ON THE GENERALIZATION OF THE ISOPARITY

by

L.C. Biedenharn, J. Nuyts and H. Ruegg
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GENEVA
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ON THE GENERALIZATION OF THE ISOPARITY

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ABSTRACT

The generalization of the isoparity (G parity) to internal symmetry groups which belong to any of the four families of compact simple and simply connected Lie groups ($SU_n, \overline{SU}_{2k+1}, \overline{SU}_2$ and $Sp_n$) is explicitly carried out. The extended isoparity is shown to depend upon the structure of the groups in question, and the required structure is developed in detail. The properties of the extended isoparity are discussed and two cases -- strong G parity and weak G parity -- are shown to arise. The relation between the isoparity extension and superselection rules is briefly discussed, and a concluding section illustrates, by example, the possibility of other extensions available in direct products. This example is of intrinsic interest since it allows an internal (symmetry) distinction between meson and baryon octets which excludes meson decuplets, for example.

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I. INTRODUCTION AND SUMMARY

The charge conjugation operation, $C$, — more properly "matter-antimatter conjugation" — is a symmetry operation of quantum mechanics which, briefly put, reverses the sign of all (generalized) charges for all states. For the special case where only ordinary charge is considered (isospin group) the operation of charge reversal, i.e., reflection in a plane containing the charge axis, can be combined in the well-known way \(^1\) with a rotation (C) in isospace to yield a reflection operator reversing all directions in isospace; this is the usual one of the two possible isoparity operators (the G parity) to use the customary but less descriptive term) defined as: $G = C \hat{C}$. Since there exists a super-selection rule \(^2\) separating states of different charge, it is clear that useful selection rules obtain only for systems of zero net charge \(^3\). (The same remark explains why the two possible isoparity operators are not essentially distinct; confer discussion below.)

The present paper is concerned with the problem of generalizing the concept of the isoparity operator from the special example of the isospin group of internal symmetries to the larger class of internal symmetries given by the unitary, rotation and symplectic families of groups (more accurately their covering groups). Although only a very few cases of these groups are seriously considered in the literature at present (notably $SU_3$, $SU_4$ and $SU_6$) it is none the less desirable to see the general features of the isoparity problem. This is particularly valuable since the isoparity problem may be seen itself as a special, and instructive, example of the more general problem of group extensions discussed by Michel \(^4\) in his Istanbul lecture (1962); in more physical language this is the problem of determining all possible inter-relations between the generalized charges and the spins of elementary particles. The isoparity problem is very much simpler than this general problem in that only the internal symmetry group is considered (complications of the Poincaré group do not enter therefore) and only a single operator ($G$) is adjoined (rather than the complete set of all possible discrete invariances). Owing to these simplifications a complete and explicit answer can be given.
In order to pose the isparity problem in a precise way it is necessary first to consider some of the technical complications connected with charge conjugation. For a given symmetry group (considered as physically applicable to one or more elementary particles), the operations (on the group defined by means of the complete set of irreducible representations) of charge conjugation, time reversal, and complex conjugation are all equivalent; only when one goes beyond the group and considers the state vectors (a Hilbert space over the complex field) does the distinction between these operators arise, namely that charge conjugation is linear unitary while time reversal is entilinear, anti-unitary. This circumstance is reflected in the well-known fact that only in a second-quantized treatment may the charge-conjugation operation be formulated consistently. We may circumvent this technical detail, however, by noting that in effect the linearity of charge conjugation is achieved in the quantized version by the explicit adjunction of a charge-conjugate Hilbert space to the original Hilbert space in such a manner as to achieve — by construction — a linear realization of the charge-conjugation operation.

We may exploit this fact by explicitly associating charge conjugation \( \mathcal{C} \) with complex conjugation \( (K_0) \) for all discussions relevant to the group itself. This reduces to the technical problem of determining the charge-conjugation matrix \( C \) for the various groups under discussion (Section III below).

Although charge conjugation and thus isparity necessarily involves this relationship (on the group) to complex conjugation, this is not a unique characterization, and there are in general several distinct operators which can equally well be called "the" isparity \( G \). Only after enumerating the complete set of all discrete invariance operators, including physical associations, can a decision as to the "right" isparity operator be made. This is a much more difficult problem and in the following we shall only enumerate all of the possible isparity operators.
We are now in a position to specify the problem precisely. Given an internal symmetry group $\mathcal{J}$, the charge conjugation operation induces an involutary automorphism on $\mathcal{J}$. Two questions then arise naturally:

a) What are the extended groups, $\mathcal{J}^{\text{ext}}$, each of which includes both the original group $\mathcal{J}$ and the charge conjugation operation?

b) Are there new quantum numbers connected with these extensions?

To be still more precise we may rephrase the first question in this way: Given $\mathcal{J}$ to find all possible groups $\mathcal{J}^{\text{ext}}$ such that $\mathcal{J}^{\text{ext}}/\mathcal{J} = C_2$, where the $C_2$ group (the cyclic group on two elements) corresponds to the involutary automorphism on $\mathcal{J}$ produced by complex conjugation.

The solution to question a) is developed in Sections II, III and IV and summarized in the Table. Representations of $\mathcal{J}^{\text{ext}}$ are given in Section V. Extensions by involutary operations other than charge conjugation also exist and are of importance in discussing the complete set of discrete extensions. We shall only sketch the answer for this case, giving the results in the Table also.

An answer to the physically interesting question b) is given in Section VI.

One further topic, not specifically relevant to isoparity, is also discussed. This is the Frobenius-Schur invariant for the extended groups. This invariant is connected with super-selection rules for the groups in question, and is discussed in Section VII.

A final section reconsiders the question of isoparity extensions as a search for new quantum numbers and shows, by example, that other possibilities exist in direct products.

One of the subsidiary results of the present paper is the discussion of the charge conjugation matrix $C$ for the $SU_n$, $R_n$ and $Sp_n$ groups. Similar results, for special cases, already exist in the literature. For the rotation groups, Pais $^5$ has given an elegant discussion, from just the physically motivated

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viewpoint which we wish to adopt. Our work is, in a sense, a sequel to his paper, for we attempt to show how his results fit into a larger scheme. For the general $SU_n$ group the matrix $C$ is implicitly contained in the results on conjugation $^6$; the relation to charge-conjugation is not discussed there, however. Let us note, too, that these results are well-known to the mathematicians — it is not this, but rather the significance of the results, which we feel is of interest to physicists and merits discussion.

The general problem of group extensions — for which the $G$ parity extension is a special case — has been treated (in the physics literature) quite outstandingly by Michel in his Istanbul lectures, as noted earlier $^4$. For the special case of $SU_3$, the $G$-parity extension has been discussed briefly by Dothan $^7$. During the final stage of preparing the present paper, we learned of an elegant treatment of the group extension problem, including $G$ parity, by Kamber and Straumann $^8$, which has just been published. The work of Kamber and Straumann is the most comprehensive of these prior discussions, but it is phrased in rather difficult mathematical terms which we feel probably renders it inaccessible, at least for the present, to the majority of physicists. More compelling perhaps is the fact that the $G$-parity problem is basically simple in structure and a more elementary treatment, such as given here, has many advantages, including accessibility.

It is essential to point out also that the problem of the "conservation properties" of the extended $G$ parity are fully as important as the extension itself and, to our knowledge, are treated in detail only in the present paper (Section VI).

Our concluding section is more physically oriented and somewhat peripheral to the $G$ parity problem; it can be understood more-or-less independently of the preceding sections.
II. THE ADJUNCTION OF AN INVOLUTION

The problem we seek to solve here has been posed in Section I: Given a symmetry group \( J \) (assumed to be a compact simple Lie group) to determine all groups extended by an involution, \( R \), such that \( J^{\text{ext}}/J = C_2 \).

