Elements of Nonlinear Quantum Mechanics (I):
Nonlinear Schrödinger equation and two-level atoms

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Starting with the same form of atomic nonlinearity Weinberg [1] and Wódziewicz and Scully [2] obtained contradictory results concerning an evolution of the atomic inversion \( w \) in a two-level atom in Weinberg’s nonlinear quantum mechanics: If the atom is initially in a ground state then either (1) the evolution of \( w \) can be linear if one uses a nonlinear generalization of the Jaynes-Cummings Hamiltonian, or (2) is always nonlinear if one uses the nonlinear Bloch equations derived from the nonlinear atomic Hamiltonian function. It is shown that the difference is rooted in inequivalent descriptions of the composite “atom+field” system. The linear evolution of \( w \) results from a “faster-than-light communication” between the atom and the field. If one applies a description without the “faster-than-light telegraph” then the calculations based on a suitably modified Jaynes-Cummings Hamiltonian lead to the same dynamics of \( w \) as this found in semiclassical calculations based on Bloch equations. A physical background of the two approaches is discussed in detail, and two, inequivalent kinds of such “telegraphs” are described. It is shown also that a nonlinear quantum mechanics based on a nonlinear Schrödinger equation does not possess a natural probability interpretation. Various definitions of nonlinear eigenvalues and probabilities are discussed and illustrated with examples. Finally, I discuss problems that appear if we apply the above results to the “object+observer” composite system, where the “observer” can evolve in a nonlinear way.

I. INTRODUCTION

The purpose of the generalization of quantum mechanics proposed by S. Weinberg [1,3,4] “was not seriously to propose an alternative to quantum mechanics but only to have some theory whose predictions would be close to but not quite the same as those of quantum mechanics, to serve as a foil that might be tested experimentally” [5]. Next, in a series of very precise experiments the Weinberg’s theory was indeed tested [6], showing no observable deviations from linearity. This was neither particularly surprising, as Weinberg admits in his recent popular book, nor the most discouraging feature of the new framework: “Even if there are small nonlinear corrections to quantum mechanics, there was no reason to believe that these corrections should be just large enough to show up in the initial round of experiments designed to search for them. What I did find disappointing was that this nonlinear alternative to quantum mechanics turned out to have purely theoretical internal difficulties. For one thing, I could not find any way to extend the nonlinear version of quantum mechanics to theories based on Einstein’s special theory of relativity.

Then, after my work was published, both N. Gisin in Geneva and my colleague Joseph Polchinski at the University of Texas independently pointed out that in the Einstein-Podolsky-Rosen thought experiment (...) the nonlinearities of the generalized theory could be used to send signals instantaneously over large distances, a result forbidden by special relativity. At least for the present I have given up the problem; I simply do not know how to change quantum mechanics by a small amount without wrecking it altogether” [5].

This quotation explains almost all purposes of this paper. In its first part I will try to point out the main internal theoretical difficulties of those versions of nonlinear quantum mechanics (NLQM) that are based on a nonlinear Schrödinger equation. We shall see, in particular, that the problem of a correct description of composite systems in NLQM is not only related to the celebrated “faster-than-light telegraphs” mentioned in the above quotation, but may also lead to different predictions for two-level atoms, a basic theoretical tool used in calculations for actual experiments.

Just to illustrate the latter let us notice the following apparent inconsistency. It has been shown in [1] that a system consisting initially of a single photon electromagnetic field and a two-level atom in a ground state, performs ordinary linear Rabi oscillations even in case the atomic Schrödinger equation contains a nonlinear term. On the other hand, the calculations of Wódkiewicz and Scully [2] starting with the same form of nonlinearity but based on the Bloch equations approach lead, with the same initial condition for the atomic inversion, to elliptic hence nonlinear oscillations of the inversion. It will be shown below that the difference is rooted simply in non-equivalent
descriptions of the composite "atom+field" system. The linear evolution occurs if we use the description with an implicit "faster-than-light telegraph", whereas the nonlinear one is found if no such "telegraphs" are present.

The two descriptions are based on different physical assumptions which shall be discussed in detail below. To make matters worse, it will be shown that there does not exist a simple alternative between the two ways of describing compound systems. In fact, we will see that there exists an infinite number of inequivalent and still "correct" possibilities of describing systems such as the the atom and the field, if one assumes that the atom alone evolves according to some nonlinear Schrödinger equation. This arbitrariness not only leads to inequivalent descriptions of optical phenomena but also to fundamental difficulties in interpretation of experiments.

Another non-unique element which additionally obscures the meaning of experiments is related to the probabilistic interpretation of NLQM based on a nonlinear Schrödinger equation. I will show that there does not exist a unique definition of eigenvalues or probabilities (propositions), and those used for interpreting experiments testing Weinberg's theory were not applied in an internally consistent way.

In the second part of the paper [7] I will propose a new framework for NLQM. The proposed generalization is based on a kind of Nambu mechanics where nonlinearities can be introduced without any modification of observables. This formulation of NLQM will be free from the difficulties of the previous approaches, no "faster-than-light telegraphs" will occur, the description of composite systems and the probabilistic interpretation will be unique, and, last but not least, such a theory can be naturally formulated in a relativistic way. The results of both parts are based on the dissertation [8].

The following sections are devoted to a critical presentation of the basic theoretical concepts of NLQM.

II. NONLINEAR QUANTUM MECHANICS À LA KIBBLE, WEINBERG, POLCHINSKI, AND JORDAN

Adding a nonlinear term to the Schrödinger equation does not mean that we have a generalization of quantum mechanics. An evolution equation is only a part of a theory. A complete theory must give a concrete prescription for a translation of theoretical formulas into the language of experiment. In the context of quantum theories we must know first of all what are the mathematical counterparts of random variables, probabilities, and averages measured in a laboratory. In linear quantum mechanics (QM) they are represented by eigenvalues and matrix elements of projectors or self-adjoint operators. In NLQM based on nonlinear Schrödinger equation none of those objects appears in a natural way.

A. States

We will discuss theories where states are represented in the same way as in ordinary quantum mechanics (the so-called normal states): by vectors or rays in a Hilbert space, or by density matrices. We should also keep in mind that possible extensions of QM may involve different manifolds of states [9,10], such as general Kähler manifolds [11-16]. A necessity of some extension of that kind is suggested, for example, by Penrose [17]. We will not include stochastic generalizations either, cf. [18].

B. Observables

In Hamiltonian formulation of the Schrödinger equation (real and imaginary parts of) components of a state vector play a role of symplectic coordinates, and observables are bilinear functionals \( \langle \psi | A | \psi \rangle \), where \( A \) is a self-adjoint operator.

The fact that quantum observables are so specifically chosen leads to a special, complex Hilbertian type of the quantum mechanical probability calculus. It seems it was Mielnik who noticed [19-22] that since there exist many different and inequivalent probability models they may be related to some nonlinear version of quantum mechanics. Such a perspective seems quite natural and suggests that one should try to investigate theories where observables, like in classical mechanics, belong to a more general set of functionals.

In the generalization proposed by Kibble [9,10] the set of observables consists of differentiable functions defined on a projective space. In the Weinberg formulation [1] one assumes that the observables are defined on a Hilbert space but satisfy the \((1,1)\)-homogeneity condition

\[
A(\lambda \psi, \tilde{\psi}) = A(\psi, \lambda \tilde{\psi}) = \lambda A(\psi, \tilde{\psi}).
\]
Averages are defined in [1] like in ordinary quantum mechanics. The (1,1)-homogeneity and the definition of averages introduces the projective structure into the set of states and, effectively, into the dynamics. This is the way the Weinberg approach can be regarded as a variant of the Kibble’s one. (We shall see later that Weinberg’s theory contains also some elements that are new with respect to the Kibble’s proposal.)

The homogeneity condition (1) can be expressed by means of the Euler formula as follows

$$\psi_n \frac{\partial A}{\partial \psi_n} = \psi_n \frac{\partial A}{\partial \psi_n} = A.$$  \hspace{1cm} (2)

With the help of (2) we can deduce some general properties of the observables. Applying (2) twice we find that

$$A = \frac{\partial A}{\partial \psi_n} \psi_n = \frac{\partial^2 A}{\partial \psi_m \partial \psi_n} \psi_n = \frac{\partial^2 A}{\partial \psi_m \partial \psi_n} \psi_n = \langle \psi | \hat{A}(\psi, \tilde{\psi}) | \psi \rangle$$  \hspace{1cm} (3)

and

$$\frac{\partial A}{\partial \psi_m} = \frac{\partial^2 A}{\partial \psi_m \partial \psi_n} \psi_n = (\hat{A}(\psi, \tilde{\psi}) | \psi \rangle)_m.$$  \hspace{1cm} (4)

We can see that the homogeneity condition leads naturally to a nonlinear operator algebra. A matrix multiplication of such nonlinear operators can be easily expressed in terms of first partial derivatives of observables since

$$A \times B := \frac{\partial A}{\partial \psi_m} \frac{\partial B}{\partial \psi_m} = \langle \psi | \hat{A}(\psi, \tilde{\psi}) \hat{B}(\psi, \tilde{\psi}) | \psi \rangle.$$  \hspace{1cm} (5)

This \(\ast\)-product conserves the (1,1)-homogeneity of observables because the nonlinear matrices defined in (3) are (0,0)-homogeneous. The (0,0)-homogeneity implies, on the other hand, that the matrices are, in certain sense, “almost constant”. Indeed,

$$\left( \frac{\partial \hat{A}}{\partial \psi_k} | \psi \rangle \right)_t = \frac{\partial}{\partial \psi_k} \left( \frac{\partial^2 A}{\partial \psi_i \partial \psi_m} \psi_m \right) = \psi_m \frac{\partial}{\partial \psi_m} \frac{\partial^2 A}{\partial \psi_i \partial \psi_k} = 0$$  \hspace{1cm} (6)

by virtue of the 0-homogeneity Euler condition. Consider now a differentiable function \(t \rightarrow \psi_n(t)\). Eq. (6) and its complex conjugate imply that \(\langle \psi | \frac{\partial^2}{\partial \psi_i \partial \psi_m} | \psi \rangle = 0\) for any observable \(A\). The algebra of observables equipped OB with the above \(\ast\)-product possesses a natural left and right unit element \(n(\psi, \tilde{\psi}) = | \psi \rangle \langle \psi |\).

Despite some similarities to ordinary linear operator algebra our algebra of observables is associative if and only if the operators are linear. This leads to several differences with respect to ordinary quantum mechanics. For example, \(\ast\)-powers of \(A\) will not, in general, \(\ast\)-commute with \(A\), a fact that will influence the integrability of the theory.

In linear QM the \(\ast\)-powers of an observable are related to higher moments of random variables measured in experiments, hence lead to the probabilistic interpretation of the theory. In nonlinear theory the product (5) leads to the first non-unique element of the formalism we shall meet later.

For let us consider an observable \(A\). We know that it can be written in a form of a matrix element \(\langle \psi | \hat{A} | \psi \rangle\) where \(\hat{A}\) is the nonlinear operator defined in (3) (for simplicity we shall, from now on, omit in our notation the dependence of \(\hat{A}\) on \(\psi\)). The \(\ast\)-square of \(A\) can be defined as \(\frac{\partial^2 A}{\partial \psi_m \partial \psi_m} = \langle \psi | \hat{A} \hat{A} | \psi \rangle\). If we want to define the third power we encounter several possibilities. The apparently most natural choice of \((A \ast A) \ast A\) or \(A \ast (A \ast A)\) is not acceptable as they are in general complex and non equal even if \(A\) is real. In Subsec. II D we shall see also that \(\ast\) does not lead to a consistent probability interpretation of \(A\). The second definition can be chosen as \(\langle \psi | \hat{A} \hat{A} \hat{A} | \psi \rangle := A^\ast A \hat{A} A\). We see that once we have defined \(\hat{A}\), which is unique, all its \(\ast\)-powers are unique as well. Notice that since \(A \ast B = A \ast B\) both products lead to the same Poisson bracket (generate the same evolution of observables).

One of the most important questions of nonlinear quantum mechanics concerns a definition of observables that correspond to subsystems of a larger system. Let us begin the discussion with recalling the ways we do it in ordinary QM.

By saying that \(H(\psi_{sub}, \tilde{\psi}_{sub})\) is a Hamiltonian function of a subsystem we mean that \(H(\psi_{sub}, \tilde{\psi}_{sub})\) describes a subsystem which is noninteracting and noncorrelated with the rest of the Universe (otherwise the subsystem is not described by a state vector). Quantum mechanically this means that the state of the Universe is \(| \psi \rangle = | \psi_{sub} \rangle \otimes | \psi_{rest} \rangle\) (or \(\rho = | \psi_{sub} \rangle \langle \psi_{sub} | \otimes \rho_{rest} \)).

