MAGNETS WITH IRON YOKES AND FIELDS ABOVE 20 KG

by

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ABSTRACT

The magnetic field in conventional beam transport magnets is limited by the fact that iron saturates at 21.5 kGauss. This report proposes a method of using iron in magnets with uniform fields and fields with a constant gradient which are higher than this saturation level. These magnets can handle larger flux because the flux does not enter the aperture via restricted pole-faces but enters and leaves across the whole iron boundary. Apart from being a magnetic equipotential line the iron boundary is chosen as a line of constant flux density so as to allow the highest possible flux to pass through it. The coils cover the iron boundary and bend the field lines thereby increasing the flux density and producing the desired uniform or linearly varying field. The report also discusses the details of the field configuration in the corners of the aperture and presents numerical examples of 30 kGauss magnets taken from an earlier report.

RESUME

Dans les aimants de transport de faisceau habituels le champ magnétique est limité par le fait que le fer est saturé à 21.5 kGauss. Dans ce rapport on propose une méthode d'utilisation du fer dans des aimants à champ homogène et à champ à gradient constant, qui dépasse cette limite de saturation. Ces aimants sont capables d'un flux plus grand parce que le passage du flux n'est pas limité aux pièces polaires mais peut s'effectuer à travers tout le bord du fer. Outre le fait d'être une ligne équipotentielle le bord du fer est choisi comme ligne de densité de flux constante de manière à permettre le passage d'un flux maximum. Les bobines couvrent le fer, incurrivent et concentrent en même temps les lignes de champ produisant ainsi le champ homogène ou à gradient constant désiré. On discute aussi dans ce rapport les détails de la configuration du champ dans les coins de l'ouverture, et on y a inclus des exemples numériques pour des aimants de 30 kGauss copiés d'un rapport précédent.
1. **Bending magnets with uniform fields**

![Diagram of bending magnets with uniform fields](image)

Fig. 1

Fig. 1 shows an infinite plane iron boundary which is traversed perpendicularly by a uniform magnetic induction $B_x$. $B$ traverses this boundary and the conductor which covers the iron boundary without any change (div. $B = 0$). Therefore we find outside the iron ($x > 0$)

$$H_x = \frac{B_x}{\mu_o} = H_0 \quad \text{constant} \quad (1)$$

If a current flows in the conductor with current density $j$ it adds a tangential field component $H_y$

$$H_y = j \cdot x \quad (2)$$
and the field lines in the conductor have the slope

\[ \frac{dy}{dx} = \frac{H_y}{H_x} = \frac{j}{H_o} \cdot x \]  \hspace{1cm} (3)

\[ y = \frac{j}{2H_o} x^2 + \text{constant} \]  \hspace{1cm} (4)

The field lines in the conductor are parabolas. On the surface \( x = d \) we have

\[ H_y = j \cdot d \]  \hspace{1cm} (5)

Outside the conductor \((x > d)\) the field \( H \) is uniform and higher than the field \( H_x = H_o \) on the iron boundary:

\[ H_o = H \cdot \cos \alpha \]  \hspace{1cm} (6)

The current density is related to this field by

\[ j \cdot d = H \cdot \sin \alpha \]  \hspace{1cm} (7)

Fig. 2 shows magnets with square and diamond-shaped apertures in which the field is considerably higher than on the iron boundary. For instance, if we admit 21.5 kGauss on the iron boundary of the square magnet \((\alpha = 45^\circ)\) then relation (6) yields us 30.4 kGauss in the aperture. Since the flux density on the iron boundary never exceeds the constant value \( B_x \), one can pass a maximum of flux through these magnets without any field distortion by partial saturation. This design principle can also be applied to focusing magnets with wedge-shaped coils which produce a constant field gradient (synchrotron magnets).
If the conductors are thin current sheets \((d \to 0)\), the left-hand side of equation (7) represents the surface current density \(J = j \cdot d\), and the total number of ampereturns required for a field \(H\) (in \(A/cm\)) can easily be calculated from the vertical diagonal (height \(h\)) of the aperture

\[
n \cdot I = h \cdot H
\]

If the conductors are thick, we have to arrange that the boundary conditions are satisfied in the corners where two conductors join. The situation in the left wedge conductor on the bottom is drawn in Fig. 3.