The fact that the internal symmetry group is a Lie group is not essential to the logic of the construction to follow and, for clarity, we may schematize the essentials of the argument by considering an elementary example using the finite cyclic group \( C_4 \), as the "internal symmetry group" \( J \). Adjoining the involution \( R \) to \( C_4 \) produces a new group \( J^{\text{ext}} \) with 3 elements. There are only 4 finite groups of order 8 with the property that \( C_4 \) is a normal subgroup and \( J^{\text{ext}}/C_4 = C_2 \); these yield specific examples of all possible extensions.

Method 1) Construct the direct product:

\[
\begin{align*}
C_4 & : \quad A^4 = E \\
C_2 & : \quad R^2 = E, \quad A R = R A \\
J^{\text{ext}} & = C_4 \otimes C_2
\end{align*}
\]

Method 2) Construct the semi-direct product:

\[
\begin{align*}
C_4 & : \quad A^4 = E \\
C_2 & : \quad R^2 = E
\end{align*}
\]

Definition of the automorphism \( \sigma \) induced by \( R: A \rightarrow A^{-1} \). In the extended group this automorphism may be written as: \( R^{-1} A R = A^{-1} \). In the following, although it is a rather loose way of speaking, we shall always write the automorphism directly in the form \( R^{-1} A R = A^{-1} \).
\[ \mathcal{J}^{ext} = C_4 \otimes C_2 = D_4 \] (dihedral group of the square).

Every element of \( \mathcal{J}^{ext} \) is given in the form \((a, r)\) with the multiplication law:

\[ (a_1, r_1)(a_2, r_2) = (a_1 a_2^{-1}, r_1 r_2) \]

Method 3) Construct the direct Schreier product:

\[ A^4 = E \]
\[ AR = RA \]
\[ R^2 = A \]

These relations define \( \mathcal{J}^{ext} \) to be the cyclic group \( C_8 \).

Method 4) Construct the semi-direct Schreier product:

\[ A^4 = E \]
\[ R^{-1} AR = A^{-1} \]
\[ R^2 = A^2 \]

These relations define \( \mathcal{J}^{ext} \) to be the quaternion group, \( Q \). (Note in both method 3 and 4, that \( R^2 \) is not the identity but does belong to \( \mathcal{J} \).

Note, too, that \( C_2 \) is not a subgroup of \( \mathcal{J}^{ext} \).
The essential point for the present paper is the fact that these four
techniques (suitably interpreted) for an involuntary extension exhaust the possi-
abilities. All four techniques can be combined into two statements:

a) the effect of the operation $R$ on the group $\mathcal{G}$ ($R$ must be an invo-
lutary automorphism),

b) the element of $\mathcal{G}$ to which $R^2$ belongs. (It follows from a) that
$R^2$ must belong to the centre of $\mathcal{G}$, since $R^{-2}gR^2 = g$ for all $g$.)

The operation $R$ is physically motivated to be a specific symmetry
operation (charge reversal) and hence (as discussed in Section I) must act as a
complex conjugation on the group; thus we must take $R = 3Z$, where $B$ is any
automorphism such that $R$ is involuntary. This requirement is not necessarily
unique, and there may be several suitable operators $R$, depending on the group $\mathcal{G}$.

Two simplifications in this general problem exist. Firstly, we may
use the fact that automorphisms $R$ and $R'$ which differ by an inner automorphism
($R' = g_oR$) do not define different extended groups. We may thus restrict atten-
tion to only those operations $R$ which define distinct groups; this means that
various possibilities for $R$ differ by outer automorphisms. In case $\mathcal{G}$ itself
induces an inner automorphism (conjugation by $g_0$, say) we may use this fact to
write the extended group as a direct product.

The second simplification occurs from the fact that $R^2 = Z$, where $Z$
belongs to the centre of $\mathcal{G}$, as noted earlier. Applying the automorphism,
induced by $R$, to $Z$ we see that: $R^{-1}ZR = Z$, so that $Z$ is invariant under
$R$. Using the fact that $R$ involves complex conjugation one finds that $Z^2 = E$
and hence $R^4 = E$.

The problem of extending $\mathcal{G}$ has thus been reduced to two specific
tasks: (1) the determination of all outer automorphisms of $\mathcal{G}$ and (2) the

*) Inner automorphisms of the group $\mathcal{G}$ are defined by: $g \rightarrow g' = g_o^{-1}gg_o$, with $g_o$ a member of the group $\mathcal{G}$. Outer automorphisms are those not of
this form.
is an invariant under the action of the group generators, $X_A$ (assuming commutation of the $\hat{\Psi}$'s). The matrix $C$ is itself an invariant, in the sense that:

$$C' = S^t C S = C$$

for all $S$, $S = \exp (i a \cdot \chi)$; that is, for all unitary transformations of the group $SU_n$. (These results follow immediately from Ref. 6).

It is interesting to note that if $\hat{\Psi}_1 = \hat{\Psi}_2$ (and hence we have a self-conjugate representation), then the invariant vanishes identically for $C^t = -C$. For this condition to hold, we must have $(-)^{\frac{n(n-1)}{2}} m_{1\mu} = (-)$, which in turn requires:
(a) $m_{1\mu}$ is odd integer and
(b) $n = 2, 3 \mod 4$, as well as
(c) $n = 0 \mod 2$ from the requirement that $(E) = (n^2)$. Hence we see that there exists no non-vanishing invariant, $(\hat{\Psi} C \hat{\Psi})$, for the representations $(\eta) = (m)$, $m_{1\mu}$ odd in $SU_{4k+2}$. This is a familiar property of the half-integer representations in $SU_2$.

Let us next extend this result to the defining representation of rotation and symplectic groups. In order to use the results obtained above let us use the technique of embedding $R_{2k+1}$ in $SU_{2k+1}$ and $Sp_{2k}$ in $SU_{2k}$. The generators of $SU_n$ ($n = 2k$ or $2k+1$) are maps of the specific generators defined in the $n \times n$ representation. Rather than the Weyl form $X_A$, let us now use Racah's form based on the tensor operators $S^{(q)}_{(k)}$ ($k = 1, 2, \ldots, n-1$) defined on the $n$-dimensional angular momentum space having $J = \frac{n-1}{2}$. The operators $S^{(q)}_{(k)}$ are (to a constant factor) the matrix operators given by the Wigner coefficients $C_{jm} \lambda^j$, acting in the $n \times n$ dimensional space of the fundamental representation $\lambda$.

Now the charge conjugation matrix $C$ is defined for a definite group, and exists as a specific matrix for each representation. But different groups may have representations (reducible or irreducible) of the same dimensionality, and for this case the possibility arises of investigating the action of different charge conjugation matrices on the same abstract space. This possibility thus affords an interesting method of splitting a group into a subgroup, as the following example shows.
Under the action of the $SU_n$ matrix $C$ we have the relation:

$$C^{-1}_n x_A C(n) = - x_{-A}$$

just as before, (since $x_A$ are simply specific matrices, the $X_A$ for $n \times n$ representation). Now let us ask: what is the effect of the $SU_2$ charge-conjugation matrix $C(2)$ in the representation of dimension $n$ on the generators $x_A$?

