ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
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SU_2 \rightarrow SU_3 \rightarrow SU_6

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GENEVA
1966
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$SU_2 \rightarrow SU_3 \rightarrow SU_6$

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(Lectures presented during the Spring and Autumn of 1965 in Professor Weisskopf's Monday afternoon Seminars intended mainly for younger experimental physicists.)

GENEVA
1966
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I. GENERALITIES ABOUT SU₃

These lectures are intended to give a simple and primitive description of unitary symmetries as applied to high-energy physics by building the way up from SU₂ to SU₃, and eventually to SU₆.

1. **The starting point: SU₂ (scalar field)**

One could begin recalling that a scalar wave function ψ(x) describing a quantum state allows for the operation of complex conjugation which transforms it into ψ*(x).

We then know that the product ψ*ψ is the "intensity of the wave" or the "magnitude of the phenomenon", which is invariant under the transformation:

\[
\psi \rightarrow e^{i\varphi} \psi \\
\psi^* \rightarrow e^{-i\varphi} \psi
\]

In a quantized field representation ψ*(x) is the operator creating a particle (at the point x), ψ(x) is the corresponding annihilation operator. If we consider particles and antiparticles, note that ψ*(x) annihilates antiparticles, and ψ creates them. In other words, for antiparticles the roles of ψ and ψ* are exchanged.

2. **Spinors, SU₂**

When we want to describe a non-relativistic electron (we will not talk about relativity until we arrive at SU₆) we find that we can no longer use a scalar function, since the electron can exist in two different, independent, and therefore orthogonal quantum states. Let us call these basic states u₁ (spin up) and u₂ (spin down), where ~ stands for "spinor". We thus have a two-component general state \( s \) by superposition of basic states

\[ s = s₁u₁ + s₂u₂ \]  (1.1)

where \( s₁ \) and \( s₂ \) are scalar numbers or functions.
Since this description should be possible in any co-ordinate system, we expect that in going from one system to another the spinors \( u_1 \) and \( u_2 \) will be intermixed so that the new spinors \( u'_i \), (\( i = 1, 2 \)), will be

\[
 u'_i = R_{ik} u_k \quad (1.2)
\]

\( R_{ik} \) is some 2×2 matrix which will depend on the angles of rotation.

From (1.2) we can obtain the transformation law for \( s \) when we express it in the new co-ordinate system.

\[
 s' = s'u'_1 + s'u'_2 = (s'R_{11} + s'R_{21})u_1 + (s'R_{12} + s'R_{22})u_2 = s_1 u_1 + s_2 u_2 \quad (1.3)
\]

A basic physical requirement is that the intensity of the spinor wave be the same in any co-ordinate system. If we define it as

\[
|s_1|^2 + |s_2|^2 = s_1s_1^* + s_2s_2^*,
\]

we find

\[
 s_1s_1^* + s_2s_2^* = (s'R_{11} + s'R_{21})(s'R_{11}^* + s'R_{21}^*) + (s'R_{12} + s'R_{22})(s'R_{12}^* + s'R_{22}^*)
\]

We can arrange this equation in matrix form as

\[
 RR^* = I, \text{ where } R_{ik}^* = R_{ki}^*.
\]

This means that the matrix \( R \) must be unitary. The 2×2 unitary matrices representing a rotation \( \Theta \) around the \( x, y, z \) axis are well known:

\[
 R_x(\Theta) = \begin{pmatrix}
 \cos \Theta/2 & i \sin \Theta/2 \\
 i \sin \Theta/2 & \cos \Theta/2
\end{pmatrix}, \quad R_y(\Theta) = \begin{pmatrix}
 \cos \Theta/2 & \sin \Theta/2 \\
 -\sin \Theta/2 & \cos \Theta/2
\end{pmatrix},
\]

\[
 R_z(\Theta) = \begin{pmatrix}
 e^{i\Theta/2} & 0 \\
 0 & e^{-i\Theta/2}
\end{pmatrix}. \quad (1.4)
\]
From (1.3) we also see that

\[ s'_i = s_k R^{-1}_{ik} . \]

3. **The adjoint spinors**

Corresponding to the definition of a \( \phi^*(x) \) state in the scalar case, we would like to have an adjoint spinor \( \bar{s} \) corresponding to \( s \).

If we define

\[ \bar{s} = s_1 u^1 + s_2 u^2 \]

and a scalar product between the basic spinors \( u^i, u_j \)

\[ u^i u_k = \delta_{ik} \]  

we have a scalar product between any \( \bar{s}, \bar{t} \) as

\[ \bar{s} \cdot \bar{t} = s^*_1 t_1 + s^*_2 t_2 . \]

The intensity of a spinor wave can then be expressed as

\[ \bar{s} \cdot \bar{s} = |s_1|^2 + |s_2|^2 . \]

From our rules we can see that the adjoint spinors are forced to transform as

\[ u^i' = u^i R^{-1}_{ki} \]  

and that the pair \( (y^1, y^2) \) transforms as the pair \( (y^*, -y) \).

4. **A physical statement about adjoint spinors**

So far the adjoint is another way of expressing a state, in the same manner as \( \phi^* \) was another way of expressing the state \( \phi \).

But we will now assume that the adjoint states express the basic states and transformation properties of the antiparticles.
5. Direct product of spinors, or "double spinors":

Operator interpretation

Consider a two-particle system, each particle being represented by a spinor; we get the basic states $\uparrow\uparrow, \downarrow\downarrow, \uparrow\downarrow, \downarrow\uparrow$.

In case the two-particle system is a particle-antiparticle pair, the basic states can be algebraically written

$$A_k^i = u_i^* \cdot u_k^i .$$

(1.7)

The four states thus obtained can be linearly combined into an invariant $u_i^* \cdot u_k^i$. (This is invariant since in a new co-ordinate system we have

$$u_i^* \cdot u_i^i = \sum R_{ki}^{-1} R_{kj} u_j^* \cdot u_k^j$$

and a triplet

$$\begin{pmatrix}
  u^2 u_4 \\
  u^1 u_2 \\
  \frac{1}{\sqrt{2}} (u^1 u_3 - u^3 u_2)
\end{pmatrix}$$

(1.8)

which can be shown by means of (1.4) and (1.6) to transform under rotations as

$$\begin{pmatrix}
  \frac{1}{\sqrt{2}} (x + iy) \\
  \frac{1}{\sqrt{2}} (x - iy) \\
  z
\end{pmatrix} .$$

The double spinor can therefore be linearly combined to break down into a scalar $S$ and a vector $V_+, V_-, V_0$:

$$V_+ = u^2 u_4 ,$$

$$V_0 = \frac{1}{\sqrt{2}} (u^1 u_4 - u^3 u_2) ,$$

$$V_- = u^1 u_2 ,$$

$$S = \frac{1}{\sqrt{2}} (u^1 u_4 + u^3 u_2)$$

(1.9)
The $A^i_k$ or their linear combinations $V, S$ are states of a particle-antiparticle system.

There is a different way of considering $A^i_k$: in a manner resembling second quantization, the product $u^i_k u_k$ can be thought of as an operator which destroys a particle in the state $k$ and creates one in the state $i$. The $A^i_k$ are then considered as displacement operators acting on a single particle and transforming it from the state $k$ to the state $i$. They therefore can be written as $2 \times 2$ matrices. In order to separate the two interpretations of $A^i_k$ - particle-antiparticle states and displacement operators - we introduce new symbols for the operators:

\[
\begin{align*}
\text{to } A^i_1 &= u^2 u_1 \quad \text{corresponds} \quad \tau_+ \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
A^i_2 &= u^1 u_2 \quad \tau_- \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
(A^i_2 - A^i_1) &= \tau_z \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
(A^i_2 + A^i_1) &= \text{I} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

The three $\tau$ operators are equivalent to the Pauli-spin operators, of course. Clearly they transform like vector components since they are equivalent to $V_+, V_-, V_0$ in (1.9) as far as transformation properties are concerned. The $\tau$ operators are called the generators of the SU$_2$ group. We get:

\[
\begin{align*}
\tau_+ u_1 &= u_2 \\
\tau_- u_2 &= u_1 \quad \text{all other combinations} \\
\tau_z u_1 &= -u_1 \quad \text{give zero.} \\
\tau_z u_2 &= u_2.
\end{align*}
\]
The same operators act also on the adjoin states (antiparticle states). Hence, of course, \(u^1u_2\) creates the state \(u^2\) and annihilates \(u^1\). Hence, \(u^1u_2\) is a transition from \(u^1\) to \(u^2\). Recalling that the pair \((u^1,u^2)\) transforms like \((u_2,-u_1)\), we get:

\[
\begin{align*}
\tau_{u^1u^2} &= -u^1 \\
\tau_{u^1} &= -u^2 \quad \text{all other combinations} \\
\tau_{u^1} &= u^1 \quad \text{give zero.} \\
\tau_{u^2} &= u^2
\end{align*}
\quad (1.12)
\]

The eigenvalue of \(\tau_z\) determines the "spin component" of the state. In fact, it is twice that quantum number. For example, the state 1 has spin component \(-\frac{1}{2}\), the state 2 has \(+\frac{1}{2}\).

The name "generator" for the \(\tau_i's\) come from the fact that the transformations \(R_x, R_y, R_z, R_x = I + \epsilon \tau_x, R_y = I + \epsilon \tau_y, \text{ etc.} - \text{ correspond to the infinitesimal rotations by an angle } \epsilon \text{ around the } x, y, z \text{ axis.}

So far we have represented the operators \(\tau\) by \(2 \times 2\) matrices, corresponding to the action of \(\tau\) upon the basic states \(u_1, u_2\). We can also act with the operators \(\tau\) upon the four states \(A_{\pm k}\) (or \(V_+ V_0 V_-, V_0\), \(V_+ S\)) of the particle-antiparticle system. For example, we get from (1.11) and (1.12):

\[
\begin{align*}
\tau_{-A^0_1} &= -u^1 u_2 = -u^1 u_1 + u^2 u_2 = \sqrt{2} \ V_0 \\
\text{ and similar results for other states. This can be used to represent the } \tau_i's \text{ as } 4 \times 4 \text{ matrices. If we take } V_+ V_0 V_- \text{ as the base, the state } S \text{ is not changed at all. (It is a scalar.) Hence, we may take only } V_+ V_0 V_- \text{ as a base and this gives rise to a } 3 \times 3 \text{ representation of } \tau_+ \tau_- \tau_0:
\end{align*}
\]

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2
\end{pmatrix}.
\]
If, instead of particle-antiparticle pairs, we consider two particles, we got states described by \( \begin{pmatrix} u_1 \ n \\ u_k \end{pmatrix} \). Recalling that \((u', u'')\) transforms like \((u_2, -u_1)\), we got \(u_2 u_1 - u_1 u_2\) as invariant and the triplet

\[
\begin{align*}
&u_1 u_1 \\
&\frac{1}{\sqrt{2}}(u_2 u_1 + u_1 u_2) \\
&-u_2 u_2
\end{align*}
\] (1.13)

for the vector. Note that the \( z \) component is the symmetric combination here, whereas it was the antisymmetric one in (1.8). (The two-particle states \( u_1 u_k \) do not correspond to any displacement operator, of course.)

Let us introduce a graphical method of combining two spinor states by first symbolizing the spinor state by a double arrow going from a centre to the left, pointing to the "spin down" state, and to the right, pointing to the "spin up" state. We called the former state "1" and the latter "2".

Fig. 1

When constructing the states of two spinors, we place a second double arrow with its centre on either of the two ends of the first. We then got the four states. The states \((1, 2)\) and \((2, 1)\) fall on the same spot. In order to get the right linear combination for the vector triplet, one takes the combination which is orthogonal to the invariant: \((21 - 12)\).

Fig. 2
6. **SU₃**: The basic triplet representation spaces: u spin and v spin

The only difference between SU₂ and SU₃ is the fact that we start in SU₃ with three basic states u₁, u₂, u₃ instead of two. In a completely analogous way to SU₂, we have in SU₃ a general spinor q in the form

\[ q = s₁u₁ + s₂u₂ + s₃u₃ \]

The transformation laws for q will be fixed as before by \( u'_i = R₁k u_k \)
and \( u'' = u^j R⁻¹ \) for the adjoint spinor \( \bar{q} \). Here \( R₁k \) is a 3×3 matrix.

In order to fulfill the physical requirement

\[ |s₁|^2 + |s₂|^2 + |s₃|^2 = \text{invariant} \]

the matrix R will have to be unitary. The entity q will be interpreted later as a "quark" and \( \bar{q} \) as an "antiquark".

Let us now define what the three basic states are. Let us start by giving a new interpretation to the group SU₂. Instead of having as basic states spin up and spin down, we can think of p and n as basic states. "p" and "n" stand for two states which are analogous with "proton" and "neutron" but not identical as we will see later. A transformation R between these two states will no longer mean a rotation in ordinary space, but will imply taking linear combinations of p and n to form a new basis.

We can also refer to this operation as rotation in a new space called isospin space.

In SU₃ we have the same two basic states as in SU₂ but we add a third state with isotopic spin zero and a new quantum number which we call hypercharge. For convenience (in order to have zero total hypercharge as we have zero total isospin) we displace the former states and give them a hypercharge content \( \frac{1}{2} \) as compared to \( -\frac{1}{2} \) for the new state. We then get the basic triangle of SU₃ states at the end of three arrows:

\[ \text{Fig. 3} \]
If we put the centre of a co-ordinate system in the middle, the ordinate gives the hypercharge, the abscissa the spin:

![Diagram](image)

And similarly for the adjoint basic antiquarks.

We have now three basic states arranged in a regular triangle in which three axes of symmetry can be defined. Corresponding to each axis of symmetry, the figure may be considered as made of a doublet and a singlet. This means that besides the original isospin SU$_2$ doublet, to which we added a new state with zero isospin and having a new quantum number (hypercharge), there are two other pairs which define an SU$_2$ doublet, and from which an identical extension to SU$_3$ can be made. The rotations corresponding to each of the three SU$_2$ doublets mix those states, leaving the third state invariant. We find that the symmetries of our figure lead to the existence of two new kinds of spin, which we will call u spin and v spin. The u spin refers to the doublet (3,1), the v spin to the doublet (2,3). The u spin and v spin rotations are subgroups of the general SU$_3$ rotation defined by a 3x3 matrix $R$ which transforms all three states among themselves. A convention is made to define within each SU$_2$ pair which states are considered with spin up or down: 1 is up, 3 is down for u spin; 3 is up, 2 is down for v spin. (Recall: 1 is down, 2 is up for isospin.)
7. **Product of triplet-antitriplet basic quarks: The octet**

The graphical multiplication method which we used in SU₂ can be repeated here to get the basic \( u \bar{u} \) states for a \( \bar{q}q \) pair. We start with the triangle of basic antiquarks with the three states \( \bar{1}, \bar{2}, \bar{3} \). (Circles in Fig. 7.) We then put the centre of the quark triangle (Fig. 3) over each state of the antiquark triangle and then get the hexagonal arrangement of the states \((i,k)\), (crosses in Fig. 7).

\[
\begin{array}{ccc}
\times & \times & \times \\
\circ & \circ & \circ \\
\bar{1} & \bar{2} & \bar{3}
\end{array}
\]

**Fig. 7**

Of the nine resulting states, three are soon to fall at the centre. If we form the combination \( \bar{1} + \bar{2} + \bar{3} \), it will be invariant under any transformation \( R \), by an argument identical to what was said in the SU₂ case. This means that when a transformation \( R \) is made on the basic \( q \) triplet (and the corresponding transformation \( (R^T)^{-1} \) on the \( \bar{q} \) triplet) the nine states \( u \bar{u} \) change into combinations of one another; but if a new basis is chosen, with \( I = \bar{1} + \bar{2} + \bar{3} \) as one of the basic states, the remaining eight states transform mixing among themselves without contributing to the invariant \( I \). If we find an orthogonal basis for these eight states, we will call it an octet.

The noncentral basic states found in our graphical method are already orthogonal to all other basic states. At the centre, however, we have combinations of three orthogonal states. Since \( I \) has been taken away, there must still be two orthogonal linear combinations. These may
be chosen in a number of different ways according to convenience. If we consider SU₃ as an extension of ordinary SU₂ isotopic space, we should try to keep as close as possible to it. Then one of the normal states should be obtained forgetting the state 33 and choosing as in SU₂

\[ A = (\bar{7}1 - \bar{2}2) \frac{1}{\sqrt{2}} \]  

We are then forced to choose for the remaining state

\[ B = [\bar{7}1 + \bar{2}2 - 2 \times (\bar{3}3)] \frac{1}{\sqrt{6}} \]  

as the only one that is orthogonal to A and to 71 + 22 + 33. (1/\sqrt{2} and 1/\sqrt{6} are normalizing factors.) The state B is clearly an invariant under our old SU₂ (isospin) group of transformations in which the states 3, 3 played no role.