Subsystem observables in linear QM can be written as

$$A(\psi, \tilde{\psi}) = \langle \psi | \hat{A} \otimes 1_{rest} | \psi \rangle = Tr_{\psi_{sub}} \hat{A}.$$  \hspace{1cm} (7)
Let now \( \{|r\} \) denote a basis in the Hilbert space of the “rest” and \( \{|a\} \) be the one of the subsystem. A general form of the state of the whole system (here we assume for inessential simplicity that the whole system is in a pure state) is

\[
|\psi\rangle = \sum_{a,r} \psi_{a,r} |a\rangle |r\rangle = \sum_r \Phi^{(r)} |r\rangle.
\]

(8)

An average of \( A \) in the state \( |\psi\rangle \) can be expressed by means of the decomposition (8) as follows

\[
\mathcal{A}(\psi, \tilde{\psi}) = \langle \psi | A_{\text{rest}} |\psi\rangle = \sum_r \langle \Phi^{(r)} | A |\Phi^{(r)}\rangle \langle \Phi^{(r)} | \Phi^{(r)}\rangle \langle \psi | \Phi^{(r)}\rangle \langle \Phi^{(r)} | \psi\rangle
\]

\[
= \sum_r \frac{\langle \Phi^{(r)} | A |\Phi^{(r)}\rangle \langle \psi | (1_{\text{sub}} \otimes |r\rangle \langle r|) |\psi\rangle}{\langle \psi | \psi\rangle}.
\]

(9)

The physical meaning of (9) is obvious: The whole ensemble of subsystems is decomposed into subensembles corresponding to different results of measurements of an observable whose spectral family is given by \( \{|r\}\rangle \langle r| \), and the average of \( A \) in the subsystem is calculated for each subensemble separately. The form (9) is useful as an illustration of the “collapse” of a state vector phenomenon. Indeed, the average is calculated as though the ensemble of subsystems consisted of subensembles collapsed by an external observer and, in certain sense, the ensemble is treated as the one composed of individual objects each of them possessing some “property” measured (via correlations) by the external observer.

The description in terms of the reduced density matrix \( \rho_{\text{sub}} \) suggests a slightly different interpretation of the average. The form

\[
\mathcal{A}(\psi, \tilde{\psi}) = Tr \rho_{\text{sub}} A
\]

(10)

means that we do not take into account possible decompositions of the ensemble, but treat it as a whole. The presence of the reduced density matrix in (10) does not mean, of course, that we cannot use the state vector’s components — it simply restricts the form of observables to only some functions of \( \psi_{a,r} \).

Quantum mechanics when viewed from the first perspective looks as a theory of individual systems but with some form of fundamental limitation of knowledge about states of such individuals. The second possibility turns QM into a kind of a “mean field theory”. Still, since both forms of description are physically indistinguishable such discussions within the linear framework seem purely academic.

In nonlinear theories the situation is drastically changed. The nonlinearity is supposed to be fundamental, hence applying to each individual even if the individual is well isolated from other physical systems and no mean-field approximation is justified.

The problem is practically the following. Assuming that we know a form of an observable describing a given subsystem itself (that is, if no interactions or correlations with the “rest” exist), what is the form of the observable if there do exist some interactions or correlations with the “rest”? The first, explicitly non-mean-field-theoretic, guess for the subsystem’s (additive) observable is suggested by the “collapse-like” form (9) of the average, i.e.

\[
A_{\text{sub}}(\psi, \tilde{\psi}) = \sum_r A(\Phi^{(r)}, \tilde{\Phi}^{(r)})
\]

(11)

where \( A \) is a given function defined on a Hilbert space of the subsystem and the other definitions are like in (8). It is this definition that was chosen by Weinberg in his formulation of nonlinear QM. For non-bilinear \( A(\Phi^{(r)}, \tilde{\Phi}^{(r)}) \) the form and value of (11) depend on the choice of the basis \( \{|r\}\rangle \).

Example 1 Consider the following non-bilinear observable

\[
A(\psi, \tilde{\psi}) = \frac{\langle \psi | \sigma_3 | \psi \rangle^2}{\langle \psi | \psi \rangle} = \frac{(|\psi_1|^2 - |\psi_2|^2)^2}{|\psi_1|^2 + |\psi_2|^2}
\]

(12)

where \( \sigma_3 \) is the Pauli matrix and \( |\psi\rangle \) belongs to a two-dimensional Hilbert space. Let the whole system be described by \( |\psi\rangle = \sum_{a=1}^2 \sum_r \psi_{a,r} |a\rangle |r\rangle \). Definition (11) yields

\[
A(\psi_{\text{sub}}, \tilde{\psi}_{\text{sub}}) = \sum_r \frac{(|\psi_{1r}|^2 - |\psi_{2r}|^2)^2}{|\psi_{1r}|^2 + |\psi_{2r}|^2}
\]

(13)
Let

\[ |\psi\rangle = \psi_{1,1} |1\rangle_1 |1\rangle_2 + \psi_{2,2} |2\rangle_1 |2\rangle_2. \]  

(14)

In this particular state

\[ A(\psi_{sub}, \bar{\psi}_{sub}) = |\psi_{11}|^2 + |\psi_{22}|^2 \]  

(15)

so that each “collapsed” subensemble contributes to the observable separately. Let us choose now a different basis \( \{|r'\rangle_2\} \) in the Hilbert space of the “rest”, such that \( |1\rangle_2 = \frac{1}{\sqrt{2}} (|1'\rangle_2 + |2'\rangle_2) \), \( |2\rangle_2 = \frac{1}{\sqrt{2}} (|1'\rangle_2 - |2'\rangle_2) \). Then

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (\psi_{11} |1\rangle_1 + \psi_{22} |2\rangle_1) |1'\rangle_2 + \frac{1}{\sqrt{2}} (\psi_{11} |1\rangle_1 - \psi_{22} |2\rangle_1) |2'\rangle_2. \]  

(16)

Substitution of \( |\Phi(1')\rangle \) and \( |\Phi(2')\rangle \) into (11) gives

\[ A(\psi_{sub}, \bar{\psi}_{sub}) = \frac{1}{2} \left( |\psi_{11}|^2 - |\psi_{22}|^2 \right)^2 + \frac{1}{2} \left( |\psi_{11}|^2 - |\psi_{22}|^2 \right)^2 \]

\[ = \left( \frac{|\psi_{11}|^2 - |\psi_{22}|^2|^2}{|\psi_{11}|^2 + |\psi_{22}|^2} \right) \]  

(17)

which is also a sum of contributions from two subensembles, but now “collapsed to linear polarizations”.

The above example reveals two properties of (11):

1. This form is basis-dependent so one should rather write \( A_{sub}(\psi, \bar{\psi}) = A_{sub}(\psi, \bar{\psi}, \{|r\}\}) \);

2. If the “rest” is described by linear QM then no change of \( \{|r\}\) can influence observables that correspond to “the rest” itself. In particular, if one considers a total energy of the whole system then a change of \( \{|r\}\) will influence its value (we change the energy of the subsystem and maintain the energy of the “rest”). It follows that the form (11) can apply only to systems that, as a whole, are open (that is, there must exist also some another nonlinear system not contained in the “whole” one and compensating the changes of energy due to the changes of \( \{|r\}\})

3. The “Subsystem+compensating subsystem+rest” system must be described in a basis-independent way.

The last remark indicates that in order to describe systems with “isolated” nonlinearities, that is with the “rest” linear, one has to describe subsystems in a way different from Weinberg’s. The following modification was proposed by Polchinski [23].

Consider a nonlinear observable \( A(\psi, \bar{\psi}) \) corresponding to some isolated physical system, and assume that the observable can be expressed as a function of the density matrix \( \rho = |\psi\rangle \langle \psi| \), i.e. \( A(\psi, \bar{\psi}) = A(\rho) \) (this is always true in linear QM). For example, the observable from the previous example could be written in either of the forms

\[ \frac{(Tr \rho \sigma_3)^2}{Tr \rho}, \]  

(18)

\[ \frac{Tr (\rho_3 \rho_3 \sigma_3)}{Tr \rho}, \]  

(19)

\[ \frac{(Tr \rho \sigma_3)^2 Tr (\rho^2)}{(Tr \rho)^3}, \]  

(20)

or

\[ \frac{(Tr (\rho_3 \rho_3 \sigma_3))^{n+1}}{Tr \rho (Tr \rho \sigma_3)^{2n}}, \]  

(21)

5
and so on. For \( \rho = |\psi\rangle\langle\psi| \) all of them, and all their convex combinations, reduce to (12). The map \( \rho \rightarrow A(\rho) \) determines \( A(\psi, \tilde{\psi}) \) uniquely, but from the knowledge of \( A(\psi, \tilde{\psi}) \) one cannot deduce the form of \( A(\rho) \) for \( \rho \) different from \( |\psi\rangle\langle\psi| \). This trivial remark has important consequences. Returning to the “atom+field” example, we see that the knowledge of the nonlinear Schrödinger equation for the atom alone does not determine uniquely the form of the evolution equation of the “atom+field” composite system! The nonlinear equation describing the atom if \emph{there are no correlations with the field} is determined by the Hamiltonian function \( H_{AT}(\psi, \tilde{\psi}) \), but this function no longer describes the atom if some correlations with the field do appear.

Therefore, it becomes clear that, in order to have a unique formalism, one should start from the very beginning with the description of observables in terms of density matrices. Such a reformulation of Weinberg’s theory was proposed by Polchinski and Jordan [23, 24]. In the second part of this paper I will generalize the density matrix formalism, so here we shall concentrate only on its interpretational problems.

**Example 2** Let \( |\psi\rangle = \psi_{11}|1_1\rangle|1_2\rangle + \psi_{22}|2_1\rangle|2_2\rangle \). The reduced density matrix representing the subsystem is

\[
\rho = |\psi_{11}\rangle\langle\psi_{11}| + |\psi_{22}\rangle\langle\psi_{22}|. 
\]

The subsystem’s observables are now

\[
\frac{(Tr \rho \sigma_3)^2}{Tr \rho} = \frac{|\psi_{11}|^2 - |\psi_{22}|^2}{|\psi_{11}|^2 + |\psi_{22}|^2},
\]

\[
\frac{Tr (\rho \sigma_3 \rho \sigma_3)}{Tr \rho} = \frac{|\psi_{11}|^4 + |\psi_{22}|^4}{|\psi_{11}|^2 + |\psi_{22}|^2},
\]

\[
\frac{(Tr \rho \sigma_3)^2 Tr (\rho^3)}{(Tr \rho)^3} = \frac{|\psi_{11}|^2 - |\psi_{22}|^2|^2(|\psi_{11}|^2 + |\psi_{22}|^2)^4}{(|\psi_{11}|^2 + |\psi_{22}|^2)^3},
\]

and

\[
\frac{(Tr (\rho \sigma_3 \rho \sigma_3))^2 Tr (\rho^3)}{(Tr \rho)^3} = \frac{(|\psi_{11}|^2 + |\psi_{22}|^2)^2}{(|\psi_{11}|^2 + |\psi_{22}|^2)^3}. 
\]

As we can see the observables are completely different. If \( |\psi_{11}| = |\psi_{22}| \) (as in the spin-1/2 singlet state) the average of the first and third observables vanishes, is non-vanishing but finite in the second case, and infinite in the last one.

Expression (23) seems, at first glance, the most natural one since can be rewritten as

\[
\frac{\langle \psi | \sigma_3 \otimes 1_{rest} | \psi \rangle^2}{\langle \psi | \psi \rangle}. 
\]

Kibble in [9] did not explicitly define his way of describing subsystems, but the examples he discussed suggest this type of description. In the next section we will discuss problems with nonlocality of nonlinear QM and see that all the descriptions by means of reduced density matrices are more safe than the proposal of Weinberg. However, if some correlations between subsystems actually exist it seems we indeed \emph{can} “collapse” a state of a remote individual.

We may expect that an average value of this “remote” observable should be composed of the discussed averages of subensembles. The description by means of (23) does not seem to possess this property: It is the “average spin” that contributes to the observable (23). So we first calculate the “average” of \( \sigma_3 \) and then square it. In the Weinberg description we first square the “spin” and then take the “average”. (The quotation marks are necessary because we have not defined yet what we mean by eigenvalues, or values of single measurements; there exists also another subtle point that has been ignored in the definition of (11) — in nonlinear QM averages of projectors, \( \langle |\psi\rangle \langle \psi| |\psi\rangle \langle \psi| \) may not have an interpretation in terms of probability (see the section on eigenvalues).)