Since $x_A = S(q)_k$, it is clear that we seek: $C(2)_m^{-1} S(q)_k C(2)$. Now the charge-conjugation matrix in $SU_2$ is simply: $C(2)_m^{-1} = (-)^{j+m} S_m^{-m}$, as follows from the general results earlier $^\ast$. One cannot conclude that, for the mapping $S(q)_k \leftrightarrow |k,q>$, the phase is $(-)^{k-q}$, since the consistency of the $X_A$ phase relations and the $S(q)_k$ phase relations must be established. The required phase relation is that:

$$|S(q)_k> = \left\{ \begin{array}{ll}
(-)^q |k,q> & q > 0 \\
|k,q> & q < 0
\end{array} \right.$$  

and hence:

$$C(2)_m^{-1} S(q)_k C(2) = (-)^{k} S(q)_k$$

$^\ast$ Actually $(-)^{j+m}$ results, but the difference is unimportant for what follows.
In order that this transformation be an automorphism for a Lie algebra generated by the \( S^q_{(k)} \), it is then required that \((-1)^k = -1\). We have thus split the SU group into the subgroup \( \sum_{n=1}^{n-1} \) generated by the operators \( \sum_{k=1, 3, \ldots}^{n-1} \) for \( n = \text{even} \) and \( \sum_{k=1, 3, \ldots}^{n-2} \) for \( n = \text{odd} \). For \( n = 2k \), this is the symplectic group \( \text{Sp}_k \); for \( n = 2k+1 \) this is the rotation group \( R_{2k+1} \).

It is now unnecessary to calculate the explicit matrices \( C \) for the defining representation of the \( R_{2k+1} \) and \( \text{Sp}_k \) groups -- since this matrix is simply the matrix \( C_{(2)} \) of appropriate dimension, \( n = 2j+1 \). It is clear also that the matrices \( C \) have precisely the desired general properties:

\[
C^* = C, \quad C^+ = C^{-1}
\]

and

\[
C^{-1} x_A C = -x_{-A}
\]

(\( x_A \) refers to the Cartan form of the \( R_{2k+1} \) or \( \text{Sp}_k \) generators).

For the particular case of the fundamental \( (n \times n) \) representation, we have the result:

\[
C^t = (-1)^{n-1} C
\]

using \( n = 2j+1 \).

Hence for the rotation group \( R_{2k+1} \) we get: \( C^t = C \) and for the symplectic group, \( C^t = -C \). The significance of the matrix \( C \) as defining an invariant now shows that the defining representation involves a quadratic symmetric real form for the rotation group, and antisymmetric, real form for the symplectic group.

The \( R_{2k} \) result also follows, by embedding it in \( R_{2k+1} \).

*) This result stems from Racah, Ref. 9), who obtained it by direct computation from the commutation relations. The method used above shows that the underlying principle is more general.
Rather more interest attaches to the spinor representations of $R_n$. The charge conjugation matrices for these can be obtained by appropriately specializing the general results. It is precisely these cases, however, that have been treated by Pais \(^5\), and we can avail ourselves of his results. Pais finds that: for $n = 2\ell$ and $n = 2\ell+1$, the $2\ell$ component spinor representation $\bar{\Psi}$ of $\overline{R_n}$ is transformed according to

$$\bar{\Psi}' = S \bar{\Psi}$$

$$S = 1 + \frac{1}{4} \epsilon_{\mu\nu} \Gamma^\mu \Gamma^\nu$$

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2 \delta_{\mu\nu} I$$

$$\Gamma'_{\mu} = S^{-1} \Gamma_{\mu} S \quad (\mu, \nu = 1 \ldots n)$$

The matrix $C$ is defined in such a way that

$$C' = S^t C S = C$$

and $\bar{\Psi}' C \bar{\Psi}$ is invariant.

Choose a representation of $\Gamma_{\mu}$ such that $\Gamma^+_{\mu} = \Gamma_{\mu}$ and $\Gamma^t_{2\mu-1} = \Gamma_{2\mu-1}$, $\Gamma^t_{2\mu} = -\Gamma_{2\mu}$. Then one gets for $C$:

\[ n = 2\ell \]
\[ C = \prod_{\alpha=1}^{\ell} \Gamma_{2\alpha-1} \]  
\[ C^t = (-)^{\frac{\ell(\ell-1)}{2}} C \]

\[ C \Gamma_{\alpha} C^{-1} = (-)^{\ell+1} \Gamma_{\alpha} \Gamma^t, \quad \alpha = 1 \ldots 2\ell \]

\[ n = 2\ell+1 \]
\[ C = \prod_{\alpha=1}^{\ell+1} \Gamma_{2\alpha-1} \]  
\[ C^t = (-)^{\frac{\ell(\ell+1)}{2}} C \]

\[ C \Gamma_{\alpha} C^{-1} = (-)^{\ell} \Gamma_{\alpha} \Gamma^t, \quad \alpha = 1 \ldots 2\ell \]

\[ \Gamma_{2\ell+1} = (-i)^{\ell} \prod_{\alpha=1}^{2\ell} \Gamma_{\alpha} \]
Let us note for clarity, that -- as Pais has discussed -- the spinor representation for $n = 2^\ell$ is not irreducible if the group does not contain reflections (the reflection operator can be taken as $\sqrt{1/2^\ell + 1}$). In the absence of reflections, the spinor representation of dimension $2^\ell$ splits into two representations of dimension $2^{\ell-1}$ (conjugate under reflections).
IV.A. INVOLUTARY OUTER AUTOMORPHISMS OF $\text{SU}_n$, $\overline{\text{R}}_n$, $\text{Sp}_n$

The problem of determining the involutary outer automorphisms of the Lie groups $\text{SU}_n$, $\overline{\text{R}}_n$ (the covering group of the rotation group) and $\text{Sp}_n$ can be put in this way: to determine all transformations $g \rightarrow g'$ of the given group $\mathcal{G}$ which leave invariant all defining relations of the group, and which are not of the form $g \rightarrow g' = g_0^{-1} g g_0$ (an inner automorphism).

This problem may be transferred to the Lie algebra of the group by noting that every automorphism of the group is an automorphism of the algebra, with infinitesimal inner automorphisms corresponding to derivations of the algebra. Conversely, every automorphism of the algebra corresponds to an automorphism of the covering group. Since the automorphisms of the algebra are given in Ref. 10, our procedure will be to give the explicit transformations for the groups in question, and appeal to the general results for the algebra to prove completeness *).

Consider first the unimodular unitary groups $\text{SU}_n$ which have the vector diagram $A_{n-1}$ for their algebra. The operation of charge conjugation is the automorphism,

$$g = \exp \left( i \varphi \cdot \mathcal{X} \right) \rightarrow g' = \exp \left( i \varphi \cdot \mathcal{X}^c \right)$$

where $X_A^c = -X_A$.

*) The results given in Ref. 10 can be understood immediately as symmetry operations of the Schouten–Dynkin diagram, which conserve both the type of weight and the connections between them. The Dynkin diagram is composed of the positive simple root vectors, with connections specifying the angle (no connection $= 90^\circ$; single line connection $= 120^\circ$; double line $= 135^\circ$; triple line $= 150^\circ$) between the joined root vectors (see reference for a full discussion). For the classical groups one has the diagrams:

$\text{SU}_{n+1} \Rightarrow A_n$

$\text{R}_{2n+1} \Rightarrow B_n$

$\text{Sp}_{2n} \Rightarrow C_n$

$\text{R}_{2n} \Rightarrow D_n$
This is an automorphism, since the Lie algebra: \[ \frac{d}{dx} \mathcal{L} = \mathcal{L} \] is invariant under the operation \( X^A \rightarrow -X^A \). Since \( SU_n \) is simply connected it follows that this automorphism of the algebra is an automorphism of the group. Moreover the operation is clearly involutary; since the classes are not invariant under \( \mathcal{L} \) (except for \( SU_2 \)) it follows that the automorphism is outer for \( n \geq 3 \).

From Ref. 10 it is shown that for \( SU_n \), \( n \geq 3 \), the algebra has only one outer automorphism, which is of period 2; this then completes the discussion for \( SU_n \).

For the groups \( R_{2k+1} \) and \( Sp_n \), i.e., the Lie algebras \( B_k \) and \( C_{n/2} \) respectively, there are no outer automorphisms. This simplifies the extension problem, and shows that \( \mathcal{L} \) carries all representations into themselves.