Instead of starting from the isospin central triplet state to build our two central octet states, we could have started from the u spin central triplet state 33 - 71 to get as central octet states

\[ A' = (\bar{7}1 - \bar{3}3) \frac{1}{\sqrt{2}} \]  

\[ B' = [\bar{7}1 + \bar{3}3 - 2 \times (\bar{2}2)] \frac{1}{\sqrt{6}} \]  

In this case, B' is invariant under u spin. A', B' are clearly combinations of A, B

\[ A' = \frac{1}{2} (+ \sqrt{3} \ B - A) \]  

\[ B' = -\frac{1}{2} (B + \sqrt{3} \ A) \]  

\[ A = -\frac{1}{2} (A' + \sqrt{3} \ B') \]  

\[ B = \frac{1}{2} (\sqrt{3} \ A' - B') \]
We thus get an octet from the $\bar{q}q$ combination with the following components:

$$\begin{align*}
\bar{3}_1 & \quad \bar{3}_2 \\
\bar{2}_1 & \quad \frac{1}{\sqrt{2}}(\bar{7}_1 - \bar{2}_2) \\
\bar{2}_3 & \quad \frac{1}{\sqrt{6}}[\bar{7}_1 + \bar{2}_2 - 2 \times (\bar{3}_3)] \\
\bar{1}_2 & \quad \bar{1}_3
\end{align*}$$

It will be useful to introduce symbols for these eight components. We will use the following ones:

$$\begin{align*}
\nu_0 & \quad \nu_+ \\
\sigma^- & \quad \sigma_0 & \quad \sigma^+ \\
\xi^- & \quad \xi_0
\end{align*}$$

These symbols are borrowed from the baryon octet which we introduce later. The $\nu$'s are the nucleons (neutron and proton), the $\sigma$'s are the $\Sigma$ hyperons, the $\lambda$ is the $\Lambda$ hyperon, and the two $\xi$'s stand for the $\Xi$ particles. We will use these small Greek letters, however, only as symbols for the eight components, whatever they are. For example, the $\sigma$'s can be $\Sigma$, $\pi$, or $\rho$, or the corresponding member of any other octet. It should be remarked that $\nu_0 \nu_+$ have hypercharge $Y = +1$, $\sigma^+, \sigma_0, \lambda$ have $Y = 0$, and $\xi^- \xi_0 \ Y = -1$. 

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p/sb
8. Generating operators for SU₃

We already know that \( \mathbf{A}_k^i = u_k^i u_{k+1}^i \) can be interpreted as an operator on the triplet \( q \) which makes a transition from \( k \) to \( i \). It is easy to see that these operators can be represented as \( 3 \times 3 \) matrices with all their elements zero except for a 1 in the place \((i, k)\). Since there will be three matrices which only differ from our old \( \tau_+ \tau_- \tau_z \) by adding a row and a column of zeros we keep for them the same symbol. The sense can be said of the two pairs of off diagonal \( u \) and \( v \) matrices. One has therefore,

\[
\begin{align*}
\tau_+ &= \tau_1 & \tau_- &= \tau_2 & v_+ &= \tau_3 & v_- &= \tau_4 & u_+ &= \tau_5 & u_- &= \tau_6 \\
21 & \ 12 & \ 32 & \ 33 & \ 34 & \ 35 & \ 36 \\
000 & 010 & 000 & 000 & 001 & 000 \\
100 & 000 & 000 & 001 & 000 & 010 \\
000 & 000 & 010 & 000 & 000 & 000 \\
\end{align*}
\]

(1.18)

We still have to construct the three diagonal matrices coming from linear combinations of states \( u_k \). One will obviously be the unit matrix. An easy guess for \( \tau_z \) is our old \( \tau_z \) with a new row and column of zeros. \( \tau_z \) comes then forced by the requirement of orthogonality to \( \tau_z \) and to I (except for normalization). Therefore,

\[
I = \begin{pmatrix} 100 \\ 010 \\ 001 \end{pmatrix}, \quad \tau_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tau_7, \quad \tau_8 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (1.18a)
\]

These eight \( 3 \times 3 \) matrices are the "quark-representation" of \( \tau_1 \ldots \tau_8 \). There is also an "antiquark-representation" (analogue of 1.12), expressing the effect of the \( \tau \)'s on the antiquark triplet. They are the negative inverted matrices (1.18) and (1.18a). We may now ascribe these matrices to each of the elements of the octet of Table 1.

Table 3

\[
\begin{array}{ccc}
\tau_+ & \tau_- & \tau_z \\
u_- & v_+ & u_+ \\
\end{array}
\]
Since the eight operators are generalizations of the three \( r \) operators in SU\(_2\), we will sometimes call them the eight generating operators of SU\(_3\) and name them \( r_1, r_2, r_3, \ldots, r_8 \). We see that the octet plays the same role in SU\(_3\) as the vector-component-triplet in SU\(_2\). The eight quark-antiquark states of Table 1 and the eight generating operators transform in the same way, as an octet. This is analogous to the SU\(_2\) statement that the three particle-antiparticle states (1,7) and the three generating operators \( r_+, r_-, r_0 \) transform in the same way, as a vector.

9. Products of two \( q \)

The basic states will now be \( u_k^u \). In SU\(_2\) we could go over from \( u_1 u_k \) to \( u_2 u_k \) by the equivalence of pairs \( (u_1 u_2) \rightarrow (-u_2^*, u_1^*) \).

No such equivalence exists in SU\(_3\), and the results of graphic multiplication are quite different. We start with the basic triangle, Fig. 3. (Circles in Fig. 8.) We put the centre of a similar triangle at each of the circles and get the states marked by crosses in Fig. 8. We find three double points, which form a triangle. This makes us suspect that three of our nine basic states may form a special basic set such that when making a transformation they do not combine with the other six but only among themselves. We recall that this was just the case for the scalar in \( \bar{q}q \). The difficulty is to know what linear combinations should be taken for the possible triplet and for the remaining sextet. We proceed recalling always SU\(_2\). Since the top row is an isospin triplet with a double point at the centre, we choose the combinations 12 - 21 (isospin scalar) and 12 + 21 (central state of the isospin triplet). The figure is symmetric for isospin, \( u \) spin and \( v \) spin, so the same can be said for the other two doublets. We thus find three antisymmetric states:

\[
\{ \begin{align*}
12 - 21 \\
31 - 13 \\
23 - 32
\end{align*} \]
and six symmetric states:

\[
\begin{align*}
&12 + 21 \\
&13 + 31 \\
&23 + 32 \\
&11 \\
&22 \\
&33
\end{align*}
\]

Since it is known that a symmetric combination of basic states never gets mixed with an antisymmetric combination when we make a linear transformation, our triplet and our sextet transform independently. Also it is easy to see that the operators \( r_\pm, u_\pm, v_\pm \), connect the states within the sextet. Furthermore, the triplet is seen to form a \( q \) basis from our graph. Therefore, the product of two quarks gives nine states, three transforming like an antiquark, six as a sextet:

\[3 \times 3 = \bar{3} + 6\]  \hspace{1cm} (1.19)

10. Products of three quarks

Let us now multiply three basic quark sets, \( qqq \), and start for enlightening with \( SU_3 \). In the case of spinors we know that \( ss = I + \text{vector} \), so

\[ss = I \cdot s + \text{vector} \cdot s = s + \text{vector} \cdot s\]

"vector \cdot s" can be constructed graphically; we start with the three vector states, \( 11, 12 + 21, 22 \) (circles in Fig. 9) and put the centre of Fig. 1 on each of the states.

---

Fig. 9
We get six points, marked with crosses in Fig. 9, two of which are double. One would like to see whether the figure breaks down to simpler figures whose states do not internix under transformations. We observe that the end points 111, 222 are symmetric states. Therefore, taking symmetric combinations of the pair formed by the states (112) and (121+211), and also of the pair formed by the states (221) and (122+212), we expect to have a symmetric set of four states (spin $\frac{3}{2}$) which does not mix in transformations with the antisymmetric combination

$$\begin{align*}
112 - \frac{1}{2}(121 + 211) \\
221 - \frac{1}{2}(122 + 212)
\end{align*}$$

which forms a spin $\frac{3}{2}$ doublet. The factor $\frac{1}{2}$ multiplying the brackets is necessary to make the antisymmetric states orthogonal to the symmetric states in the double points. This orthogonality is of course essential in order to have the doublet and the triplet transforming independently under rotations, without getting mixed into one another.

We now do the same in SU$_3$

$$\begin{align*}
qqq = \bar{3} \times q + 6 \times q
\end{align*}$$

It has already been shown that $\bar{3} \times q = \bar{q} = I + \text{octet}$, so we only need to calculate $6 \times q$ as shown in the figure. We start with the sextet obtained in Fig. 8, (circles in Fig. 10) and put the centre of the basic triangle on each. We then read the points denoted as crosses in Fig. 10. The upper row is identical to our sss problem in SU$_2$, and this gives us the clue for our diagram. There are two double points on each external row. Taking the symmetric combination out of each of them, a symmetric combination at the centre (which is a triple point) and the three vertex points, we have a set of 10 basic symmetric states.
This immediately fixes the other linear combinations at each double point (it is the one which is orthogonal to the symmetrical). One is left then with the triple point at the centre. Here we have the symmetric one from the decuplet. The other two make up the twocentre components of the octet. We have, therefore, the full decomposition:

\[
qqq
\begin{array}{c}
3\times q \\
I + 8
\end{array}
+ \begin{array}{c}
6\times q \\
8 + 10
\end{array}
\]

antisymmetric symmetric

I is now a completely antisymmetric combination since it comes from \(3\times q\), where \(3\) is

\[
\begin{pmatrix}
12 & -21 \\
31 & -13 \\
23 & -32
\end{pmatrix}
\]

We know that the invariant in \(3\times q\) is antisymmetric. Therefore,

\[
I = (12 - 21)3 + (31 - 13)2 + (23 - 32)1
\]

The other components are given in Table 4.

11. Physics of SU\(_3\)

SU\(_3\) is a group of transformations: its generators transform one quark state into another. Any physics application of such a quark must be of the following form: one assumes that the mechanics of a system is invariant or almost invariant to the transformation of the group. This will then ensure degenerate or almost degenerate multiplets. They should be found in nature. If they are completely degenerate, the invariance is perfect. If the multiplets are split, the invariance is
imperfect and one should be able to predict the pattern of the split from some simple assumptions about the nature of the violation of invariance.

The isotopic SU$_3$ group is, of course, plainly visible in the baryon and meson states: we have the typical isotopic doublets before us in the (proton-neutron) pair, the (Ξ$^0$Σ$^-$) pair, and the (K$^0$K$^-$) and K$^0$K$^+$ pair. Isotopic triplets are found in the (K$^0$K$^+$K$^0$) set and the (π$^+$π$^0$π$^-$) set. A quartet is found in the famous N*(3/2 3/2) set. These multiplets are practically completely degenerated. The small electromagnetic split will be discussed later.

The SU$_3$ group starts with a "trichotomy" of three states, two of which represent a basic isotopic doublet with zero strangeness, and the third is an entity whose isotopic spin is zero, but it carries a unit of strangeness which is usually assumed to be negative. Obviously, in the first attempts in this direction, one identified the proton, the neutron and the hyperon as the basic states of this fundamental triplet. Soon it turned out that the situation is, in fact, more sophisticated and more interesting.

Let us look at the baryon spectrum. Here we find isotopic spins $I = \frac{1}{2}$ and $\frac{3}{2}$ with strangeness $S = 0$, we find $I = 0$ and 1 with $S = -1$, and $I = \frac{1}{2}$ with $S = -2$, and $I = 0$ with $S = -3$. What does this mean? It points towards the fact that the baryons behave as if they were combinations of three entities which are members of a basic trichotomy. These entities have received the ugly name of "quarks". (Let us not be confused by the fact that the number "three" is used in two different ways: firstly, the "quark" exists in three basic states; secondly, the baryon is supposed to be a combination of three quarks.) Let us see how it works. There are three kinds of quarks: a pair of isotopic spin $I = \frac{1}{2}$ with $S = 0$, and a third one with $I = 0$ but $S = -1$. Strangeness $S$ is defined here as $\bar{Y} - \frac{1}{2}$, where $\bar{Y}$ is the hypercharge. What kind of baryons can we build if each baryon must be a combination of three quarks?
If all three are of the \((I = \frac{1}{2}, S = 0)\) kind, we get systems with \(I = \frac{1}{2}\) or \(\frac{1}{2}\) but \(S = 0\); if two are of the \((I = \frac{1}{2}, S = 0)\) kind and one of the \((I = 0, S = -1)\) kind, we get systems with \(I = 0\) or \(1\) and \(S = -1\); if two are of the \((I = 0, S = -1)\) kind, we get obviously \(I = \frac{1}{2}\) and \(S = -2\); if all three are of the \((I = 0, S = -1)\) kind, we get \(I = 0\) and \(S = -3\). This is exactly what we find in the baryon spectrum.

Let us now look at the boson spectrum; here we observe the following characteristic properties: (a) a symmetry in regard to positive and negative strangeness, any particle in one group has its antiparticle in the other; (b) the fact that the \(S = 0\) bosons are their own antiparticles; (c) only \(I = \frac{1}{2}\) for \(|S| = 1\). This points to the fact that the bosons behave as if they were combinations of one quark and one antiquark. If the two quarks are of the type \((I = \frac{1}{2}, S = 0)\), such a combination would yield entities with \(I = 0\), or \(1\), \(S = 0\), and they would be their own antiparticles. If one quark is of the second type \((I = 0, S = -1)\), one gets a meson of isotopic spin \(\frac{1}{2}\) with \(S = +1\) or \(-1\), depending upon whether the quark or the antiquark is of the second type. The \(S = +1\) and \(S = -1\) combinations then are each other's antiparticles.

If this is so, and if the dynamics of elementary particles is at least approximately \(SU_3\) invariant, we should expect the following multiplets: for the baryons: singlets, octets, and decuplets; for the mesons: singlets and octets. These multiplets do in fact exist and contain the correct isotopic multiplets as subgroup. Here they are:

**Table 5: Mesons**

(angular spin 0, odd parity)

<table>
<thead>
<tr>
<th>Octet</th>
<th>K^- K^+</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi^-)</td>
<td>(\pi^0) (\pi^+)</td>
</tr>
<tr>
<td>(\eta^0)</td>
<td>(K^0)</td>
</tr>
</tbody>
</table>

(angular spin 1, even parity)

<table>
<thead>
<tr>
<th>Octet</th>
<th>(K_{-}) (K^*_{+})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho^-)</td>
<td>(\rho^0) (\rho^+)</td>
</tr>
<tr>
<td>(\omega_0)</td>
<td>(\overline{\rho^0})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Singlet</th>
<th>(\varphi_0)</th>
</tr>
</thead>
</table>

\[\text{55/1280/10}\]
It is seen that particles and antiparticles are in symmetric position relative to the centre as one would expect from Table 1. Here it should be added that \( \omega_0 \) and \( \Phi_0 \) are not the real \( \omega \) and \( \Phi \) mesons, but linear combinations of them. We come back to this point later on.

Table 6: Baryons

(angular spin \( \frac{1}{2} \))

<table>
<thead>
<tr>
<th>Octet</th>
<th>Singlet</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>( \Sigma )</td>
</tr>
<tr>
<td>( P )</td>
<td>( \Sigma^0 )</td>
</tr>
<tr>
<td>( \Sigma^- )</td>
<td>( \Sigma^0 \Lambda )</td>
</tr>
<tr>
<td>( \Sigma^+ )</td>
<td>( \Sigma^+ )</td>
</tr>
<tr>
<td>( \Xi^- )</td>
<td>( \Xi^- )</td>
</tr>
<tr>
<td>( \Xi^0 )</td>
<td>( \Xi^0 )</td>
</tr>
<tr>
<td>( \Omega^- )</td>
<td>( \Omega^- )</td>
</tr>
</tbody>
</table>

Decuplet: angular spin \( \frac{3}{2} \)

| \( N^*^- \) | \( N^*0 \) | \( N^*+ \) | \( N^{*++} \) |
| \( \Xi^- \) | \( \Xi^0 \) | \( \Xi^+ \) |
| \( \Xi^{*-} \) | \( \Xi^{*0} \) |

Since the baryons are made up of three quarks, the antiparticles form different octets made up of three antiquarks. These multiplets obviously are not degenerate, except for the isotopic subsets. There is therefore a \( SU_3 \) violating effect present which we will discuss very soon.

Note that our analysis of the three-quark system \( qqq \) has given us two octets. So far only one baryon octet has been clearly identified. One should not yet try to identify the octet of Table 6 with either one of the octets in Table 4. The considerations of \( SU_6 \) will clear up this point, and will show that the baryon octet is a well-defined linear combination of the two octets in Table 4.
A word about the electric charge of the quarks. If we apply the relation for the charge \( Q \)

\[ Q = I_z + Y/2 , \]

where \( I_z \) is the third component of the isospin and \( Y \) is the hypercharge, we would find that the quarks have fractional charges. In fact, the quark in the states 1 and 3 has the charge \(-\frac{1}{3}\), and in the state 2 it has \(+\frac{2}{3}\):

\[
\begin{array}{c|c}
Y \\
\hline
Q = -\frac{1}{3} & Q = +\frac{2}{3} \\
1 & 2 \\
-\frac{1}{2} & \frac{1}{2} \\
3 & \\
Q = -\frac{1}{3} \\
\end{array}
\]

Fig. 11

The states of the meson and baryon multiplets have then just the right charge, since \( Q = I_z + Y/2 \) will always be fulfilled. The fractional charge of the quark makes it somewhat more likely that the quark is only a fictitious entity.
II. BREAKING THE SU$_3$ SYMMETRY

It is a well-known fact in many fields of physics that some of the most interesting results appear when a theory, which is supposed symmetrical under some group as a starting point, is broken by a non-symmetrical perturbation. The all important question is then to derive the form of these perturbations, and the beauty of SU$_3$ lies in the fact that it provides us with some very natural ways of perturbing the initial symmetry in different ways. We will see now two examples. In both of them the perturbation will be supposed to possess a lesser degree of symmetry. More precisely, it will be only SU$_2$ symmetric: in one case U spin symmetric, and in the other I spin symmetric.

1. Electromagnetic properties

In our diagrams of Table 5 and 6 we observe that the U-spin multiplets are aligned in rows of equal charge. This makes it plausible that the electromagnetic properties are invariant with respect to this spin. There are many good tests of this hypothesis.

a) Since electromagnetism is invariant with respect to U spin, the photon should be a scalar under this group, i.e., under a U-spin transformation it should transform as U = 0. A consequence of this is that there may be transitions from the decuplet states to the octet states, which are allowed or forbidden by U-spin conservation. Since the photon carries spatial spin 1, these transitions are possible. Typical examples in remarkable agreement with experiment are:

$$
\begin{align*}
\Sigma^* &\rightarrow \Sigma^- + h\nu \\
Y^- &\rightarrow \Sigma^- + h\nu \\
Y^+ &\rightarrow \Sigma^+ + h\nu \\
N^+ &\rightarrow p + h\nu
\end{align*}
$$

(allowed by U-spin conservation).  