It is interesting that the “collapse”-like property of the average can be seen also in (24) where both values of “spin” are averaged separately. Finally, let us notice that the last observable (26) is unbounded, so that the values of single measurement cannot be identified with eigenvalues of \( \sigma_3 \).
C. Dynamics

Both Kibble and Weinberg assumed that the dynamics of pure states is given by Hamilton equations with Hamiltonian functions belonging to the algebra of generalized observables discussed in Subsec. II B. More generally, all one parameter flows of canonical transformations are assumed to be integral curves of the Hamilton equations of motion $i\gamma = dH$.

Since the evolution is, in general, nonlinear no dynamical separation of states and observables is possible (the ordinary Heisenberg and Schrödinger pictures can be formulated only if the dynamics is linear). Instead, the evolution of observables is governed by the Poisson bracket resulting from the Hamilton equations.

The Hamilton equations in Kibble-Weinberg theory can be written in a Schrödinger-like form

$$i\frac{d}{dt}\psi = \hat{H}(\psi, \bar{\psi})\psi$$  \hspace{1cm} (28)

where $\hat{H}(\psi, \bar{\psi})$ is Hermitian. An interesting class of nonlinear Schrödinger equations which are not equivalent to Weinberg’s equations, even though the Hamiltonians are (0,0)-homogeneous, is given by equations of the form

$$i\frac{d}{dt}\psi = \left(\hat{H} + \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle} \hat{A} - \frac{\langle \psi | B | \psi \rangle}{\langle \psi | \psi \rangle} \hat{B} \right)\psi := \hat{H} \psi.$$  \hspace{1cm} (29)

In such a case all the nonlinear terms involving $\hat{A}$ and $\hat{B}$ cancel out in $\langle \psi | \hat{H} | \psi \rangle = \langle \psi | \hat{H} | \psi \rangle$, which, for this reason, cannot be regarded as a Hamiltonian function for (29).

Example 3 Consider a two-level atom irradiated by an external classical light and assume that the Hamiltonian function of the atom takes the Weinberg form

$$\langle \psi | \hat{H}_L + \hat{H}_{NL} | \psi \rangle$$  \hspace{1cm} (30)

where the nonlinear $2 \times 2$ matrix is

$$\hat{H}_{NLmn} = \frac{\partial^2 \hat{H}_{NL}}{\partial \bar{\psi}_m \partial \psi_n} = (h_0 + \bar{h}_{NL} \sigma)_{mn}.$$  \hspace{1cm} (31)

Let $\hat{s}(\psi, \bar{\psi}) = \langle \psi | \sigma | \psi \rangle$. We find

$$\hat{s} = \{\hat{s}, \hat{H}\} = -i\langle \psi | [\sigma, \hat{H}_L + \hat{H}_{NL}] | \psi \rangle = 2(\hat{h}_L + \bar{h}_{NL}) \times \hat{s} = (\bar{\omega}_L + \omega_{NL}) \times \hat{s}.$$  \hspace{1cm} (32)

Defining

$$\begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \\ \bar{\omega}_3 \end{pmatrix} = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_{NL1} \\ \omega_{NL2} \\ \omega_{NL3} \end{pmatrix},$$

with $\omega$ being the frequency of light, we obtain after RWA the following nonlinear Bloch equations

$$\begin{aligned}
\dot{u} &= -\Delta v - \bar{\omega}_3 v + \bar{\omega}_2 w \\
\dot{v} &= \Delta u + \Omega w + \bar{\omega}_3 u - \bar{\omega}_1 w \\
\dot{w} &= -\Omega v - \bar{\omega}_2 u + \bar{\omega}_1 v.
\end{aligned}$$  \hspace{1cm} (33)

Choosing $(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3) = (-A/2, A/2, 2\epsilon)$ we get

$$\begin{aligned}
\dot{u} &= -\Delta v - 2\epsilon w + \frac{A}{2} uw \\
\dot{v} &= \Delta u + \Omega w + 2\epsilon w u - \frac{A}{2} v w \\
\dot{w} &= -\Omega v - \frac{A}{2} u^2 - \frac{A}{2} v^2.
\end{aligned}$$  \hspace{1cm} (34)
Eqs. (34) are identical to Jaynes’ neoclassical Bloch equations [2] where $\epsilon$ is the neoclassical Lamb shift and $A$ is equal to Einstein’s coefficient of spontaneous emission. So let us check what kind of the KW-Schrödinger equation can lead to the neoclassical description.

Returning to the non-rotated reference frame we obtain

$$\left(\omega_{NL1}, \omega_{NL2}, \omega_{NL3}\right) = \left(-\left(A/2\right)\ddot{\varphi}, \left(A/2\right)\ddot{\varphi}, 2\epsilon\ddot{\varphi}\right)$$

and the nonlinear Schrödinger equation we are looking for is

$$i\frac{d}{dt} |\psi\rangle = \left(\hat{H}(\psi, \bar{\psi}) + \frac{A}{4} \frac{\langle \psi | \sigma_3 | \psi \rangle}{\langle \psi | \psi \rangle} \sigma_1 + \frac{A}{4} \frac{\langle \psi | \sigma_1 | \psi \rangle}{\langle \psi | \psi \rangle} \sigma_2 + \epsilon \frac{\langle \psi | \sigma_3 | \psi \rangle}{\langle \psi | \psi \rangle} \right) |\psi\rangle$$

$$= \hat{H}(\psi, \bar{\psi}) |\psi\rangle.$$  

(36)

If $\hat{H}$ is some $(0,0)$-homogeneous function then the Hamiltonian is also $(0,0)$-homogeneous. However

$$\langle \psi | \hat{H}(\psi, \bar{\psi}) |\psi\rangle = \langle \psi | \hat{H}(\psi, \bar{\psi}) + \epsilon \frac{\langle \psi | \sigma_3 | \psi \rangle}{\langle \psi | \psi \rangle} \rangle |\psi\rangle$$

(37)

cannot play the role of the Hamiltonian function for (36) with $A \neq 0$. For $A = 0$ the relevant KW Hamiltonian function is

$$E\langle \psi | \psi \rangle + \frac{1}{2}\epsilon \frac{\langle \psi | \sigma_3 | \psi \rangle^2}{\langle \psi | \psi \rangle^2}$$

(38)

and

$$\hat{H}(\psi, \bar{\psi}) = E - \frac{1}{2}\epsilon \frac{\langle \psi | \sigma_3 | \psi \rangle^2}{\langle \psi | \psi \rangle^2}.$$  

(39)

D. Probabilities and Results of Single Measurements

Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\hat{H}$ a Hermitian operator acting in $\mathcal{H}$. If $H = \langle \psi | \hat{H} | \psi \rangle$ is the associated observable, the values of single measurements of $H$ can be defined in at least three equivalent ways. Since in nonlinear QM the three options will lead to different results we will use here different names for each of them.

a) Eigenvale $E$ of $H = \langle \psi | \hat{H} | \psi \rangle$ is the number satisfying for some eigenstate $\psi$ the equation

$$\hat{H} |\psi\rangle = \lambda |\psi\rangle$$

(40)

or

$$\frac{\partial H}{\partial \psi_m} = \lambda \psi_m.$$  

(41)

b) Diagonal values are solutions of

$$\det(\hat{H} - \lambda I) = 0$$

(42)

or

$$\det \left( \frac{\partial^2 H}{\partial \psi_m \partial \psi_n} - \lambda \delta_{mn} \right) = 0.$$  

(43)

c) The third possibility comes from the fact that, since eigenstates form a complete orthogonal set of vectors in $\mathcal{H}$, any solution of the Schrödinger equation $i\frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$ can be expressed as

$$\begin{pmatrix}
\psi_1(t) \\
\vdots \\
\psi_N(t)
\end{pmatrix} = \begin{pmatrix}
\psi_1(0)e^{-i\omega_1 t} \\
\vdots \\
\psi_N(0)e^{-i\omega_N t}
\end{pmatrix}$$

(44)

and the frequencies $\omega_n$ can be termed the eigenfrequencies. The three possibilities can be used for definitions of the results of single measurements also in nonlinear QM.

Definitions (41) and (43) do not explicitly use the bilinearity of $H$, so can be naturally adapted also in the nonlinear framework. It is interesting that then they are not equivalent.
Example 4 The so-called “simplest nonlinearity” considered in experiments designed as tests of Weinberg’s nonlinear QM corresponds to the following Hamiltonian function

\[ \langle \psi | \hat{H}_0 | \psi \rangle + \epsilon \left( \frac{\langle \psi | \sigma_3 | \psi \rangle}{| \psi \rangle} \right)^2 \]  

where \( \hat{H}_0 = (E_{1,0}) \). Denote \( \psi_1 = \sqrt{A_1} e^{i \alpha_1} \), \( \psi_2 = \sqrt{A_2} e^{i \alpha_2} \). The eigenvalue condition reads

\[ \left\{ E_1 + \epsilon \left[ \frac{2(A_1 - A_2)}{A_1 + A_2} - \left( \frac{A_1 - A_2}{A_1 + A_2} \right)^2 \right] \right\} \sqrt{A_1} = \lambda \sqrt{A_1} \]  

\[ \left\{ E_2 + \epsilon \left[ -2 \frac{A_1 - A_2}{A_1 + A_2} - \left( \frac{A_1 - A_2}{A_1 + A_2} \right)^2 \right] \right\} \sqrt{A_2} = \lambda \sqrt{A_2}. \]

For \( A_2 = 0 \) we find \( \lambda_+ = E_1 + \epsilon \), for \( A_1 = 0 \) \( \lambda_- = E_2 + \epsilon \). If \( A_1 \neq 0 \) and \( A_2 \neq 0 \) we get, for \( \epsilon \neq 0 \),

\[ \frac{A_1 - A_2}{A_1 + A_2} = \frac{E_2 - E_1}{4\epsilon} \]

which implies (\( A_1 \) and \( A_2 \) are positive)

\[ |E_2 - E_1| < 4|\epsilon| \]

and

\[ \lambda_0 = \frac{1}{2} \left( E_1 + E_2 - \frac{1}{8\epsilon} (E_1 - E_2)^2 \right) \]

with the eigenvector

\[ | \psi \rangle = \left( \frac{1 + \frac{E_2 - E_1}{4\epsilon}}{1 - \frac{E_2 - E_1}{4\epsilon}} \right)^{-1/2} e^{i\alpha_1} \]

\[ \left( \frac{1 + \frac{E_2 - E_1}{4\epsilon}}{1 - \frac{E_2 - E_1}{4\epsilon}} \right)^{-1/2} e^{i\alpha_2} \]

We find three eigenvalues although the dimension of the Hilbert space is two. The third eigenstate exists only for \( \epsilon \) satisfying the condition (48).

Of course, all the eigenstates are stationary. The eigenstate corresponding to \( \lambda_0 \) is not orthogonal to the remaining two eigenstates, which excludes the ordinary probability interpretation in terms of projectors.

Let us now calculate the diagonal values of this Hamiltonian function. The nonlinear Hamiltonian is

\[ \hat{H} = \left( \begin{array}{cc} E_1 + \epsilon (8p^2 - 20p^2 + 16p - 3) & 8\epsilon |\psi_1 \rangle |\psi_2 \rangle \bar{\psi}_1 \bar{\psi}_2 \\ 8\epsilon |\psi_1 \rangle |\psi_2 \rangle \bar{\psi}_1 \bar{\psi}_2 & E_2 + \epsilon (-p^3 + 4p^3 + 1) \end{array} \right) \]

where the state is assumed normalized and \( p = |\psi_1 |^2 \). The matrix is Hermitian and its eigenvalues (\( \pm \) diagonal values of \( \hat{H} \)) are

\[ E_{\pm} = \frac{1}{2} \left( E_1 + E_2 - 2\epsilon (8p^2 - 8p + 1) \right. \]

\[ \left. \pm \left[ (E_1 - E_2)^2 - 8\epsilon \left( (E_1 - E_2)(4p^3 - 6p^3 + 4p - 1) \right) \right] + 2\epsilon (20p^3 - 40p^3 + 28p^2 - 8p + 1) \right]^{1/2} \]

The diagonal values, in opposition to eigenvalues, are in the nonlinear case functions and the number of them, again as opposed to eigenvalues, is always equal to the dimension of the suitable Hilbert space.

The solution of the respective nonlinear Schrödinger equation is

\[ \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} \psi_1(0)e^{-i(E_1 + 2\epsilon \sigma_3) - i(\sigma \sigma_3)^2}t \\ \psi_2(0)e^{-i(E_2 - 2\epsilon \sigma_3) - i(\sigma \sigma_3)^2}t \end{pmatrix} \]

where the averages in the exponents are integrals of motion. It follows that the eigenfrequencies are also state dependent functions but differ from the diagonal values.
In linear QM mechanics observables correspond to averages hence, at experimental level, to random variables. With any random variable one can associate its higher moments. The higher moments, on the other hand, can be used to deduce a probability interpretation of the theory. Therefore, one of the essential points of any generalized algebra of observables is the question of the representability of higher moments corresponding to some given observable. In linear QM the problem is solved by the spectral theorem.