The remaining family of Lie groups, with the covering group \( \tilde{R}_{2k} \), all have at least one involutary outer automorphism; the discussion below shows that there are three cases to consider:

(a) the groups \( \tilde{R}_{4,\ell} \)

(b) the groups \( \tilde{R}_{4,\ell+2} \) and

(c) the highly exceptional (and therefore very interesting!) case \( \tilde{R}_3 \).

Consider first the case \( \tilde{R}_{4,\ell} \). The charge conjugation operator, as we know from Section IV, induces the transformation: \( \mathcal{L} = \frac{1}{2} D(2\ell) \mathcal{L} = C^{-1} D(2\ell) C \) on the fundamental (spinor) representation \( D(2\ell) \) of dimension \( 2\ell \), where \( C = \prod_{i=1}^{2\ell} \gamma_i \). The charge conjugation matrix \( C \) therefore is a product of an even number of \( \gamma_i \) matrices; using the fact that \( \gamma_i \gamma_j = \exp(\frac{n}{2} \gamma_i \gamma_j) \) and the mapping

\[ -\frac{i}{2} \frac{\gamma_i \gamma_j}{i} \rightarrow R_{ij} \]
(the abstract generators of the \( \tilde{R}_{4\ell} \) group) one sees that the charge conjugation matrix may be written, in general case, as a product of group elements:

\[
C = \prod_{i=1}^{\ell} \exp \left( i \prod R_{4i-3, 4i-1} \right)
\]

It follows that the charge conjugation operation induces an inner automorphism on the group \( \tilde{R}_{4\ell} \). The Dynkin diagram shows there exists an outer automorphism for the group (exchange of the two left-hand circles; it is easily seen that this automorphism is generated by the reflection operator, \( P = \gamma_{4\ell} \gamma_{4\ell+1} \).

Let us consider next the rotation covering groups of the form \( \overline{R}_{4\ell+2} \). Just as above we examine the explicit charge conjugation matrix and see that now \( C \) consists of a product of an odd number of \( \gamma_i \) operators. In this case the charge conjugation matrix must be written as a product of a rotation (i.e., group element) and the reflection operator \( P \); that is:

\[
C = \bar{P} \prod_{i=1}^{\ell} \exp \left( i \prod R_{4i-2, 4i} \right)
\]

where \( P = \gamma_{4\ell+2} \gamma_{4\ell+3} \). The charge conjugation operation \( C \) is therefore an outer automorphism for the group \( \overline{R}_{4\ell+2} \) and from the Dynkin diagram, the only such.

Finally let us examine the exceptional group \( \overline{R}_6 \). It follows from the earlier discussion of \( \overline{R}_{4\ell} \) that the operation \( C \) is an inner automorphism.

