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STUDIES ON CAPTURE EFFICIENCIES OF

SYNCHRO-CYCLOTRONS

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G E N E V A

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ABSTRACT

The starting phase of the particles in relation to the radio frequency affects the capture time considerably. This is found from numerical computations in which both the equilibrium phase and the starting phase have been varied. The total capture time per acceleration cycle has been derived from an integration over the starting phase interval 0° - 60° . The phase oscillations are damped with increasing energy; this fact has been used to minimize the acceleration time. The frequency range will be reduced considerably if radially increasing magnetic fields are employed. The capture efficiency, however, does not differ compared to that which can be obtained for actual linearly sloping fields. Intensities are estimated by means of the presented results and from available space-charge theories. A synchro-cyclotron of 600 MeV energy can be improved to deliver at least 60 μ A of protons.

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1. INTRODUCTION

In general, the capture efficiency, ϵ , determines the current obtainable in synchro-cyclotrons, as compared to that obtainable in fixed-frequency cyclotrons. In this paper the existing theory for ϵ is reviewed and additional features are included.

Bohm and Foldy¹⁾ give expressions for the range of stable frequencies, $\Delta\omega$, that can be captured, under the assumption of an initially full energy gain, eV, per turn. In order to obtain ϵ they assume that the acceleration time is proportional to the frequency time derivative at the capture moment. Henrich, Sewell and Vale²⁾ include the adiabatically damped phase oscillations in their theory. They show that ϵ will actually be greater due to this. Neither of these theories is very applicable to synchro-cyclotrons with large dee gaps. This is because the phase equation is different in a uniform electric field, as has been pointed out by Keller, Fidecaro and Barbier³⁾. Therefore, ϵ , which is calculated from the phase equation, ought to be different for such a case. The theory of Bohm and Foldy should, however apply well, with certain modifications, to the narrow gap geometries, which are considered for synchro-cyclotrons⁴⁾.

In the following the capture time, Δt , and the acceleration time, T , will be treated separately. Δt is calculated for different starting phase-angles, ϕ_0 , with respect to the RF, the maximum available energy gain per turn being constant. The calculation of T is based on the constancy of the action integral, which states that the density of particles in $\dot{\phi}$ - ϕ space increases as the particle energy grows. Therefore, the frequency can be made to change faster with time than if constant equilibrium phase had been used throughout the acceleration⁵⁾. In order to calculate the internal currents it is necessary to know the output currents from the central regions. Those currents are space charge limited⁶⁾ rather than ion source limited. A brief discussion is therefore given about two space charge theories, which can be of help in calculating the internal currents.

Symbols and definitions are adopted from Bohm and Foldy's paper, with two exceptions. The phase, ϕ , is defined from when the energy gain per turn is maximum, and the phase-time derivative, $\dot{\phi}$, is equal to the electric

frequency, ω_e , minus the ion revolution frequency ω_i . Some of their basic equations are therefore changed with regard to angular quantities.

2. CAPTURE TIME

2.1 The range of stable frequencies

The capture time is given as the range of frequencies possible to capture, $\Delta\omega$, over the frequency-time derivative, $\dot{\omega}$, at the capture moment. The expression for $\Delta\omega$ is given by Bohm and Foldy as

$$\Delta\omega = 2 \left(\frac{eV\omega^2 K}{\pi E_0} \right)^{1/2} L(\varphi_0, \varphi_s) \quad (1)$$

where

$$L(\varphi_0, \varphi_s) = \frac{1}{2} (1 + \lambda) F(\varphi_0, \varphi_s)^{1/2} \quad (2)$$

and

$$F(\varphi_0, \varphi_s) = \sin \varphi_0 + \sin \varphi_s - (\varphi_s + \varphi_0) \cos \varphi_s. \quad (3)$$

They calculate L for the case where $\varphi_0 = 0$, which is shown with continuous lines in Fig. 1. The decrease of L to the left of $\cos \varphi_s = 0.5$ depends on the return of initially phase stable particles to the centre.