The first interesting result concerning the nonlinear eigenvalues, noticed by Weinberg in [1], is the following

**Lemma 1** Let $F$ and $G$ be two $(1,1)$-homogeneous observables possessing a common eigenstate $|\psi\rangle$, i.e. $\frac{\partial F}{\partial \psi_k} = f\psi_k$ and c.c., $\frac{\partial G}{\partial \psi_k} = g\psi_k$ and c.c., and let $F \ast G = \left(\frac{\partial F}{\partial \psi_k}\frac{\partial G}{\partial \psi_k}\right)^{\frac{1}{2}}$. Then $\frac{\partial (F \ast G)}{\partial \psi_k} = fg\psi_k$ and c.c.

The homogeneity implies that an average of $F$ in an eigenstate is equal to the respective eigenvalue. For finite dimensional Hilbert spaces we know also that a number of eigenvalues is not smaller than the dimension of the space (a result from the Morse theory) and the eigenvalues are critical points of averages. For averages defined on the whole Hilbert/projective space their maxima and minima must be found at critical points (averages defined on a finite dimensional projective space are smooth functions defined on a compact set). These facts suggest that the eigenvalues are correct candidates for the results of single measurements of the generalized observables. However, the following examples show that the problem is not that simple.

**Example 5** The observable

$$A = \frac{\langle\psi|\psi\rangle^2}{\langle\psi|\sigma_3|\psi\rangle} \tag{53}$$

satisfies the Weinberg’s homogeneity requirements. Its eigenvalues are $\pm 1$ hence do not bound the averages. It follows that such singular observables must be excluded if we want to have the probability interpretation in terms of the eigenvalues. The algebra of observables would have to be restricted to $(1,1)$-homogeneous smooth functions defined of the whole Hilbert/projective space. This requirement is not very restrictive. Notice that no difficulties will appear if we will apply to $A$ the interpretation in terms of diagonal values or eigenfrequencies.

**Example 6** In QM we often encounter observables whose eigenstates are degenerate. Consider now

$$H = E\langle\psi|\psi\rangle + \epsilon\frac{\langle\psi|\sigma_3|\psi\rangle^3}{\langle\psi|\psi\rangle^2} \tag{54}$$

Its eigenvalues are $E \pm \epsilon$ and $E$. The number of them is hence greater than the dimension of the Hilbert space no matter how small $\epsilon$ is. Again, no problems appear if we apply the diagonal values or eigenfrequencies interpretation. Had we substituted the first term with some $\langle E_1, E_2 \rangle$, where $E_1 \neq E_2$, we would have obtained the third eigenvalue provided $|E_1 - E_2| \leq \epsilon\langle\psi|\psi\rangle$ so that the eigenvalue will not appear for sufficiently small $\epsilon$. This phenomenon seems to be related to the Kolmogorov-Arnold-Moser theorem where a dimension of an invariant torus may not change with nonlinear perturbation if a nonperturbed Hamiltonian system is nondegenerate and the perturbation is not too large [25].

**Example 7** Let

$$H = \langle\psi|\hat{H}|\psi\rangle + \epsilon\frac{\langle\psi|\sigma_3|\psi\rangle^3 N}{\langle\psi|\psi\rangle^{2N-1}} \tag{55}$$

where $\hat{H}_0 = \langle E_0, E_0 \rangle$. Two eigenvalues corresponding to the eigenstates $|\pm\rangle$ of $\sigma_3$ are $E_+ = E_1 + \epsilon$ and $E_- = E_2 + \epsilon$ so are shifted by the same amount. The third eigenstate occurs if

$$|\epsilon| > \frac{|E_1 - E_2|}{4N} \tag{56}$$

The third eigenvalue

$$E_0 = \frac{1}{2}(E_1 + E_2) + \epsilon(1 - 2N)\left(\frac{E_2 - E_1}{4N\epsilon}\right)^{\frac{2N}{2N-1}}$$

(57)

tends to $E_1$ with $N \to \infty$. This example is important. It shows that the ordinary projection postulate cannot be consistently applied, for calculation of probabilities corresponding to nonlinear eigenstates, in experiments whose goal is elimination of all nonlinear corrections to QM.
Example 8 Consider

\[ H = E\langle \psi | \psi \rangle + \epsilon \frac{\langle \psi | \sigma_3 | \psi \rangle^2}{\langle \psi | \psi \rangle} \tag{58} \]

The eigenvalues are \( E \) and \( E + \epsilon \). Probabilities calculated by means of

\[ \langle H \rangle = (E + \epsilon)p_{E+\epsilon} + Ep_E \tag{59} \]

and the normalization of probability are

\[ p_{E+\epsilon} = |\sigma_3|^2, \quad p_E = 1 - |\sigma_3|^2. \tag{60} \]

We know that \( H \ast H \) has eigenvalues \((E + \epsilon)^2\) and \( E^2 \), so we can calculate the probabilities by means of \( \langle H \ast H \rangle = (E + \epsilon)^2p_{E+\epsilon} + E^2p_E \) and the normalization condition. Since

\[ \langle H \ast H \rangle = E^2 + (4\epsilon^2 + 2E \epsilon)|\sigma_3|^2 - 3\epsilon^2|\sigma_3|^4 \tag{61} \]

we obtain

\[ p_{E+\epsilon} = \frac{(4\epsilon^2 + 2E \epsilon)|\sigma_3|^2 - 3\epsilon^2|\sigma_3|^4}{2E \epsilon + \epsilon^2} \tag{62} \]

which is, of course, inconsistent with the previous result. The troubles arise because of the nonassociativity of \( \ast \). An analogous result was derived by Jordan [26], who showed that propositions cannot be defined consistently within the \( \ast \)-algebraic approach to Weinberg’s observables. We conclude that \( \ast \) cannot be applied for a definition of higher moments and the above lemma is useless from the viewpoint of the probability interpretation.

Next candidate for a single result of a measurement of some \( H \) is the diagonal value. We have already seen that the diagonal values are state dependent functions. There is no general guarantee that a diagonal value of an integral of motion is itself an integral of motion. In fact, whereas the number of the diagonal values is equal to the dimension of \( \mathcal{H} \), the diagonal values, for \( N > 2 \), can be time independent only for integrable systems. In addition, as diagonal values are roots of algebraic equations, they even do not have to be differentiable functions of states. Therefore, we cannot find for them any general equation of motion.

In spite of these disadvantages the diagonal values are in some respects superior to eigenvalues. If for two observables \( A \) and \( B \) the commutator \( AB - BA = 0 \), then \( A \) and \( B \) can be diagonalized simultaneously and the product \( AB \) has eigenvalues being products of the eigenvalues of \( A \) and \( B \). This leads to the following unique probability interpretation. Let \( U(\psi, \bar{\psi}) \) be the unitary transformation diagonalizing \( H \). Then

\[ H = \sum_n E_n(\psi, \bar{\psi})(U|\psi\rangle_n)\langle U|\psi\rangle_n. \tag{63} \]

The functions \( p_n = |(U|\psi\rangle_n)\langle U|\psi\rangle_n| \) are probabilities resulting from the higher moments procedure.

Let us recall that for two observables \( F \ast G = F \hat{\ast} G \). The nonassociativity of \( \ast \) is reflected in the \( \hat{\ast} \)-algebra of observables in the following non-uniqueness of \( \hat{\ast} \). Although always

\[ \langle \psi | (H \hat{\ast} \ldots \hat{\ast} H) | \psi \rangle = \langle \psi | \hat{H} \ldots \hat{H} | \psi \rangle \tag{64} \]

but for non-bilinear observables

\[ (H \hat{\ast} \ldots \hat{\ast} H) \neq \hat{H} \ldots \hat{H}. \tag{65} \]

The eigenfrequencies approach can be applied only to systems that are close to integrable [25]. For finite dimensional Hamiltonian systems some general KAM theorems are known. It is known that components of state vectors exhibit in such a case, unless some resonant initial conditions occur, a quasi periodicity described by [1]

\[ \psi_z(t) = \sum_{n_1 \ldots n_N} c_h(n_1 \ldots n_N)e^{i\sum_{\nu=1}^N \nu \omega_{\nu} t} \tag{66} \]

where both the amplitudes and the frequencies are dependent on initial conditions, as shown in our first example in this subsection, and the sums run over all positive and negative integers. The homogeneity condition implies that
\[ \langle \psi | \psi \rangle = \sum_{k} \sum_{n_1 \ldots} |E_k(n_1 \ldots)|^2 \]  

and

\[ H = \sum_{k} \sum_{n_1 \ldots} |E_k(n_1 \ldots)|^2 \sum_{\nu} n_\nu \omega_\nu. \]

These formulas lead to the natural interpretation of the eigenfrequencies \( \sum_{\nu} n_\nu \omega_\nu \) as the results of single measurements and the respective normalized coefficients as probabilities. The interpretation can be naturally extended to other observables by assuming that all observables generate one parameter groups of canonical transformations \( \text{via} \) the Hamilton equations of motion. The same result would be also obtained by means of the “standard” theory of measurement where we assume that the so-called pre-measurement is described by an interaction Hamiltonian function which is proportional to the observable measured, and that all the measuring procedures are based on measurements of observables that are bilinear like in linear QM. To be precise, we must remark that this procedure (chosen finally by Weinberg) is not unique either because there is no unique description of the composite “object+observer” system even if the observer is assumed to be linear (see remarks in the section devoted to observables). Another difficulty is that the assumption about measurements based only on bilinear observables cannot be formulated in a relativistic way. In standard measurements one measures positions or momenta. A theory where moments are always linear observables whereas the Hamiltonian function is a nonlinear observable cannot be relativistically covariant. Perhaps it was this point that made Weinberg admit that he “could not find any way to extend the nonlinear version of quantum mechanics to theories based on Einstein’s special theory of relativity” [5]. My own proposal presented in part II avoids this problem, because I introduce the nonlinearity through \textit{entropy}, which is not an observable, hence without any modification of the algebra of observables. To better understand mutual relations between eigenvalues and eigenfrequencies it is interesting to compare the two notions in situations where a number of eigenvalues is different from this of eigenfrequencies.

**Example 9** We know that

\[ H = E \langle \psi | \psi \rangle + \epsilon \frac{\langle \psi | \sigma_3 | \psi \rangle^3}{\langle \psi | \psi \rangle^2}. \]

has three eigenvalues \( E \pm \epsilon \) and \( E \). The eigenfrequencies are

\[
E_1(\psi, \tilde{\psi}) = E + \epsilon (3\langle \sigma_3 \rangle^2 - 2\langle \sigma_3 \rangle^2) \\
E_2(\psi, \tilde{\psi}) = E + \epsilon (-3\langle \sigma_3 \rangle^2 - 2\langle \sigma_3 \rangle^2)
\]

The three eigenvalues correspond to eigenvectors satisfying

\[
\langle \sigma_3 \rangle = \begin{cases} 
0 & \text{for } \psi^0 \\
1 & \text{for } \psi^1 \\
-1 & \text{for } \psi^{-1}
\end{cases}
\]

In this notation

\[
E_1(\psi^0, \tilde{\psi}^0) = E \quad \text{with probability } p_1(\psi^0, \tilde{\psi}^0) = 1/2 \\
E_2(\psi^0, \tilde{\psi}^0) = E \quad \text{with probability } p_2(\psi^0, \tilde{\psi}^0) = 1/2 \\
E_1(\psi^1, \tilde{\psi}^1) = E + \epsilon \quad \text{with probability } p_1(\psi^1, \tilde{\psi}^1) = 1 \\
E_2(\psi^1, \tilde{\psi}^1) = E - 5\epsilon \quad \text{with probability } p_2(\psi^1, \tilde{\psi}^1) = 0 \\
E_1(\psi^{-1}, \tilde{\psi}^{-1}) = E + 5\epsilon \quad \text{with probability } p_1(\psi^{-1}, \tilde{\psi}^{-1}) = 0 \\
E_2(\psi^{-1}, \tilde{\psi}^{-1}) = E - \epsilon \quad \text{with probability } p_2(\psi^{-1}, \tilde{\psi}^{-1}) = 1.
\]

As expected even in an eigenstate there are two eigenfrequencies. The ones which are not equal to the eigenvalues occur with probabilities 0.