From the Dynkin diagram:

```
  3
  2
  1
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it is clear that the outer automorphisms of the group \( \overline{R}_8 \) (each corresponding to a symmetry of the diagram) form the symmetric group on 3 objects, \( S_3 \). The group \( S_3 \) may be generated by 2 operators \( P_{12} \) and \( P_{23} \), each operator corresponding to the interchange of the numbered circles in the Dynkin diagram. It is quickly seen that one of these operators is the reflection operator, i.e., \( P_{12} = \overline{1} \overline{9} \). There are then three involutory outer automorphisms in \( \overline{R}_8 \), generated by the operators \( P_{12}, P_{23} \) and \( P_{13} = P_{12} P_{23} P_{12} \).

Pais \(^{11}\) has examined this unusual case in some detail and (extending an earlier idea of Tjonno \(^{12}\)) based a dynamical scheme for elementary particles upon its properties. Briefly put the "threefold symmetry" of \( \overline{R}_8 \) allows one to consider the two projected spinors \( P_+ \Psi \) and \( P_- \overline{\Psi} \) where \( P_\pm = \frac{1}{2}(1 \pm P_{12}) \) each having eight components as basis vectors equivalent to the basis vectors of the defining \( 8 \times 8 \) representation. This is a "triality" principle which defines a unique trilinear interaction.

For the present purpose the case \( \overline{R}_8 \) offers an unusual freedom of defining very many (ten!) inequivalent isoparity operators.
IV.B. THE CENTRES OF THE GROUPS SU\(_n\), \(\mathbb{R}_n\) AND SP\(_n\)

The remaining task is the determination of the central elements of the various groups. This is easily done if one notes that on every representation the central elements are diagonal, and that on the fundamental representation (the faithful representation of lowest dimension) all central elements are distinct.

For the SU\(_n\) groups, the fundamental representation is \(\times n\); the central elements are of the form \(e^{i\Psi} \mathbb{I}\), and to be unimodular \(\Psi\) must be \((2\pi k/n)\). The centre is therefore \(C\) and is generated by the element: \(g_c = e^{2\pi i/n}\).

This element may be equivalently written as:

\[
g_c = \exp \left( \frac{2\pi i}{n} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -(n-1) \end{pmatrix} \right)
\]

in which form one sees that the general form of \(g_c\) is:

\[
g_c = \exp \left[ i \frac{2\pi}{n} (n-1) \mathbb{H}_{n-1} \right]
\]

using the result:

\[
\mathbb{H}_{n-1} = \frac{1}{n(n-1)} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -(n-1) \end{pmatrix} \mathbb{H}_{n-1}
\]

In this latter form the central element \(g_c\) is readily evaluated for any representation, using the result:

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\[ \mathcal{H}_{m,n} \mid (m) \rangle = \left( \frac{1}{\eta} \sum_{i=1}^{n-1} m_i, n - \frac{1}{\eta} \sum_{i=1}^{n} m_i, n \right) \mid (m) \rangle \]

where \((m)\) is a Gelfand state vector of the representation having the Young pattern labels \(\sum_{i=1}^{m} \cdots m_{n-1} \). It follows that:

\[ g_c = 0 \exp \left( i \frac{2\pi}{n} \sum_{i=1}^{n} m_i \right) \]

We have thus determined the explicit form of the centre of \(SU_n\) evaluated on every irreducible representation. (We see in particular that this element separates the representation into \(n\) types specified by \(\sum_{i=1}^{n} m_i \) modulo \(n\). This is the generalization of the triality type of \(SU_3\).

The condition on the possible choices \(R^2 = Z \in \text{Centre}\) was shown (in Section II) to be \(R^{-1}ZR = Z\). Under the operation \(R = \mathcal{C}\) for \(SU_n\) the diagonal generators reversed sign; it follows that \(R^{-1}g_c R = g_c^{-1}\) and hence the only admissible elements \(Z\) must be of period 2. This greatly simplifies the extension problem for \(SU_n\) and shows that \(R^2 = E\) for \(SU_{2k+1}\) and \(R^2 = E\) or \(R^2 = Z\) \((Z^2 = E)\) are the only two choices for \(SU_{2k}\).

Let us turn next to the covering group of the rotation group, \(\mathcal{R}_n\). There are two cases: \(n\) even and \(n\) odd. For the \(n = \text{odd}\) case the centre has, for the spinor representation, the central element \(g_c = \exp \left( \eta \sum_{i=1}^{n} \sum_{j} \right)\) which is of period 2. (This result is seen immediately from the fact that no element other than the identity commutes with all \(\sum_{i=1}^{n} \sum_{j} \).) Since \(R^{-1}g_c R = g_c\), both \(g_c\) and \(E\) are suitable choices for \(R^2\).
For the group $R_{2k}$, the centre is more complicated. From the fundamental representation one sees that the operators $E$ and $\gamma_{2k+1}$ commute with all the generators $i \gamma_j$, $i \neq k$, $ij1...2k$. Hence the centre contains the four elements $\pm E$, $\pm \gamma_{2k+1}$, with $\gamma_{2k+1}^2 = (-E)^k$. (One must verify that these elements can be written as group elements in general, i.e., in the form $\exp (i\phi \cdot X)$.) For $E$ one may use simply $\exp (\prod_i \gamma_i)$; for $\gamma_{2k+1}$ one may use:

$$\gamma_{2k+1} = \prod_{i=1}^{2k} \gamma_i = \prod_{j=1}^{k} \exp \left( \frac{\pi}{2} \gamma_{2j-1} \gamma_{2j} \right)$$

To simplify the discussion to follow let us denote the group element $\exp \eta \gamma_1 \gamma_2$ $\rightarrow \exp (2\pi i R_{12})$ by $A$, and the group element $\gamma_{2k+1}^k = \prod_{j=1}^{k} (\exp i\eta R_{2j-1,2j})$ by $B$. Then one sees that for $R_{4\ell}$, $\{A\} \otimes \{B\}$ generate the "vierergruppe" $C_2 \otimes C_2$, while for $R_{4\ell+2}$, the single element $B$ generates $C_4$.

In order to decide which of the elements of the centre are admissible for the square of $R$, that is (confer Section III), the condition: $R^2 = Z \in \text{Centre}$ $\rightarrow R^{-1}ZR = Z$, one must next examine the effect of the automorphisms on the centre. There are three cases:

(a) $R_{4\ell}$: ($R_8$ excluded)

The effect of charge conjugation $C$ is given by the charge conjugation matrix $C$. For $R_{4\ell}$ we recall that $C$ is a group element, hence $C$ leaves every element of the centre unchanged. Since all elements have period 2, it follows that every element of the centre is admissible.

For the remaining automorphism $P$, one sees that:

$P$: $A \rightarrow A$
$B \rightarrow AB$
$AB \rightarrow B$. 

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Thus if we take $R$ to be the automorphism generated by $C \cdot P$, only the element $A$ (and of course $E$) is admissible as the square of $C \cdot P$.

(b) $E$:

We have two additional involutory outer automorphisms, $P_{23}$ and $P_{13}$.

It is easily seen that the effect of these operations is given by:

- $P_{12} = P$: $B \leftrightarrow AB$
- $P_{23}$: $AB \leftrightarrow A$
- $P_{13}$: $A \leftrightarrow B$

For each operation there is only one admissible element of the centre (besides $E$): $A$ for $C \cdot P_{12} = R$, $B$ for $C \cdot P_{23}$ and $AB$ for $C \cdot P_{13}$.

(c) $E_{4L+2}$:

For this case charge conjugation itself generates the only outer automorphism and one finds that under $C$ the centre $\{B\} = C_2$ transforms as: $C \Rightarrow B \Rightarrow B^{-1}$. Thus one sees that the only admissible elements of the centre are $E$ and $B^2 = A$.

For the symplectic group $Sp_n$ one may use the operators $S_{(k)}^q$ ($k = 1, 3, ..., n-1$) as the generators of the fundamental $nxn$ representation, and thereby explicitly obtain the elements of the centre. The only diagonal matrices which are group elements are the identity and the element $g_c = \exp(2\pi i S_{(1)}^3)$; the centre is the group $C_2$. Both elements of $C_2$ are easily seen to be admissible for $E_{4L}$.

It is helpful to assemble the results of this section in tabular form. This is given in the table. In this table we have also included results for the exceptional groups $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$.  

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V. THE REPRESENTATIONS OF THE EXTENDED GROUPS, \( \mathcal{U}_{\text{ext}} \)

In the preceding three sections we have developed the necessary technical information to carry out explicitly the most general possible adjunction of charge-conjugation \( \mathcal{C} \) to an internal symmetry group; the extended group \( \mathcal{U}_{\text{ext}} \) is thus completely defined from these results. It is helpful, however, to carry out these results in detail, in order to obtain a clearer view of the isoperity operator \( G \). This is most conveniently done by explicitly constructing the representations of \( \mathcal{U}_{\text{ext}} \).