In all these transitions, strangeness is conserved, but the first two are forbidden because U spin changes. ($\Sigma^*$ and $Y^-$ belong to the U-spin quartet, $\Sigma^-$ and $\Sigma^-$ to U-spin doublets.)
b) **Electromagnetic mass differences**

Let us make the additional assumption that any difference in mass in an isotopic multiplet is due to electromagnetism. This means that nuclear forces are exactly invariant to isotopic spin rotation.

Since electromagnetic effects are supposed to be U-spin scalars, one would then expect that the following electromagnetic masses be equal among states of equal U spin:

\[
\begin{align*}
    m_e(\Sigma^+) &= m_e(p) \\
    m_e(n) &= m_e(\Sigma^0) = m_e(A') \\
    m_e(\Sigma^-) &= m_e(\Xi^-).
\end{align*}
\]

(A' stands here for an appropriate U-spin combination of \(\Sigma^0, A'\).)

Therefore,

\[
\Delta(pn) = m_e(p) - m_e(n) = m_e(\Sigma^+) - m_e(\Sigma^0) - m_e(\Xi^-) + m_e(\Xi^-)
\]

so,

\[
\Delta(pn) = \Delta(\Sigma^+\Xi^-) - \Delta(\Sigma^0\Xi^-)
\]

which is in remarkable agreement with experiment.

2. **Magnetic moments of baryons**

Here, we make use of the fact that the electromagnetic properties are represented by operators which are invariant under U spin, and transform like a combination of scalar and vector in I spin. (We recall that the light quantum changes I spin either by zero or by unity.)

It now follows from this, that electromagnetic properties of baryons belonging to the same isotopic multiplet must obey the "equal spacing rule". It comes from the fact that an operator, which is a scalar or a vector in I spin, has the form \(a + b \tau_3\) and only \(\tau_3\) has expectation values different from zero in the states of an isotopic multiplet. The operator \(0 = a + b \tau_3\) between two eigenstates
of $\tau_3 \alpha, \beta <\alpha|0|\beta> = \delta_{\alpha\beta}(a+mb)$ where $m$ is such that $\tau_3(\beta) = m(\beta)$.

Since, for a multiplet, $m$ takes the $2m+1$ integer values $-m,\ldots, +m$, it follows that in particular we will have for the magnetic moments:

$$\mu_{\Sigma^+} - \mu_{\Sigma^0} = \mu_{\Sigma^0} - \mu_{\Sigma^-} \quad \text{or}$$

$$\mu_{\Sigma^+} + \mu_{\Sigma^-} = 2\mu_{\Sigma^0} \quad \text{(2.1)}$$

Because of invariance under U spin we also have:

$$\mu_n = \mu_{\Sigma^0} = \mu_{A'}$$
$$\mu_p = \mu_{\Sigma^+}$$
$$\mu_{\Sigma^-} = \mu_{\Sigma^-} \quad \text{(2.2)}$$

A further relation which we will prove later is given by the fact that the sum of all magnetic moments of the octet of particles must be zero:

$$\mu_{\Sigma^+} + \mu_{\Sigma^-} + \mu_{\Sigma^0} + \mu_A + \mu_p + \mu_n + \mu_{\Sigma^-} + \mu_{\Sigma^0} = 0 \quad \text{(2.3)}$$

From these relations and with some simple manipulations one can obtain important predictions.

Let us first express $\mu_{A'} = \mu_n$ as a function of other magnetic moments. Since we remember

$$+B = A = \frac{1}{4}(\sqrt{3} A' - B')$$
$$-A = \Sigma = \frac{1}{4}(A' + \sqrt{3} B')$$

we have (0 as a U-spin scalar has a zero matrix element between $A'$ and $B'$)

$$\mu_A = \frac{1}{4}(\sqrt{3} A' - B' | 0 | \sqrt{3} A' - B') = \frac{1}{4} \mu_{A'} + \frac{1}{4} \mu_{B'}$$

$$\mu_{\Sigma^0} = \frac{1}{4}(A' + \sqrt{3} B' | 0 | A' + \sqrt{3} B') = \frac{1}{4} \mu_{A'} + \frac{1}{4} \mu_{B'}$$

$$2\mu_{A'} = 3\mu_A - \mu_{\Sigma^0} = 2\mu_n \quad \text{(2.4)}$$
Relation (2.3) gives then, using (2.1) and (2.2)

\[ 2(\mu_{\Sigma^+} + \mu_{\Sigma^-}) + \mu_{\Xi^0} + \mu_\Lambda + \mu_{2n} = 0, \]

or again from (2.1)

\[ 5\mu_{\Xi^0} + \mu_\Lambda + 2\mu_n = 0, \]

and from (2.4)

\[ \mu_\Lambda = \frac{\mu_n}{2}. \quad (2.5) \]

Similarly, we have from (2.3) and (2.1)

\[ 3\mu_{\Xi^0} + \mu_\Lambda + \mu_p + 2\mu_n + 2\mu_{\Xi^-} = 0. \]

From (2.4) and (2.5) then

\[ \mu_p + \mu_n = -\mu_{\Xi^-}. \quad (2.6) \]

Relation (2.5) is well verified by experiment (within the large 30% error) and so is relation (2.6).

In order to prove Eq. (2.3) we need an assumption which is not contained in our former ones, but seems reasonable. From the fact that the charge operator \( Q \) may be written \( Q = \frac{2}{3}(2\tau_3 + u_3) \), we assume that the current operator \( j \) can also be written as (or has the same transformation properties as)

\[ j = j_1(x)\tau_3 + j_2(x)u_3. \quad (2.6a) \]

Since summing over the whole octet the expectation value of the operators \( \tau_3 \) and \( u_3 \) is clearly zero, we obtain the desired (2.3) relation.

3. Mass Formulae

There must be a "fifth force" in nature which is \( SU_3 \) violating in order to explain the large differences in mass between different isotopic multiplets. It should be a scalar under I spin, since apart from electromagnetic differences the masses within these multiplets are equal.
The simplest assumption (complementary to the electromagnetic properties) would be to assume that this fifth force must be a combination of scalar and vector in U spin. Therefore, we also have an "equal spacing rule" for the mass differences in the U-spin multiplets. We then write, assuming $H = H_0 + m$, where $m$ is the mass correction,

$$m_{\Sigma^0} - m_{A'} = m_{A'} - m_n,$$

hence

$$m_{\Sigma^0} + m_n = 2m_{A'}.$$

Identical considerations as were made before give us from

$$A' = \frac{1}{2} (-3\Lambda + \Sigma^0)$$
$$B' = \frac{1}{2} (\Lambda + \sqrt{2} \Sigma^0)$$

$$m_{A'} = (-\sqrt{2} \Lambda + \Sigma^0 | m | -\sqrt{2} \Lambda + \Sigma^0) = \frac{3}{4} m_{\Lambda} + \frac{1}{4} m_{\Sigma} \quad (m \text{ is mass operator})$$

since $m$, as an isoscalar, has no matrix element between $\Lambda$ and $\Sigma^0$. So we get

$$m_{\Sigma^0} + m_n = 2m_{A'} = \frac{3m_{\Lambda} + m_{\Sigma^0}}{2} \quad (2.7)$$

or

$$\frac{m_{\Sigma^0} + m_n}{2} = \frac{m_{\Sigma^0} + 3m_{\Lambda}}{4} \quad \text{(Gell-Mann/Okubo formula)}. \quad (2.8)$$

The same can be done for bosons, only for some obscure reason the squares of the masses take the place of the linear masses, and one obtains

$$m_k^2 = \frac{3m_n^2 + m_\Xi^2}{4}. \quad (2.8 a)$$

These formulae are in excellent agreement with the actual observed values.

For the decuplet of baryons the rules are even simpler since we have no double points. The masses of baryons of equal I spin must be the same; (horizontal direction in the decuplet scheme, Fig. 6.)
According to the equal spacing rule, the masses should be equally spaced along the direction of equal U spin (from upper left to centre down). This means that there is equal spacing between the different I-spin multiplets, which is in excellent agreement with the facts.
III. BARYON-Meson Interactions:
The Question of the Two Octets

1. Baryon-meson $f$ and $d$ couplings

The typical interaction term for the energy of strong processes has the well-known form $\bar{N}_i N_j \phi_k$, where $\bar{N}_i N_j$ are baryons and antibaryons, and $\phi_k$ is a boson. In the isotopic spin theory of strong interactions, $\phi$ has three different states (corresponding to the three pions) and transforms like a vector in isospace. Therefore, if the whole interaction term $\bar{N}N\phi$ is to be invariant under I-spin rotations, $\bar{N}N$ must also transform as a vector in order to provide an invariant when multiplied by $\phi$.

As is well known, the problem of building an entity, which contains $\bar{N}, N$ and transforms as a vector, is solved by sandwiching the $\tau$ matrices between the two spinors $\bar{N}, N$. The interaction energy, therefore, has the form

$$(\bar{N} \tau N) \phi,$$

where the operator $\tau$ is an isovector, multiplied into the isovector $\phi$ of the meson field.

In SU3, we face the problem of constructing an $\bar{N} \times N$ entity which, multiplied by the octet of bosons $\phi$, gives us a scalar. Here, $\bar{N} \times N$ will have to transform as an octet, so we must get an octet out of the two octets represented by $\bar{N}, N$.

Here, we face a different problem compared to the isotopic theory of strong interactions. The reason is easy to understand: in the isotopic theory of strong interactions, the baryons ($p, n$) form a spin in isospace to be contrasted with the vectorial (or "regular") representation to which the three pions are ascribed. It is very easy to form a vector out of two spinors by means of the $\tau$ matrices. Now, as we already know, it happens that in SU3 both the baryons and the mesons are ascribed to the octet representation of SU3, which is precisely the "regular" representation of this group and corresponds to the vectorial representation in SU2.
So we must use the matrix method to build invariants, instead of taking recourse to expressions like $N \tau N$ as can be done in the $I$-spin case.

In order to understand the matrix formalism for $SU_3$, we start developing it for $SU_2$. When we want to build a vector out of two vectors $V_1, V_2$, we simply take the vector product of these vectors, $V_1 \times V_2$. However, the prescription for forming a vector product in 3-space cannot be extended to vectors in more complicated spaces. We seek, therefore, a more general prescription.

We have seen that a vector in 3-space can be expressed as a bi-linear combination of bi-dimensional quarks [see (1.8)]. We will now introduce a new matrix arrangement to express the components of that vector.

Let us call $B^i_k = u^i u_k^*$ and consider $B^i_k$ as a $2 \times 2$ matrix. The trace of $B^i_k$ is our known invariant

$$ I = \text{tr} B^i_k. $$

We then construct

$$ A^i_k = B^i_k - \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3.1} $$

This is the traceless matrix

$$ A^i_k = \begin{pmatrix} \frac{1}{2} (u^i u_1 - u^2 u_2) & u^2 u_1 \\ u^1 u_2 & -\frac{1}{2} (u^i u_1 - u^2 u_2) \end{pmatrix}, \tag{3.2} $$

which according to (1.9) has the same transformation properties as

$$ A^i_k = \begin{pmatrix} \frac{z}{\sqrt{2}} & \frac{1}{\sqrt{2}} (x + iy) \\ \frac{1}{\sqrt{2}} (x - iy) & -\frac{z}{\sqrt{2}} \end{pmatrix}. \tag{3.3} $$

We now call

$$ \begin{align*}
  x + iy &= \sqrt{2} x_+ \\
  x - iy &= \sqrt{2} x_-
\end{align*} \tag{3.4a} $$

so

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$p/cm$
\[ A_{k}^{i} = \begin{pmatrix} \frac{z}{\sqrt{2}} & \frac{x_{+}}{\sqrt{2}} \\ \frac{x_{-}}{\sqrt{2}} & -\frac{z}{\sqrt{2}} \end{pmatrix} \]  

(3.4b)

\( A_{k}^{i} \) has three independent elements which transform as a vector.

From two of these traceless matrices, \( A_{k}^{i}, A_{l}^{j} \), can we obtain a new traceless matrix representing something which also transforms like a vector? We get it by constructing

\[ P_{k}^{l} = \sum_{i} A_{k}^{i} A_{l}^{i} - \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \]  

(3.5)

where \( I \) is the invariant

\[ I = A_{k}^{i} A_{k}^{i}. \]

In fact, \( P_{k}^{l} \) is

\[ P_{k}^{l} = \begin{pmatrix} \frac{1}{2} zz' + x_{+} x_{+}' & \frac{1}{\sqrt{2}} (z x_{+}' - x_{+} z') \\ \frac{1}{\sqrt{2}} (z x_{+}' - x_{+} z') & x_{+} x_{+}' + \frac{1}{2} zz' \end{pmatrix} - \frac{1}{2} \begin{pmatrix} zz' + x_{+} x_{+}' + x_{-} x_{-}' & 0 \\ 0 & zz' + x_{+} x_{+}' + x_{-} x_{-}' \end{pmatrix} \]

so that

\[ P_{k}^{l} = \begin{pmatrix} \frac{z}{\sqrt{2}} & \frac{x_{+}}{\sqrt{2}} \\ \frac{x_{-}}{\sqrt{2}} & -\frac{z}{\sqrt{2}} \end{pmatrix}, \]

where

\[ z = \frac{1}{\sqrt{2}} \left( x_{+} x_{+}' - x_{-} x_{-}' \right) \]

\[ x_{+} = \frac{1}{\sqrt{2}} \left( z x_{+}' - x_{+} z' \right) \]

\[ x_{-} = \frac{1}{\sqrt{2}} \left( x_{-} z' - z x_{-}' \right) \]

which correspond to the well-known components of the vector product of two vectors. (Up to a normalization factor.)
Now comes a very important point. In our construction of \( P_k^l \) there was another very similar, yet different possibility. Instead of writing \( A_k^i A_k^i \), one could also write \( A_k^l A_k^i \), and define a new matrix

\[
Q_k^l = A_k^l A_k^i - \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
\]  

(3.6)

\( Q_k^l \) can be obtained from \( P_k^l \) simply by exchanging primed and unprimed components. Since \( P_k^l \) is antisymmetric in respect to this exchange, we get \( Q_k^l = -P_k^l \).

We are now ready to follow the same steps in \( SU_3 \). Let us now define

\[
B_k^i = u_k^i u_k^i, \quad \Sigma B_k^i = I,
\]

(3.6a)

and the octet as the traceless matrix

\[
A_k^i = B_k^i - \frac{1}{3} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}
\]

(3.7)

\[
= \begin{pmatrix}
\frac{2}{3} u_1^i u_1^i - \frac{1}{3} (u_1^1 u_1^1 + u_1^2 u_2^2 + u_1^3 u_3^3) & u_1^2 u_1^1 & u_1^3 u_1^1 \\
u_1^2 u_1^2 & \frac{2}{3} u_1^2 u_1^2 - \frac{1}{2} (u_1^1 u_1^1 + u_1^3 u_3^3) & u_1^3 u_1^2 \\
u_1^2 u_1^3 & u_1^3 u_1^3 & \frac{2}{3} u_1^3 u_1^3 - \frac{1}{2} (u_1^1 u_1^1 + u_1^2 u_2^2)
\end{pmatrix}
\]

It is better to use the shorthand introduced in Table 2; \( A_k^i \) may be written

\[
A_k^i = \begin{pmatrix}
\frac{1}{\sqrt{2}} \sigma^0 + \frac{1}{\sqrt{6}} \lambda & \sigma^+ & \nu_+ \\
\sigma^- & \frac{1}{\sqrt{6}} \lambda - \frac{1}{\sqrt{2}} \sigma^0 & \nu_0 \\
\xi^- & \xi^0 & -\frac{i}{\sqrt{3}} \lambda
\end{pmatrix}
\]

(3.8)
Perhaps a word of warning is in order here. The fact that we are using quark-antiquark combinations to build our octet in SU_3 is purely a convenient mathematical way to arrive at our matrix result. Therefore, once we have our octet in matrix form, it can be interpreted as any physical octet. That is, it bears no relation to a quark-antiquark physical structure, which is typical of the bosons, and can be used as well as an octet of baryons and antibaryons coming from \(qq\) and \(\bar{q}\bar{q}\). We are, after all, only interested in the transformation properties.

Our next step is to define an "octet multiplication" which gives another octet.

As an obvious extension to what was done with vectors, we define the product of octets \(A^i_k A'^j_i\) as

\[
0^\ell_k = A^i_k A'^j_i - \frac{1}{3} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array} \right),
\]

(3.9)

where \(I\) is the invariant \(A^i_k A'^j_i\)

\[
I = \sigma^0 \sigma^0 + \lambda \lambda' + \sigma^+ \sigma^- + \sigma^- \sigma^+ + \xi^0 \xi'^0 + \nu^0 \nu'^0 + \nu^+ \nu'^+.
\]

We write explicitly some of the components of \(0^\ell_k\),

\[
0^\ell_i = A^i_1 A^2_1 + A^i_2 A^3_2 + A^i_3 A^2_3 =
\]

\[
= \frac{1}{\sqrt{6}} (\lambda \sigma^0) + \frac{1}{\sqrt{2}} (\sigma^0 \sigma^0) + \frac{1}{\sqrt{6}} (\sigma^+ \lambda') - \frac{1}{\sqrt{2}} (\sigma^+ \sigma^0) + (\nu^+ \xi^0).
\]

(3.9a)

Now, it is clear that \(0^\ell_i\) does not simply change sign when we interchange primed and unprimed components as was the case with vectors. \(0^\ell_i\) is antisymmetric in the \(\sigma\), symmetric in the \(\lambda\), and has no symmetry for the \(\nu^+ \xi^0\) term.