A reader of the main Weinberg’s paper may be a little bit confused with what is finally understood as a result of a single measurement. A half of the paper suggests that this role will be played by eigenvalues, then probabilities are defined in terms of eigenfrequencies and, finally, in the analysis of a two-level atom a difference of eigenvalues is treated as the energy difference while the difference of the eigenfrequencies is treated as the frequency of the emitted photon. It seems that from the viewpoint of the analysis above the eigenfrequency difference should be treated as the energy difference of atomic levels. This is one of the subtleties that have been omitted in the analysis of experiments.

I think that each of the three possibilities lacks elegance and generality, and this is one of the reasons one should look for a possibility of introducing nonlinearities without altering observables.
III. COMPOSITE SYSTEMS AND NONLOCALITY

One of the most often quoted “impossibility theorems” about nonlinear QM is that any such theory must imply faster than light communications. This statement is evidently too strong. In fact, we will show in part II that there exists a rich class of theories where the mentioned phenomenon does not occur. As we shall see the theories must satisfy two conditions.

1. Their interpretation must not be based on the “collapse of a state vector” postulate. It means that any reasoning based on this postulate has to be regarded as unphysical. (In linear QM there exist such limitations. For example, all “counterfactual” problems like EPR paradox or Bell theorem can be eliminated trivially by rejecting reasonings involving alternative measurements.) Interpretations of QM that do not introduce the collapse (projection) postulate exist, to mention the many worlds one [28].

2. Observables corresponding to subsystems must be functionals depending on density matrices of those subsystems (the Polchinski postulate [23]).

In the second part of the paper I shall prove some general, quite strong theorems related to the latter condition. Here, we shall limit ourselves to several simple examples showing different ways of making quantum mechanical nonlocality “malignant”, to use the marvelous phrase of Bogdan Mielnik.

It seems it was Nicolas Gisin who was the first to observe that a nonlinear evolution can lead to a faster-than-light communication between two separated systems. His argument was the following [29]. Let \( \mathcal{H} \) be a finite dimensional Hilbert space, \( |\psi_i\rangle, |\phi_j\rangle \in \mathcal{H}, i = 1, \ldots, n, j = 1, \ldots, m, \langle \psi_i | \psi_j \rangle = \langle \phi_i | \phi_j \rangle = \delta_{ij} \). Then the following lemma holds.

**Lemma 2** If for some nonvanishing probabilities \( x_i, y_j \)

\[
\sum_i x_i |\psi_i\rangle \langle \psi_i | = \sum_j y_j |\phi_j\rangle \langle \phi_j | 
\]

then there exist orthonormal bases \( \{|\alpha_i\rangle\}, \{|\beta_j\rangle\} \) in some Hilbert space \( \mathcal{H}' \) and the state

\[
|\chi\rangle = \sum_i \sqrt{x_i} |\psi_i\rangle \otimes |\alpha_i\rangle = \sum_j \sqrt{y_j} |\phi_j\rangle \otimes |\beta_j\rangle.
\]

The proof can be found in [29]. The meaning of the lemma is that the two decompositions of the density matrix can be obtained by means of the EPR correlations. Indeed, we can take the density matrix \( |\chi\rangle \langle \chi | \) and trace out \( \mathcal{H}' \). The resulting density matrices are these appearing in the lemma. Assume now that we have a nonlinear evolution of pure states

\[
|\psi_i\rangle \langle \psi_i | \rightarrow g_t (|\psi_i\rangle \langle \psi_i |).
\]

Then, in general,

\[
\sum_i x_i g_t (|\psi_i\rangle \langle \psi_i |) \neq \sum_j y_j g_t (|\phi_j\rangle \langle \phi_j |)
\]

even if initially the two decompositions were equal. The important assumption leading to the faster-than-light communication is that each of the pure state sub-ensembles evolves according to (74), even if the whole state is given by their convex combination. This seems reasonable if we assume that the ensemble consists of the collapsed sub-ensembles. If we apply some “no-collapse” interpretation then the argument cannot be consistently formulated. However, it has to be stressed that the form of the evolution appearing in (75) can be derived from quite general assumptions. It was shown in a rigorous way in the language of theory of categories by Posiewnik [30] that this form is implied by Mielnik’s definition of mixed states as probability measures on the set of pure states. This remark has quite nontrivial consequences: The description of mixed states in terms of probability measures (Mielnik’s “convex approach”) combined with putative nonlinearity of the Schrödinger equation leads to faster that light communication. If one wants to get rid of such difficulties one cannot keep the figure of states convex.

Gisin, in his second paper [31] considered an example of a nonlinear evolution taken from the Weinberg paper [1]. Consider an ensemble of pairs of spin-1/2 particles in the singlet state and assume that in one arm of the experiment the evolution is given by the Hamiltonian function \( \langle \psi | \sigma_3 | \psi \rangle \omega /\langle \psi | \psi \rangle \). An experimenter in the other arm
chooses between two settings of his Stern-Gerlach device and decomposes, by means of the EPR correlations, the ensemble in the other arm into subensembles corresponding to, say, spins up or down in the z direction, or spins up or down in some \( u \) direction tilted at 45° with respect to the z-axis. The evolution equation, up to an overall phase, is given by \( i \hbar \frac{d}{dt} |\psi\rangle = 2\langle \sigma_3 \rangle \sigma_3 |\psi\rangle \). If the initial state is either up or down in the z direction the evolution is stationary and the average of \( \sigma_3 \) is always zero. If the state is either up or down in the \( u \) direction the spin is precessing around the \( z \) axis but the sense of the rotation is opposite for the up and down states. After a time of a quarter of the “Larmor period” the spin will have the same positive value of \( \langle \sigma_3 \rangle \). And this result can be detected by an observer in the arm with the nonlinearity.

As we can see Gisin in both of his proofs ignored details of the description and the evolution of the “large” system. To be precise he should rather define a Hamiltonian function of the two arms, then solve the nonlinear Schrödinger equation for the whole system or calculate the evolution of the average of \( \sigma_3 \) in one arm. This is the crucial point. We know already that the description of composite systems is not unique as long as we know only the evolution of pure states of subsystems. It will be shown below that the Gisin’s telegraph will never work if we describe the whole system in the way proposed by Pochinski. It will be shown also that the proof is valid for the specific choice of the description proposed by Weinberg.

But before we shall pass on to the details of the calculations let us consider another “general proof” which was formulated by myself independently and simultaneously with this of Gisin [32]. Consider a polarizing Mach-Zehnder whose first beam splitter is an analyzer of circular polarizations (for a detailed description of the polarizing Mach-Zehnder interferometer see [33]). A reflected beam is righthanded and the transmitted one — lefthanded. The lefthanded beam passes through a half-wave plate and then the two beams are phase shifted by \( \alpha \) with respect to each other and recombined.

If a source produces a linear polarization state, say

\[
|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |\rangle).
\]

the interferometer transforms the whole state into

\[
|\psi'\rangle = \frac{1}{2}((e^{i\alpha} + 1)|+\rangle + (1 - e^{i\alpha})|\rangle).
\]

and we observe the interference between the two outgoing channels. If, however, the source produces a singlet state [34]

\[
|\phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle |\rangle + |\rangle |\rangle),
\]

the whole state transforms as follows

\[
|\phi'\rangle = \frac{1}{2}(i e^{i\alpha} |+\rangle |+\rangle - e^{i\alpha} |\rangle |\rangle + |\rangle |+\rangle + |+\rangle |\rangle).
\]

and there cannot appear the interference in the interferometer (subsystem \( I \)): The linear dependence of the states which interfered in the linear polarization case is destroyed by orthogonality of their singlet state partners. The lack of the interference in \( I \) is therefore a nonlocal phenomenon.

We know that nonlinear evolutions in a Hilbert space do not conserve scalar products (the “mobility phenomenon” [22]) and states that are initially orthogonal may loose their orthogonality during the course of the evolution. It is clear that if we will violate in some system \( II \) separated from \( I \) the orthogonality of \(|+\rangle \) and \(|\rangle \rangle \) because of some nonlinearity then the photons in \( I \) will start to interfere and the interference will be the stronger the more violation of the orthogonality has been obtained. Putting it differently, tracing over a space where the evolution does not conserve scalar products may result in some “remains” of the traced out system in the reduced density matrix describing the remote, separated system.

This argument looks general but a careful reader has probably noticed that one additional assumption has been smuggled here: It is implicitly assumed that a nonlinear evolution in \( H \) can be extended to \( H \otimes H' \) in such a way that a solution describing the composite system is of the form

\[
|\phi\rangle \otimes |+\rangle + |\phi'\rangle \otimes |\rangle
\]

where \(|\phi\rangle \) and \(|\phi'\rangle \) are solutions of the subsystem’s nonlinear evolution equation that are “in mobility”, that is whose scalar product is not conserved. This assumption is a strong limitation. In fact, it never occurs if separated systems are described in a Polchinski’s way. However, the proposal of Weinberg does allow for such pathological solutions.
In what follows we shall formulate the arguments of Gisin and myself in a precise way within the framework of Weinberg’s approach.

Let us assume that we have two separated systems, I and II, described by Hamiltonian functions

\[ H_1(\varphi, \tilde{\varphi}) = E_1(\varphi|\varphi), \quad H_2(\chi, \tilde{\chi}) = E_2(\chi|\chi) + \epsilon \frac{\langle \chi | \sigma_3 | \chi \rangle^2}{(\chi|\chi)}. \]  

(81)

According to Weinberg, the Hamiltonian function of the whole I+II system is given by

\[ H(\psi, \tilde{\psi}) = \sum_l H_1(\varphi_l, \tilde{\varphi}_l) + \sum_k H_2(\chi_k, \tilde{\chi}_k), \]  

(82)

where \( \varphi_{k(l)} = \chi_{k(l)} = \psi_{k(l)} \).

A general solution of the nonlinear Schrödinger equation corresponding to (82) takes the required form

\[ |\psi\rangle = |\xi_1\rangle \otimes |\phi\rangle + |\xi_2\rangle \otimes |\psi\rangle \]  

(83)

where \( |\xi_1\rangle = (\zeta^a_1, \zeta^b_1) \), \( |\xi_2\rangle = (\zeta^a_2, \zeta^b_2) \), \( \alpha = \exp(-iE_1t) \) and \( |\phi\rangle = (\zeta^a_y, |\psi\rangle = (\zeta^b_y) \) are some solutions of the single particle nonlinear Schrödinger equation.

The Hamiltonian functions

\[ H_1(\psi, \tilde{\psi}) = \sum_l H_1(\varphi_l, \tilde{\varphi}_l) \quad \text{and} \quad H_2(\psi, \tilde{\psi}) = \sum_k H_2(\chi_k, \tilde{\chi}_k) \]  

(84)

commute because \( H_1(\psi, \tilde{\psi}) = E_1(\psi, \tilde{\psi}) \). Bearing in mind that there is no interaction part in \( H \), we conclude that there is no flow of energy between I and II.

Let us now calculate the reduced density matrix of the linear subsystem I. With the notation of (83), we find \( \rho_1 = \frac{1}{2} (|\xi_1\rangle \langle \xi_1 | + |\xi_2\rangle \langle \xi_2 | + |\psi\rangle \langle \psi | + \langle \xi_1 | \langle \xi_2 | + \langle \psi | \langle \psi |) \). The off-diagonal elements vanish in the linear theory \( (\epsilon = 0) \) if initially the states are orthogonal, and we get a “fully mixed” state. For \( \epsilon \neq 0 \) these coherences oscillate with the mobility frequency. There therefore exist observables whose average values oscillate in this way. For example, the components of spin satisfy \( \langle \sigma_1 \rangle = \langle \sigma_3 \rangle = 0 \), but \( \langle \sigma_2 \rangle \) is proportional to \( \sin(4e\langle \sigma_3 \rangle t) \) hence depends on the parameter characterizing the nonlinearity of the other, separated and noninteracting system.

Let us complete the analysis of this part with two comments.

1. The telegraph is based on the fact that an average of an observable in the linear system I depends on parameters of the Hamiltonian function of the nonlinear system II. Recalling the form of the Poisson bracket equation for observables \( A_t \) related to \( I \), \( A_t = \{ A_1, H_1 + H_2 \} \) we can see that such a dependence is possible if and only if \( \{ A_1, H_2 \} \neq 0 \). It is an easy exercise to check that this condition is met indeed in the case of \( A = \langle \psi | \sigma_2 | \psi \rangle \) and \( H_2 \) from (82). It follows that the necessary an sufficient condition for elimination of telegraphs based on the mobility phenomenon is that any observables corresponding to separated systems must be in involution with respect to the Poisson bracket generating the evolution of observables. The observation that such observables are not necessarily in involution in the Weinberg approach is due to J. Polchinski [23].