![Fig. 3](image)

The field component

\[
H_y = j \cdot x
\]

is found along BC and ensures that the field is normal to AC. Along the boundary AB, given by

\[
y = x \cdot \tan \alpha
\]

we want to have a tangential field

\[
\frac{H_y}{H_x} = \tan \alpha
\]
Thus,

\[ H_x = \frac{i}{\tan^2 \alpha} \cdot y \]

satisfies all our boundary conditions. These field lines are hyperbolas. In the region ABC we need the current density

\[ j' = \text{curl} \, H = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j \left(1 - \frac{1}{\tan^2 \alpha}\right) \]

In the square magnet (\(\alpha = 45^\circ\)) it vanishes and for \(\alpha > 45^\circ\) it is always smaller than \(j\).

In a similar way one can show that there is no field distortion if the conductors in the right and left corner have the current density \(2j\) and the field lines become circles (for any \(\alpha\)). As the coil dimensions and current densities are known, it is easy to calculate the currents, but it can be shown that formula (8) applies also to thick conductors if \(h\) is replaced by \((h_1 + h_2)/2\) where \(h_1\) and \(h_2\) refer to inner and outer diamond. The edge conductors with the high current density \(2j\) could probably be replaced by thicker conductors which carry the same current, without seriously disturbing the field conformity.

An interesting consequence of the uniform induction on the straight iron boundary is that a small parallel displacement of the boundary does not significantly deteriorate the field. The situation is similar to the change of the height of an air gap and the decrease in field strength can be compensated by additional current turns in the corners. This may be useful in the case of superconducting coils which need thermal insulation from the iron.
2. **Focusing bending magnets with straight iron boundaries**

a) **Thick conductors with straight boundaries**

![Diagram of thick conductor with straight iron boundaries](image)

**Fig. 4**

Consider a straight iron boundary covered by a wedge-shaped conductor (Fig. 4) in which we have the parabolic field lines

\[
\vec{H} = \begin{pmatrix} H_x \\ H_y \end{pmatrix} = \begin{pmatrix} H_0 \\ j \cdot x \end{pmatrix}
\]

Then we shall show that the field lines which come out of the sloping conductor surface

\[ y = x / \tan 2\beta \]
can be matched to a quadrupole field with the gradient
\[ g = j \cdot \sin 2\beta \]
which is centred at
\[ x_0 = \frac{H_0}{j \cdot \tan 2\beta} \]
\[ y_0 = \frac{-H_0}{j} \]
and has the axes \( \bar{x}, \bar{y} \) which are turned by the angle \(-\beta\) with respect to \( x, y \). We observe that the axis \( \bar{y} \) of the quadrupole field is parallel to the line of symmetry of the wedge, and that the line \( V0 \) forms the angle \( \beta \) with the \( \bar{x}\)-axis. If \( x_0 > 0 \), we call the wedge "positive".

We introduce appropriate co-ordinates \( \bar{x}, \bar{y} \) by means of the transformations
\[
\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}
\]
in which the quadrupole field is expressed by
\[
\begin{pmatrix} H_{\bar{x}} \\ H_{\bar{y}} \end{pmatrix} = g \begin{pmatrix} -\bar{x} \\ -\bar{y} \end{pmatrix} = g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}
\]
\[
= j \sin 2\beta \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}
\]
The field in the conductor is also transformed into the new co-ordinates
\[
\begin{pmatrix} H_{\bar{x}} \\ H_{\bar{y}} \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} H_x \\ H_y \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} H_0 \\ jx \end{pmatrix}
\]
Now we compare the two fields along the boundary $x = y \cdot \tan 2\beta$ of the conductor

$$
\begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}
\begin{pmatrix}
H_0 \\
jy \cdot \tan 2\beta
\end{pmatrix} =
\begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}
\begin{pmatrix}
y \cdot \tan 2\beta \\
y + \frac{H_0}{j}
\end{pmatrix}
$$

and find that this is in fact an identity.

Let again be

$$
\begin{pmatrix}
H_x \\
H_y
\end{pmatrix} = \begin{pmatrix} H_0 \\ jx \end{pmatrix}
$$

inside the conductor, but put $\beta = \pi/4$. Then the axis of the quadrupole field will also be turned by $\pi/4$, the centre $0$ will shift along the line $y_0 = -\frac{H_0}{j}$ onto the iron boundary $x_0 = 0$ and the gradient will be $g = j$. We obtain the quadrupole field

$$
\begin{pmatrix}
H_x \\
H_y
\end{pmatrix} = \begin{pmatrix} jy + H_0 \\ jx \end{pmatrix}
$$

which fits well to the field in the conductor. Especially on the line $y = -\frac{H_0}{j}$ the field is normal and we can replace this line by an iron boundary thereby obtaining a corner in $0$. Such a situation occurs in the upper and lower corner of Fig. 6.