This means that a different octet comes out if we define

\[
\tilde{0}^\ell_k = A^i_k A'^j_i - \frac{1}{3} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array} \right),
\]

(3.10)
The $\tilde{0}_i^2$ component is now

$$\tilde{0}_i^2 = A_i^2 \Lambda_i' + A_i^2 \Lambda_i'^2 + A_i^2 \Lambda_i'^3 =$$

$$= \frac{1}{\sqrt{6}} (\sigma^\tau \Lambda') - \frac{1}{\sqrt{2}} (\sigma^\tau \sigma') + \frac{1}{\sqrt{6}} (\lambda \sigma') + \frac{1}{\sqrt{2}} (\sigma^0 \sigma') + (\varepsilon^0 \nu'^+) \tag{3.10a}$$

What we are really interested in is constructing an octet which is a bi-linear form — not of two baryons as the above octets, $0_k^\ell$ and $\tilde{0}_k^\ell$, but of an antibaryon-baryon pair as we find it in the baryon-boson interaction. Since the baryon octet $\lambda \sigma^\tau \sigma^0 \nu^0 \nu' \varepsilon \varepsilon^0$ transforms like the antibaryon octet $\overline{\lambda} \sigma^\tau \sigma^0 \nu^0 \nu' \overline{\varepsilon} \overline{\varepsilon}^0$, we get what we want if we replace in the bi-linear forms the first baryon by the corresponding antibaryon.

Thus, we have for example

$$0_i^2 = \frac{1}{\sqrt{6}} (\overline{\Lambda} \sigma'^+) + \frac{1}{\sqrt{2}} (\overline{\sigma} \sigma'^+) + \frac{1}{\sqrt{6}} (\overline{\sigma} \sigma') - \frac{1}{\sqrt{2}} (\overline{\Lambda} \sigma'^0) + (\overline{\nu} \varepsilon'^0) \tag{3.10b}$$

$$\tilde{0}_i^2 = \frac{1}{\sqrt{6}} (\overline{\Lambda} \sigma'^+) - \frac{1}{\sqrt{2}} (\overline{\sigma} \sigma'^+) + \frac{1}{\sqrt{6}} (\overline{\sigma} \sigma') + \frac{1}{\sqrt{2}} (\overline{\Lambda} \sigma'^0) + (\overline{\nu} \nu'^+) \tag{3.10b}$$

Both octets could be coupled to the octet of bosons, but it is more convenient to form from them symmetrical and antisymmetrical combinations.

We accordingly call

$$P_i^k = \frac{1}{2} (0_i^k - \tilde{0}_i^k) \quad \text{(antisymmetric octet)}$$

$$Q_i^k = \frac{1}{2} (0_i^k + \tilde{0}_i^k) \quad \text{(symmetric octet)} \tag{3.11}$$

As an example we give their (2.1) components

$$P_i^0 = \frac{1}{\sqrt{2}} (\overline{\sigma} \sigma' - \frac{1}{2} (\overline{\sigma} \sigma') + \frac{1}{2} (\overline{\Lambda} \lambda') - \frac{1}{2} (\overline{\nu} \nu'^+) \tag{3.11a}$$

$$Q_i^0 = \frac{1}{\sqrt{6}} (\overline{\lambda} \sigma'^+) + \frac{1}{\sqrt{6}} (\overline{\sigma} \lambda') + \frac{1}{2} (\overline{\varepsilon} \varepsilon'^0) + \frac{1}{2} (\overline{\nu} \nu'^+)$$
Here, we have two ways of constructing an octet as a bi-linear form of the octet and anti-octet. Let us apply this to the baryon-meson coupling. Here, we must construct a form \( H = (\bar{B}B) \Phi \), where \( B \) are baryons and \( \Phi \) are bosons. The bosons, \( \Phi \), form an octet so we must get a form \( (\bar{B}B) \) which is an octet too. This can be done in two ways. All we have to do is to write:

\[
H = \left( f \, P_{\ell}^{k} + d \, \bar{Q}_{\ell}^{k} \right) \bar{\phi}_{\ell}^{k},
\]

where \( P_{\ell}^{k} \) and \( \bar{Q}_{\ell}^{k} \) are given by (3.11), but in which the symbols \( \lambda, \sigma, \nu, \xi \) are replaced by the baryons: \( \lambda = \Lambda, \sigma = \Sigma, \nu_0 = n, \nu_+ = \rho, \xi = \Xi \).

The properties of "f" and "d" couplings are different. For example, in the \( f \) coupling the first term of \( P_{\ell}^{2} \), considered as an operator, would destroy a \( \sigma^+ \) and create a \( \sigma^0 \), and the second term would destroy a \( \sigma^0 \) and create a \( \sigma^- \). Therefore, there is no change of I spin. However, the \( \bar{Q}_{\ell}^{2} \) operator makes us "travel" in the octet in the same direction, but through the \( \lambda \) state, with a consequent change of I spin. Also, the second and third terms contribute with the same strength but with equal phase in one case, and opposite in the other.

Transitions with \( \pi^- \) absorption or \( \pi^+ \) emission.

Fig. 12
It is interesting to point out that one can understand, in a simple "isospin way", the general features of $P_i^2$ and $Q_i^2$, though not the numerical factors.

Assume that we want to couple the hyperons $\Sigma$ and $\Lambda$ with a pion field. The $\Sigma$'s are an isovector, the $\Lambda$'s an isoscalar. We must construct an isovector which is bi-linear in the hyperons and multiply it with the isovector of the pions. This can be done by a "vector product" of the vector $\vec{E}$ and $\Sigma$, or by the product $\vec{E} \Lambda$ or $\vec{E} \Sigma$. Let us look at the $(\pm)$ components. In the first case we get $\Sigma_0 \Sigma_+ - \Sigma_+ \Sigma_0$ which is - in its transformation - equivalent to $\vec{E}_0 \vec{E}_+ - \vec{E}_+ \vec{E}_0$; in the second case we get $\Sigma_+ \Lambda'$ or $\Lambda \Sigma_+'$, which is equivalent to $\vec{E}_0 \Lambda'$ and $\vec{E}_0 \Sigma_+$. These are just the combinations appearing in $P_i^2$ and $Q_i^2$ respectively.

2. Remarks on "two octets"

We wish to express the bi-linear forms $P_i^j$ in a compact form. This can always be accomplished if we write

$$P_i = \sum_k \bar{A}_k t_i^k A_k \quad k = 1 \ldots 8 \quad (3.13)$$

where $\bar{A}_k, A_k$ correspond to the $\sigma^+ \sigma^- \sigma^0 \ldots$ states of the octet (and anti-particle octet), and the $t_i^k$ are some $8 \times 8$ matrices such that

$$t_i^k A_k = t_i^{k\ell} A_\ell \quad i,k,\ell = 1 \ldots 8. \quad (3.14)$$

The expression $P_i \bar{A}_k A_k$ is an invariant. Since the $P_i$ forms an octet, it is the matrices $t_i^k$ that bring in the octet transformation properties. That means that the eight matrices $t_i^k$ form an octet.

In a parallel way we can define, by means of the bi-linear forms $Q_i$, some other matrices $s$, such that

$$Q_i = \Sigma \bar{A}_k s_i^k A_k$$

$$s_i^k A_k = s_i^{k\ell} A_\ell \quad (3.14a)$$
the $s_1$ again being eight by eight matrices. All this is interesting because it allows us to write the octets $P_1, Q_1$ in a simple form if we know the matrices $t_1, s_1$.

It is interesting that one can construct the eight by eight matrices $t_1$ and $s_1$ by using the generating three by three matrices $\tau_1, \ldots, \tau_8$ defined on page 13. These are, in fact, the operators $\tau_+ \tau_- \tau_0 u_+ u_- v_+ v_- \tau_8$. One does this as follows. Consider the octet components $\sigma^+, \sigma^- \ldots$ represented by a $\tilde{q}$ product, such as $\sigma^+ = u^i u_2$, $\sigma^0 = 1/\sqrt{2} (u^i u_1 - u^2 u_2)$ etc. (see Table 1 on page 12). Then remember the definition of the $\tau_+$ – see page 13, which describes their action on the $u_k$'s. For example, let us take the operators $u_1^+$ or $\tau_+$ which are two of the eight $\tau$'s:

\[
\begin{align*}
    u_+ u_3 &= u_1, & \tau_+ u_1 &= u_2 \\
    u_+ u_4 &= 0, & \tau_+ u_2 &= 0 \\
    u_+ u_2 &= 0, & \tau_+ u_3 &= 0 
\end{align*}
\]

We also need the effect of the $\tau$ on the adjoint states $u_k^\dagger$. They should have the effect of an inverse matrix, and change the sign.*

\[
\begin{align*}
    u_+ u_1^\dagger &= -u_3 \\
    u_+ u_2^\dagger &= 0 \\
    u_+ u_3^\dagger &= 0
\end{align*}
\]

\[
\begin{align*}
    \tau_+ u_1^\dagger &= -u_2 \\
    \tau_+ u_2^\dagger &= 0 \\
    \tau_+ u_3^\dagger &= 0
\end{align*}
\]

\[
(3.15)
\]

and as a third example

\[
\begin{align*}
    \tau_8 u_1^\dagger &= -1/\sqrt{6} u_1 \\
    \tau_8 u_2^\dagger &= -1/\sqrt{6} u_2 \\
    \tau_8 u_3^\dagger &= +2/\sqrt{6} u_3
\end{align*}
\]

* The effect of the $\tau$ on the adjoint states can be deduced from the transformations (1.6) of the adjoint states. After all, we know that the $\tau$ represents infinitesimal transformations. Hence, the matrices which represent the action of $\tau$ on the adjoint states are the reciprocal inverted ones of the ordinary ones; if \( n_1 = R_{1k} u_k \), then \( n_1^i = R^{-1}_{ki} u_k \).
The effect of the operators \( \tau_1 \ldots \tau_8 \) on the triplet \( u_1 u_2 u_3 \) can be expressed by ascribing to each \( \tau_\alpha \ (\alpha = 1 \ldots 8) \) a 3x3 matrix \((\tau_\alpha)_\ell^k\), \( k, \ell = 1, 2, 3 \). This was done on page 13, expression (1.18). Then

\[
\tau_\alpha u_\ell = \sum_\ell (\tau_\alpha)_\ell^k u_k . \tag{3.15a}
\]

The effect of the \( \tau \)'s on the adjoint triplets is given by

\[
\tau_\alpha u_\ell^i = - \sum_\ell (\tau_\alpha)_\ell^i u_\ell . \tag{3.15c}
\]

We can now calculate the effect of the \( \tau_\alpha \)'s on an octet. We represent the octet by the matrix \( A_\ell^k \) [see (3.7)]; each element is a bilinear form of a triplet and an adjoint triplet. Then

\[
\tau_\alpha A_\ell^k = \sum_m (\tau_\alpha)_m^k A_\ell^m - \sum_n (\tau_\alpha)_n^k A_\ell^n . \tag{3.15b}
\]

For example, let us calculate \( \tau_+ A_1^2 = \tau_+ \sigma^+, \) this is the effect of the operator \( \tau_+ \) on \( \sigma^+ = u^1 u_2 \). We can use expressions (3.15) and get

\[
\tau_+ u_1^i u_2 = - u^1 u_2 + u_1^1 u_1 = - \sqrt{2} \sigma^+ .
\]

We can also use expression (3.15b), bearing in mind that in the matrix \((\tau_+)_\ell^k\) all elements are zero except \((\tau_+)_1^1 = 1\). If we insert this in (3.15b), we get

\[
\tau_\alpha A_1^2 = A_2^2 - A_1^1 = - \sqrt{2} \sigma^+ .
\]
If we express all these relations in the form

\[ \tau_i A_k = t_{ik}^{\ell} A_{\ell} , \quad i, k, \ell = 1 \ldots 8 \tag{3.150} \]

the resulting eight 8\times8 matrices \( t_{ik}^{\ell} \) are identified with the ones which we have defined above.

The other set of matrices \( s_{ik}^{\ell} \) cannot be constructed in such a simple way.

There is another relation between these matrices which is important. (It is the way Gell-Mann first introduced them.)

Let us take the operators \( \tau_1 \ldots \tau_8 \) and construct their commutation relations. We find for the commutators and for the anticommutator

\[
\begin{align*}
[\tau_i, \tau_k] &= 2i t_{ik}^{\ell} \tau_{\ell} \\
\{\tau_i, \tau_k\} &= 2 s_{ik}^{\ell} \tau_{\ell} + \gamma^8 \delta_{ik} .
\end{align*}
\]

Clearly, the \( t_{ik}^{\ell} \) and \( s_{ik}^{\ell} \) are eight 8\times8 matrices. They are identical to the ones defined above.

Those matrices \( t_{ik}^{\ell} \) or \( s_{ik}^{\ell} \), which do not contain diagonal elements, can be graphically represented in the form of arrows connecting different states of the octet. An arrow going from \( i \) to \( k \) represents a matrix element \( ik \), and the value of this element is written on the arrow. We represent \( \tau_+ \) and the corresponding component of \( s \) -- \( s_+ \) -- in this way in Fig. 13 and Fig. 14. (The values are correct only up to a normalization factor.)
The matrix \( t_4 \) which corresponds to \( v_- \) and the corresponding component of \( s - s_4 \) can be represented as follows: (They will be required in the next chapter.)

It is seen that \( v_- \) and \( s_4 \) change the charge by +1 in each transition.
Let us now go back to (3.12) and rewrite the baryon-meson interaction with the help of (3.13) and (3.14):

$$H = \sum_{k\ell} \tilde{A}_{k}(f \cdot t_{i}^{k\ell} + d \cdot s_{i}^{k\ell}) A_{\ell} \Phi_{i} = 1 \ldots 8.$$  

If we want to compare this with the actual baryon-meson interaction, we must take into account that the meson field $\Phi$ is a pseudoscalar. Therefore, there must be a $\gamma_{5}$ incorporated.

$$H = \sum_{k\ell} \tilde{A}_{k}(f \cdot t_{i}^{k\ell} + d \cdot s_{i}^{k\ell}) \gamma_{5} A_{\ell} \Phi_{i}.$$  

which in the non-relativistic approximation gives:

$$H = \sum_{k\ell} \tilde{A}_{k}(f \cdot t_{i}^{k\ell} + d \cdot s_{i}^{k\ell}) \sigma^{(\text{bar})} A_{\ell} \Phi_{i}. \quad (3.15d)$$

When this expression is compared with the experimental knowledge about coupling constants between various baryon-octet members and the pseudoscalar mesons, one finds

$$f/d \sim \frac{2}{3}. \quad (3.15c)$$

(The symbol $\sigma^{(\text{bar})}$ means "spin operator", operating on the spin of the baryon. This is to distinguish it from a spin operator which acts on the spin of the quark, which we will introduce later.)
3. The two octets and the mass formula

We now go back to the mass formula, which we can formulate in a better way, having learned about the two octets. On pages 26 and 27 we derived the Gell-Mann/Okubo mass formula on the assumption that the SU₃ splitting mass operator is scalar in I spin and vector in U spin. It should also be diagonal in the octet and decuplet states since it does not produce any mixing among particles. (The $\Phi - \omega$ mixing will be treated later,) We can find among the eight generating operators (see p. 13) one, namely $\tau_8$, which has all these properties. It is diagonal and it can be expressed in the form

$$\tau_8 = \frac{2}{3}(u_z - v_z),$$

where $u_z$ and $v_z$ are then z components of the u spin and v spin. Hence, it is a vector in u spin. Therefore, the SU₃ violating operator $\Delta M$, which causes the mass split, should transform like the eighth component of an octet. The value of the mass split for a member $A_i$ of an octet is:

$$\Delta M_i = (\bar{A}_i \Delta M A_i) \quad i = 1 \ldots 8. \quad (3.16)$$

Let us now go back to the bilinear forms of octets which, in themselves, transform like an octet. We know that there are two of them. Hence, if we construct the eighth component, it will have two possible forms:

$$P_\delta = \sum_k \bar{A}_k t_\delta A_k \quad (3.17)$$

$$Q_\delta = \sum_k \bar{A}_k s_\delta A_k.$$
where \( t^b A_k \) and \( s^b A_k \) are defined by (3.14), (3.14A). In our repre-
sentation \( t^b \) and \( s^b \) are diagonal. The first term in (3.17) 
corresponds to the part of the mass operator which is equal to \( \tau^b \).
Its eigenvalues are proportional to the hypercharge \( Y \). Hence, the
mass formula contains a term \( aY \). Now, let us look at \( s^{ii}_b \); the
result is:

\[
s^i_b = \frac{1}{3} \begin{pmatrix}
1 & -2 & -2 \\
-2 & -2 & +2 \\
-2 & +2 & 1
\end{pmatrix}
\]

where the order of the rows is: \( \nu^0, \nu^\pm, \sigma^0, \sigma^\pm, \lambda \xi^-, \xi^+ \). We will
prove this on page 47. Here, we see that this matrix splits the \( \lambda \)
from the \( \sigma \).

The eigenvalues of \( s^i_b \) can be expressed by:

\[
s^{ii}_b = \frac{1}{3} (2 - 2 I (I + 1) + \frac{1}{2} Y^2),
\]

where \( I \) is the I spin (\( \frac{1}{2} \) for \( \nu, \xi \), 1 for \( \sigma, \lambda \)) and \( Y \) is the hypercharge.
This can be checked by trial. Therefore, for the octet mass formula
we get

\[
\Delta_m = a Y + b (I(I+1) - \frac{1}{4} Y^2),
\]

where \( a, b \) are some constants. A constant term equal for all octet
members is omitted. This expression is identical with the formula
(2.8). Two constants appear, since one can make two octets out
of bilinear forms (\( \bar{A}A \)).