2. If one of the systems is linear, then the telegraph based on the mobility phenomenon can be used for sending information only from the nonlinear system to the linear one. In Gisin’s telegraph one utilizes the remote preparation of mixtures entering the nonlinear system. Accordingly, this kind of telegraph works in the opposite direction hence cannot be equivalent to this based on the mobility phenomenon.

Let us describe now the Gisin’s telegraph more precisely [35]. Consider again the same Hamiltonian functions of the subsystems and the Weinberg’s description of the whole one (Eqs. (81) and (82)).

The sender is free to choose a basis in his Hilbert space by a rotation of his Stern-Gerlach device. Let the unitary matrix with unit determinant

\[ \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) \]  

(85)

describe the freedom in the choice of bases in the linear subsystem and the relation between the components of the whole state in the chosen basis and in the spin up-down one is given by

\[ \left( \begin{array}{cc} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{array} \right) = \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) \left( \begin{array}{cc} \psi_{++} & \psi_{+-} \\ \psi_{-+} & \psi_{--} \end{array} \right). \]  

(86)
The state is initially the singlet, which means that
\[
\begin{pmatrix}
\psi_{11}(0) \\
\psi_{12}(0) \\
\psi_{21}(0) \\
\psi_{22}(0)
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\alpha & \beta \\
-\beta & \bar{\alpha}
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-\beta & \bar{\alpha} \\
\bar{\alpha} & -\beta
\end{pmatrix}.
\] (87)

Finally, the solution for the whole system is
\[
|\psi\rangle = \frac{1}{\sqrt{2}} e^{-i(E_1 + E_2 - \epsilon x^2) t} \begin{pmatrix}
-\beta e^{-i\epsilon x t} & \bar{\alpha} e^{i\epsilon x t} \\
-\bar{\alpha} e^{i\epsilon x t} & -\beta e^{-i\epsilon x t}
\end{pmatrix}
\] (88)

where \( X = |\beta|^2 - |\alpha|^2 \). The reduced density matrix of the nonlinear system \( \mathbf{II} \) reads
\[
\rho^{II} = \frac{1}{2} I + Re(\bar{\alpha} \beta) \sin(4\epsilon(|\alpha|^2 - |\beta|^2)t) \sigma_2 + Im(\bar{\alpha} \beta) \sin(4\epsilon(|\alpha|^2 - |\beta|^2)t) \sigma_1.
\] (89)

The average of \( \sigma_2 \) in the nonlinear system is
\[
\langle \sigma_2 \rangle = 2 Re(\bar{\alpha} \beta) \sin(4\epsilon(|\alpha|^2 - |\beta|^2)t)
\] (90)

hence depends on the choice of basis made in the linear one.

Notice that since the Hamiltonian function of the linear subsystem is proportional to \( \langle \psi | \psi \rangle \) it is in involution with any observable. In particular \( \{ \langle \sigma_2 \rangle, H_1 \} = 0 \) which means that the commutability of observables corresponding to separated systems is not a sufficient condition for the nonexistence of faster-than-light telegraphs. The dependence of \( \langle \sigma_2 \rangle \) on \( \alpha \) and \( \beta \) follows from the dependence of \( H_2 \) on these parameters. We have remarked already in the section dealing with observables that the Weinberg’s choice is basis dependent: By a change of basis in \( \mathbf{I} \) we can change a value of energy in \( \mathbf{II} \).

It becomes clear now under what conditions this kind of pathology can be eliminated. Consider a density matrix \( \rho \) describing a “large” system. A change of basis in a subsystem \( \mathbf{I} \) is represented by the unitary transformation
\[
\rho \rightarrow U_I \otimes 1_{II} \rho U_1^{-1} \otimes 1_{II}.
\] (91)

Any function of \( \rho \) that can be written in a form of a series
\[
\hat{f}(\rho) = \sum_k s_k \rho^k
\] (92)
transforms in the same way. It follows that all expressions like
\[
Tr \left( \hat{f}(\rho) 1_{II} \otimes \hat{A}_{II} \right)
\] (93)
are \( U_I \)-independent. Also all functionals depending on reduced density matrices of subsystems are independent of changes of bases outside of those subsystems. In the second part of the paper I shall prove a general theorem stating that if two observables depend on reduced density matrices of different subsystems then they are in involution with respect to a large class of (Poisson or generalized Nambu) brackets. In such a case both kinds of telegraphs will be eliminated.

The last point that has to be explained is the uniqueness of the dynamics of subsystems. We have shown already that a knowledge of observables on pure states of subsystems does not determine their form if the subsystems are correlated with something else. Here we will show that observables that differ only on mixed states will, in general, generate different evolutions of subsystems. In the “atom+field” case the evolution of the atomic inversion will depend on our (arbitrary) choice of the description.

Consider our “canonical” example (81) but rewritten in a form including density matrices. For the sake of clarity let us also use the more general matrix \( \hat{e} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \) instead of \( \sqrt{\epsilon} \sigma_3 \).

We have now an infinite number of possibilities. The simplest nontrivial ones are
\[
H_1(\rho_1) = E_1 Tr \rho_1, \quad H_2(\rho_2) = E_2 Tr \rho_2 + \frac{(Tr \rho_2 \hat{e})^2}{Tr \rho_2}
\] (94)

and
\[
H_1(\rho_1) = E_1 Tr \rho_1, \quad H_2(\rho_2) = E_2 Tr \rho_2 + \frac{(Tr \rho_2 \hat{e})^2}{Tr \rho_2} \frac{Tr (\rho_2 \hat{e})}{(Tr \rho_2)^2}.
\] (95)

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Let $\rho_{kl} = \sum_m \psi_{mk} \bar{\psi}_{ml}$ denote the components of the reduced density matrix $\rho_2$. The Hamiltonian function of the whole system is

$$H(\rho) = H_1(\rho_1) + H_2(\rho_2). \tag{96}$$

The two forms of $H_2$ lead, respectively, to the following two evolution equations

$$\frac{d}{dt} \rho_{kl} = -2i \frac{Tr \rho_2 \hat{c}}{Tr \rho_2} \rho_{kl}(\epsilon_k - \epsilon_l) \tag{97}$$

and

$$\frac{d}{dt} \rho_{kl} = -2i \frac{Tr \rho_2 \hat{c} Tr(\rho_2^2)}{Tr \rho_2 (Tr \rho_2)^2} \rho_{kl}(\epsilon_k - \epsilon_l) \tag{98}$$

where the expressions involving traces are integrals of motion. The equations are different.

There is only one way out of the above dilemma: The description must from the outset be given in terms of density matrices.

### IV. A TWO-LEVEL ATOM IN NONLINEAR QM

The aim of the analysis below is to clarify some elementary features of the nonlinear formalism involved in calculations of optical phenomena in two-level atoms. The contents of this section should not be understood as a complete, unique solution or systematization of all the questions encountered. Still, I hope that I have managed to point out some elements essential for correct computations in practical, experimenタル situations.

In linear QM a two-level system is mathematically equivalent to a spin-1/2 nonrelativistic particle. All the examples discussed in this work were based on a two-dimensional Hilbert space and the reader may have a feeling that they could be applied equally well to both spin-1/2 particles and two-level atoms. This kind of conviction has been shared by all the authors dealing with theoretical and experimental aspects of nonlinear QM. The main result of this section, as we shall see later, is that, paraphrasing G. Orwell’s words, in nonlinear QM all two-level systems are two-level but some of them are more two-level than others.

We shall assume that the putative nonlinearity is of purely atomic origin. This means that we shall consider the atom as a nonlinear subsystem of the larger “atom+field” system where both the field and interaction Hamiltonian functions are linear in density matrix. (The papers discussing the problem can be divided into two groups: Either the authors do not care about the description of the “atom+field” composite system or treat it in the way proposed by Weinberg. We know that none of them can be correct unless the atom and the field are in a product state, which is typical for semiclassical treatments. It follows that no really quantum description of the problem has been given as yet.) We shall assume also the dipole and rotating wave approximations. To clarify the role of the latter we shall briefly treat both $\Delta m = 0$ and $\Delta m = \pm 1$ cases.

We begin with the form of the atomic Hamiltonian function. The simplest one (at lest from the point of view of simplicity of calculations) is

$$H_{AT}[\rho] = Tr \rho \hat{H}_L + \frac{(Tr \rho \hat{c})^2}{Tr \rho} \tag{99}$$

where $\hat{H}_L$ is the linear Hamiltonian of the atom and $\hat{c}$ is an operator commuting with $\hat{H}_L$. Assuming that we consider the atom in a pure state $\rho = \langle \psi | \psi \rangle$ we find that a general solution of the resulting nonlinear Schrödinger equation is (in this section we shall use the ordinary units with $\hbar = 1$)

$$\psi_k(t) = \psi_k(0) \exp \left[ -\frac{i}{\hbar} (E_k + 2\langle \hat{c} \rangle \epsilon_k - \langle \hat{c} \rangle^2) t \right]. \tag{100}$$

The averages of $\hat{c}$ are integrals of motion and, of course, depend on all nonvanishing components of $\langle \psi \rangle$. This is an important point. In the analysis of a coupling between the atom and an external electromagnetic field we meet two difficulties. First of all we have to decide which states will be involved in the absorption-emission process. In linear QM the situation is simple: We take two stationary states of the noninteracting atom. In nonlinear QM the atomic nonlinearity may lead to stationary states that are not orthogonal between one another. Atomic creation and annihilation operators corresponding to such levels cannot satisfy ordinary anticommutation relations.
To avoid such complications the interaction term we shall choose will be defined in the ordinary way, that is, in terms of creation and annihilation operators corresponding to the levels of the linear Hamiltonian.

Second, we cannot a priori restrict the atomic Hilbert space to two dimensions because a $k$-th eigenfrequency depends on amplitudes of all other components of $|\psi\rangle$ and the evolution cannot be naturally “cut into $N$-dimensional pieces”. Of course, we cannot also assume that only these $\epsilon_k$ are nonvanishing which we are interested in (a probability that in a given case we will find just those only nonvanishing ones is $0$). Therefore, to make the analysis perfectly consistent we should give up the two level approximation in the interaction term. We shall not, however, consider such complications although this approximation will further restrict the physical validity of the calculations presented below.

Let $b_k, b_k^\dagger$ be the $k$-th level atomic annihilation and creation operators satisfying the fermionic algebra $[b_k, b_l^\dagger] = \delta_{kl}$ and $a, a^\dagger$ be the annihilation and creation operators of a monochromatic photon field whose frequency is $\omega$. The choice of the creation-annihilation operators language leads naturally to the following Hamiltonian function of the whole “atom+field” composite system

$$H_{A+F}(\psi, \bar{\psi}) = \langle \psi | \sum_k \hbar \omega_k b_k^\dagger b_k + \hbar \omega a^\dagger a + \frac{i \hbar q}{2} (b_1^\dagger b_1 a - a^\dagger b_1^\dagger b_2) | \psi \rangle$$

$$+ \langle \psi | \sum_k \epsilon_k b_k^\dagger b_k | \psi \rangle^2$$

(101)

The state in the Fock basis is $| \psi \rangle = \sum_{kn} \psi_{kn} | k \rangle | n \rangle$. The nonlinear term is therefore equivalent to

$$\langle \psi | \epsilon \circ 1_F | \psi \rangle^2 = \frac{(\text{Tr} \rho_{AT} \epsilon)^2}{\text{Tr} \rho_{AT}}$$

(102)

which seems natural but, as we have seen before, is only one out of the whole variety of inequivalent possibilities. Such a description is consistent with the definition of the atomic Hamiltonian function (99) (in this sense it is unique) and free from any “malignant” nonlocalities. On the other hand it is in a mean-field style; we have remarked in the section on observables that

$$\frac{\text{Tr} \rho_{AT} \epsilon \rho_{AT} \epsilon}{\text{Tr} \rho_{AT}}$$

(103)

would look more “fundamental”. Anyway, although the conclusions drawn on a basis of such an analysis are limited in their generality, some choice has to be made and calculations with (103) would be more complicated.