If we change the sign of $j$ inverting the direction of the current in the conductor

$$
\begin{pmatrix}
H_x \\
H_y
\end{pmatrix} = \begin{pmatrix} H_0 \\ -jx \end{pmatrix}
$$

still keeping $\beta = \pi/4$, then this field fits to another field

$$
\begin{pmatrix}
H_x \\
H_y
\end{pmatrix} = \begin{pmatrix} jy + H_0 \\ -jx \end{pmatrix}
$$
which is also normal on the line \( y = -H_0/j \) so that we can replace this line by an iron boundary and form a corner in \( O \). However, now the current density \( \text{curl} \ H = -2j \) has to be twice that in the conductor and the field lines are circles around \( O \). Therefore they are perpendicular on any straight iron boundary intersecting in \( O \). Such a circular field in a conductor with current density \( 2j \) can be used to match the fields in the left and the right corner of the magnet in Fig. 6. A similar situation occurred already in 2 corners of the bending magnet in Fig 2.

---

**Fig. 5a**
Fig. 5a illustrates two cases: the so-called "positive" wedge with its quadrupole centre on the air side, and the so-called "negative" wedge with its quadrupole centre on the iron side of the wedge conductor.

In the next sections we shall combine the wedge conductors to design focusing magnets (see Figs. 6 and 7).

b) Corner formed by two "positive" wedges

Fig. 5b

In Fig. 5b consider the two wedges with vertices V and V' intersecting at AB'CB. Let VC, V'C be iron boundaries and let the
two wedges with vertices \( V \) and \( V' \) be conductors with the same current density \( j \). They overlap on the area \( AB'CB \) where the current density is \( 2j \) and the field lines are circles centred at \( C \).

We have to show that,

if the two parabolic fields match along the lines \( VA \) and \( V'A \) respectively to the same quadrupole field of centre \( 0 \), then they match along the lines \( BA \) and \( B'A \) to the same circular field with centre \( C \), the lines \( BA \) and \( B'A \) being normal on the iron boundaries. In fact, we know that the distance between the normal projection \( N \) of the quadrupole centre \( 0 \) onto the iron boundary \( VB \) of the parabolic field and the wedge vertex \( V \) is simply

\[
|y_0| = \frac{H_0}{j}
\]

This same value is the measure of the distance \( CB \) between the centre \( C \) of the circular field associated with the cut \( BA \) and \( B \).

So having \( j \) equal for both wedges and using \( H_0 \) as a mapping factor (field strength at \( VB \) : field strength at \( V'B' \)) the result is established by anti-similarity of \( VBCB'A \) and \( V'B'CBA \).

c) Corner formed by two "negative" wedges

In Fig. 5c consider the two wedges with vertices \( V \) and \( V' \) intersecting at \( AB'CB \).

Let \( CF' \) and \( CF \) be iron boundaries and let the two wedges with vertices \( V \) and \( V' \) be conductors with current density \( -j \). They overlap on the area \( ABCB' \) where the current density is \( -2j \) and the field lines are circles centred at \( C \).
We have to show that,

if the two parabolic fields match the same quadrupole field of centre O along the lines AB' and AE respectively, then they match the same circular field of centre C along the lines AB' and AB respectively, the lines B'A and BA being normal on the iron boundaries. In fact we know that the distance between the normal projection N of the quadrupole centre O onto the iron boundary VF' of the parabolic field and the wedge vertex V is simply \( y_0 = \frac{E_0}{j} \). This is also the value of the distance CB' between the centre C of the circular field associated with the cut AB' and this cut.
So, having \(-j\) equal for both wedges and using \(H_0\) as the mapping factor \((H_0\text{ on } B'F' : H_0\text{ on } BF)\) the result is established by anti-similarity of \(V'\text{B}'\text{ABC}\) and \(V'B'\text{ABC}\).

\(d)\text{ Corner formed by one "positive" and one "negative" wedge}\)

Since again the distance \(\left|\frac{H_0}{j}\right|\) between the normal projection of the quadrupole field centre onto the iron boundary of the parabolic field is a characteristic of this field and hence independent of the wedge angle, we can argue as follows:

![Diagram](image)

**Fig. 5d**

Consider in Fig. 5d the wedges with vertices \(\bar{V}\) and \(\bar{V}'\), iron boundaries \(\bar{V}V\) and \(\bar{V}'C\), and current densities \(+j\) and \(-j\).
Let thus the parabolic fields of these wedges match along AE and \( V'A \) to the same quadrupole field of centre 0.

By equality of \( N\bar{N} \) and \( CV = V'A \) the quadrupole field defined by the cut VA has the centre C. By equality of \( N'\bar{N}' \) and \( V'C = AV \) the quadrupole field defined by the cut \( V'A \) has the centre C. The axes of these quadrupole fields are turned by \( \frac{\pi}{4} \) with respect to CV, CV'. As they agree in A, they are identical.