Let us assume that the representations of the simple Lie groups \( \text{SU}_n^\prime, \text{Sp}_n \) are all explicitly available (this is certainly true for \( \text{SU}_n^\prime \), and essentially complete for the other cases). Denote these representations by \( \mathcal{D}(a)(g) \).

Under the action of the automorphism \( R \) these representations will be carried into another representation which we denote as the conjugate representation \( \mathcal{D}(\overline{a})(g) \), to within an equivalence transformation. We have defined as the "charge-conjugation matrix" the matrix \( C \) in the equation: \( R^{-1} \mathcal{D}(a)(R) = C^{-1} \mathcal{D}(\overline{a})(C) \), noting that only for \( R = \mathcal{C} \) (the most important case) is this the matrix \( C \) of Section III.

Finally we denote by \( Z \) an admissible distinct element of the centre such that \( R^2 = Z \).

In every case we may then write the representations of the extended group as:

\[
\mathcal{D}^{[a]}(g, e) = \begin{pmatrix}
\mathcal{D}^{(a)}(g) & 0 \\
0 & C^{-1} \mathcal{D}^{(\overline{a})}(g) C
\end{pmatrix}
\]

(1)

\[
\mathcal{D}^{[\overline{a}]}(g, r) = \begin{pmatrix}
0 & Z \mathcal{D}^{(a)}(g) \\
C^{-1} \mathcal{D}^{(\overline{a})}(g) C & 0
\end{pmatrix}
\]

(2)

*) One may discuss (in the physical case, baryonic charge say) the \( \text{U}_n \) group instead of \( \text{SU}_n \), but then the only self-conjugate representations are those with \( m_m = 0 \), i.e., \( \text{SU}_n^\prime \).
There are two possibilities:

(a) If \((a) \neq (\overline{a})\) (that is, a non-self-conjugate representation of \(\mathfrak{J}\)), then \(\mathfrak{D}[A]\), above, is irreducible. Denote such representations by \(\mathfrak{D}[A,0]\).

(b) If \((a) = (\overline{a})\) (self-conjugate representation of \(\mathfrak{J}\)) then \(\mathfrak{D}[A]\) is reducible and is the sum of two inequivalent representations \(\mathfrak{D}[A,+]\) and \(\mathfrak{D}[A,-]\). These two representations have the explicit form:

\[
\begin{align*}
\mathfrak{D}[A,+] (\mathfrak{g}, e) &= \mathfrak{D}^{(a)} (\mathfrak{g}), \quad (a) \\
\mathfrak{D}[A,+] (\mathfrak{g}, r) &= \varepsilon \mathfrak{D}^{(a)} (\mathfrak{g}) \mathcal{C}^{-1} \\
\mathfrak{D}[A,-] (\overline{\mathfrak{g}}, e) &= \mathcal{C}^{-1} \mathfrak{D}^{(a)} (\mathfrak{g}) \mathcal{C} \\
\mathfrak{D}[A,-] (\mathfrak{g}, r) &= -\varepsilon \mathcal{C}^{-1} \mathfrak{D}^{(a)} (\mathfrak{g}) \mathcal{C}^{-1} \end{align*}
\]

where \(\varepsilon = \pm (Zc^2)^{1/2}\), \(\varepsilon^4 = E\).

Let us note once again that the matrices \(Z, C\) are functions of the specific representations involved, \(\mathfrak{J}\), i.e., the labels \((a)\) and moreover \(C\) depends upon the automorphism \((\mathcal{R})\).

The above set of representations is easily seen to define the complete set of irreducible representations of \(\mathfrak{J}^{\text{ext}}\). There are three types of representations denoted by \([A, 0]\), \([A, +]\) and \([A, -]\).
What now is the extended isoparity operator $G$? It is clear from these representations that the isoparity operator $G^{\text{ext}}$ is to be defined now as:

$G^{\text{ext}} = CR$. This operator is diagonal only for self-conjugate (in $\mathcal{H}$) representations. For the remaining representations, the $G^{\text{ext}}$ parity is not defined (that is, $G^{\text{ext}}$ cannot be made simultaneously sharp along with the complete set of observables of $\mathcal{H}$). To summarize:

\begin{equation}
(a) = (\tilde{a}) : \quad G^{\text{ext}} \Rightarrow \pm 1 \quad \text{for} \quad ZC^2 = E \quad (7)
\end{equation}

\begin{equation}
\Rightarrow \pm i \quad \text{for} \quad ZC^2 = -E \quad (8)
\end{equation}

\begin{equation}
(a) \neq (\tilde{a}) : \quad G^{\text{ext}} \quad \text{not defined.} \quad (9)
\end{equation}

The implications of $G^{\text{ext}} \Rightarrow \tilde{u}$ will be discussed in Section VII.
VI. THE PROPERTIES OF THE $G$ PARITY

In the preceding sections we have explicitly established the complete set of irreducible representations of the extended group, and have thus, in effect, defined the meaning of $G$ parity in the extended group. The purpose of the present section is to examine the $G$ parity so defined more closely.

There are two possibilities that arise — which we shall call "weak" and "strong" $G$ parity.

(a) Strong $G$ parity

The simplest possibility occurs for those cases (and only those) where charge conjugation is an inner automorphism \((\text{Sp}_n, R_{2k+1}, R_{4n})\), which means that all representations are self-conjugate. Taking the operation $R$ to be inner then shows that every representation of the extended group is of the form given in Eq. (V,3,6), and hence the $G$ parity is well defined:

\[
G \left| \begin{bmatrix} A \pm \end{bmatrix} \right> = \pm \epsilon \left| \begin{bmatrix} A \pm \end{bmatrix} \right>
\]

It is easily shown, using the explicit representations, that this $G$ parity is a **conserved multiplicative "parity"**. This is quite clear in the cases where $\epsilon^2 = 1$ for in these cases the extended group is a direct product, $\mathfrak{U}^{\text{ext}} = C_2 \otimes \mathfrak{U}$, and the $G$ parity is just the group $C_2$. (This corresponds to the usual $G$ parity for the isospin group, where $G \equiv C_2 \otimes C_2 \otimes C_2$ and $G^2 = +1$. This latter implies that $\mathfrak{u}^2 = C_2 \otimes C_2 = Z$, that is, the centre of the isospin group $Z = e^{2\pi i/2}$ is used to make the eigenvalues $\pm 1$.) Depending upon whether or not the centre of the symmetry group possesses one or more elements $\epsilon^2$ of period 2 (see the Table) — (remember that $C_2$ is always an element of the centre of period 2) — one may define other $G$ parities \((G \rightarrow \pm \epsilon)\) in which $G \rightarrow \pm i$ as well as $G \rightarrow \pm 1$. These parities are also multiplicatively conserved, although this is probably less obvious now since the extended group is no longer a direct product.
but rather a direct Schreier product (Method 3, Section II). (Recall that $\varepsilon^2 = 2\varepsilon^2$ is a function of the specific representation $D^{(a)}$ of $\mathcal{G}$ and whether or not $\varepsilon = +1$ or $+1$ is fully determined from the representations of $\mathcal{U}$.) The possibility of such a direct Schreier product exists for the $\mathcal{G}$ parity in the isospin case; half-integer isospins correspond to $G = \mp 1$ and integer isospins to $G = \pm 1$. As will be discussed in Section VII, the two systems ($G = \mp 1; \ G = \pm 1$) are separated by a super selection rule and hence the "extra freedom" represented by the two possible iso-parities is not of any consequence.

(b) **Weak $G$ parity**

As was seen already in the finite group example of Section II, the $G$ parity may be well defined only on certain representations of the original group $\mathcal{G}$, namely those representations which are self-conjugate. For such representations, the extended representations are of the form of Eqs. (V.3, ...6), and correspond to a sharp $G$ parity: $G [a^+] = \pm \varepsilon [A^+]$, Where $\varepsilon = \varepsilon (\{A\})$.

For the non-self-conjugate representations a $G$ parity is not defined, a situation we indicate symbolically by $G' = 0$.

The crucial question now is this: Is this $G$ parity conserved multiplicatively?

To determine the answer let us consider all direct products of the systems $\pm \varepsilon, 0$. It is easily established (using the characters of the representations in Section (V)) that:

(a) the direct product, $[A, 0] \otimes [B, 0]$, contains only representations with $G' = 0$ and/or equal numbers of representations $G' = 4\varepsilon$ and $G' = -\varepsilon$.

*) This is not to be taken as an eigenvalue equation — which does not exist! — but rather as a classification. More precisely: $\text{tr} D^{[A]}(g, r) = 0$ for representations $[A, 0]$, whereas for representations $[A^\pm]$, $\frac{\text{tr} D^{[A^\pm]}(g, r)}{|\text{tr} D^{[A^\pm]}(g, r)|} = \pm \varepsilon (\{A\})$

In this sense the use of $G \rightarrow G' = \pm \varepsilon, 0$ is a correct notation.
The direct product, $[A, 0] \otimes [B, \pm]$, also behaves in precisely the same way.

Thus the representations with $G' = 0$ behave somewhat like a zero in that the $G$ parity of the product adds to zero.