The Hamiltonian resulting from (101) is

$$\hat{H}_{A+F} = \sum_k \hbar \omega_k b_k^\dagger b_k + \hbar \omega a^\dagger a + \frac{i \hbar q}{2} (b_1^\dagger b_1 a - a^\dagger b_1^\dagger b_2)$$

$$+ 2 \langle \psi | \sum_k \epsilon_k b_k^\dagger b_k | \psi \rangle^2 \sum_k \epsilon_k b_k^\dagger b_k - \langle \psi | \sum_k \epsilon_k b_k^\dagger b_k | \psi \rangle^2$$

(104)

and the nonlinear Schrödinger equation is

$$i \hbar \dot{\psi}_{1n} = \left\{ \hbar \omega_1 + 2 \sum_{bn} \epsilon_l | \psi_{bn} |^2 - \left( \sum_{bn} \epsilon_l | \psi_{bn} |^2 \right)^2 + n \hbar \omega \right\} \psi_{1n}$$

$$- \frac{i}{2} \hbar q \sqrt{n} \psi_{2,n-1}$$

$$i \hbar \dot{\psi}_{2n} = \left\{ \hbar \omega_2 + 2 \sum_{bn} \epsilon_l | \psi_{bn} |^2 - \left( \sum_{bn} \epsilon_l | \psi_{bn} |^2 \right)^2 + n \hbar \omega \right\} \psi_{2n}$$

$$+ \frac{i}{2} \hbar q \sqrt{n+1} \psi_{1,n+1}$$

$$\vdots$$

$$i \hbar \dot{\psi}_{kn} = \left\{ \hbar \omega_k + 2 \sum_{bn} \epsilon_l | \psi_{bn} |^2 - \left( \sum_{bn} \epsilon_l | \psi_{bn} |^2 \right)^2 + n \hbar \omega \right\} \psi_{kn}$$

for $k > 2$.

(105)
Writing $\psi_{kn} = A_{kn} \exp(-i a_{kn}/\hbar)$ we find that for $k > 2$ $A_{kn} = \text{const}$ for all $n$. Since also $\|\psi\|$ is time independent (hereafter we put $\|\psi\| = 1$) it follows that for $k > 2$ the exponents depend on time also via $\langle \psi | \psi_{12} | \psi \rangle = \sum_n (\epsilon_1 | \psi_{1n} \rangle + \epsilon_2 | \psi_{2n} \rangle)$ whose explicit form has to be determined. Decomposing

$$\langle \psi | \psi_{12} | \psi \rangle = \frac{1}{2} \langle \psi | (\epsilon_1 + \epsilon_2)(b_1^1 b_1^1 + b_2^1 b_2^1)| \psi \rangle + \frac{1}{2} \langle \psi | (\epsilon_1 - \epsilon_2)(b_1^1 b_1^2 - b_1^1 b_2^1)| \psi \rangle$$

$$= (\epsilon_1 + \epsilon_2) \langle \psi | R_2 | \psi \rangle + (\epsilon_2 - \epsilon_1) \langle \psi | R_2 | \psi \rangle,$$  \hspace{1cm} (106)

where the first expression is an integral of motion, we see that the problem reduces to calculating $\langle \psi | R_3 | \psi \rangle$ which is one half the atomic inversion. Denoting $\varepsilon = \epsilon \odot 1_F$, the total Hamiltonian can be decomposed now into two parts

$$\hat{H}_1 = \hbar (\omega_2 - \omega_1) R_3 + 2(\epsilon_2 - \epsilon_1) \langle \psi | \psi \rangle R_3 + \hbar \omega a^\dagger a$$

$$+ \frac{i \hbar q}{2} (b_1^1 b_1^1 - a^\dagger b_1^1 b_3^1),$$

$$\hat{H}_2 = \hbar (\omega_1 + \omega_2) R_0 + \sum_{k > 2} \hbar \omega_k b_1^k b_k^1 + 2 \langle \psi | \psi \rangle (\epsilon_1 + \epsilon_2) R_0$$

$$+ \sum_{k > 2} \hbar \epsilon_k b_1^k b_k^1) - \langle \psi | \psi \rangle \epsilon.$$

Operators, $R_3, R_1 = \frac{1}{2} (b_1^1 b_1^1 + b_2^1 b_2^1)$, $R_2 = \frac{i}{\hbar} (-b_1^1 b_1^2 + b_1^1 b_2^1)$, $a$ and $a^\dagger$ commute with $\hat{H}_2$ so that the evolution of the atomic operators $R_j$ is generated by the following nonlinear generalization of the Jaynes-Cummings Hamiltonian

$$\hat{H}_1 = \hbar \omega_0 R_3 + 2\epsilon_0 \langle \psi | \psi \rangle R_3 + \hbar \omega a^\dagger a + \hbar \epsilon \hat{H} a^\dagger R_\sigma - a^\dagger R_\sigma),$$

where $\omega_0 = \omega_2 - \omega_1$ and $\epsilon_0 = \epsilon_2 - \epsilon_1$. To explicitly distinguish between initial conditions and dynamical objects we shall decompose $\langle \psi | \psi \rangle$ as follows

$$\langle \psi | \psi \rangle = \sum_{k > 2, n} \epsilon_k | \psi_{kn} \rangle^2 + (\epsilon_1 + \epsilon_2) \langle \psi | R_0 | \psi \rangle + \epsilon_0 \langle \psi | R_3 | \psi \rangle := A + \epsilon_0 \langle \psi | R_3 | \psi \rangle$$

\hspace{1cm} (109)

where $A$ is a constant depending on initial conditions. Denoting further $\hbar \omega_0 + 2\epsilon_0 A = \hbar \omega_0', 2\epsilon_0 = \hbar \epsilon$ we finally obtain

$$\hat{H}_1 = \hbar \omega_0' R_3 + \hbar \epsilon \langle \psi | R_3 | \psi \rangle R_3 + \hbar \omega a^\dagger a + \frac{i \hbar q}{2} (R_+ a - a^\dagger R_\sigma).$$

\hspace{1cm} (110)

It is clear now what is the actual meaning of the two-level approximation in nonlinear QM. The evolution of the atomic operators $R_j$ is generated by a two-dimensional Hamiltonian in analogy to the linear case, but the parameters of $\hat{H}_1$ depend on components of the wave function corresponding to the levels being outside of the two-dimensional Hilbert space. On the other hand, the phases of the remaining components depend on the average of the atomic inversion of the two levels.

(It should be stressed that one can consider a quantum system whose Hamiltonian function is a sum of a linear term and some nonlinearity which involves only spinor components of the wave function. Formally, such a case would be equivalent to a composite system whose constituents are a scalar particle that evolves according to the laws of the linear QM and some nonlinear two-level system noninteracting with the particle. We know that various solutions of the nonlinear Schrödinger equation, including the particle one, will exist and no components other than the spinor ones will be involved in the nonlinear part of the evolution. Systems with nonlinearities of this kind would be “truly two-level” as opposed to the systems which are two-level in the sense specified above.)

Our next task is to find and solve an equation for the atomic inversion. The Poisson bracket evolution equation reads

$$i \hbar \frac{d}{dt} \langle \psi | R_3 | \psi \rangle = \langle \psi | R_3, \hat{H}_1 | \psi \rangle = \frac{q}{2} \langle \psi | R_+ a + a^\dagger R_\sigma | \psi \rangle$$

\hspace{1cm} (111)

so is just like in linear QM [36]. Following the notation of [36] the second derivative is found equal

$$\frac{d^2}{dt^2} \langle \psi | R_3 | \psi \rangle = -q^2 \langle \psi | R_3 (\hat{N} + \frac{1}{2}) | \psi \rangle$$

$$+ \frac{i q}{2} (\Delta' + \epsilon \langle \psi | R_3 | \psi \rangle) \langle \psi | R_+ a - a^\dagger R_\sigma | \psi \rangle$$

\hspace{1cm} (112)
where \( \hat{N} = R_3 + a^\dagger a \) and \( \Delta' = \omega' - \omega \). Define

\[
\hat{H} = \sum_{k \geq 2} \hbar \omega_k |\psi_k\rangle \langle \psi_k| + \hbar (\omega_1 + \omega_2) \langle \psi | R_0| \psi \rangle.
\] (113)

\( \langle \psi | \hat{N} | \psi \rangle \), like in the linear case, is constant. In order to get rid of the average in the last row of (112) we rewrite the whole Hamiltonian function as follows

\[
\hat{H}_{A+F} = \hbar \omega_{A+F} = A^2 + B + \hbar \omega \langle \psi | \hat{N} | \psi \rangle
\]

\[
+ \hbar \Delta'(\langle \psi | R_0| \psi \rangle + \epsilon_2^2 \langle \psi | R_0| \psi \rangle^2 + \frac{\hbar q}{2} \langle \psi | R_+a - a^\dagger R_-| \psi \rangle).
\] (114)

Denoting \( w = 2 \langle \psi | R_3| \psi \rangle, \omega_A = A^2/\hbar, \omega_B = B/\hbar, \omega_{RW A} = \omega_{A+F} - \omega_B \) we finally get

\[
\tilde{w} = 2 \Delta'(\omega_{RW A} - \omega \langle \hat{N} \rangle - \omega_A)
\]

\[
+ \left( \epsilon \omega_{RW A} - \omega \langle \hat{N} \rangle + \omega_A - \Delta' \right) w - q^2 \langle \psi | R_3(\hat{N} + \frac{1}{2})| \psi \rangle
\]

\[
- \frac{3}{4} \epsilon \Delta' w^2 - \epsilon^2 \frac{w^3}{8}.
\] (115)

We have met here the characteristic inconvenience of the Poisson bracket formalism of nonlinear QM: The nonexistence of the Heisenberg picture. In the linear case we can solve the Jaynes-Cummings problem completely, independently of any particular initial conditions for states. Here the term \( \langle \psi | R_3(\hat{N} + \frac{1}{2})| \psi \rangle \) involves correlations between the atom and the field and I have not managed to express it solely in terms of constants and \( w \) unless the state is an eigenstate of \( R_3 \), or a semiclassical decorrelation is assumed. So let initially the state of the system be a common eigenstate of \( R_3 \) and \( a^\dagger a \) with respective eigenvalues \( n' \) and \( n \). The atomic inversion satisfies then the general elliptic equation [37]

\[
\tilde{w} = 2 \Delta'(\Delta' n' + \frac{1}{8} \epsilon) + (\epsilon \Delta' n' + \frac{1}{8} \epsilon) - \frac{q^2}{2} (N + \frac{1}{2}) w - \frac{3}{4} \epsilon \Delta' w^2 - \epsilon^2 \frac{w^3}{8}.
\] (116)

Although we could try to find a general expression for the atomic inversion following from (116) it seems more instructive to make here some simplifying assumptions. First of all we can take the “two-level initial conditions”, that is, assume that the initial state of the system is such that the only nonvanishing components of the wave function are those with \( k = 1, 2 \). Then \( \hbar \Delta' = \hbar \Delta + \epsilon_2^2 - \epsilon_1^2 \) where \( \Delta = \omega_B - \omega \). (Let me remark here that in most of the papers dealing with two-level systems (cf. [2,1]) their authors assumed that for the “simplest” nonlinearities \( \epsilon_2 = 0 \) which seemed to suggest that the nonlinearity must shift the resonant frequency. As we can see, the more symmetric “\( \sigma_3 \)” choice does not change the detuning.) Further, choosing the “detuning” \( \Delta' = 0 \) and denoting \( \epsilon' = 8 = 2 \epsilon \) we get

\[
\tilde{w} = (2 \epsilon^2 - \Omega^2) w - 2 \epsilon^2 w^3.
\] (117)

This equation can be solved immediately. For example, with the initial condition \( w(0) = -1 \) we find

\[
w(t) = \begin{cases} -cn(\Omega t, \epsilon/\Omega) & \text{for } \Omega > \epsilon \\ -sech(\Omega t) & \text{for } \Omega = \epsilon \\ -dn(\epsilon t, \epsilon/\Omega) & \text{for } \Omega < \epsilon \end{cases}
\] (118)

The result is analogous to this of Wódkiewicz and Scully [2] who chose the Bloch equations approach.