In Figs. 6 and 7 we have combined the wedges of Figs. 5b, 5c, and 5d to form symmetrical magnets, with \( \tan \beta = 0, \frac{\pi}{2} \).

Fig. 6 gives the field lines for the following case:

We define the two quantities

\[
\begin{align*}
x_1 & := \frac{x}{\bar{x}}, = \frac{\bar{y}}{y} \quad \text{(of Fig. 5c)}; \quad x_2 := \frac{x}{\bar{x}}, = \frac{\bar{y}}{y} \quad \text{(of Fig. 5b)} \\
& = \frac{x}{\bar{x}} \quad \text{(of Fig. 5d)} \quad \quad \frac{\bar{y}}{y} \quad \text{(of Fig. 5d)}
\end{align*}
\]

Fig. 6 illustrates the choice

\[ x_2 : x_1 = 2 : 1. \]

Fig. 7 gives the field lines for

\[ x_1 = 0 \]

where the wedges having the quadrupole centre as vertex contain a degenerate "parabolic" field (\( H_0 = 0 \))

\[
\begin{pmatrix}
H_x \\
H_y
\end{pmatrix} = \begin{pmatrix}
0 \\
Jx
\end{pmatrix}
\]

The cross-section of each of the four conductors (overlapping in the corners) is

\[
q = \frac{\sin \frac{4\beta}{\beta}}{4 \cos^2 (x_2^2 - x_1^2)}
\]
The window corner with the highest field has

\[ \bar{x}_{\text{max}} = \bar{y}_{\text{max}} = x_2 \frac{1}{\cos \beta (\cos \beta + \sin \beta)} \]

the one with the lowest field

\[ \bar{x}_{\text{min}} = \bar{y}_{\text{min}} = x_1 \frac{1}{\cos \beta (\cos \beta - \sin \beta)} \]

So, if \( d \) is the distance between the right and left corner of the window, and if the fields \( H_{\text{max}} \) and \( H_{\text{min}} \) and the current density \( j \) are specified, we calculate first

\[ g = \frac{H_{\text{max}} - H_{\text{min}}}{d}, \quad \sin 2\beta = \frac{g}{j} \]

\[ x_1 = \frac{H_{\text{min}} \cos \beta (\cos \beta - \sin \beta)}{\sqrt{2} g}, \quad x_2 = \frac{H_{\text{max}} \cos \beta (\cos \beta + \sin \beta)}{\sqrt{2} g} \]

then we can calculate the cross-section \( q \) and the current-turns \( nI \) and the resistance per unit length,

\[ nI = 2 j q, \quad R^* = 2 \frac{1}{\kappa q}, \quad (\kappa = \text{conductivity}) \]

Finally, the power dissipation per unit length of the magnet is

\[ N^* = (nI)^2 R^* = 8 \frac{1}{\kappa} \frac{\sin \frac{4\beta}{2}}{\cos^2 \beta} (x_2^2 - x_1^2) \]
3. **Focusing bending magnets with curved iron boundaries**

   a) **Current sheets**

   The problem of a magnet with curved iron boundaries is simplified if we replace the wedge shaped conductors by thin current sheets. In a given quadrupole field

   \[ H_x = x \quad \quad \quad H = \sqrt{x^2 + y^2} = r \]

   \[ H_y = -y. \]

   with magnetic potential

   \[ u = \frac{x^2 - y^2}{2} \]

   and flux function

   \[ v = xy \]

   We want to find curves which intersect the flux lines in such a way that the number of flux lines per line element is constant (e.g. = 1). We let this line be our iron boundary covered by a current sheet in which we adjust the surface current density \( J \) in such a way that our constant flux can enter perpendicularly into the iron. We split the field \( H \) into a normal component \( H_0 = 1 \) and a tangential component which is equal to our current density \( J \)

   \[ H_0 = J = \sqrt{H^2 - H_y^2} = \sqrt{r^2 - 1} \]

   The correct surface current density depends simply on the radius and we are only interested in \( r \gg 1 \).