A simple way to get $L(\varphi_0, \varphi_s)$ is to integrate the phase equation

$$\frac{2\pi E_0}{eV\omega^2 K} \frac{d^2\varphi}{dt^2} + \cos \varphi = \cos \varphi_s, \quad (4)$$

step by step, during the first phase oscillation. The initial values are $t = 0$, $\varphi_a = \varphi_0$ and $\dot{\varphi}_a = \lambda \dot{\varphi}_c$ where the ordinate of the separatrix, $\dot{\varphi}_c$, is

$$\dot{\varphi}_c = \left[\frac{eV\omega^2 K}{\pi E_0} F(\varphi_0, \varphi_s) \right]^{1/2}. \quad (5)$$

The letter a in Fig. 2 denotes the starting point in phase space for the trajectory running from a to b. Particles following this trajectory gain energy until $\varphi = 90^\circ$ is crossed. The particles are then decelerated until they reach point b, where the first radial minimum occurs. The particle will not return to the centre if

$$\lambda \dot{\varphi}_0 - \dot{\varphi}_b + \frac{d\omega}{dt} \cdot \tau \leq 0 \quad (6)$$

where τ is the time to move from a to b. When the equal sign is valid in expression (6) λ gives the range of frequencies possible to capture

$$\Delta\omega = \dot{\varphi}_0 (1 + \lambda) . \quad (7)$$

A Runge-Kutta integration of Eq. (4) from point a to b has been made. In the computer programme developed for the purpose λ is cycled till the equality (6) is satisfied. The result of the calculation for $\varphi_0 = 0$ coincides, of course, with the Bohm and Foldy curve above $\cos \varphi_s = 0.5$ (see Fig. 1), but is considerably different for small values of $\cos \varphi_s$. The function L is shown for different values of φ_0 in Figs. 3 and 4. The point where particles start to return to the origin moves towards higher $\cos \varphi_s$ when φ_0 is increased. At small $\cos \varphi_s$ L increases rapidly with φ_0 , which depends mainly on three facts

- that $\dot{\varphi}_0$ is increased,
- that $\lambda \dot{\varphi}_0 - \dot{\varphi}_b$ becomes smaller for the same λ ,
- that τ becomes larger for the same λ .

Hence λ must be increased in order to satisfy Eq. (6) and the function L will be increased correspondingly.

2.2 The integrated capture time

In order to obtain the capture time, Δt , the formula in Ref. 1 which connects frequency-time derivative, $\dot{\omega}$, to $\cos \varphi_s$ has to be used.

$$\dot{\omega} = - \frac{eV\omega^2 K}{2\pi E} \cos \varphi_s . \quad (8)$$

The capture time Δt which is defined from

$$\Delta t = \frac{\Delta \omega}{|\dot{\omega}|} \quad (9)$$

can then be calculated by using Eqs. (1) and (8).

$$\frac{\Delta t}{4 \left(\frac{\pi E_0}{e V \omega^2 K} \right)^{1/2}} = \frac{L(\varphi_0, \varphi_s)}{\cos \varphi_s} \quad (10)$$

This set of curves is given in Fig. 5 for rather small $\cos \varphi_s$. It is seen that Δt increases with φ_0 , which suggests large phases should be utilized.

The over-all capture time can be integrated over some suitable starting-phase interval with the function $L/\cos \varphi_s$ being the capture-time density. With hooded-arc sources and anticipated dee voltages it will be difficult to clear the source on the first turn, if φ_0 is larger than 60° . Negative phases are electrically defocused and are in addition captured during a comparatively short time. It is therefore thought convenient to assume the phase range $0^\circ - 60^\circ$. The capture time can be integrated over the considered interval.

$$\Delta t(0^\circ < \varphi_0 < 60^\circ) = \frac{2}{\cos \varphi_s} \left(\frac{E_0}{\pi e V \omega^2 K} \right)^{1/2} \cdot \int_0^{\pi/3} L(\varphi_0, \varphi_s) d\varphi_0 \quad (11)$$

The result is shown in Fig. 6. The maximum at small $\cos \varphi_s$ is mostly the contribution from large φ_0 - values. Note that Eq.(11) takes into account that capture is possible during a small fraction of each RF cycle. With a saturated ion source yielding a d.c. current of $I_{d.c.}$ the charge captured per modulation cycle is $I_{d.c.} \cdot \Delta t$. This assumes no later restrictions in the vertical motion.