It is an appropriate point for a brief comparison of our results with those of Weinberg who chose his own, basis dependent description of the “atom+field” system. Consider the same form of the atomic nonlinearity. The Hamiltonian function of the composite system in the \( Fock \) basis (this basis was chosen by Weinberg) is

\[
\hat{H}_{A+F}(\psi, \tilde{\psi}) = \langle \psi | \sum_k \hbar \omega_k b_k^\dagger b_k + \hbar (\omega_1 + \omega_2) \langle \psi | R_0| \psi \rangle + \frac{\hbar q}{2} \langle \psi | R_+a - a^\dagger R_-| \psi \rangle + \sum_n \langle \psi | \sum_k \epsilon_k b_k^\dagger b_k \hat{P}_n | \psi \rangle^2.
\] (119)

where \( \hat{P}_n \) project on \( n \)-photon states. Assume now that initially the state of the whole system has only one nonvanishing component \( \psi_{11} \) (one photon and the atom in the ground state). It follows that only \( \psi_{11} \) and \( \psi_{20} \) will appear in the nonlinear Schrödinger equation. Our “nonmalignant” choice leads to the Schrödinger equation of the form
Simplifying the fractions we obtain

\begin{align}
\dot{\psi}_{11} &= \left\{ \hbar \omega_1 + 2 \epsilon_1 |\psi_{11}|^2 / |\psi_{11}|^2 \epsilon_1 - \left( \epsilon_1 / |\psi_{11}|^2 + |\psi_{20}|^2 \right)^2 + \hbar \omega \right\} \psi_{11} \\
&\quad - i \hbar \dot{q} \psi_{20} \\
\dot{\psi}_{20} &= \left\{ \hbar \omega_2 + 2 \epsilon_2 |\psi_{20}|^2 / |\psi_{20}|^2 \epsilon_2 - \left( \epsilon_2 / |\psi_{20}|^2 + |\psi_{28}|^2 \right)^2 \right\} \psi_{20} \\
&\quad + i \hbar \dot{q} \psi_{11},
\end{align}

while the Weinberg’s one leads to

\begin{align}
\dot{\psi}_{11} &= \left\{ \hbar \omega_1 + 2 \epsilon_1 |\psi_{11}|^2 / |\psi_{11}|^2 \epsilon_1 - \left( \epsilon_1 / |\psi_{11}|^2 \right)^2 + \hbar \omega \right\} \psi_{11} \\
&\quad - i \hbar \dot{q} \psi_{20} \\
\dot{\psi}_{20} &= \left\{ \hbar \omega_2 + 2 \epsilon_2 |\psi_{20}|^2 / |\psi_{20}|^2 \epsilon_2 - \left( \epsilon_2 / |\psi_{20}|^2 \right)^2 \right\} \psi_{20} \\
&\quad + i \hbar \dot{q} \psi_{11}.
\end{align}

Simplifying the fractions we obtain

\begin{align}
\dot{\psi}_{11} &= (\hbar \omega_1 + \epsilon_1^2) \psi_{11} - i \hbar \dot{q} \psi_{20} \\
\dot{\psi}_{20} &= (\hbar \omega_2 + \epsilon_2^2) \psi_{20} + i \hbar \dot{q} \psi_{11}
\end{align}

which are linear and the only modification with respect to ordinary QM is that the energy levels of the atom are the nonlinear eigenvalues corresponding to the atom in the absence of radiation. In addition, the “\( \sigma_3 \)” nonlinearity satisfies \( \epsilon_1^2 = \epsilon_2^2 \) so that for this choice of the “simplest” nonlinearity neither the energy difference nor the shape of the inversion’s oscillation would be affected, whereas we know already that the “correct” Polchinski’s description leads to elliptic oscillations even for \( w(0) = -1 \) and \( N + 1/2 = 1 \).

This result explains the difference between the calculations of Weinberg and those of Wódkiewicz and Scully. Both of them are based, more or less implicitly, on different assumptions about the description of composite systems. Nevertheless, it is clear that we have chosen some other “correct” form of the total Hamiltonian function, we would have obtained some other solution for the atomic inversion.

In the calculations in this section we have not needed any assumption about the “smallness” of the nonlinearity. Moreover, as we have shown before, it is not evident what should be actually meant by a small nonlinearity. This lack of uniqueness is related to the fact that there exist singular nonlinearities that are negligible in the lack of correlations, but can become dominant if the nonlinear system in question correlates with something else. Anyway, even ignoring these subtle points it seems reasonable to expect that a physical nonlinearity, if any, should be in ordinary situations small in some sense. Therefore, it becomes interesting to understand in what respect the solutions we have found depend on approximations. In particular the role of the rotating wave approximation should be clarified.

The easiest way of doing that is to consider transitions with the selection rules \( \Delta m = \pm 1 \) involving circularly polarized light. It can be shown easily that the only difference with respect to the \( \Delta m = 0 \) transitions discussed above is the necessity of substituting \( q \dot{q} \) for \( q^2 \) in the equations for \( w \), so that no qualitative change in the time dependence of \( w \) will appear.

V. WIGNER’S FRIEND REVISITED

One of the essential elements of the argumentation against the description of composite systems proposed by Weinberg was its dependence on a particular basis in a Hilbert space. In ordinary quantum situations such a dependence is difficult to accept. However, the situation changes if we enter the domain of the measurement theory. Notice, that if the role of the composite system is played by an “object” and an “observer”, and the nonlinear evolution occurs in the state space of the observer, then there exists a privileged basis in this space, the pointer basis [38], and this is the basis that has to be chosen for calculation of averages of observed quantities, unless one uses the many-worlds interpretation, which does not select any basis.
The idea that the domain of (conscious) observations is a natural arena for a nonlinear evolution of quantum states was introduced by Wigner in his paradox of a friend [39]. Wigner considered a physical system that can be described as “observed object + observer + observer.” The hypothetical nonlinearity was to make the off-diagonal elements of the density matrix of the “object + observer + observer” system disappear, since once the result is consciously observed, no further interferences should be possible. Such a goal can be achieved either by a “collapse” of a state vector, or in some “no-collapse” way (decoherence approach). The suggestion of Wigner, contrary to the opinion of Penrose expressed in [40], belonged to the latter class of theories.

A modification of the observed (linear) system can exert an influence on the observer; the Gisin’s telegraph belongs to such a class of phenomena, so may not be very pathological in the context of measurements. On the other hand, the telegraph based on the mobility phenomenon would lead to telekinetic phenomena [41]. Indeed, consider a pre-measurement that produces an entangled state of a system and an observer. Let the observed system be a two-level atom and an electromagnetic field, initially in the state

\[ |1\rangle|1\rangle + |2\rangle|0\rangle = |11\rangle + |20\rangle \]  

(123)

where the notation is analogous to this from the previous section. Assume that a Hilbert space of the observer is also spanned by two states \(|±\rangle\). After the pre-measurement the state of the joint system is

\[ |11\rangle|+\rangle + |20\rangle|−\rangle. \]  

(124)

If now the observer’s state space undergoes a nonlinear evolution with the “\(σ_3\)” nonlinearity like in (81) then two possibilities occur. First, if only the \(|±\rangle\) start to rotate with the mobility frequency then the atomic reduced density matrix is like in linear QM. However, if the nonlinearity violates the orthogonality of the states \(|1\rangle|+\rangle := |1+\rangle\) and \(|0\rangle|−\rangle := |0−\rangle\) (i.e. when the observer makes both his own consciousness and the interacting photons evolve in a nonlinear way) then the atom starts to “feel” it. It follows that, at least in principle, a suitable form of his “own” nonlinearity can enable the observer influence in a statistically observable way a behavior of a random generator just by watching it! Notice that there is also some limitation on the possibility of the influence. For consider a situation where there is a number of intermediate states between the random generator and the observer, so that the entangled state takes the form

\[ |1\rangle |+1\rangle |+2\rangle \ldots |+n\rangle |+\rangle + |2\rangle |−1\rangle |−2\rangle \ldots |−n\rangle |−\rangle. \]

The random generator will “feel” the mobility only provided the mobility will involve the underbraced states

\[ |1\rangle |+1\rangle |+2\rangle \ldots |+n\rangle |+\rangle + |2\rangle |−1\rangle |−2\rangle \ldots |−n\rangle |−\rangle. \]

In case the nonlinearity is more localized, say

\[ |1\rangle |+1\rangle |+2\rangle \ldots |+n\rangle |+\rangle + |2\rangle |−1\rangle |−2\rangle \ldots |−n\rangle |−\rangle. \]

then the generator’s density matrix does not contain the “malignant” terms (one cannot influence the generator by watching it on TV; or, it is much easier to influence with my thoughts my own finger than someone else’s).

The above phenomenon is present only if we assume the kind of description à la Weinberg. One may hope that at least in the “correct” description no ways of exerting observer’s influence on external world by “thoughts” or “intentions” exist. The following surprising example is a slightly modified version of the phenomenon noticed by J. Polchinski [23]. This is the only paradox that I was not able to eliminate in the Nambu-like generalization of QM proposed in [7].

Consider a process involving four steps. (1) A spin-1/2 ion enters a Stern-Gerlach device, which couples to the linear spin \(σ_3\) component and the beam splits into two, \(+1\) and \(−1\) sub-beams. (2) The “+1” beam evolves freely; in the path of the “−1” beam a macroscopic observer (or a random generator) takes one of two actions: \((a)\) does nothing (say, with probability \(\lambda_1\)), or \((b)\) rotates the spin into the “+1” direction with a magnetic field coupled to \(σ_3\) (with probability \(\lambda_2\)). (3) The two beams are rejoined and the ion enters a region of field coupled to the nonlinear observable

\[ f \frac{\langle Tr ρ σ_3 \rangle^2}{Tr ρ}. \]  

(125)

(4) The observer again measures the spin with a Stern-Gerlach device coupled to \(σ_3\).
The steps 1 and 2 prepare the initial condition for the nonlinear evolution, and the reduced density matrix of the ion after step 2 is

\[ \rho_0 = \lambda_1 \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}. \] (126)

The ion’s nonlinear Hamiltonian for the step 3 is

\[ \dot{H} = 2f \frac{Tr \rho \sigma_1}{Tr \rho} \sigma_1 \] (127)

where we have assumed the “correct” form of the evolution, i.e. that it is generated by the Hamiltonian function depending only on the reduced density matrix of the ion. The reduced density matrix satisfies

\[ i\dot{\rho} = 2f \frac{Tr \rho \sigma_1}{Tr \rho} [\sigma_1, \rho] \] (128)

whose solution is

\[ \rho(t) = e^{-2if(\sigma_1)e_{1,t}t} \rho(0)e^{2if(\sigma_1)e_{1,t}t} \]

\[ = \frac{\lambda_1}{2} 1 + \frac{\lambda_3}{4} \left( 21 + \sigma_1 + \sigma_3 \cos 2\lambda_2 ft + \sigma_2 \sin 2\lambda_2 ft \right). \] (129)

In the analogous manner we can calculate the evolution of the projector \( P_\pm = \frac{1}{2} (1 \pm \sigma_3) \). We find in the “Heisenberg picture” (i.e. we solve the Heisenberg equations of motion with the nonlinear Hamiltonian)

\[ P_\pm(t) = \frac{1}{2} \left( 1 \pm \sigma_3 \cos 2\lambda_2 ft \mp \sigma_3 \sin 2\lambda_2 ft \right). \] (130)

The linear case, where in step three the ion couples with the same coupling constant linearly to \( \sigma_1 \), would yield

\[ P_\pm(t) = \frac{1}{2} \left( 1 \pm \sigma_3 \cos 2ft \mp \sigma_3 \sin 2ft \right). \] (131)

Assuming that the time of the interaction during the third step satisfies \( 2ft = \pi \) we get, in the linear case,

\[ P_\pm(t) = \frac{1}{2} \left( 1 \mp \sigma_3 \right) = P_\mp(0), \] (132)

which means that the spin changes its sign during the evolution. In the nonlinear case, however,

\[ P_\pm(t) = \frac{1}{2} \left( 1 \pm \sigma_3 \cos \lambda_2 \pi \mp \sigma_3 \sin \lambda_2 \pi \right) \] (133)

hence, in particular, the evolution of \( P_\pm \) depends on \( \lambda_2 \) — the probability of one of the two actions taken by the observer or the random generator in case the spin turned out to be \(-1\). The result means that the evolution of the ion depends on “intentions” of the observer concerning his possible actions he would have undertaken had the spin of the ion turned out to be \(-1\) — even in case the spin is \(+1\) and the observer is passive! For example, for \( \lambda_2 = 0 \), that is when the observer is decided not to take any actions, the spin state of the ion would remain unchanged. In the opposite case, \( \lambda_2 = 1 \), the evolution would be like in linear QM.

It seems that the essential point of this argumentation is the assumption that a single member of a beam of ions is described by the same density matrix as the whole ensemble. This is exactly opposite to the reasoning leading to Gisin’s telegraph. And I think this example indicates one of the most fundamental conceptual, or practical, difficulties of nonlinear QM: The fact that we do not really know how to treat beams of single objects. Intuitively, weak beams should be “linear” while strong ones could be, perhaps, “nonlinear” in some mean-field sense. The second hint for further generalizations of linear QM is a possibility, suggested by Wigner, that the only domain of fundamental nonlinearities could be the consciousness of observer. Then the density matrix representing the consciousness could evolve nonlinearly and no decompositions of the observer into sub-ensembles would make any sense. Still, the Polchinski’s phenomenon can describe something like intuition: A perception of a single event depends on the overall property of an ensemble of such events since the whole density matrix is involved.

A question that arises immediately is how we can distinguish between systems that are “conscious” (observers) and “non-conscious”. If one wants to make such distinctions more formal and without introducing the state vector reduction postulate, one has to use the language of the information theory. In the second part of this paper I shall discuss a generalization of QM based on Rényi entropies. Such entropies naturally select a class of systems that can gain information. It will be shown that the possibility of gaining information by a quantum system is naturally related to the nonlinearity of evolution.
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