The situation is simpler when the \( A_1 \) in (3.16) are members
of a decuplet. A decuplet member is a symmetric combination of three
quarks, as given in the table on page 18. A combination of three
quarks can be written in analogy to (3.6a) in the form

\[ B_{ikl} = u_i^k u_l^k. \]

There are \(3 \times 3 \times 3 = 27\) of them. The linear combinations which transform like singlets, octets and decuplets can be easily formed:

\[ \text{singlet} = (B_{ikl})_{ikl}, \quad (3.21) \]

where \((B_{ikl})_{ikl}\) signifies the linear combination, which is antisymmetric in all three indices. Then we can form combinations which are antisymmetric only in \((ik)\) or \(il\), or \((kl)\):

\[ (B_{ikl})_{ik}, (B_{ikl})_{kl}, (B_{ikl})_{il}. \quad (3.22) \]

Since the antisymmetric combination of two quarks transforms like an antiquark, \((B_{ikl})_{ik}\) transforms as \(B_{ij}^m\), where \(m \neq i, n \neq k\). Hence, the three forms in (3.22) are analogous to the form \(B_{ij}^m\) as in (3.6a) and - after subtraction of the invariant [which is (3.21)] - lead to octets as seen in (3.7), and the subsequent paragraphs on page 32. We get three octets from the forms (3.22) but only two are linearly independent because the sum of the three forms (3.22) vanishes. These correspond to the two octets on page 18, although they are, in fact, different linear combinations. (See more of this in Chapter V.)

The decuplet of course comes from the symmetric combinations:

\[ \text{decuplet: } [B_{ikl}]_{ikl} = A^\alpha \quad \alpha = 1 \ldots 10, \quad (3.23) \]

where \([B_{ikl}]_{ikl}\) means the linear combinations, symmetric in all three indices. There are ten of them which are denoted here as \(A^\alpha\). We have, of course, also the adjoint states:

\[ A^\alpha = [A_{ikl}]_{ikl}. \]
We get, as usual, an invariant by summing over all states:

$$\sum_{\alpha=1}^{10} A^\alpha A_\alpha = I, \quad (3.24)$$

We now wish to construct an octet from a bilinear form of decuplets. If we had the unsymmetric B_{ikl} to play with, we would have nine ways to make an octet, C^m_n, as defined by (3.8):

$$\sum_{ik} A^{ikm} A_{ikn}, \sum_{ik} A^{ikm} A_{ink}, \text{ etc.},$$

not all of them being necessarily different. Fortunately, we have to deal with the symmetric linear combinations (3.23) only, where the position of the index is irrelevant. One therefore gets, in fact, only one form which transforms like an octet. It can be shown that it is:

$$C^m_n = \sum_{ik} \left( \epsilon^m_{ik} \epsilon^n_{ik} \right)^{1/2} [B_{ikm}^{ikm} B_{ikn}^{ikn}] - \frac{1}{2} \delta^m_n,$$

where

$$\epsilon^m_{ik} = \begin{cases} 
3 & \text{if } m \neq i \neq k \\
2 & \text{if } m = i \neq k \text{ or } m = k \neq i \\
1 & \text{if } m = i = k 
\end{cases}$$

The symmetry of the decuplet allows only one octet to be constructed in a bilinear form $A^\alpha A_\beta$. In the mass formula we are interested in the $\lambda$ component. According to (3.8) this is $C^3_\lambda$:

$$C^3_\lambda = \sum_{kl} \epsilon^3_{kl} [B_{kl3}^{kl3} B_{kl3}^{kl3}],$$

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/\p/cm
\( \epsilon_{ik}^3 \) is the number of "threes" in ik, and therefore we can also write it in the following way

\[
C_3^i = \sum_{i=1}^{10} (Y_i - \frac{1}{3}) A_i^i A_i^1,
\]

(3.25)

where \( Y_i \) is the hypercharge of the state \( i \). This, by the way, is identical with

\[
C_3^3 = \sum_{\alpha=1}^{10} A_\alpha^\alpha \tau_8 A_\alpha,
\]

(apart from a constant) which obviously transforms like \( \tau_8 \) since (3.24) is an invariant. This shows that the mass-correction term of the Hamiltonian must have the form:

\[
H_{\text{dec.}}' = C \cdot \sum Y_i A_i^1 A_i^1,
\]

(3.26)

which means that for the decuplet the mass corrections for the particle \( i \) will have the form \( C Y_i A_i^1 A_i^1 \); they will be proportional to the hypercharge. It is interesting to note that the mass formula (3.20) would also apply to the decuplet, since the second term becomes linear in \( Y \) for the decuplet. However, SU3 does not provide any reason why the factors in the octet-mass formula (3.20) should have anything to do with the factor \( C \) in (3.26).

4. Derivation of the diagonal matrix \( s_8 \)

In the matrix representation (3.8) of the octet, the eighth octet component is essentially identical with \( A_3^3 \). We therefore construct \( 0_8 \) and \( 0_8^\dagger \) in analogy to (3.9a) and (3.10). We then put them into the form corresponding to (3.10b), by replacing the first symbol by the corresponding antisymbol. We then symmetrize and antisymmetrize according to (3.11) and finally get, in analogy to (3.11a),
\[ P_3 = \xi^- \xi^- + \xi^0 \xi^- + \nu_+ \nu_+ - \nu_0 \nu_0 \]

\[ Q_3 = \xi^- \xi^- + \xi^0 \xi^0 + \nu_+ \nu_+ + \nu_0 \nu_0 + \frac{1}{2} \lambda \lambda \]

Comparing these expressions with (3.13) and (3.14a), we get (again using the index 8 for the \( \frac{3}{3} \) component):

\[
t_{ik}^{\lambda} \rightarrow \begin{pmatrix}
-1 & 0 \\
-1 & 0 \\
0 & 0 \\
0 & +1 \\
0 & +1 \\
\end{pmatrix}
\]

where the order of rows and columns is: \( \nu^+ \nu^0 \sigma^+ \sigma^0 \xi^0 \xi^- \). We get for \( s_{ik}^{\lambda} \)

\[
s_{ik}^{\lambda} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

After subtracting a unit matrix such that \( s_{ik}^{\lambda} \) is traceless, one arrives at (3.16).
IV. WEAK INTERACTIONS

For those weak processes involving only n, p and any pair (e, ν) (μ, ν), the interaction between hadrons and leptons is known to be described by

\[ G(\bar{a} | \gamma_\mu (1 + \gamma_5) | b) (\bar{a}' | \gamma_\mu (1 + \gamma_5) | b') , \]

where \( a, b \) represent n, p and \( a', b' \) a pair of leptons.

The baryon current may be written

\[ G(\bar{N}\gamma_\mu (1 + \gamma_5) \tau_+ N) + c.c. , \]  \hspace{1cm} (4.1)

where \( N = \begin{pmatrix} p \\ n \end{pmatrix} \).

If we wish to consider a more general scheme including the eight baryons, an appealing generalization to (4.1) is to assume that \( N \) consists of an octet column, and \( \tau_+ \) is a generator of \( SU_3 \) in its \( 8 \times 8 \) matrix representation [see (3.13)].

This has clear experimental consequences. It predicts that the ratios for transition amplitudes

\[ \frac{n \rightarrow p}{\Sigma^- \rightarrow \Sigma^0} \]

must be equal [both are transitions in a \((T = \frac{1}{2})\) doublet], and furthermore \( 1/\sqrt{2} \) weaker than transitions

\[ \frac{\Sigma^- \rightarrow \Sigma^0}{\Sigma^0 \rightarrow \Sigma^+} . \]

This is the factor between \( (\tau_+) \) transitions in a triplet and a doublet. It would also forbid transitions \( \Sigma \rightarrow \Lambda \), since \( \tau_+ \) does not connect \((T = 1)\) and \((T = 0)\) states. (We will see later that this is actually the case for the so-called vector part of the interaction only.)
The assumption (4.1) implies that weak interactions are not SU₃ invariant. It assumes that the baryonic part of the interaction is not an SU₃ scalar. Specifically, it assumes that the baryon current (4.1) transforms like an octet component. This is a very special assumption, which seems to be correct.

The conclusions from (4.1) are not as simple as they seem because of the fact that a baryon is a complicated mass surrounded by mesons and baryon pairs. It is "dressed" and not "bare". We will, therefore, show that the significance of γₜ, τ and γₜ, γₜ are not so obvious, and that one cannot, as we have done above, only consider the effects of τ alone.

We will only be interested in small momentum transfers, i.e., we can consider the baryons as non-relativistic. Hence, in (4.1) the vector part is approximated by:

\[
G \left( \bar{N} Y_\mu \tau_+ N \right) \begin{cases} 
0 & \text{for } \mu = 1, 2, 3 \\
G(\bar{N} \tau_+ N) & \text{for } \mu = 0
\end{cases} \quad (4.2)
\]

and the axial vector part by:

\[
G \left( \bar{N} Y_\mu \gamma_5 \tau_+ N \right) \begin{cases} 
G(\bar{N} \sigma_\mu \tau_+ N) & \text{for } \mu = 1, 2, 3 \\
0 & \text{for bare nucleons.}
\end{cases} \quad (4.3)
\]

We will now see what happens if we have to deal with dressed baryons. The vector part does not change as long as we restrict ourselves to the zero-momentum part (4.2). This is because the isotopic spin by itself is an invariant of the strong interaction. A state of a given isotopic spin quantum number remains a state of this quantum number when interaction is switched on. Hence, the form (4.2) remains correct for dressed nucleons; also, the conclusions of before remain correct. There will now be Σ-Λ transitions induced by the vector part, the n-p and Σ⁻-Σ⁰ transitions will have the same matrix element, the Σ-Σ transitions will have a larger matrix element by \( \sqrt{2} \). Note that the operators \( \tau_+ \) in (4.2), (4.3) and the following equations are 8×8 matrices as defined in (3.150c). They act on octet members.
The axial situation is more complicated because of the $\sigma$. The nuclear forces are not necessarily independent of the spin direction; hence $\sigma\tau$ is not an invariant. When we go from the bare to the dressed nucleon, $\sigma\tau$ has no definite value anymore. (For example, when a nucleon emits a virtual meson, it changes its spin direction.) Hence, nothing so simple can be said for the axial part. As long as we consider $N$ to be only neutrons or protons, this uncertainty manifests itself only in the fact that the matrix element $<\bar{N}\sigma_\mu\tau_+N>$ has a different factor than the vector part. (Note, we are only speaking of zero-momentum transfer processes.) It must have the form $\sigma_\mu\tau_+$, since there is no other combination (with $q = 0$) which gives a change of charge, and an axial vector.

However, if we consider $N$ to be the SU$_3$ octet, then we can make use of the fact that the interactions are SU$_3$ invariant. If $(4.3)$ is an octet component before switching on the strong interactions, it will also remain as such afterwards. We remember, however, that $<\bar{N}|\tau_+|N>$ is not the only way to construct that octet component. There is another combination: $(\bar{N}s_+N)$ where $s_+$ is the corresponding $8 \times 8$ matrix defined in (3.14). Hence, the axial part of the weak interaction will contain two parts with arbitrary components $f$ and $g$:

$$<\bar{N}|\sigma_\mu(f\tau_++gs_+)|N>$$

(4.4)

We know that $s_+$ connects $\Sigma$ with $\Lambda$ and has different signs for $p$-$n$ and $\Sigma^0$-$\Sigma^-$ transitions. Hence, for axial currents the bare rules are no longer valid; there will be $\Sigma$-$\Lambda$ transitions, and $p$-$n$ will not be equal to $\Sigma^0$-$\Sigma^-$. We thus get instead of (4.2) and (4.3) for dressed nucleons and zero momentum transfer:

$$G(\bar{N}|Y_\mu\tau_+|N) = \begin{cases} 0 & \text{for } \mu = 1,2,3 \\ G(\bar{N}\tau_+N) & \text{for } \mu = 0 \end{cases}$$

(4.5)

$$G(\bar{N}|Y_\mu Y_5\tau_+|N) = \begin{cases} G(\bar{N}|\sigma_\mu(f\tau_++gs_+)|N) & \mu = 1,2,3 \\ 0 & \mu = 0 \end{cases}$$
So far we have only dealt with weak transitions in which \( \Delta S = 0 \). This is clearly so if in (4.5) we use the operators \( \tau_+ \) and \( s_+ \) which only connect states of equal strangeness. In nature, we do find \( \Delta S \neq 0 \) transitions. They are very much weaker, however. We can incorporate this in our theory by changing it in the following way. Instead of using the operators \( \tau_+ \) and \( s_+ \), which are both the same octet component, we use another two operators \( \tau'_+ \) and \( s'_+ \) which are both a slightly different octet component, such that they involve a little transition of the \( \Delta S \neq 0 \) kind. This can be done by an admixture of \( \nu \) to \( \tau_+ \), and a corresponding admixture of the component of \( s \) which corresponds to \( \nu_- \) (it is \( s_+ \)). So we get

\[
\begin{align*}
\tau'_+ &= \cos \theta \tau_+ + \sin \theta \nu_- \\
s'_+ &= \cos \theta s_+ + \sin \theta s_-
\end{align*}
\]

\( \theta \) is the Cabibbo angle. We have turned the "direction" of the operators in the octet space away from the \( \tau_+ \) direction.

Our baryon weak current is then

\[
\begin{align*}
\mu = 0 \text{ component } & \quad G<\bar{N}|\cos \theta \tau'_+ + \sin \theta \nu_-=|N> \\
\mu \neq 0 \text{ components } & \quad G<\bar{N}q_1|f(\cos \theta \tau'_+ + \sin \theta \nu_-) + \\
& \quad + d(\cos \theta s'_+ + \sin \theta s_-)|N>
\end{align*}
\]

(4.6)

As was shown at the end of Chapter III, the matrices \( \nu_- \) and \( s_+ \) also change the charge by +1, just as \( \tau_+ \) and \( s_+ \) did. This is why one must use these matrices and no others in order to describe the \( \beta \) decay (and, of course, also their c.c. which are \( \tau_-, s_-, \nu_+, s_+ \)).

Let us be sure that (4.6) contains a lot of assumptions which may or may not be true. The most important is that the weak baryon current is a definite octet component, and that it is the same component for the vector and axial part. (The Cabibbo angle could be different in the two parts.) The only way to check these very natural and simple assumptions is to see whether the results fit the observations.
There are four unknown constants, \( G, f, g, \theta \) in (4.6). The values of \( G \cos \theta \) and \( G(f + d) \) are determined by the usual (p-n) \( \beta \) decay. \( G \cos \theta \) is the coupling constant of the vector part, and \( G(f + c) \cos \theta \) is the coupling constant of the axial part. One finds \( (f + d) \cos \theta = 1.15 \).

The value of \( d \cos \theta \) could be obtained from comparing \( \Sigma \rightarrow A \) decay with \( N \rightarrow P \) decay. The angle \( \theta \) could be obtained from comparing \( \Delta S = 0 \) transitions with \( \Delta S \neq 0 \) ones.

By such comparison we roughly get the values

\[
\frac{f}{d} = \frac{2}{3}, \quad \theta \sim 23^\circ.
\]

So far, these values seem to fit all known results.
V. SU\(_6\)

When one enlarges the groups SU\(_3\) to SU\(_6\) it turns out that one obtains even more surprising results. What does this mean? The SU\(_3\) results were obtained by assuming that the mechanics of elementary particles is almost invariant to the replacement of one quark state by another. There were assumed to be three quark states: a pair of \(I = \frac{1}{2}, S = 0\), and a third state, \(I = 0, S = -1\).

In SU\(_6\), we again assume that the mechanics of elementary particles is almost invariant to the replacement of one quark state by another. But we also take the ordinary spin into account. We assume that the quarks have in all three states a mechanical spin \(\sigma = \frac{1}{2}\). We then get six quark states, since each of the previous three states can be realized with mechanical spin up or down.

The assignment of \(\sigma = \frac{1}{2}\) to the quarks seems most natural. After all, most of the boson states have \(\sigma\) spin 0 or 1, as expected from two-quark systems, and the baryon states belonging to the lowest multiplets have \(\sigma\) spin \(\frac{1}{2}\) or \(\frac{3}{2}\), as expected from three-quark systems.

Let it be said, however, that the invariance in respect to those six states is a most curious assumption. There are two reasons for this: a) The mechanical spin is always coupled with orbital-angular momenta; therefore an invariance to spin alone is most abnormal. For example, we know that in the process \(p \rightarrow n + \pi\) there is a transfer of angular momentum to the \(\pi\) meson; the spin of the nucleon changes and is transformed into the orbital-angular momentum of the pion. Hence, the spin alone is far from being invariant. The second trouble is b): The mechanical spin \(\sigma\) is a non-relativistic concept. A relativistic particle has no definite spin, the so-called small components having an opposite spin to the spin of the large ones. Still, the assumption seems to work. Maybe -- with regard to (a) -- the strong coupling overshadows the importance of the \(p \rightarrow n + \pi\) process by many-pion emissions, and the spin of the nucleons remains on the average reasonably well conserved. Maybe -- with regard to (b) -- for some
reason the motions of the quarks, or what corresponds to them, are, in effect, non-relativistic, which would be surprising, since their binding energy ought to be of the order of their mass. As yet, the real meaning of SU$_6$ is not sufficiently understood.

1. Quark pairs in SU$_6$

We now start from six fundamental states $u_1 u_2 ... u_6$, where $u_1, ... u_3$ are the three SU$_3$ states with spin down, and $u_4, ... u_6$ with spin up.