2.3 A practical case

The preceding analysis applies to a theoretical case with an infinitely narrow accelerating gap. This assumption breaks down for any design during the first particle turns, when the ratio dee gap to radius d/r is rather large.

A minor effect due to this is that the maximum available energy gain per turn will be less than twice the dee voltage.. The difference, however, is usually small even for the first particle turn and will be neglected. A more serious effect is the grouping towards $\phi = 0^\circ$, which takes place as long as d/r is not small. An expression for the grouping has been calculated by Cohen⁷⁾. Neglecting the positive shift his equation is

$$\frac{d\phi}{dv} = - \frac{2}{3} \frac{Vd^2}{r^2 \omega^2} \sin \phi, \quad (12)$$

where the left side is the angular shift per gap crossing due to the finite gap d . As the phase equation (4) has been derived neglecting the bunching effect, the initial starting angle ϕ_0 has to be reduced. A decreased capture time is not expected due to this (at least not theoretically), because when the particle returns towards zero radius it is again rapidly pulled towards the voltage maximum.

To see the effects quantitatively, particles have been traced in magnetic and electric fields by means of a modified Pinwheel programme. The data used were

$$B = 1.9 (1 - 0.2r^2)$$

$$eV = 24 \text{ keV} \quad d = 0.02 \text{ m}$$

$$\cos \phi_s = 0.0375 \quad \phi_0 = 60^\circ.$$

The electric field in the 0.02 m gap was assumed to be uniform. The phase and radius as functions of the number of turns are plotted in Figs. 7 and 8 for two representative particles. The starting frequencies were taken close to the edges of the $\Delta\omega$ interval (Fig. 2) given for the narrow gap theory. Particle a starts earliest in time with $\omega_e/\omega_i = 0.9984$, and particle b, moving closest to the separatrix, starts with $\omega_e/\omega_i = 0.9951$. Both particles are, due to the grouping effect, initially pulled $10 - 15^\circ$ towards the 0° -phase. Particle b has a first radial minimum $r_{\min} = 0.035$ and very little grouping is found there. The phase of particle a, however, is rapidly pulled towards 0° -phase in its first $r_{\min}$. If that had not been the case, it should have

returned to the origin. Thus the narrow gap theory should have approximate validity even for realistic cases as the excessive initial phase motion is regained in the first radial minimum. In practice $r_{\min}$ is not wanted as low as 1 cm so the initial phase slip cannot be regained. In order to utilize large ϕ_0 the dee gap must therefore be made small.

An interesting point worth noting is that the dee gap can be increased rapidly at a somewhat larger radius. If the phase at that radius is around 0° , when the particles are accelerated the phases will not be very affected according to Eq. (12). When the particles are decelerating past that radius they will quickly be pulled toward 0° phase and move out again. This directional effect might be used to save particles, which otherwise would have returned to the centre.

2.4 The first radial minimum

A particle captured at one of the edges of the $\Delta\omega$ -interval (see Fig. 2) will have its first $r_{\min}$ equal to zero. Particles starting at the other edge, which is close to the separatrix have the largest $r_{\min}$ in the interval. The intermediate range of $r_{\min}$ can be determined by the distance, $\Delta\dot{\phi}$, from the edge for which particles just return to the centre. Such a calculation has been performed in Appendix 1, the main results being Eqs. (A15) and (A17). Equation (A15) has been solved when $\cos \phi_s = 0.1$, and $\phi_0 = 60$. The result is

$$r_{\min} = 1.08 \text{ c } \left(\frac{\Delta\dot{\phi}}{\omega^3 K} \right)^{1/2} \quad (13)$$

In Fig. 9 radius versus time is given for a numerically performed integration in fields which are characterized by

$$eV = 36 \text{ keV} \quad \omega = 2\pi \cdot 29 \cdot 10^6 \quad K = 2 .$$

As $\Delta\dot{\phi}$ decreases with time at the same rate as the electric frequency, the envelope for $r_{\min}$ can be plotted by means of Eq. (13). The envelope, shown by a dashed line in Fig. 9, agrees well with the numerical results.

