result, σ_H is no more divergent. On the other hand in the eikonal unitarization all average quantities remain divergent, since they are given by the average evaluated with the single scattering term. These average quantities may however be related again to average numbers of interactions. In the self shadowing approach, the quantity of physical interest, is obtained by replacing also in this second case the average number of interactions with the corresponding interaction probability and the infrared problem is avoided as a consequence. Obviously the overall average numbers of interactions are still expressed as in Eq.(47) and in Eq.(57) and remain divergent. The point

is however that one has the possibility of defining a physical quantity, which is obtained from the interaction probability rather than from the average number of interactions and, since in the self-shadowing approach the interaction probability is explicitly bounded by one, this last physical quantity remains meaningful also in the infrared limit.

VII. Concluding summary

In the present note the cutting rules have been discussed in the context of perturbation theory. An explicit derivation for a simple case of multi-parton interactions is the argument of the first part of the article, while in the second part some of the consequences of the cutting rules are outlined. A preliminary section is devoted to the behaviour of the parton amplitudes in the large- s fixed- t limit and to gluon Reggeization. The derivation of the cutting rules is sketched in a simplest case of three-body-quark interaction in the third paragraph. The case considered corresponds to the color term, in the three-body-quark amplitude, whose space-time factor is given by the sum of the space-time factors of those Feynman diagrams which have an analogous in the Abelian theory. The main point to obtain the cutting rules is the symmetry of the outcome of the cut amplitude with respect to the different cuts. The resulting leading term at high energy is always the same, apart from the sign and a multiplicity factor. The multiplicity factor is the consequence of counting the possibilities of having a definite kind of cut. The sign may be positive, when the cut amplitude corresponds to a modulus squared, or negative, when it can be understood as an absorptive correction. These two are the only possibilities for every cut amplitude which is leading in the large- s fixed- t limit. To select the leading cut amplitudes and to obtain the actual dominant term a major role is played by the longitudinal dynamics of the projectile quark. Each dominant term corresponds to the kinematical configuration which leads to the eikonal approximation: the projectile is on shell between successive interactions and the momentum exchange has mainly transverse components. The dominance of the eikonal is the result of summing coherently all diagrams with the quanta exchanged in all possible orderings. When the configuration with the intermediate projectile on shell is allowed by the cut of the amplitude, one obtains a leading contribution. When the cut does not allow the intermediate projectile on shell the leading configuration cannot be reached and the distinct diagrams, which contribute to the given cut amplitude, interfere destructively giving rise to a subleading result. Differently with respect to the case of the cutting rules in soft interactions[1], in the simplest example discussed here there is no splitting of the projectile.

Two of the main features which follow from the cutting rules, namely the probabilistic picture for the multiple interaction and the AGK cancellation are outlined in the fourth section. The representation of the nuclear interaction as a

probabilistic superposition of nucleon-nucleon events allows also the identification of physical cross sections which self-shadow: A given class of events, the events of kind $\mathcal{C}$, self-shadow when the nuclear cross section σ_C^A depends only on σ_C , namely on the cross section for events of kind $\mathcal{C}$ at the level of hadron hadron interactions. There is no effect on σ_C^A from events which are not of kind $\mathcal{C}$ at the nucleon level. Semi-hard interactions are the argument of the fifth part of the note. The semi-hard cross section is a self-shadowing cross section and the nuclear semi-hard cross section can be represented as a probabilistic distribution of multiple semi-hard partonic interactions. The non-perturbative inputs to the multi-parton interactions are the multi-parton distributions[17], which are quantities containing an independent information on the parton-bound state with respect to the information provided by the parton structure functions of large p_t physics. To approach the problem of multi-parton distributions in the most general way a functional formalism is introduced in the fifth paragraph. With the help of the functional approach and without any need of assumptions on the multi-parton distributions, the average number of semi-hard interactions, both in nucleus-nucleus and in parton-nucleus collisions, are shown to be expressed exactly by the single scattering term, namely by the perturbative QCD parton model expression for the cross section. The relation is analogous to the AGK cancellation: in the average number of interactions all unitarity corrections cancel and the naive expression holds exactly.

A discussion on the most used eikonal unitarization of the nuclear semi-hard cross section[23] is the argument of the last part. The major point is the observation that, from the point of view of the infrared behaviour, the eikonalization is able to give only one quantity which is infrared safe, actually the integrated semi-hard cross section itself. Every differential cross section, which is singular at the level of single interaction, is singular also after eikonalization. The reason is that the eikonal expression for the semi-hard cross section is equivalent to a Poisson distribution of multiple semi-hard elementary interactions and in a Poisson distribution the average is given by the single scattering term. All average quantities evaluated after eikonalization are, therefore, equal to the quantities evaluated before eikonalizing. This feature is very different in the self-shadowing picture of semi-hard interactions. Although one may find quantities which become very large in the limit of interest, self shadowing allows one to obtain physically meaningful quantities which remain bounded and therefore are well behaved in the infrared limit. With an explicit example the result obtained in the self-shadowing picture of the interaction is compared with the result obtained after the eikonal unitarization in the last paragraph.

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