Let us look at the quark-antiquark pairs: $u_i^a u_{\bar{k}}$. We have 36 states but we know that there is an invariant:

$$I = u_1^a u_i + u_2^a u_2 ... + u_6^a u_6.$$  \hspace{1cm} (5.6)

It corresponds to the singlet state in SU$_3$ and has $\sigma$-spin zero (it is invariant). The remaining states form a 35-plet. (This is in analogy to the octet in SU$_3$.) It is easy to see that this multiplet consists of an SU$_3$ octet with the two $\sigma$ spins combining to $\sigma = 0$ (8 states); an SU$_3$ octet with the two $\sigma$ spins combining to $\sigma = 1$ (3x8 states), and an SU$_3$ singlet with the two $\sigma$ spins combining to $\sigma = 1$ (3 states). The SU$_3$ singlet with $\sigma = 0$ is, of course, the invariant (5.6). We can, therefore, write for SU$_6$

$$\bar{q} q \rightarrow \text{singlet (}\sigma = 0\text{)} + 35\text{-plet}$$

$$35\text{-plet} = \text{octet (}\sigma = 0\text{)} + \text{octet (}\sigma = 1\text{)}$$ \hspace{1cm} (5.1)

$$+ \text{singlet (}\sigma = 1\text{)}.$$

This includes the most important meson multiplets:
Table 7

<table>
<thead>
<tr>
<th>Singlet</th>
<th>Singlet</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma = 0 )</td>
<td>( \sigma = 1 )</td>
</tr>
</tbody>
</table>

Octet

\( \mathcal{S} = 0 \)

\( \mathcal{C} = -1 \)

\( K^0 K^+ \)

\( K^{*0} K^{*+} \)

\( \pi^- \pi^0 \pi^+ \)

\( \rho^- \rho^0 \rho^+ \)

\( \eta \)

\( \omega \)

\( K^- K^0 \)

\( K^{*0} K^{*+} \)

Let us give an explicit example of the composition of some of these states. We use Table 1 on page 12 and remember the spins:

\[
\pi^-: \quad \frac{1}{\sqrt{2}} \left( \begin{array}{c} \uparrow \downarrow \\ \downarrow \uparrow \end{array} \right)
\]

\[
\pi^0: \quad \frac{1}{2} \left( \begin{array}{cc} \uparrow \downarrow & \downarrow \uparrow \\ \downarrow \uparrow & \uparrow \downarrow \end{array} \right)
\]

\[
\rho^-: \quad \frac{1}{2} \uparrow \downarrow, \quad \rho^0: \quad \frac{1}{\sqrt{2}} \left( \begin{array}{c} \uparrow \uparrow \\ \downarrow \uparrow \downarrow \uparrow \end{array} \right)
\]

The last two examples, of course, represent the \( m = 1 \) component of the \( \sigma = 1 \) state.

2. Generating operators for SU₆

We interpret again the 36 expressions \( u_i^a u_k \) as operators. In SU₆, the operator \( u_i^a u_k \) operates in the quark-sextet \( q_1 \ldots q_6 \) by transforming \( q_k \) into \( q_i \). These operators can be represented as 6x6 matrices with all elements zero but with one in place of \((i,k)\). As in SU₃, we use slightly different matrices. One of them is the 6x6 unity matrix. The others are [in analogy with \((1,18)\)] as follows:
\[
\begin{pmatrix}
\tau_i & 0 \\
0 & \tau_i
\end{pmatrix}
\quad i = 1 \ldots 8 \tag{5.2}
\]

\[
\begin{pmatrix}
\tau_i & 0 \\
0 & -\tau_i
\end{pmatrix}
\quad i = 1 \ldots 8 \tag{5.3}
\]

\[
\begin{pmatrix}
0 & \tau_i \\
0 & 0
\end{pmatrix}
\quad i = 1 \ldots 8 \tag{5.4}
\]

\[
\begin{pmatrix}
0 & 0 \\
\tau_i & 0
\end{pmatrix}
\quad i = 1 \ldots 8 \tag{5.5}
\]

\[
\begin{pmatrix}
0 & I \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
I & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}
\tag{5.6}
\]

Here, the \( \tau_i \ (i = 1 \ldots 8) \) are the \( 3 \times 3 \) matrices given in (1.18), \( I \) is the \( 3 \times 3 \) unity matrix. The operators (5.2) and (5.3) perform SU\(_3\) operations only, and do not change the \( \sigma \) spin. The operators (5.6) change only the \( \sigma \) spin; the operators (5.4), (5.5) change both \( \sigma \) spin and other spins (I,u,v).

The generating operators correspond, of course, to antibaryon-baryon pairs, and therefore one can ascribe each operator to a member of a 35-plet, that is to one of the mesons. It is obvious that the operators (5.2) correspond to (transform like) the \( (\sigma = 0) \) octet; the operators (5.3), (5.4) and (5.5) to the three substates of the \( (\sigma = 1) \) octet; the three operators (5.6) to the substates of the \( (\sigma = 1) \) singlet. The \( (\sigma = 0) \) singlet corresponds to the unity operator \( \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \).

At this point it is interesting to observe the reasons why the 35-plet of the \( \bar{q} q \) system is irreducible, that is, it does not split into separate octets and singlets with a given \( \sigma \) spin. We may ask which of the operators (5.2) - (5.6) would transfer a member of,
say, the octet with $\sigma = 0$ into a member of, say, the octet with $\sigma = 1$. The operators (5.6) would not since they are pure $\sigma$-spin operators, and, therefore, do not change the $\sigma$ spin. The operators (5.2),(5.3) are pure SU$_3$ operators and, therefore, stay within the octet. But the operators (5.4),(5.5) do the job. One might object by saying that they are products of $\tau$'s and $\sigma$'s (they are $\tau_1 \times \sigma_1$) and therefore cannot change $\sigma$ spin, but this is not so. They are products of operators acting on a single quark. When they act on the two-quark system $\bar{q}q$, one must really write them in this form:

$$ (\tau_1)_1 \times (\sigma_1)_1 + (\tau_1)_2 \times (\sigma_2)_2, \quad (5.7) $$

where $(\tau_1)_n$ is the operator acting on the $n^{th}$ quark, $n = 1,2$. The operators which would not change the $\sigma$ spin or would not lead out of the octet are:

$$(\sigma_1)_1 + (\sigma_2)_2 \text{ or } (\tau_1)_1 + (\tau_1)_2; \quad (5.7)$$

is not the product of those operators!

3. **Boson spectrum**

It is most astonishing to see that this 35-plet is, in fact, realized in nature among the observed bosons. We do find among the bosons the two octets with $\sigma = 0$ and $\sigma = 1$, and a singlet with $\sigma = 1$. This 35-plet comprises the most important boson states. The singlet ($\sigma = 0$) is not unambiguously identified. There are, of course, additional states which may belong to more complicated multiplets arising not from $\bar{q}q$ but from things like $\bar{q}q \bar{q}q$.

However, we have the spin-zero octet, the spin-one singlet, and the spin-one octet in one multiplet. This brings about an interesting situation between the spin-one singlet $\Phi_0$, and the $T = 0$ member of the spin-one octet $\omega_0$. They have the same quantum numbers ($\sigma$ spin, isospin). According to SU$_3$ they should also have the same energy if there is no SU$_3$ violating force present. We know that
the mass operator has a term $\Delta M$, which is not SU$_3$ invariant. This part may lead to a splitting and a mixing of the two states.

Let us call $\phi_0$ the spin-one singlet, and $\omega_0$ the T = 0 member of the spin-one octet before splitting; these are the states if SU$_3$ violating perturbation is present. Now, we introduce such a perturbation in the form of $\Delta M$ (see page 42) which is a scalar in respect to I spin but also, of course, to ordinary spin. It connects only states of equal I, equal $\sigma$, such as $\phi_0$ and $\omega_0$.

Between the states $\phi_0$, $\omega_0$, there exists a 2 x 2 matrix, $\Delta M$:

$$
\begin{pmatrix}
\phi_0 \\
\omega_0
\end{pmatrix}
\begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
\phi_0 \\
\omega_0
\end{pmatrix}
$$

We will now try to determine the matrix elements. If we diagonalize this matrix into

$$
\begin{pmatrix}
M_{\phi} & 0 \\
0 & M_{\omega}
\end{pmatrix}
$$

we get as diagonal elements the masses of the true $\phi$ and $\omega$. We also get the "mixing angle", that is given by the linear combinations

$$
\phi = a \phi_0 + b \omega_0 \quad |a|^2 + |b|^2 = 1
\omega = c \phi_0 + d \omega_0 \quad |c|^2 + |d|^2 = 1
a^*c + b^*d = 0
$$

The mixing angle $\eta$ is defined as

$$
\tan \eta = \frac{b}{a} = -\frac{c}{d}
$$

Simple matrix algebra tells us that

$$
\tan \eta = \frac{M_{22} - M_{\phi}}{M_{22} - M_{\omega}} = \frac{M_{11} - M_{\omega}}{M_{11} - M_{\phi}}.
$$

(5.8)
We can guess $\eta$ theoretically and we can calculate it empirically: we get the same result.

Let us first guess it: we look at the quark-antiquark definition of $\phi_0$ and $\omega_0$ (singlet and $T = 0$ octet members) [see line 8 from bottom of page 10, and formula (1.15)]

$$\phi_0 = (\bar{1}1 + \bar{2}2 + \bar{3}3) \frac{1}{\sqrt{3}}$$

$$\omega_0 = [\bar{1}1 + \bar{2}2 - 2 \times (\bar{3}3)] \frac{1}{\sqrt{6}}$$

Since, the mass operator separates in energy the third quark from the other two which remain degenerate ($T$-spin doublet), we would guess that the new states are:

$$\phi = (\bar{1}1 + \bar{2}2) \frac{1}{\sqrt{2}} = \sqrt{\frac{2}{3}} \phi_0 + \sqrt{\frac{1}{3}} \omega_0$$

$$\omega = \bar{3}3 = \sqrt{\frac{1}{3}} \phi_0 - \sqrt{\frac{2}{3}} \omega_0,$$

hence

$$\tan \eta = \sqrt{\frac{2}{3}} \quad (\eta = 35^\circ).$$

Now, the empirical determination: $M_{\omega^2}$, the hypothetical mass of $\omega_0$, is known from the Gell-Mann/Okubo mass formula (2.8), applied to the spin-one octet:

$$M_{\omega^2} = \frac{4}{3} m_K^2 - \frac{1}{3} m_p^2.$$

Now, we can use our experimental knowledge of $M_\omega$ and $M_\phi$ to find $\tan \eta$ from (5.8) and get also $1/\sqrt{2}$. Hence, the actually observed $\phi$ and $\omega$ meson are really linear combinations of the spin-one scalar $\phi_0$, and the spin-one octet component $\omega_0$, the mixing being caused by the SU$^c_3$ breaking-mass operator.
4. Quark triplets in $SU_3$

There are evidently $6 \times 6 \times 6$ quark triplet states, $q \times q \times q$, which is much more than the known number of baryon states. It seems, however, that the main baryon states all belong to one $SU_3$ multiplet, namely, the completely symmetric one. Here, nature was most kind and simple. In $SU_3$ the only fully symmetric multiplet is the decuplet.

What is the number of symmetric states one can get from three quarks, each quark having $n$ basic states? The answer is:

$$\text{Number of symmetric states } N = \frac{n(n+1)(n+2)}{6}.$$ 

Proof 1) There are $n$ states $iii$, obviously symmetric.
2) There are $n(n-1)$ states of the type "iik", that is, in the symmetrized form: $iik + ik + ki$. [Here, $n(n-1)$ is just the number of unequal pairs $i,k$, where $ik$ and $ki$ is counted separately, as should be done, since "iik" $\neq "kki".$]
3) There are $n(n-1)(n-2)$ ways of having ikt, where $i \neq k \neq t$. But a symmetric state is $ikt + kti + tit + tik$, hence, the above number must be divided by six. The number of symmetric states is therefore:

$$N = n + n(n-1) + n(n-1)(n-2)/6.$$ 

In our case $n = 6$, and $N = 56$. Instead of the decuplet ($N = 10$ for $n = 3$), we get a 56-plet in $SU_3$. Let us analyse what states it contains.

In $SU_3$, we had the following $qqq$ states:

1. Singlet, antisymmetric
2. Decuplet, symmetric
3. Two octets, mixed.

In $SU_3$, we also ascribe a spin to each quark. That means that the $SU_3$ states must be multiplied with the spin states of three quarks. This may help in creating states which are fully symmetric. The $SU_3$ singlet...
is out. One would need a fully antisymmetric state of the three spins in order to get a symmetric state. This is impossible.

The decuplet is easy. We put the three spins in a fully symmetric state \((\uparrow \uparrow \uparrow)\) which is a state of spin \(\frac{3}{2}\) with four substates. Hence, we get in our 56-plet a decuplet with spin \(\frac{3}{2}\), and 40 states are accounted for.

For example, the \(\Omega^-\) is (in the substate \(J_z = \frac{3}{2}\))

\[
\Omega_{J_z}^- = \frac{3}{2} \times 333
\]

meaning that all three quarks are in the state 3 with spin up. The \(S^*\) would be

\[
S_{J_z}^* = \frac{3}{2} \rightarrow \frac{1}{\sqrt{3}} \left( 331 + 313 + 133 \right).
\]

Later on we need the form of the \(N^{*0}\):

\[
N_{J_z}^{*0} = \frac{3}{2} \rightarrow \frac{1}{\sqrt{3}} \left( 112 + 121 + 211 \right).
\]

We will need that state in the \(J_z = -\frac{3}{2}\) substate:

\[
N_{J_z}^{*0} = \frac{3}{2} \rightarrow \frac{1}{\sqrt{3}} \left( 112 + 121 + 211 \right) \left( \downarrow \downarrow \uparrow + \downarrow \uparrow \downarrow + \uparrow \downarrow \downarrow \right) = \\
= \frac{1}{\sqrt{3}} \left( \downarrow \downarrow \uparrow + \downarrow \uparrow \downarrow + \uparrow \downarrow \downarrow \right) = \\
= \frac{1}{\sqrt{3}} \left( 112 + 112 + 112 \right) \\
+ \downarrow 121 + \downarrow 121 + \downarrow 121 \\
+ \downarrow 211 + \downarrow 211 + \downarrow 211 \\
(5.8a)
\]

We will also need \(S^*\) in the \(J_z = -\frac{3}{2}\) substate:
\[ S^z = -\frac{1}{2} \rightarrow \frac{1}{\sqrt{3}} \left( \begin{array}{c}
\uparrow \uparrow \uparrow \\
\uparrow \uparrow \downarrow \\
\uparrow \downarrow \downarrow \\
\downarrow \downarrow \uparrow \\
\downarrow \downarrow \downarrow \\
\downarrow \uparrow \downarrow \\
\downarrow \uparrow \uparrow \\
\downarrow \downarrow \uparrow \\
\downarrow \uparrow \uparrow \\
\downarrow \downarrow \downarrow \\
\downarrow \uparrow \downarrow \\
\downarrow \uparrow \uparrow \\
\downarrow \downarrow \uparrow \\
\downarrow \uparrow \uparrow \\
\downarrow \downarrow \downarrow \\
\downarrow \uparrow \downarrow \\
\downarrow \uparrow \uparrow \\
\downarrow \downarrow \downarrow \\
\downarrow \uparrow \downarrow \\
\downarrow \uparrow \uparrow \\
\downarrow \downarrow \downarrow \\
\downarrow \uparrow \downarrow \\
\downarrow \uparrow \uparrow \\
\downarrow \downarrow \downarrow \\
\end{array} \right) \]  
(5.16)

The consideration of the two SU\(_3\) octets takes more time.

We went in two steps when we analysed the q\(q\)q multiplets on SU\(_3\).

First we analysed the qq states and found \(3^+ 6\) (see page 14), that

is a triplet transforming like an antiquark, which was antisymmetric

in the two quarks, and a sextet that was symmetric. We then added

a third quark and got (see page 16):

\[ 3 \times q = \text{singlet (antis.)} + \text{octet} , \]  
(5.9)

\[ 6 \times q = \text{decuplet (sym)} + \text{octet} . \]  
(5.10)

The octet form (5.9) is antisymmetric in the first two quarks, the

second is symmetric. Let us call the first quark \(a\), the second \(b\),

the third \(c\). Then one can symbolize the octet from (5.9) by \((ab)c\)

and the octet from (5.10) by \([ab]c\), where ( ) signifies antisymmetry

and [ ] symmetry. These octets have no definite symmetry in respect

to the exchanges \(a \leftrightarrow c\) and \(b \leftrightarrow c\).

In order to clarify matters let us give examples taken from

the table on p. 18. Take the \((ab)c\) octet: its normalized \(\nu^+\) component

(transforming like the proton or the \(K^+\)) is, according to page 18:

\[ \{(ab)c\}_{\nu^+} = \frac{1}{\sqrt{2}} (12 - 21)2 \]  
(5.11)

[122 means quark \(a\) in state 1, \(b\) in state 2, \(c\) in state 2.] Its \(\sigma_0\)

component is:

\[ \{(ab)c\}_{\sigma_0} = \frac{1}{\sqrt{4}} \left( (32 - 23)1 - (13 - 31)2 \right) . \]  
(5.12)
The $\nu_+^0$ component of the $[(ab)c]$ octet is:

$$[[ab]c]_{\nu_+} = \frac{1}{\sqrt{5}} \left( 221 - \frac{1}{2}(12+21)2 \right). \quad (5.13)$$

The $\sigma_0^0$ component of this octet is:

$$[[ab]c]_{\sigma_0} = \frac{1}{\sqrt{5}} \left( (12+21)3 - \frac{1}{2}(31+13)2 - \frac{1}{2}(32+23)1 \right). \quad (5.14)$$

Those who want to look at details should watch how (5.13) is a combination of an isospin one triplet [components 11, $1\sqrt{5}/(21+12)$, 22] and an isospin $1/2$ doublet (components 1,2), combining to a state of isospin $1/2$.