Destructive electrodes, or bad focusing inside a certain radius, will cause particle losses. In order to get the fraction, x , of the captured particles, which have their minima below 4 cm radius (arbitrarily chosen),

Eq. (A17) can be used. With $K = 2$, $eV = 50 \text{ keV}$, $\omega = 2\pi \cdot 29 \cdot 10^6$ and $\cos \varphi_s = 0.1$, the fraction falling inside 4 cm is 10% for the 60° and 70% for the 0° starting phase. As the contribution to the integrated capture time curve comes mostly from large φ_0 the situation is not as bad as it seems. If the whole interval $0^\circ - 60^\circ$ is considered 20% of the particles have their minima below 4 cm.

3. THE ACCELERATION TIME

3.1 Equations

In general present tuning condensers limit the acceleration time T due to a too low repetition rate f_n . A second limitation is the frequency-time programme, $\omega(t)$, which is not designed for optimum conditions. If a new RF system were to be constructed one should determine the $\omega(t)$ programme, which optimizes ϵ . The acceleration time T has, therefore, been calculated with ω always being as high as possible. This yields the $\omega(t)$ programmes, which have to be used in order to minimize T .

From the average energy gain per turn, $eV \cos \varphi_s$, an expression for T is obtained

$$T = \int_{E_0}^{E_0 + E_k} \frac{2\pi}{\omega eV \cos \varphi_s} dE \quad (14)$$

In order to carry out an integration it is necessary to know the energy dependence of the parameters. It will be assumed that eV is constant during the whole acceleration. The frequency, ω , is dependent on magnetic field and energy

$$\omega = \frac{Bec^2}{E} \quad (15)$$

When the magnetic field is given as a function of radius it is related to the energy through

$$B^2 r^2 e^2 c^2 = E^2 - E_0^2 \quad (16)$$

As the phase oscillations are damped during the acceleration it is possible to make $\cos \varphi_s$ increase with energy. The action integral,

$$J = \frac{E}{\omega^2 K} \oint \dot{\varphi} d\varphi \quad (17)$$

which must be constant during adiabatic changes, gives $\cos \varphi_s$ as a unique function of E. This can be seen by rewriting Eq. (17).

$$\frac{eV \cdot E}{\omega^2 K} [\oint F(\varphi, \varphi_s)^{1/2} d\varphi]^2 = \text{const.} \quad (18)$$

The normalized bucket area inside the brackets is plotted in Fig. 10. As K can be shown to be as follows:

$$K = 1 - \frac{E}{B} \frac{dB}{dE} \quad (19)$$

$\cos \varphi_s$ is dependent only on E. The integration of Eq. (14) can be performed by using the relations (15) to (19) for $\cos \varphi_s$ and ω .

3.2 Applications to the CERN Synchro-cyclotron

To get numerical values of T it is necessary to know the magnetic field law. A good approximation for the present Synchro-cyclotron magnetic field is

$$B = B_0(1 - 0.0237 r) \quad (20)$$

where $r_{\text{max}} = 2.25$ m.

Some field laws have also been developed for semi-isochronous cases with the aim of seeing how a reduced frequency range affects T and hence ϵ . In order to meet many design considerations^{s)} the defocusing force, n, must not be too strong. As n has its largest value at full radius, some sort of limit should be set there. The chosen figure $n_F = -0.14$ makes $K = K_F = 0.8$. The frequency range can be further reduced for this particular n_F by making K smaller initially. This can be seen from the definition of K

$$\frac{\Delta\omega}{\omega} \frac{S}{S} = -K \frac{\Delta E}{E} \frac{S}{S} \quad (21)$$

Quite small values of K can be adopted without compromising phase stability.