   If our line element is given by

   \[ ds = \sqrt{1 + (y')^2} \, dx \]

   *) NB: To obtain this simple expression for \( v \) we have adopted a coordinate system which is turned by \( \pi/4 \) with respect to the usual notation.
then we look for solutions of the differential equation

\[
\left| \frac{dy}{ds} \right| = \left| \frac{dy}{dx} \frac{ds}{dx} \right| = \left| \frac{y + xy'}{\sqrt{1 + (y')^2}} \right| = 1
\]

or

\[(y + xy')^2 = 1 + y'^2\]

Particular solutions of this differential equation are the straight boundaries

\[
y = +1 \text{ and } y = -1 \text{ with } y' = 0
\]

\[
x = +1 \text{ and } x = -1 \text{ with } y' = 0
\]

which we know already (wedge with \( \beta \rightarrow 0 \)). If we turn our square aperture by \( \pi/4 \) with respect to the field, we obtain a Panofsky quadrupole \(^1\) in which not the flux but the surface current density is constant on the boundary. To return to our problem, in general the differential equation remains unchanged if we change simultaneously the sign of \( x \) and \( y' \) or change simultaneously the sign of \( x \) and \( y \).

This means that, if we know one set of curves, we can repeat it symmetrically to both axes as shown in fig. 8. If we interchange the roles of \( x \) and \( y \) then \( y' \) is replaced by \( 1/y' \). Therefore we need to consider only curves with positive slope \( 0 < y' < 1 \) and repeat them by symmetry to the oblique lines \( y = x \) and \( y = -x \).

Now we apply the Legendre transformation

\[
p = y' \quad \quad x = q'
\]

\[
q = xy' - y \quad \quad y = q'p - q
\]
to get

\[(2p \, q' - q)^2 = 1 + p^2\]

The homogeneous equation \(2p \, q' - q = 0\) can be solved by \(q = C \sqrt{p}\).

By variation of the constant \(C(p)\) we obtain

\[C = \int \frac{\sqrt{1 + p^2}}{2p \sqrt{p}} \, dp, \quad 0 \leq p \leq 1\]

whence

\[x = \frac{1}{2\sqrt{p}} \int \frac{\sqrt{1 + p^2}}{2p^{3/2}} \, dp + \frac{\sqrt{1 + p^2}}{2p}\]

This can be simplified by partial integration

\[x = \frac{1}{2\sqrt{p}} \left( \int \frac{\sqrt{p} \, dp}{\sqrt{1 + p^2}} + \text{const.} \right)\]

The parameter \(p = y'\) is the slope and \(y\) is obtained from our differential equation

\[y = -p \cdot x(p) + \sqrt{1 + p^2}\]

The integral can be expressed in terms of elliptic integrals:\n
\[\int \frac{\sqrt{p} \, dp}{\sqrt{1 + p^2}} = \int \frac{dp}{\sqrt{(1+p^2)/p - 2}} = \int \frac{\sin^2 \varphi \, d\varphi}{(1 + \cos \varphi) \sqrt{1 - \frac{1}{2} \sin^2 \varphi}}\]
Substitutions:

\[ \sin^2 \varphi = \frac{4p}{(1+p)^2} \]

\[ p = \frac{1 - \cos \varphi}{1 + \cos \varphi}, \quad dp = \frac{2 \sin \varphi}{(1 + \cos \varphi)^2} \]

After some calculation we can obtain

\[ = \int \frac{dp}{\sqrt{1 - \frac{1}{2} \sin^2 \varphi}} - \int \frac{2 \sin \varphi \sqrt{1 - \frac{1}{2} \sin^2 \varphi}}{1 + \cos \varphi} \]

\[ = F(\varphi, \sqrt{2}) - 2 E(\varphi, \sqrt{2}) + \frac{2 \sqrt{p(1+p^2)}}{1+p} \]

The complete set of curves is shown in Fig. 8 (for \( g = 1, \alpha < \varphi \)). To the right we have also drawn a possible "diamond"-shaped magnet which resembles the magnets of Figs. 6 and 7, but it should be noted that our curved boundaries can now have any angle \( \alpha > 0 < 45^\circ \), and that now the flux density \( B_0 \) is the same on the right and on the left iron boundaries. (This was not the case in the square magnets of Figs. 7 and 6).

The next section formulates the same problem for thick conductors.

b) Power series procedure for curved wedge conductors

Given a quadrupole field

\[ Q(x, y) = \begin{pmatrix} q_x \\ q_y \end{pmatrix} = \begin{pmatrix} 2(y+y_0) \\ 2(x+x_0) \end{pmatrix} \]

with gradient 2, centre \((-x_0, -y_0)\) and axes turned by \( \pi/4 \) with
respect to the $x,y$ axes, we look for an iron boundary $y = f(x)$ and an air boundary $y = g(x)$ intersecting at $(0,0)$, i.e. $f(0) = 0$, $g(0) = 0$, and a field $H'$ inside the wedge conductor situated between these boundaries and having constant current density $j$ (so $\text{div}H' = 0$, $\text{curl} H' = j$), so that on the boundary $f$, $H'$ matches $Q$:

$$H'(x, g(x)) = \begin{pmatrix} 2(y + y_0) \\ 2(x + x_0) \end{pmatrix}.$$

On the boundary $f$, $H'$ is normal to $y = f(x)$ and of constant absolute value:

$$H'(x, f(x)) = \frac{2x}{\sqrt{1 + f'^2(x)}} \begin{pmatrix} -f'(x) \\ 1 \end{pmatrix}.$$