According to (5.9) and (5.10), one could also have constructed states by starting with the first and the third quark, to which was added the second; or the second and the third to which was added the first. One would then have obtained the octets $(ca)b$, $[ca]b$ and $(bc)a$, $[bc]a$. Are they different from the octets $(ab)c$ and $[ab]c$? Yes, but they are linear combinations of those two octets. Let us write down three relations which indicate this:

$$\begin{align*}
(ab)c + (bc)a + (ca)b &= 0 \\
[ab]c + [bc]a + [ca]b &= 0 \\
[ab]c &= \frac{1}{\sqrt{5}} \{(ab)c - 2(ac)b\} .
\end{align*} \quad (5.15)$$

These relations can be checked with the examples (5.11) to (5.14). If they are correct for one octet component, they must also be correct for the others. The relation (5.16) shows that one can always express the octets which are symmetric in a pair by those which are antisymmetric in this and other pairs. The relations (5.15) show that the three pair-wise symmetric (or the three antisymmetric) octets are not linear independent. Hence, there are only two linear independent octets of this kind.
By adding the ordinary $\sigma$ spin it is now easy to construct fully symmetric octet states for SU$_3$. Take, for example, the SU$_3$ octet $(ab)c$ and put the quarks $a,b$ into a singlet spin state, and then put the $c$ spin up. We write this as follows:

\[
\begin{align*}
\uparrow \downarrow
\quad (ab)c .
\end{align*}
\]

Here, the $\sigma$-spin state is written above the quark symbol, and $\uparrow \downarrow$ means the singlet state, $1/\sqrt{2}(\downarrow \downarrow - \downarrow \uparrow)$. The state (5.17) is symmetric for the exchange of $a$ and $b$, but not in $c$. We can get an over-all symmetric state by writing:

\[
\text{Octet: } \frac{1}{\sqrt{3}} \left\{ (ab)c + a(bc) + b(ca) \right\} .
\]

There is a comparison state in which the third quark has the spin down. Together, they form a doublet with $\sigma$ spin $\frac{1}{2}$. Note that (5.18) does not vanish because of (5.15), because the quarks may now have different spins.

We could also get a fully symmetric octet by starting with $[ab]c$ and putting the quarks $a,b$ into a symmetric spin state, the triplet. Then we have to combine this triplet with the spin of $c$. It combines to $a$ a total $\sigma$ spin $\frac{3}{2}$ or $\frac{1}{2}$. Let us first do the combination to $\frac{3}{2}$ and look at the $m = \frac{3}{2}$ component of this quartet:

\[
\uparrow \uparrow \uparrow \uparrow
\quad [ab]c .
\]

This is symmetric in $a,b$. If we want to symmetrize it with respect to $c$, we would get zero because of (5.15). [Here, all spins are equal, and (5.15) can be applied.] We cannot get an all symmetric octet with $\sigma = \frac{3}{2}$.

We now do the combination to $\frac{1}{2}$ and look at the $m = \frac{1}{2}$ component:

\[
\sqrt{\frac{3}{2}} \quad (ab)c - \sqrt{\frac{1}{2}} [ab]c .
\]

\[5.19\]
Here, $\Uparrow$ stands for the $m = 0$ component of the triplet: $1/\sqrt{2}(\Uparrow \downarrow + \downarrow \uparrow)$. The linear combination $(\sqrt{2/3} \Uparrow \downarrow \downarrow - \sqrt{2/3} \Uparrow \downarrow \uparrow)$ is, of course, the way one combines a spin 1 and a spin $1/2$ to a spin $1/2$. Eq. (5.19) is symmetric in $ab$, but not in respect to $c$ and the other quarks. Hence, we must symmetrize:

$$\sqrt{2/3} \left[ \begin{array}{c} \uparrow \uparrow \downarrow \\ [ab]c + [bc]a + [ca]b \end{array} \right]$$

$$- \sqrt{2/3} \left[ \begin{array}{c} \uparrow \uparrow \downarrow \\ [ab]c + [bc]a + [ca]b \end{array} \right].$$

Equation (5.20)

This is again a fully symmetric octet which does not vanish on the basis of (5.15) because the quarks have different spin. It is, however, identical with (5.18), as a detailed comparison shows. So, we get only one fully symmetric octet in SU$_6$ as a $\sigma$ spin $1/2$ doublet. This gives us the residual 16 states, in order to make up the 56-plet:

$$56\text{-plet} = \begin{cases} \text{Decuplet } \sigma = \frac{3}{2} & \text{40 states} \\ \text{Octet } \sigma = \frac{1}{2} & \text{16 states} \end{cases}$$

Hence, SU$_6$ gives all the basic baryon states as members of a 56-multiplet, and it also determines the spin of these states as they were found in nature.

### Table 3

<table>
<thead>
<tr>
<th>Octet $\sigma = \frac{1}{2}$</th>
<th>Decuplet $\sigma = \frac{3}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^-$, $\Sigma^-$, $\Xi^-$</td>
<td>$N^*$, $N^{<em>0}$, $N^+$, $N^{</em>+}$</td>
</tr>
<tr>
<td>$\Lambda$, $\Sigma^0$, $\Xi^0$</td>
<td>$\Sigma^<em>$, $\Xi^</em>$</td>
</tr>
<tr>
<td>$\Omega$</td>
<td></td>
</tr>
</tbody>
</table>
It is useful to study the octet states in more detail. Let us first look at the $\nu_+^c$ component. We use (5.11) in order to write down the $\nu_+^c$ component of (5.17):

\[
\begin{align*}
\left( \begin{array}{c}
\uparrow \\
(ab)^C \\
\downarrow \\
\end{array} \right)_{\nu_+^c} &= \frac{1}{\sqrt{4}} \left( \begin{array}{c}
\uparrow \\
12 - 12 - (21 - 21) \\
\end{array} \right) \end{align*}
\]

(5.21)

The $\nu_+^c$ component of $a(b^c)$ would be:

\[
\begin{align*}
\left( \begin{array}{c}
\uparrow \uparrow \\
(a^b)^c \end{array} \right)_{\nu_+^c} &= \frac{1}{\sqrt{4}} \left[ \begin{array}{c}
\uparrow \uparrow \\
2 \uparrow 12 - 12 - (21 - 21) \end{array} \right].
\end{align*}
\]

(5.22)

(all we do is to change the order of symbols, having $\uparrow$ first.) The $\nu_+^c$ component of $b(c^a)$ would be (here we have $2$ in the middle):

\[
\begin{align*}
\left( \begin{array}{c}
\uparrow \uparrow \downarrow \\
b(c^a) \end{array} \right)_{\nu_+^c} &= \frac{1}{\sqrt{4}} \left[ \begin{array}{c}
\downarrow \uparrow \uparrow \\
221 - 221 - (122 - 122) \end{array} \right].
\end{align*}
\]

(5.23)

Adding it all together and normalizing we get [watch that some of the terms are identical, such as the second in (5.21) and the third in (5.23)]

\[
\left[0\right]_{\nu_+^c} = \text{Proton} = \frac{1}{\sqrt{18}} \left[ \begin{array}{c}
\uparrow \uparrow \downarrow \\
122 + 212 + 221 \\
\end{array} \right],
\]

\[
\begin{align*}
\uparrow \uparrow \downarrow - 122 - 222 - 212 \\
\end{align*}
\]

(5.24)
The symbol \([0]\) means fully symmetric octet. By the substitutions which are below, we get the following other components from (5.24):

\[
[0]_{\nu_0} = \text{Neutron}: \quad 1 \rightarrow 2 \quad \quad [0]_{\xi_0} = \Sigma^0: \quad 2 \rightarrow 3 \\
[0]_{\sigma^-} = \Sigma^-: \quad 2 \rightarrow 1 \quad \quad [0]_{\xi^-} = \Xi^-: \quad 2 \rightarrow 3 \\
[0]_{\sigma^+} = \Sigma^+: \quad 1 \rightarrow 3
\]

The \([0]_{\sigma_0}\) and \([0]_{\lambda}\) components are more complicated. They can be calculated by similar methods from (5.12) and corresponding expressions.

5. Magnetic moments

The magnetic moments of baryons and mesons are determined in SU\(_6\), apart from one constant only. This is the great advantage of SU\(_6\) and one of its most striking successes is the correct prediction of the ratio of the proton and neutron moment.

It is obvious that SU\(_6\) has to say more about magnetic moments than SU\(_3\), since it includes statements about the direction of the quark spin. We start with the quark triplet itself. According to the conclusions on page 26 (see 2.6a), we expect the sum of the moments in the triplet to be zero. Let us call \(\mu_1\) the magnetic moment of a quark in the state \(u_1\); we get

\[
\mu_1 + \mu_2 + \mu_3 = 0.
\]

Since the electromagnetic properties are scalar in the u spin, we also get (\(u_1^z = -\frac{1}{2}\) in state 3, \(u_2^z = +\frac{1}{2}\) in state 1)

\[
\mu_1 = \mu_3.
\]

It follows from this that

\[
\mu_1 = \mu_3 = -\zeta, \quad \mu_2 = 2\zeta, \quad (5.26)
\]
where \( C \) is an unknown value of magnetic moment. As expected, one finds here the magnetic moment of the quark proportional to its "charge". In fact, we have (see p. 22)

\[
\mu_i = 3Cq_i \quad .
\]

(5.27)

The magnetic moment of baryons and bosons can now be determined very simply from the quark hypothesis. The results are more general than the quark hypothesis: they result directly from the transformation properties. We make the assumption that the quarks are always bound in \( S \) states and that their motion is non-relativistic. The total magnetic moment is then just the vector sum of the quark moments.

We deal first with the bosons. Clearly, the \( \sigma = 0 \) bosons have no magnetic moment. In the \( (\sigma = 1) \) octet, the spins are parallel; the magnetic moments add. (Antiquarks have, of course, the opposite spin and charge.) Because of the relation (5.27), we get simply:

\[
\mu_\alpha = 3Cq_\alpha \quad \alpha = 1 \ldots 8, \quad (\sigma = 1) \text{ octet } \\
\mu_\alpha = 0 \quad \begin{cases} (\sigma = 0) \text{ octet} \\ (\sigma = 1) \text{ singlet} \end{cases}.
\]

(5.28)

We now deal with the baryons. The simplest case is the decuplet members. Here, the spins are also parallel; hence the magnetic moments of the quarks add as do the charges. We get again

\[
\mu_\alpha = 3Cq_\alpha \quad \alpha = 1 \ldots 10, \quad (\sigma = 3/2) \text{ decuplet}.
\]

(5.29)

The octet presents a different situation because of the fact that the three-quark spins are not parallel here. However, we can take advantage of the fact that the octet states belong to an all symmetric 56-plet. Let us look at the "outside" states of the octet, \( n, p, \Xi^+, \Xi^0, \Xi^- \) and their quark composition (which is defined
by charge and strangeness). We understand, by "composition", the combination of SU₃-quark states 1,2,3 which make up the particle:

\[ n \rightarrow 112, \quad \Sigma^- \rightarrow 113, \quad \Xi^0 \rightarrow 332 \]
\[ p \rightarrow 221, \quad 
\]

(5.30)

We see that in each case two quarks are in the same SU₃ state. Since the system should be symmetric, it is necessary that these two quarks must be in a triplet-spin state (σ = 1). Only then is the system symmetric for an exchange of these two quarks (triplet is symmetric, identical SU₃-quark state is also symmetric). All six particles in (5.30) have σ spin \( \frac{1}{2} \). Hence, the third quark (σ = \( \frac{1}{2} \)), together with the triplet (σ = 1), add up to a total of σ = \( \frac{3}{2} \).

If we have two entities: A with σ = 1 and B with σ = \( \frac{1}{2} \), adding up to a total \((AB)\frac{1}{2}\) with spin \( \frac{1}{2} \), the magnetic moment of the total is:

\[ \mu_{(AB)\frac{1}{2}} = \frac{1}{2} \mu_A - \frac{1}{2} \mu_B \].

(5.31)

Here, A is the pair of quarks in the same SU₃ state, with parallel spin (triplet), and B is the third quark. Looking at (5.30) and (5.26), we get

\[ n \rightarrow \begin{cases} \mu_A = -2C, \\ \mu_B = +2C \end{cases}, \quad \Sigma^- \rightarrow \begin{cases} \mu_A = -2C, \\ \mu_B = -C \end{cases}, \quad \Xi^0 \rightarrow \begin{cases} \mu_A = -2C, \\ \mu_B = +2C \end{cases} \]

\[ p \rightarrow \begin{cases} \mu_A = +4C, \\ \mu_B = -C \end{cases}, \quad \Sigma^+ \rightarrow \begin{cases} \mu_A = +4C, \\ \mu_B = -C \end{cases}, \quad \Xi^- \rightarrow \begin{cases} \mu_A = -2C, \\ \mu_B = -C \end{cases} \]

By inserting this into (5.31) we get

\[ \mu_n = -2C, \quad \mu_{\Sigma^-} = -C, \quad \mu_{\Xi^0} = -2C \]
\[ \mu_p = +3C, \quad \mu_{\Sigma^+} = +3C, \quad \mu_{\Xi^-} = -C \].

(5.32)
This gives us directly the famous ratio \(-\frac{2}{3}\) for \(\mu_n/\mu_p\). The magnetic moments of \(\Sigma_c\) and \(A\) can be deduced from (2.1) and (2.4) with the following result:

\[
\mu_{\Sigma_c} = C, \quad \mu_A = -C.
\] (5.33)

Let us add that the constant \(C\) need not be the same for bosons and baryons. One can see this even in a quark model: the effective magnetic moment of the quark may depend on its kinetic energy, which might be different in the \((q\bar{q})\) system and in the \((qqq)\) system.

Derivation of formula (5.31): entity \(A\) exists in three substates: \(A_+^1, A_0^1, A_-^1\); entity \(B\) in two substates: \(B_{1/2}, B_{-1/2}\).

Clebsch-Gordan gives:

\[
(AB)_{\sigma=1/2, \sigma_x=1/2} = \sqrt{3/2} A_{+1} B_{1/2} - \sqrt{3/2} A_{0} B_{1/2}.
\]

Hence, we have the magnetic moment of "\(A\) up and \(B\) down" with probability \(\frac{2}{3}\), and "\(A\) perpendicular (zero) and \(B\) up" with probability \(\frac{1}{3}\). This leads to (5.31).

6. Mass formula in \(SU_6\)

The situation with regard to the mass formula is not as simple in \(SU_6\) as it was in \(SU_3\). Remember (see page 42) that in \(SU_3\) the mass operator was assumed to transform like the 8-component of an octet; that is, like one of the eight generating operators of the group, namely \(\tau_8\). This gave a two-constant mass formula (5.20) for the octets and a one-constant mass formula (proportional to hypercharge) for the decuplet, because of the latter's symmetry in the three quarks. If we again assume that the mass operator \(\Delta m\) transforms like one of the generating operators, we will get a one-constant mass formula for the 56-plet; that is, for both octets and decuplets. This is because of the fact that in \(SU_6\) the 56-plet...
assumes the role of the decuplet in $SU_3$: it is fully symmetric. The only bilinear combination which transforms like $\tau_8$ would be $(\bar{A}\tau_8 A)$; this would give a mass formula with $\Delta m$ proportional to the hypercharge, which is false for the octets. The additional form $Q_8$ in (3.17) does not exist in $SU_6$, which means that $Q_8$ would not transform like $\tau_8$ in the wider group $SU_6$.

We must try, therefore, a more complicated mass formula. Expression (3.19) suggests terms which are quadratic in the spins (generating operators). It is not unusual that mass splits should transform like bilinear expressions in spins. For example, in nuclear structure, Wigner has assumed that the forces are in first order spin independent (iso and angular). He then obtains supermultiplets which are split by perturbations of the form $(\vec{\tau}_1 \cdot \vec{\tau}_k)$ or $(\vec{\tau}_1 \cdot \vec{\tau}_k)$ where the indices refer to different particles.

What mass formula should we get? For baryons in the 56-plet the following three-constant formula works:

$$\Delta m = aY + b[I(I+1) - \frac{1}{4} Y^2] + c \cdot s(s+1). \quad (5.34)$$

Here, $a, b, c$ are constants and $s$ is the angular spin. The third term guarantees the split between the centre of mass of octet and decuplet. It is remarkable that this formula works both for the octet and the decuplet with the same values of $a$ and $b$. In spite of the quadratic appearance of $Y$ in the second term, this formula again gives equidistant levels for the decuplet.

Can we find an operator whose expectation value gives the mass formula (5.34)? Since the operator $\tau_8$ has the eigenvalues $Y$, we can write the mass operator in the following form:

$$(\Delta m)_{\text{op}} = a\tau_8 + b \left[ (I)_{\text{op}}^2 - \frac{1}{4} Y^2 \right] + c \cdot \vec{\sigma}^2. \quad (5.35)$$

Here, $(I)_{\text{op}}^2$ is the operator of the square of the isotopic spin which has the eigenvalues $I(I+1)$; $\vec{\sigma}$ is the angular spin operator.
Equation (5.35) is a perfectly reasonable mass operator containing terms linear and quadratic in the generating operators of SU$_6$, and it leads to the mass formula (5.34) when used in first approximation as small perturbation. But it has an interesting feature: the ratio 1 to 4 in the coefficients of $\tau_8^2$ and $(1)_{0p}^2$ is arbitrary and unexplained. Nevertheless, that ratio is most important since it is essential for equidistant levels in the decuplet. Any other ratio would give non-equidistant levels for the decuplet. Hence, this equidistence is not explained but arbitrarily put into our mass operator. So much for baryons.

We now turn to the mesons. In this case the antiparticles are contained in the multiplets. Hence, the members with opposite $Y$ must have the same mass: formula (5.34) can be valid only with $a = 0$. It is remarkable that the formula

$$\Delta(m^2) = b' [I(I+1) - \frac{1}{4} Y^2] + c' \cdot a(s+1) \quad (5.34a)$$

does in fact work well for the two octets. This formula can also be written in terms of operators, bilinear in the generating operators of SU$_6$.

7. Baryon-meson interaction

Remember that in SU$_3$ we found that there are two constants in the interaction between the members of the baryon octet and the meson octet. The reason was that one can construct two bilinear forms of baryons, $P_1$ and $Q_1$ (formulae (3.13) and (3.14)), which transform like an octet member. How is the corresponding situation in SU$_6$? Now, the mesons form a 35-plet, and the question arises: how can one form a bilinear form of 56-plet members which transforms like a 35-plet member? Let us call the 56 baryons $A_\alpha$, $\alpha = 1 \ldots 56$, and the 35 mesons $\phi_\beta$, $\beta = 1 \ldots 35$. Then the bilinear form would have the form

$$P_\beta = \sum_{\gamma} \sum_{\alpha = 56} A_\alpha t_\beta^{\alpha \gamma} A_\gamma \quad (5.36)$$
Since the 56-plet and the 35-plet in $SU_6$ play the same role as the decuplet and the octet in $SU_3$, the question is related to the following $SU_3$ question: how can one construct a bilinear form of decuplet members, which transforms like an octet? Because of the full symmetry of the decuplet in the three quarks, the answer is that one can do it only in one way, as was already discussed in Chapter 3 near formula (3.23). The same is true for $SU_6$: there is only one expression (5.36) which transforms like a 35-plet.