The linear law

$$K = \frac{K_f - K_i}{E_k} (E - E_0) + K_i \quad (22)$$

is well behaved if the starting value K_i is not too small. If $B(E)$ is integrated from Eqs. (19) and (22) the result is

$$\langle B \rangle = \text{const. } E^{[1 - K_i + E_0(K_f - K_i)/E_k]} \cdot e^{-[E(K_f - K_i)/E_k]} \quad (23)$$

The fields in Fig. 11 have been computed for a final energy E_k of 600 MeV. The T-plots in Fig. 12 are obtained from an integration of Eq. (14) using the magnetic field laws given from Eqs. (20) and (23). Due to the different field laws $\cos \varphi_s$ increases in different ways from its starting value $\cos \varphi_{si}$ (see Eq. 18 and Fig. 10). This is the main reason for the discrepancy between the T-curves. The $\omega(t)$ curves in Fig. 13 are computed with $\cos \varphi_{si} = 0.32$. The fields II and IV require the same amount of sector-focusing at full radius, but field IV gives a smaller frequency range than field II, due to the smaller starting value K_i .

4. THE CAPTURE EFFICIENCY

4.1 With fixed frequency modulation rate

The efficiency ϵ is

$$\epsilon = f_m \cdot \Delta t = f_m \cdot \left(\frac{\pi E_0}{e V \omega^2 K} \right)^{1/2} \cdot M \quad (24)$$

M is the function plotted in Fig. 6. Because of the small values of $r_{\min}$ it might be hard to utilize the maximum of the function M . $\cos \varphi_s = 0.1$ seems to be a good compromise between M and $r_{\min}$. The operating point of the present RF system is, however, far below this value. Hecken⁹⁾ has pointed out that it is possible to increase $\dot{\omega}$ and hence $\cos \varphi_s$ by raising the upper frequency of the tuning fork system. A further increase of $\dot{\omega}$ can be accomplished by making a "diving" tuning fork as has been put forward

by Beger¹⁰⁾. A third way of increasing $\cos \varphi_s$ at the capture moment is to have an electronically modulated RF programme, which is applied to a small central dee. A difficult problem would then be to synchronize the two systems.

The mentioned schemes admit ϵ to be optimized in the case of fixed frequency modulation rate. With $\cos \varphi_s = 0.1$, $eV = 30$ keV/turn, $\omega = 2\pi \cdot 29$ MHz and $K = 2$ the required $\dot{\omega}$ is $2\pi \cdot 5400$ MHz/s^{*)}. M , which is 2.8, ought to be reduced by 20% to account for small $r_{\min}$. With the given figures and $f_m = 55$ c/s, ϵ is $1.5 \cdot 10^{-4}$. To calculate an anticipated current it is necessary to use space charge limited currents rather than ion source currents. The Berkeley group has reported¹¹⁾ that 4 mA can be focused by means of wires extending out to 8 cm radius with 20 kV dee voltage in a magnetic field $B = 1.6$ Wb/m². Space charge conditions are harder for the CERN Synchro-cyclotron due to a higher B and a smaller eV . A current of 3 mA might, therefore, be more realistic. The charge is assumed to be in a 60° phase interval, which gives an equivalent d.c. current of 18 mA. As $\epsilon = 1.5 \cdot 10^{-4}$, the optimized beam intensity is expected to be 2.7 μ A.

4.2 With adjustable frequency programmes

Without considering possible dead-times inherent in the RF system the optimum capture efficiency is given by

$$\epsilon = \frac{\Delta t}{T} \quad . \quad (25)$$

The plots of ϵ in Fig. 14 are derived from the Δt and T curves in Figs. 6 and 12. Figure 14 shows that the maximum value of ϵ is about the same for the linear sloping magnetic field as for the two kinds of radially increasing fields. It is to be noted, however, that larger $\cos \varphi_{si}$ can be employed with field IV, which might preserve the beam quality. The capture efficiency which can be attained for the cases considered, is $0.9 \cdot 10^{-5} \sqrt{eV}$. If eV is assumed to be 80 keV/turn $\epsilon = 2.6 \cdot 10^{-3}$. A linear voltage dependence on the Berkeley figure results in an equivalent d.c. current of 48 mA and an

*) These values of eV and $\dot{\omega}$ seem to be the highest which can possibly be achieved with the present RF system.

expected internal current of $120 \mu\text{A}$. If the dead-time of the RF system is 50%, $60 \mu\text{A}$ remain, which should be compared with the $2.7 \mu\text{A}$ obtained in the previous section.