![Diagram](image)

we set $H'(x,y) = H(x,y) + \begin{pmatrix} 0 \\ jx \end{pmatrix}$ and hence can write:

$$H(x, g(x)) = \begin{pmatrix} 2(g(x) + y_0) \\ 2(x + x_0) \end{pmatrix} - \begin{pmatrix} 0 \\ jx \end{pmatrix}.$$
\[
H(x, f(x)) = \frac{2r_0}{\sqrt{1 + f'^2(x)}} \begin{pmatrix} -f'(x) \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ jx \end{pmatrix}
\]

\[\text{div } H = \text{curl } H = 0\]

Introducing the abbreviations:

\[
B(g) = \begin{pmatrix} 2(g + y, ) \\ 0 \end{pmatrix}
\]

\[
A(f') = \frac{2r_0}{\sqrt{1 + f'^2}} \begin{pmatrix} f' \\ 1 \end{pmatrix}
\]

\[
b(x) = \begin{pmatrix} 0 \\ (2-j)x + 2x_0 \end{pmatrix}
\]

\[
a(x) = \begin{pmatrix} 0 \\ jx \end{pmatrix}
\]

We can state the problem in the following form:

Given (2-vector valued) functions \( A, B, a, b \) (of one scalar real variable) find (scalar real) functions \( f, g \), and a (2-vector valued) function \( H \) (on one 2-vector variable) so that

\[
H(x, g(x)) = B(g(x)) + b(x)
\]

\[
H(x, f(x)) = A(f'(x)) + a(x)
\]

\[f(0) = g(0) = 0\]

\[\text{div } H = \text{curl } H = 0\]
In order to have at hand the notations of both vectors/matrices and complex numbers, we shall ambiguously use the symbols

\[ E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} x \\ y \end{pmatrix} = x + iy, \quad (x,y) = x - iy \]

\[ xE + yI = x + iy \]

Set \( z = x + iy, \quad c = a + ib \), then we may write inter alia:

\[ \begin{pmatrix} y \\ -x \end{pmatrix} = -i \begin{pmatrix} x \\ y \end{pmatrix}, \quad (z_1, z_2)^T = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \]

\[ (xE + yI) \begin{pmatrix} a \\ b \end{pmatrix} = z \cdot c, \quad (a, -b) (xE + yI) = c \cdot z \]

\[ \det (xE + yI) = |z|^2, \quad \det (z_1, z_2) = \text{Im} (\bar{z}_1 \cdot z_2) \]

\[ \det \left( \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} \right) = -\det (z_1, z_2), \quad \det (-i z_1, z_2) = \text{Re} (\bar{z}_1 \cdot z_2) \]

Then expand \( H \) into a power series:
Write

\[ \text{div } H = \text{curl } H = 0 \quad \text{as} \quad \frac{\partial H}{\partial x} = I \frac{\partial H}{\partial y} \quad \text{to get} \]

\[ H(x, y) = \sum_{n=0}^{\infty} \frac{(x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y})^n}{n!} \quad H(0, 0) = \sum_{n=0}^{\infty} \frac{(x e^{-y})^n}{n!} \frac{\partial}{\partial x} \quad H(0, 0) \]

Define the symbols \([n_i]\) by

\[ (\frac{d}{dx})^n B(g(x)) = \sum_{i=0}^{n} \binom{n}{i} B^{(i)}(g(x)) \cdot g^{i-1}(x) \cdot g^{(n+1-i)}(x) \]

with arbitrary functions \(B, g,\) or equivalently by

\[ (\frac{\partial}{\partial t})^n_{t=0} \frac{1}{1-x} = \sum_{i=0}^{n} \binom{n}{i} i! (n+1-i)! x^i \]

or

\[ (\frac{\partial}{\partial t})^n_{t=0} \frac{1}{1+x \log(1-t)} = \sum_{i=0}^{n} \binom{n}{i} i! (n-i)! x^i \]

Then the comparison of coefficients for

\[ H(x, g(x)) = B(g(x)) + b(x) \quad f(0) = g(0) = 0 \]

\[ H(x, f(x)) = A(f'(x)) + a(x) \]