It is easy to write down a combination $P_{\beta}$ which has this property and that will then be the only one. It is

$$P_{\beta} = \sum_{\alpha} \bar{A}_\alpha (\tau_{\beta}^{\dagger} A_{\alpha}),$$  \hspace{1cm} (5.37)

where $\tau_{\beta}^{\dagger}$ is one of the 35 generating operators of $SU_6$. Clearly, then $P_{\beta}$ transforms like $\tau_{\beta}^{\dagger}$ since $\bar{A}_\alpha \tau_{\beta}^{\dagger} A_{\alpha}$ is an invariant. As the operation of a generating operator on an octet was defined in (3.15c), $\tau_{\beta} \alpha$ is defined here in a similar way:

$$\tau_{\beta}^{\dagger} A_{\alpha} = \tau_{\beta}^{\dagger} A_{\alpha} = \sum_n (\tau_{\beta}^{\dagger})^n A_{\alpha} + \sum_n (\tau_{\beta}^{\dagger})^n A_{\alpha}$$

$$+ \sum_n (\tau_{\beta}^{\dagger})^n A_{\alpha}.$$  \hspace{1cm} (5.38)

Here, $A_{\alpha}$ is the three-quark representation of $A_{\alpha}$ and $(\tau_{\beta}^{\dagger})^n$, $k,i = 1 \ldots 6; \beta = 1 \ldots 35$ is the matrix form of the generating operators operating upon the fundamental sextet:

$$\tau_{\beta}^{\dagger} u_i = \sum_k (\tau_{\beta}^{\dagger})^k u_k$$

$$\tau_{\beta}^{\dagger} u_i = \sum_k (\tau_{\beta}^{\dagger})^k u_k, \hspace{1cm} i,k = 1 \ldots 6; \beta = 1 \ldots 35.$$  \hspace{1cm} (5.39)

In other words, in (5.38) $\tau_{\beta}$ acts on each of the three quarks:
\[ \tau_{\beta} = \tau_{\beta}^{(1)} + \tau_{\beta}^{(2)} + \tau_{\beta}^{(3)} , \quad (5.39a) \]

where \( \tau_{\beta}^{(i)} \) is an operator working on quark \( N^0 \) i.

It would then seem natural to write the baryon-meson interaction in the form:

\[ H = \sum_{\alpha \beta} \left( \vec{A}_\alpha \gamma_5 (\tau_{\beta} \vec{A}_\alpha) \right) \phi_\beta \quad \alpha = 1 \ldots 56 \quad \beta = 1 \ldots 35 . \quad (5.40) \]

The terms in this sum, which determine the interaction with pseudoscalar mesons, would be the first eight terms in the sum over \( \beta \). They contain \( \tau_1 \ldots \tau_8 \) which, in the sense of (5.39a), are made up of the eight generating operators (5.2), which are the generating operators of \( SU_3 \), only repeated for each spin direction. Hence, for the coupling with pseudoscalar mesons we get the same form as in (3.13). That means that \( SU_3 \) gives us \( f = 1 \) and \( d = 0 \) in the terminology of (3.12), which is in disagreement with experiments.

It is no surprise that the interaction (5.40) does not work well. We know that pseudoscalar mesons are emitted in \( p \) states, and that they are coupled with a spin operator of the baryons. As was mentioned at the beginning of this chapter, it is in contradiction to \( SU_3 \), because that group does not allow the transformation of spin into orbital momentum.

There is a way in which one can cheat so that one gets the correct result. One can say that since the meson field must be coupled with the spin operator, the baryon operator in (5.40) should not be \( \tau_1 \ldots \tau_8 \) but \( \vec{\sigma} \tau_1 \ldots \vec{\sigma} \tau_8 \). This is again meant in the sense of (5.39a):

\[ \vec{\sigma} \tau_{\beta} = \vec{\sigma}^{(1)} \tau_{\beta}^{(1)} + \vec{\sigma}^{(2)} \tau_{\beta}^{(2)} + \vec{\sigma}^{(3)} \tau_{\beta}^{(3)} , \quad (5.41) \]
where the $\tau^{(i)}_\beta$ is an operator working on quark $N^0$ i in the baryon ($i = 1, 2, 3$). This operator is one of the $3 \times 8$ operators (5.3), (5.4), (5.6) [three components of $\bar{\tau}$, and $\beta = 1 \ldots 8$]. These 24 operators are thus generating operators of SU$_6$.

For the coupling with scalar mesons ($\beta = 1 \ldots 8$), we would then get instead of (5.40):

$$ P_\beta = \sum_{\alpha\beta} \bar{A}_\alpha (\bar{\tau}_\beta) A_\alpha \bar{\psi}_\beta, \quad \alpha = 1 \ldots 56, \quad \beta = 1 \ldots 8. \quad (5.41a) $$

Let us be sure that this is cheating: (5.41a) is not SU$_6$ invariant; $\bar{\psi}_\beta$ transforms differently from $\bar{\tau}_\beta$. The $\psi$ is not considered in SU$_6$. The physical assumption made in (5.41a) is this: each quark in the baryon is coupled to the pseudoscalar meson field $\bar{\psi}_\beta$ by

$$ \left( \psi^*_1 \tau^{(i)}_\beta \psi^*_1 \right) \bar{\psi}_\beta, \quad (5.41b) $$

where $\psi_1$ is the state of the quark $N^0$ i. This interaction is not SU$_6$ invariant.

In the interaction (5.40), the operators $\bar{\tau}_\beta$, $\beta = 1 \ldots 8$ ($\tau_\beta$ being SU$_3$ operators) would be the ones which are coupled to the vector mesons. The cheating proposed here is to assume that the coupling to the pseudoscalar meson-octet is the same which (5.40) would give for the vector-meson octet.

Let us see what result we get. In spite of the cheating, the result is of interest because we can use it without cheating in the SU$_6$ treatment of weak interactions (see next section). First remark that the operator (5.41) is not $\bar{\sigma}_{\text{bar}} (\tau)_{\text{bar}}$ where

$$ \bar{\sigma}_{\text{bar}} = \bar{\sigma}^{(1)} + \bar{\sigma}^{(2)} + \bar{\sigma}^{(3)} \quad \tau_{\text{bar}} = \tau^{(1)} + \tau^{(2)} + \tau^{(3)}, \quad (5.42) $$

65/1280/10
/p/cm
where the superscript \( \text{bar} \) means the total \( \sigma \) or \( \tau \) for the baryon in a three-quark system. The operator (5.39a), however, is \( \tau_\beta^{(\text{bar})} \). This remark is similar to the one made on account of formula (5.7).

Let us apply one of the operators (5.41), namely, \( \sigma_- \tau_2 \) on the proton. Remember that \( \sigma_- \) is the operator which lowers the angular spin, and \( \tau_2 \) is the isotopic spin operator \( \tau_- \) which changes a proton into a neutron (see 1.13). Let us express the proton in the form (5.24), and apply the operator \( \sigma_- \tau_2 \) on it. It transforms any \( \uparrow \) into \( \uparrow \). Hence, we get: \( |P> \) stands for proton

\[
(\sigma_- \tau_2)|P> = \frac{1}{\sqrt{18}} \left\{ \begin{array}{c}
\uparrow\uparrow\uparrow \\
\downarrow\uparrow \uparrow
\end{array} \right\} \begin{array}{c}
121 + 112 + 211
\end{array}
\]

\[
\uparrow\downarrow \downarrow \downarrow
-112 - 112 - 121
\]

\[
\uparrow\uparrow\uparrow
- 121 - 211 - 211
\].

Compare this with (5.24) and (5.8a), then you see that

\[
(\sigma_- \tau_2)|P> = \frac{5}{3} |\bar{N}>, + \frac{\sqrt{2}}{3} |N^{*0}>
\]  

(5.43)

Here, \( N^{*0} \) is one of the decuplet members. (The arrows above \( P, N, N^{*0} \)

mean \( \sigma_z = +\frac{1}{2} \) or \( -\frac{1}{2} \)). A similar calculation gives [use (5.25) and (5.8b)]:

\[
(\sigma_- \tau_2)|S^0> = \frac{1}{3} |S>, + \frac{\sqrt{2}}{3} |S^{*-}>
\]  

(5.44)

Here \( S^{*-} \) is a decuplet member.

It is good to keep in mind that

\[
\sigma_-^{(\text{bar})} \tau_2^{(\text{bar})} |P> = |\bar{N}>
\]

and

\[
\sigma_-^{(\text{bar})} \tau_2^{(\text{bar})} |S^0> = |\frac{1}{2}>
\]
Let us now see what we get when we compare the baryon-
reson interaction (5.41c) with the one we got in SU1 in formula
(3.15d). We will see that now the ratio \( f/d \) is determined.
The easiest way to see this is as follows. First, we realize that
(5.41c) must be equivalent to (3.15d) with a given value of \( f/d \),
since \( SU_3 \) is contained in \( SU_6 \). We can find \( f/d \) by equating (3.15d)
with (5.41c), which gives the operator equation:

\[
(f t_+ + d s_-) \sigma_{(\text{bar})} = (\sigma_{- \tau_2}) \quad (5.46)
\]

Let us calculate the matrix elements of this operator between \( N \) and
\( P \), and also between \( \Sigma^- \) and \( \Sigma^0 \). We remember (see figure on page 40)
that the matrix \( t_- \) connects \( P \to N \) and \( \Sigma^0 \to \Sigma^- \) with opposite phase:

\[
t_- |P> = |N>, \quad t_- |\Sigma^0> = -|\Sigma^->,
\]

whereas the matrix \( s_- \) connects them with equal phase:

\[
s_- |P> = |N>, \quad s_- |\Sigma^0> = +|\Sigma^->.
\]

From this we conclude [remember, the operator \( \sigma_{(\text{bar})} \) switches the
spin of the baryon from \( \uparrow \) to \( \downarrow \)]:

\[
\begin{align*}
\downarrow \left< N | (f t_+ + d s_-) \sigma_{(\text{bar})} \right| \uparrow P & = f + d \\
\downarrow \left< \Sigma^- | (f t_+ + d s_-) \sigma_{(\text{bar})} \right| \Sigma^0 & = -f + d.
\end{align*}
\quad (5.47)
\]

According to (5.43) and (5.44), the operator \( \sigma_{- \tau_2} \) fulfills the
relations:

\[
\begin{align*}
\downarrow \left< N | \sigma_{- \tau_2} \right| \uparrow P & = \frac{2}{3}, \quad \downarrow \left< \Sigma^- | \sigma_{- \tau_2} \right| \Sigma^0 = \frac{1}{3}.
\end{align*}
\quad (5.48)
\]

The matrix elements (5.47) and (4.48) must fulfill relation (5.46).
This gives

\[
\begin{align*}
f + d = \frac{2}{3}, \quad \frac{f}{d} = \frac{2}{3}.
\end{align*}
\]

\[65/1280/10 \quad \text{p/cm}\]
Hence, the interaction (5.41c) is equivalent to a ratio $f/a = \frac{3}{2}$ in the form (3.15d), which is in agreement with experiments.

8. Weak interactions

SU$_6$ is a much better group for weak interactions than SU$_3$. Let us look back at the situation in SU$_3$ as described in Chapter IV. The vector interaction current is given by (4.2) for strangeness non-changing transitions (Cabibbo angle 0), and by (4.6) (first line) for finite Cabibbo angle. The operators $\tau_+$ and $\sigma_+$ have to be understood as:

$$\tau_+ = \tau_+^{(1)} + \tau_+^{(2)} + \tau_+^{(3)}, \quad \sigma_+ = \sigma_+^{(1)} + \sigma_+^{(2)} + \sigma_+^{(3)}, \quad (5.49)$$

where $\tau_+^{(i)}$, $\sigma_+^{(i)}$ are the $(3 \times 3)$ matrices acting on quark $N^0 = 1$ within the baryon. Since both $\tau_+$ and $\sigma_+$ are generating operators of SU$_3$, the vector interaction matrix elements (4.2) and (4.6) (first line) are of the same form for "bare" and "dressed" nucleons. The reason is that the strong interactions do the "dressing".

The axial-vector interaction current for bare nucleons is given by (4.3) (\(\Delta S = 0\)) and by the second line in (4.6) for \(\Delta S \neq 0\), by putting $f = 1$, $d = 0$.

The operators have to be understood as:

$$\sigma_+^{(1)} \tau_+ = \sigma_+^{(1)} \tau_+^{(1)} + \sigma_+^{(2)} \tau_+^{(2)} + \sigma_+^{(3)} \tau_+^{(3)}$$

$$\sigma_+^{(1)} \sigma_+ = \sigma_+^{(1)} \sigma_+^{(1)} + \sigma_+^{(2)} \sigma_+^{(2)} + \sigma_+^{(3)} \sigma_+^{(3)}. \quad (5.50)$$

In contrast to (5.49) this operator is not a generating operator of SU$_3$, which is why the strong interactions change these matrix elements. All we can claim is that it must be of the same transformation properties as $\tau_+$, and this gives rise to (4.4) for \(\Delta S = 0\) or to (4.6) (second line) for \(\Delta S \neq 0\). The ratio $f/a$ is undetermined.
In \(SU_6\), however, there is an essential advantage: Eq. (5.50) is a generating operator of \(SU_6\). This is why the strong interactions leave the matrix elements of the operator (5.50) unchanged. Therefore, for the axial vector current matrix element, we get, instead of (4.6) (second line):

\[
G < \bar{N} | \cos \theta \sigma_\mu^+ + \sin \theta \sigma_\mu^- | N > , \tag{5.51}
\]

where \(\sigma_\mu^+\) and \(\sigma_\mu^-\) are understood in the sense of (5.50).

Now, from the last section we know that

\[
< \bar{N} | \sigma_\mu^+ | N > = < \bar{N} | (t^+ + s^+) \sigma_\mu^{(\text{bar})} | N >
\]

with \(f/d = \frac{3}{2}\). Here, \(t^+\) and \(s^+\) are the \(8 \times 8\) matrices corresponding to the operators \(\sigma^+\) and \(s^+\) as used in (4.6). That means that in \(SU_6\) the ratio \(f/d\) in (4.6) is well determined and is the same \(\frac{3}{2}\) as in the baryon-meson interaction.

\(SU_6\) covariance also tells us something about the relative strength of the vector and axial interaction. As indicated in (5.51), the constant \(G\) will be the same for the vector current (4.6) (first line) and the axial vector current (5.51). We assume that for the bare nucleon the two matrix elements have the same constant, and the "dressing" will not change the ratio, because of \(SU_6\) invariance. (Both are matrix elements of a generating operator.) Hence, the two interaction current matrix elements can be written:

\[
G < \bar{N} | \cos \theta \tau^+ + \sin \theta \nu^+ | N > , \tag{5.52}
\]

\(\text{(for the vector interaction)}\)

\[
G < \bar{N} | \cos \theta \sigma_\mu^+ + \sin \theta \sigma_\mu^- | N > \tag{for the axial vector interaction}
\]

In this form both interactions have the same constant. However, that does not mean that they have the same constant in the form which is commonly used and which contains operators such as \(\sigma^{(\text{bar})}\) and
and \( \tau^{\text{bar}} \). Take as an example the interaction matrix elements between the neutron and the proton. From (5.43) and (5.45) we know the effect of these operators on the neutron and proton. In fact, we can write (5.43) and (5.45) as follows:

\[
\downarrow N|\sigma^+_{\tau}P> = \gamma_5 \downarrow N|\sigma^\text{bar}_+\tau_{\text{bar}}^+P> ; \tag{5.53}
\]

a similar expression holds for \( \sigma^+_{\tau} \). (N and P exchanged.) From (5.39a) and (5.42) we get the obvious relation: [it is a definition of \( \tau^{\text{bar}} \)]

\[
\uparrow N|\tau^-_P = \downarrow N|\tau^{\text{bar}}_+P> \tag{5.54}
\]

and the inverse for \( \tau^+_P \). (N and P exchanged!) Hence, if \( \mathcal{N} \) stands for proton or neutron, we can write (5.52) in the form

\[
\begin{align*}
\mathcal{E} \downarrow \mathcal{N} & |\cos \theta \tau^{\text{bar}}_+ + \sin \theta \nu^+ \mathcal{N} > \rightarrow (\text{vector interaction}) \\
\epsilon \downarrow \mathcal{N} & |\cos \theta \sigma^{\text{bar}}_{\mu} \tau^{\text{bar}}_+ + \sin \theta \sigma_{\nu}^{\text{bar}} \nu^{\text{bar}} \mathcal{N} > \rightarrow (\text{axial vector}).
\end{align*}
\]

Because of (5.53) and (5.54) we get \( \epsilon = \gamma_5 \). If \( \mathcal{N} \) stands for the \( \mathbb{Z} \) pair, \( \epsilon = \frac{1}{2} \). Experimentally \( \epsilon = 1.15 \) for protons and neutrons. Here, the numerical agreement is poor.

The essence of what is going on here is this: the weak interaction currents are for the quarks and not for the baryons. These currents are simple only for the quarks: they are generating operators of \( SU_6 \) (\( G \cdot \tau^+ \), \( G \cdot \tau^+ \sigma^+ \)) with equal constants for vector and axial. (Renormalization does not impair this equality because of \( SU_6 \) invariance of strong interactions.) For systems of several quarks, such as baryons, the situation is well-defined but somewhat different, as shown in the last formula.