With a decreasing magnetic field it is possible to increase the capture efficiency further by having a magnetic field bump, which makes K larger at small radii. The radial extension of the bump must be sufficient so that the adiabatic theorem (18) is applicable for radii, where K assumes its normal value. Particles can thus be captured during a long time with relatively small values of $\cos \varphi_s$. After the capture time the phase oscillations are damped due to the decreasing value of K . $\cos \varphi_s$ can, therefore be increased (Fig.10). The increase of $\cos \varphi_s$ takes place automatically, when $\dot{\omega}$ and eV are constant (Eq.8). The acceleration can, therefore, start with a larger $\cos \varphi_{si}$ than that which was used during the capture moment. The acceleration time, which will be shorter due to this, makes the capture efficiency larger.

A magnetic bump extending to 30-40 cm should be sufficient to utilize the damping effect for normal values of the parameters. Assume that particles are captured with $\cos \varphi_s = 0.1$. Without the bump $eV \cdot T = 84$ (Fig.12). A bump giving a twofold increase of K cuts down the capture time by 30%. The damping effect, however, makes it possible to start the acceleration with $\cos \varphi_{si} = 0.27$, which yields $eV \cdot T = 57$. The net gain in ϵ will be 4%. This result has to be compared with the normal 30% loss obtained with the described cone, if the adiabatic damping is not utilized.

5. VERTICAL FOCUSING CONSIDERATIONS

Space charge defocusing makes the beam profile blow up before sufficient vertical focusing is attained. Formulae for the limiting current, I_c , have been derived for various focusing methods^{6, 12-14}). These formulae apply actually to cyclotrons, but can also be used for synchro-cyclotrons. For this case it can be supposed that particles occupy the whole phase interval 2π . The capture time, which applies to d.c. currents can then be used to get the captured charge $q = I_c \cdot \Delta t$. This charge is contained in the "sausage", which is accelerated to full radius. Rainwater¹⁵) recently pointed out that a

space charge limit, q_n , exists for the filled sausage, which might be as severe as the limit at smaller radii. If $q_n < q$ the obtainable intensity will be q_n/T rather than q/T .

Rainwater's approach can be adopted from the moment when the sausage is no longer able to accept phase stable particles from the centre. Particles are at that time occupying radii between 0 and $r_{\max}$. An expression for $r_{\max}$ can be found by using Eqs. (1), (2) and (A14).

$$r_{\max} = 2 \frac{c}{\omega} \left(\frac{F \cdot eV}{\pi K E_0} \right)^{1/4} \quad (26)$$

where $F = 2 \sin \phi_s - 2\phi_s \cos \phi_s$. Except for the very first one can assume that the focusing forces vary slowly with radius. With a rather uniform distribution of particles in phase space the full "sausage" is the strongest defocusing one. When particles are accumulating inside $r_{\max}$ and 0 space charge conditions are less severe than when the sausage is full. Rainwater's theory should, therefore, cover all cases where the focusing force varies slowly with radius.

When the sausage starts to travel away from the centre, the radial extension Δr is approximately given as

$$\Delta r = \frac{r_{\max}^2}{r_0} \quad (27)$$

where r_0 is the average radius. As the azimuthal extension is proportional to r_0 , the charge density on the median plane will be independent of r_0 , which calls for a constant focusing force, if a beam "blow-up" is to be prevented. This can be achieved by means of a magnetic cone extending to a rather large radius in order to maintain the focusing there at small radii. The advantages of such a cone can be summarized as follows:

- i) Increased vertical focusing makes the limit q_n larger.
- ii) Larger $1-Q_r$ will preserve beam quality in regions where perturbations from electric and magnetic fields are serious.
- iii) No correcting coils are required if central sectors are installed to get magnetic focusing at small radii. The sectors can be designed to give a suitable field shape.

- iv) The adiabatic damping of the phase oscillations can be utilized to give an increased rather than a decreased capture efficiency.

At very small radii where the focusing forces change rapidly, MacKenzie's approach leading to a space charge limited current, I_c , has to be used. Due to the many parameters which are involved, quantitative figures must be established by means of experiments on central region models. It is believed that the focusing at larger radii can be improved so that $q_n > q$. The quantities A_t and T presented in this paper should, therefore, be of help in predicting intensities of synchro-cyclotrons.