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runs as follows:

Put \( \left( \frac{d}{dx} \right)^n_{x=0} = D^n \), then

\[
D^n H(x, f(x)) = \sum_{i=0}^{n} \binom{n}{i} P^{i-1} P^{n+1-i} H^{(i)}
\]

where

\[
P(x) = x \cdot E - f(x) \cdot I, \quad P(k) = P(k)(0),
\]

\[
H^{(i)} = \left( \frac{\partial}{\partial x} \right)^n H(0,0)
\]

Again with

\[
G(x) = x \cdot E - g(x) \cdot I, \quad G(k) = G(k)(0),
\]

\[
D^n H(x, g(x)) = \sum_{i=0}^{n} \binom{n}{i} G^{i-1} G^{n+1-i} H^{(i)}
\]

Finally,

\[
D^n A(f'(x)) = \sum_{i=0}^{n} \binom{n}{i} f^{i-1} f^{n+2-i} A^{(i)}
\]

where

\[
f^{(k)} = f^{(k)}(0), \quad A^{(i)} = A^{(i)}(f'(0)),
\]

as well as later

\[
a^{(k)} = a^{(k)}(0), \quad b^{(k)} = b^{(k)}(0).
\]

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The 0-th derivative then gives:

\[ H(0,0) = B(0) + b(0) = A(f'(0)) + a(0) \]

which defines \( f'(0) \), and so \( H(0,0) \).

The 1-st derivative gives:

\[ G'H(1) = B'g' + b' \quad F'H(1) = A't'' + a' \]

or

\[ H(1) = G^{-1}B'g' + b' = F^{-1}(A't'' + a') \]

Since \( F' \) and \( G' \) as complex numbers commute:

\[ (1 - if'(0))(B'g' + b') = (1 - ig'(0))(A't'' + a') \]

\[ f''g'(-ia') + f''(A') + g'(-ia' - (1-if')(b')) + l(a' - (1-if')(b')) = 0 \]

Given any equation

\[ f \cdot g \cdot A + f \cdot B + g \cdot C + D = 0 \]

with \( f, g, A, B, C, D \) 2- columns, the solution is found from:

\[ g = -\frac{f.B + D}{f.A + C} \quad \text{(where this is true component-wise)} \]

and

\[ f'^2, \text{Im} (\overline{A}.B) - f' \cdot \text{Im}(\overline{B}.C - \overline{A}.D) + l.\text{Im}(\overline{C}.D) = 0 \]

In our case this equation writes:

\[ f''^2 |A'|^2 - f' \cdot \text{Im}(A'(-2i\overline{E}' + (1 + if')(\overline{B} + i\overline{E}')) + \\
+ l \cdot (|a'|^2 + (1 + f'^2) \cdot \text{Im}(\overline{E}' \cdot b') - \text{Im}(a'(1 + if')(\overline{B} + i\overline{E}'))) = 0, \]

an equation for \( f'', g', H(1) \).
For $n \geq 2$ the $n$-th derivative gives

$$g^{(n)}_H(1) + \sum_{i=2}^{n-1} \binom{n}{i} g^{(i-1)} g^{(n+1-i)} H(i) + g^{(n)} H(n) =$$

$$= g^{(n)} B' + \sum_{i=2}^{n} \binom{n}{i} g^{(i-1)} g^{(n+1-i)} B(i) + g^{(n)} H(n) =$$

$$= r^{(n+1)} A' + \sum_{i=2}^{n} \binom{n}{i} r^{(i-1)} r^{(n+2-i)} A(i) + a(i)$$

Eliminating $H(n)$ we have (since again $F'$ and $G'$ commute):

$$g^{(n)} (B' - iH(1)) + b(n) + \sum_{i=2}^{n} \binom{n}{i} g^{(i-1)} g^{(n+1-i)} B(i) - \sum_{i=2}^{n-1} \binom{n}{i} g^{(i-1)} g^{(n+1-i)} H(i) =$$

$$= (G'/F')^{n} (f^{(n+1)} A' + a(n) + \sum_{i=2}^{n} \binom{n}{i} f^{(i-1)} f^{(n+2-i)} A(i) - \sum_{i=1}^{n-1} \binom{n}{i} F^{(i-1)} f^{(n+1-i)} H(i))$$