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APPENDIX 1

Calculation of $r_{\min}$

When the starting point in phase space $\dot{\phi}_a = \lambda \dot{\phi}_0$, the first radial minimum $r_{\min} = 0$. If an increment $\Delta\dot{\phi}_a$ is added to $\dot{\phi}_a$ the particle has an $r_{\min}$ different from zero. As the point ϕ_a , $\dot{\phi}_a$ is usually well inside the separatrix the trajectory can be assumed to be circular. This assumption holds since only small deviations around $\dot{\phi} = \dot{\phi}_a$ will be studied. The definitions of quantities is shown in Fig. 15. The starting point is denoted by a, and b denotes the point at which $r_{\min}$ occurs. The time dependence of the electric frequency gives the relation between the electric frequencies at a and b

$$\omega_{eb} = \omega_{ea} (1 - \alpha\tau) . \quad (A1)$$

τ is the time it takes for a particle to move from a to b in phase space. The relation between corresponding ion frequencies is

$$\omega_{ib} = \omega_{ia} + \nu_i . \quad (A2)$$

The quantity $\dot{\phi}_a$ is

$$\dot{\phi}_a = \omega_{ea} - \omega_{ia} \quad (A3)$$

and

$$\dot{\phi}_b = \omega_{eb} - \omega_{ib} . \quad (A4)$$

The pertinent parameter ν_i can be derived from Eqs. (A1) to (A4).

$$\nu_i(\dot{\phi}_a) = \dot{\phi}_a - \dot{\phi}_b - \omega_{ea} \cdot \alpha \cdot \tau . \quad (A5)$$

When an increment $\Delta\dot{\phi}_a$ is given to $\dot{\phi}_a$, there are corresponding increments $\Delta\dot{\phi}_b$ and $\Delta\tau$.

$$\nu_i(\dot{\phi}_a + \Delta\dot{\phi}_a) = \dot{\phi}_a + \Delta\dot{\phi}_a - \dot{\phi}_b - \Delta\dot{\phi}_b - \omega_{ea} \cdot \alpha (\tau + \Delta\tau) . \quad (A6)$$

An expression for $dv_i/d\dot{\phi}_a$ can be derived from Eqs.(A5) and (A6)

$$\frac{dv_i}{d\dot{\phi}_a} = 1 - \frac{\Delta\dot{\phi}_b}{\Delta\dot{\phi}_a} - \omega_{ea} \cdot \alpha \cdot \frac{\Delta\tau}{\Delta\dot{\phi}_a} \quad . \quad (A7)$$

This equation can be simplified by using relations between quantities in phase space

$$R^2 = \left(\frac{\pi}{2} - \varphi_s\right)^2 + \left(\frac{\dot{\phi}_b}{\Omega}\right)^2 = \left(\varphi_s - \varphi_a\right)^2 + \left(\frac{\dot{\phi}_a}{\Omega}\right)^2 \quad (A8)$$

where

$$\Omega^2 = \sin \varphi_s \frac{eV\omega^2 K}{2\pi E_0} \quad . \quad (A9)$$

Eq. (A8) can be differentiated with constant values of φ_a , φ_b and φ_s

$$\frac{\Delta\dot{\phi}_b}{\Delta\dot{\phi}_a} = \frac{\dot{\phi}_a}{\dot{\phi}_b} \quad . \quad (A10)$$

The time increment $\Delta\tau$ is

$$\Delta\tau = \frac{\Delta\theta}{\Omega} = -\frac{1}{R\Omega^2} \left[\left(\varphi_s - \varphi_a\right) \Delta\dot{\phi}_a + \left(\frac{\pi}{2} - \varphi_s\right) \Delta\dot{\phi}_b \right] \quad . \quad (A11)$$

Eq. (8) gives for $\omega_{ea} \cdot \alpha$

$$\omega_{ea} \cdot \alpha = \frac{K\omega^2 eV}{2\pi E_0} \cos \varphi_s \quad . \quad (A12)$$

The expressions (A9) to (A12) can be used to simplify Eq. (A7)

$$\frac{dv_i}{d\dot{\phi}_a} = 1 - \frac{\dot{\phi}_a}{\dot{\phi}_b} + \frac{\cot \varphi_s}{R} \left[\varphi_s - \varphi_a + \frac{\dot{\phi}_a}{\dot{\phi}_b} \left(\frac{\pi}{2} - \varphi_s\right) \right] \quad . \quad (A13)$$

From Eq. (21) a relation is obtained for the range of frequencies, $\Delta\omega_s$, which is contained between zero radius and r .

$$\Delta\omega_s = - \frac{K\omega^3}{2c^2} r^2 . \quad (A14)$$

This is inserted in Eq. (A13) with $\Delta\omega_s = d\nu_i$, $r = r_{\min}$ and $d\dot{\phi}_a = \Delta\dot{\phi}$

$$r_{\min} = \frac{c}{\omega} \left[\frac{2\Delta\dot{\phi}}{\omega K} \left(\frac{\dot{\phi}_a}{\dot{\phi}_b} - 1 - \frac{\cot \varphi_s}{R} \left(\varphi_s - \varphi_a + \left(\frac{\pi}{2} - \varphi_s \right) \frac{\dot{\phi}_a}{\dot{\phi}_b} \right) \right) \right]^{1/2} . \quad (A15)$$

An expression for $\dot{\phi}_a = \lambda\dot{\phi}_0$ can be found from Eqs. (2) and (5)

$$\dot{\phi}_a = \left[\frac{cV\omega^2 K}{\pi E_0} \right]^{1/2} \left[2L(\varphi_0, \varphi_s) - F(\varphi_0, \varphi_s) \right]^{1/2} . \quad (A16)$$

As R and $\dot{\phi}_b$ are calculated from Eq. (A8), Equation (A15) is uniquely given by φ_0 and φ_s .

The fraction, x , of the range of frequencies $\Delta\omega$ whose radial minima are below $r_{\min}$ is $-\Delta\dot{\phi}/\Delta\omega$. One can rewrite Eq. (A15) as a function of x

$$r_{\min} = 2 \frac{c}{\omega} \left(\frac{cV}{\pi K E_0} \right)^{1/4} \left[x \cdot L(\varphi_0, \varphi_s) \cdot \frac{d\nu_i}{d\dot{\phi}_a}(\varphi_0, \varphi_s) \right]^{1/2} . \quad (A17)$$

For the purpose of illustration this equation is solved with $\cos \varphi_s = 0.1$ for two different values of the starting angle φ_0

$$r_{\min} = 2 \frac{c}{\omega} \left(\frac{cV}{\pi K E_0} \right)^{1/4} x^{1/2} \cdot \begin{matrix} 0.30(\varphi_0 = 0^\circ) \\ 0.84(\varphi_0 = 60^\circ) \end{matrix} . \quad (A18)$$

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FIG.1 $L(\cos \psi_s)$, $\psi_0 = 0^\circ$

FIG.2 Phase space showing the initial motion
 $\cos \psi_s = 0.1$ $\psi_0 = 0^\circ$

FIG. 3 The normalized range of stable frequencies, $L(\psi_0, \psi_s)$

FIG. 4 $L(\varphi_0, \varphi_s)$

FIG.5 Normalized capture times with ϕ_0 as parameters

$$M = \frac{\Delta t(0 < \varphi_0 < 60^\circ)}{(\pi E_s / eV K \omega_s^2)^{1/2}}$$

FIG.6 Capture time per acceleration cycle for the phase range $0-60^\circ$

FIG.7 Radial motion with a 2cm accelerating gap $\cos \varphi_s = 0.0375$, $\varphi_0 = 60^\circ$

FIG.8 Phase versus number of turns for the particles in fig.7

FIG.9 Radial motion for first starting stable particles $\cos \varphi_s = 0.1$, $\varphi_0 = 60^\circ$

FIG.10 The square of bucket area

FIG.11 Magnetic field laws

FIG.12 Acceleration times T as functions of initial equilibrium phases φ_{si}

FIG. 13 Frequency versus time

$\cos \varphi_{si} = 0.32$

FIG. 14 Capture efficiencies for three different field laws. See fig. 11

FIG.15 Initial phase motion for a small amplitude particle