This is an inductive linear equation for $f^{(n+1)}$, $g^{(n)}$, $(H(n))$ with determinant:

$$-\text{Im} \left( (B' - iH(1)) \cdot (G'/F')^{n} A' \right).$$
We now substitute the special $A, a, B, b$; putting $x, y, r$ instead of $x_0, y_0, r_0$, we find:

$$f' = -\frac{v}{x}, \quad a' = -ji, \quad b' = (2-j)i, \quad B' = 2,$$

$$A' = 2r \frac{(-1-if'')(1+f'^2)^{3/2}}{(1+f'^2)^{3/2}} = 2 \frac{x^2}{r^2} (-x + iy).$$

$$f''^2 \frac{4x^4}{r^6} - f'' \cdot 2 \cdot \frac{8yx^2}{r^6} + j \frac{2(x^2 - y^2)}{x^2} + \frac{4r^2}{x^2} = 0$$

Hence for real $f''$:

$$-j \geq \frac{2(x^2 - y^2)}{r^2}, \quad \text{and} \quad f''' = \frac{2y}{x^2} \quad \text{for equality}.$$
\[ H^{(1)} = \left( \frac{2 \, xy}{x - y^2} - f'' \frac{2x^3(y + y')}{(x - y^2)^2} + f''^2 \frac{2x}{r^4(x - y^2)} \right) + \]

\[ + i \left( \frac{2 \, x^2}{x - y^2} - f'' \frac{4x \, y^3}{r^4(x - y^2)} + f''^2 \frac{2 \, x}{r^6(x - y^2)} \right) \]

\[ \left( B' - iH^{(1)} \right) A' = \frac{1}{r^4(x - y^2)} \left( -2r^4 \, x^3 + i(x^2-y^2)f'' \right) + \]

\[ + r^6 \left( x^4y + x^4(y^2-x^2)f'' - 0 \cdot f''^2 \right) \]

and

\[ G' / F' = \frac{x}{f'' \, x - r^2 \, y} \left( f'' x^2 - 2xy - i(x^2 - y^2) \right) \]

The easiest way of satisfying all these conditions is, of course, to set \( f'' = 0 \). Then

\[ -j = \frac{2r^2}{x^2 - y^2} \], \quad \frac{g'}{y} = \frac{x}{y}, \quad H^{(0)} = \left( \frac{2y}{2x} \right), \quad H^{(1)} = \frac{2x}{x^2 - y^2} \left( \frac{y}{x} \right) \]

This is the straight solution for section 2. Cases different from this could be considered later by using a computer programme.
4. **Numerical examples**

To choose examples of practical value it is useful to consider the power consumption and the current density \((\alpha = 45^\circ; \ w\ \text{and} \ d\ \text{see Fig. 2})\).

With the ampere-turns for thick conductors

\[
nI = H \cdot \sqrt{2} (w + d)
\]

and the resistance per unit length (winding overhang and space factor neglected):

\[
R^* = \frac{R}{\ell} = \frac{2}{\kappa q} = \frac{2}{kd (2w + 2d)} = \frac{1}{kd (w + d)}
\]

one gets for the power per unit length

\[
N^* = \frac{N}{\ell} = (nI)^2 \frac{R^*}{\kappa} = \frac{2H^2}{\kappa} \left(\frac{w}{d} + 1\right)
\]

and for the volume current density:

\[
j = \frac{H \sin \alpha}{d} = \frac{H}{\sqrt{2} d}
\]

These equations are plotted in Figs. 9 and 10 with \(B = 30 \text{ kG}, \ H = 24,000 \text{ A/cm}, \ \kappa = 56 \cdot 10^2 \Omega \text{ cm}^2; \ w\ \text{and} \ d\ \text{in cm}\).

The power does not depend on the size of the window but only on the ratio \(\frac{w}{d}\). The current density depends only on \(d\).
Example 1 (Beam transport of ejected beam) :

\[ w = 4,0 \text{ cm} \quad d = 2,0 \text{ cm} \quad B = 30 \text{ kG} \]

\[ \frac{w}{d} = 2 \quad B_o = 21,2 \text{ kG} \]

From this results:

\[ N^* = 642 \text{ kW/m} \quad j = 84,5 \text{ A/mm}^2 \]

This current density can easily be achieved in pulsed operation. Then the average power is decreased correspondingly.

Example 2 (spark chamber magnet) :

\[ w = 100 \text{ cm} \quad d = 30 \text{ cm} \quad B = 30 \text{ kG} \]

\[ \frac{w}{d} = 3,3 \quad B_o = 21,2 \text{ kG} \]

From this results:

\[ N^* = 884 \text{ kW/m} \quad j = 5,6 \text{ A/mm}^2 \]

These are rather conservative values for dc-excitation.
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POWER CONSUMPTION AT 30 K GAUSS

FIG. 9

CURRENT DENSITY AT 30 K GAUSS

FIG. 10