More difficult to establish, and more interesting, is the rule for a direct product of non-zero $G$ parities. Let us state the rule first, and then establish it.

(b) The direct product, $[A, \varepsilon_1] \otimes [B, \varepsilon_2]$, where $\varepsilon_1 = \pm \varepsilon$ contains representations having $\varepsilon_3 = \varepsilon_1 \varepsilon_2$ and, in addition, the possibility of equal numbers of representations with $\pm \varepsilon_1 \varepsilon_2$ as well as representations with $G' = 0$.

These rules for the properties of "weak $G$ parity" are not very satisfying from the point of view of defining a quantum number, and show that the most interesting case is probably that of strong $G$ parity. Nevertheless weak $G$ parity does represent a parity-like restriction on the extended group -- for one sees that the rules, (a) and (b), state that the weak $G$ parity is conserved in the sense that the product of weak $G$ parities is conserved as a sum of weak $G$ parities. This is a new kind of quantum number which behaves neither additively nor multiplicatively but rather mixes both behaviours. Weak $G$ parity exists formally, as these examples show -- whether or not it is a useful classification remains to be seen.

It is useful to note explicitly that the isoparity in $SU_3^{\text{ext}}$ is necessarily a weak $G$ parity, with the $F$ and $D$ operators having $\varepsilon = -$ and $+$ respectively.

The proof of rules (a) and (b) is given in an appendix.
VII. SUPERSELECTION RULES AND THE PROBENIUS-SCHUR INVARIANT

Super-selection rules were introduced by Wigner \(^{14}\) and by Wigner, Wightman, and Wick \(^{15}\) partly as an answer to problems raised by the possibility of parity \(\uparrow\). Similar considerations apply to the possibility, noted in Section V, of an isoparity having \(\uparrow\) eigenvalues. Briefly put, super-selection rules occur whenever a quantum mechanical system can be divided into two parts which can be connected by no observable operations whatsoever. The "canonical" example of such a possibility occurs in the rotation group: the two systems, half-integer and integer spins, are separated by a super-selection rule. Since the possibility of \(\uparrow\) parity arises only for half-integer systems, which are already separated by a super-selection rule from the \(\uparrow\) parities, there is no possibility of a real and imaginary parity occurring in the same system (or in the reduction of a product) and the use of parity \(\uparrow\) is completely consistent. Similar remarks apply to the extended isoparity, but this is not obviously true since more than just the rotation group enters and one must first examine the idea of super-selection rules for more general groups.

How can one extend the idea of super-selection rules to arbitrary groups? There are two ways the generalization might proceed, based upon each of two underlying properties of the rotation group. The first property is the result (denoting integer systems by 0, half-integer systems by 1) that the systems \(x_1\) and \(x_2\) combine additively modulo 2. This result can be generalized at once for the group \(SU_n\); two representations \([m_{in}]\) and \([m'_{in}]\) combine (direct product) such that the integers \(z = \sum_i m_{in}\) and \(z' = \sum_i m'_{in}\) add modulo \(n\). This property can be proved directly but is most easily seen from examining the centre of \(SU_n\). We have seen (Section IV.B) that for \(SU_n\) the centre is generated by \(A\), \(C_n = \{A\}\), with \(A\) having the explicit value \(A = \exp(\frac{2i}{n}z)1\) for any representation \([m_{in}]\). Clearly the centre is responsible for the behaviour modulo \(n\). (A similar result can be given by considering the centre of the remaining covering groups of the simple Lie groups, and shows that the centre is the crucial element to consider for factor groups of the covering group.)

\(^{14}\) This has been pointed out, for example, by Yang and Tichno \(^{16}\).
Returning to the problem of super-selection rules, the second property of the rotation group \(R_3\), which could be invoked, is the Frobenius-Schur invariant (FSI). The FSI is defined most easily from the characters of the group:

\[
\text{FSI} = \frac{\mathcal{M} \mathcal{C}}{3} \left[ \chi(g^2) \right] = \begin{cases} 
+ \frac{1}{4} \\
- \frac{1}{4} \end{cases} 
\]

(Here \(\mathcal{M}\mathcal{C}\) denotes the mean value over the group; note that the square of the elements \(g\) enters.)

For non-self-conjugate representations, the FSI is zero; for self-conjugate representation \(\text{FSI} = +1\) ("integer representations", representations that can be brought to real form) or \(\text{FSI} = -1\) ("half-integer representations", self-conjugate representations which cannot be brought to real form). For \(R_3\), only self-conjugate representations occur; in such a case the FSI then is multiplicatively conserved under direct products. Clearly the FSI invariant is an equally good characterization of the super-selection rule for angular momentum, and is equivalent to the modulus 2 rule obtained from the centre. However, for general groups the FSI result is not equivalent to the modular law of the centre, and thus the FSI is an additional discrete property characterizing the representations.

The explicit representations given in Section V suffice to make the calculation of the FSI for \(\mathfrak{g}^{\text{ext}}\) a matter of direct computation. Using the general definition, Eq. (1) above, for the FSI one finds the following results:

\[
\text{FSI} = \frac{1}{2 \dim (a)} \left\{ \mathcal{M} \mathcal{C} \sum_{g} \left[ (\mathcal{D}^{(a)}(g,e))^2 \right] + \mathcal{M} \mathcal{C} \sum_{g} \left[ (\mathcal{D}^{(r)}(g,r))^2 \right] \right\}
\]
(Using here the fact that the extended group consists of the two dis-
connected pieces \((g,e)\) and \((g,r)\), of equal volume.)

The first of these two terms is just the FSI of the representations
\((a)\) and \((\tilde{a})\). Since these representations were defined to be non-self-
conjugate the first term vanishes. The second term, by multiplying out
the representations given in Eqs. (V,1, 2), consists of two equal terms,
and hence:

\[
\text{FSI} = Z \left( \frac{\text{dim}(\tilde{a})}{2} \right) \text{tr} \left[ D^{(3)}(g) C^{-1} D^{(\tilde{a})}(g) C \right]
\]

(where we use the fact that \(Z\) is a multiple of the unit matrix.)

Using next the fact that \(C^{-1}D^{(\tilde{a})}(g)C = D^{(a)}(g^{-1})^t\), and also the
orthonormality of the representations \(D^{(a)}(g)\) averaged over the group,
one sees that:

\[
\text{FSI}(A, o) = Z
\]

(Note that \(Z^2 = 1\), so that FSI = ±1 as required.)

(b) \(\mathcal{D}[A, ±]\)

\[
\text{FSI}(A, ±) = \frac{1}{2 \text{dim}(a)} \left\{ \text{tr} \left[ \mathcal{D}^{[A, ±]}(g,e) \right] \right\}
+ \text{tr} \left[ \mathcal{D}^{[A, ±]}(g,r) \right]
\]
The first term, once again, is (to within a constant) the FSI of the representation \( (a) \) in the original group -- this time it no longer vanishes. Using the explicit form for the representations, one thus finds:

\[
FSI(A, z) = \frac{1}{\lambda} FSI(a) + \frac{e^2}{2 \dim(a)} \sum_j \text{Tr} \left[ D^{(a)}(j) C^{-1} D^{(a)}(j) C^{-1} \right]
\]

The second term is evaluated exactly as before, upon noting that \( C^2 \) belongs to the centre and therefore a multiple of the unit matrix which may be removed from the trace. The final result is then:

\[
FSI(A, z) = \frac{1}{\lambda} \left[ FSI(a) + Z(a) \right]
\]

In order to apply these results let us first note that if all representations of a group are self-conjugate \((FSI \neq 0)\), then the FSI is a multiplicative quantum number. The adjunction of the charge conjugation operator to the group \( \mathcal{J} \) has the effect of producing self-conjugate representations in \( \mathcal{J} \) out of two associated non-self-conjugate representations of the original group \( \mathcal{J} \). If now all representations are self-conjugate in the extended group, we have another conserved property with which to classify the physical systems described by \( \mathcal{J} \).

When will \( \mathcal{J} \) possess only self-conjugate representations? The answer is contained in the above results; the representations \( [A, 0] \) are always self-conjugate, the representations \( [A, \pm] \) will be self-conjugate if \( Z(a) = FSI(a) \).
To make the situation clearer, let us consider the $SU_n$ system in more detail. The representation $[m_{in}]$ of $SU_n$ will be self-conjugate if and only if: $m_{in} = m_{n-1-n,n} - m_{n+1-n,n}$. (Note that, in particular, $m_{nn}$ must be 0, i.e., $SU_n$.) This condition is very restrictive and shows, for example, that:

(a) $m_{1,n} = \text{even integer for } n \text{ odd}$

(b) $\sum_{i=1}^{n} m_{in} = \frac{n}{2} m_{n,n}$, in general.

It follows from these results that any element of the centre, (cf., Section IV,B), $Z(a)$ is always $+1$ for self-conjugate representations in $SU_n$ for $n$ odd; and $Z(a) = \pm 1$ for $n$ even.

The FSI for $SU_n$ has been determined \(\ast\) to be:

(a) FSI = $+1$ for all self-conjugate representations of $SU_n$ for $n = 2k+1$ and $n = 4\ell$,

(b) FSI = $\pm 1$ for $SU_n$ for $n = 4\ell + 2$.

The two cases in $SU_{4\ell+2}$ are distinguished by $m_{1,n}$: FSI(a) = $(-)m_{1,n}$.

With these results one may now determine which extended groups of $SU_n$ are completely self-conjugate. The condition is: $Z(a) \cdot \text{FSI}(a) = +1$. One sees that $SU_n$, $n$ odd, always satisfies this condition; similarly in $SU_{4\ell+2}$, the extension using $Z$ satisfies this condition. For $SU_{4\ell}$ the extension using $E$ (and not $Z$) satisfies the condition. In these particular extended groups, the multiplicative quantum number defined by FSI is clearly trivial for $SU_{2k+1}$ and $SU_{4\ell}$ — since all representations are assigned the number $+1$! However, in $SU_{4\ell+2}$ both cases, $\pm 1$, occur; the separation into two sub-systems split by a super-selection rule is then a non-trivial result. (This result is easy to understand if one views it as formally equivalent to the split into bosons and fermions.)

Let us remark that the validity of a super-selection rule is only as good as the validity of the original group. For example, the isospin group has a super-selection rule formally identical to the fermion-boson super-selection rule — but the isospin rule is much the weaker since it is violated by the $|\Delta I| = \frac{1}{2}$ weak interaction decays.

VIII. ANOTHER POSSIBILITY FOR AN INTERNAL PARITY

We have seen in Section VI that the most important G parity — "strong G parity" which is multiplicatively conserved — occurs only for the simplest type of extended group, the direct product or direct Schreier product. This case is of such importance that it is useful to study it in the hope of a further generalization.

Let us consider the SU₂ example; here the isparity is formally identical to the ordinary (spatial reflection) parity (since charge-conjugation is an inner automorphism). The representations of the extended group are (using \( R^2 = E \)) of the form \((0)^{\frac{1}{2}}, (\frac{1}{2})^{\frac{1}{2}}, \ldots \). The parity \((\frac{1}{2})\) is multiplicatively conserved and the assignment of a parity to a given realization of a representation \( J \) is completely arbitrary.

This is not the whole story, however, since the usual discussion concludes with the specific parity assignment \((-)^{\ell}\) for systems with positive internal parity; no such assignment is made for half-integer angular momentum. This poses a question: what property of the group \( SU_2/O_2 \) (since integer angular momenta are singled out) is responsible for the consistency of the assignment \( \text{parity} = (-) \)? The answer is not far to seek, for it lies in the well-known property of the specific "parity-preserving" Wigner coefficient \( \epsilon_{\ell_1 \ell_2 \ell_3}^{\ell_1 \ell_2 \ell_3} \) which vanishes unless \( \ell_1 + \ell_2 + \ell_3 = 0 \) mod 2.

If we examine this answer more closely, however, we see that it depends in an essential way on the following ideas:

(a) The integer angular momentum states of the original group \( SU_2 \) are put in one-to-one correspondence with the states of a larger group \( SU_2 \times SU_2 \). Specifically

\[
\begin{align*}
SU_2 & \quad |\ell \, m\rangle \quad \rightarrow \quad SU_2 \otimes SU_2 \\
|D^\ell_{m,0}\rangle
\end{align*}
\]

(where \( D^\ell \) is a rotation matrix);
(b) the parity \((-)^\ell\) is assigned to these associated states;

(c) parity conservation is now an automatic consequence of the mapping onto the matrices:

\[
\prod_{m_{1,0}}^\ell \times \prod_{m_{2,0}}^\ell = \sum_{m_{3,0}}^\ell C_{m_{1,0} m_{2,0}}^{\ell_1 \ell_2 \ell_3} \prod_{m_{3,0}}^\ell C_{000}^{\ell_1 \ell_2 \ell_3}
\]

using now the property, mentioned above, of the parity-preserving Wigner coefficient \(C_{000}^{\ell_1 \ell_2 \ell_3}\).

This technique thus assigns the parity \((-)^\ell\) to an abstract state \(|\ell m\rangle\) of the original group in a manner which is automatically consistent. The merit of our discussion of this very familiar result is that it suggests that a similar technique may be applied to a more general situation. That such a possibility does exist in general is clear from the fact that the representation matrices of an arbitrary group \(\mathcal{G}\) are themselves state vectors of the direct product group \(\mathcal{G} \otimes \mathcal{G}\). It is equally clear why this technique is not contained as a special case of the general isoparity extension problem treated at length earlier; the point being that the group \(\mathcal{G}\) has now been enlarged to \(\mathcal{G} \otimes \mathcal{G}\) which lies outside the purview of the earlier extension problem.

To illustrate the possibilities contained in this new technique let us consider the \(\text{SU}_3\) group as an example. The isoparity for the (unique) extended group, \(\text{SU}_{3}^{\text{ext}}\), was of the "weak G parity" type, and thus not very satisfying. In essence, what we would like to obtain for \(\text{SU}_3\) is a sort of parity for octets, say, which distinguishes two octets in a manner similar to the distinction between the \(F\) and \(D\) octet (tensor) operators. This is just the distinction which the weak \(G\) parity makes for octets, but at the price of adjoining conjugation and hence obtaining only weak \(G\) parity conservation rules.

By the technique discussed above, however, we have now the possibility — for the eightfold way \((\text{SU}_3/C_3)\) representations only — of introducing a new parity by the association:
\[
\begin{bmatrix}
[n' \quad q' \quad 0]
\end{bmatrix}
\begin{pmatrix}
(m)
\end{pmatrix}
\rightarrow
\begin{bmatrix}
[n' \quad q' \quad 0]
\end{bmatrix}
\begin{pmatrix}
(m) ; (m')
\end{pmatrix}
\]

(SU_3/C_2 state)  

(state vector of SU_3xSU_3)

By analogy to the rotation group example which had \( m' = 0 \) we restrict the \( (m') \) in this association to have the zero additive quantum numbers \( I_z = Y = 0 \). (It is this restriction which is responsible for the limitation to \( SU_3/C_2 \) -- only the representations of the eightfold way have always at least one state with \( I_z = Y = 0 \).) The restriction to \( I_z = Y = 0 \) does not determine \( (m') \) completely; we have now several possibilities in general, namely the number of possible (integer) \( I \) spins contained in the representation \( (pq0) \) having \( Y = 0 \). (This number is easily determined. Every representation of the eightfold way is either of the form \( [2k+3\ell \quad k0] \) or of the conjugate form \( [2k+3\ell \quad k*3\ell \quad 0] \). In either case the number of possibilities is \( k+1 \).

To determine the multiplication law of this system one uses the direct product reduction:

\[
\begin{align*}
\mathcal{D}^{[2k'+3\ell' \quad k' \quad 0]}_{(m')} ; \begin{pmatrix}
k' + \ell' + f' \k' + \ell' - f' 
\end{pmatrix} \times \mathcal{D}^{[2k+3\ell \quad k \quad 0]}_{(m)} ; \begin{pmatrix}
k + \ell + f \k + \ell - f 
\end{pmatrix}
\end{align*}
\]

\[
= \sum_{\mathcal{P} \mathcal{Q} \mathcal{M}} \left\langle \begin{array}{ccc}
\mathcal{P} & \mathcal{Q} & 0 \\
(m) & (\wedge) & (m')
\end{array} \right| \left[ \begin{array}{c}
[2k+3\ell \quad k \quad 0] \\
(m)
\end{array} \right] \left[ \begin{array}{c}
[2k'+3\ell' \quad k' \quad 0] \\
(m')
\end{array} \right]
\]

\[
\mathcal{D}^{[\mathcal{P} \mathcal{Q} \mathcal{O}]}_{(m)} ; \begin{pmatrix}
\frac{4}{3}(\mathcal{P} + \mathcal{Q}) + \mathcal{M} \\
\frac{1}{3}(\mathcal{P} + \mathcal{Q}) - \mathcal{M}
\end{pmatrix}
\]

\[
\left\langle \begin{array}{ccc}
\mathcal{P} & \mathcal{Q} & 0 \\
(\frac{1}{3}(\mathcal{P} + \mathcal{Q}) + \mathcal{M}) & (\frac{1}{3}(\mathcal{P} + \mathcal{Q}) - \mathcal{M})
\end{array} \right| \left[ \begin{array}{c}
[2k+3\ell \quad k \quad 0] \\
(k + \ell + f \quad k + \ell - f)
\end{array} \right] \left[ \begin{array}{c}
[2k'+3\ell' \quad k' \quad 0] \\
(k' + \ell' + f' \quad k' + \ell' - f')
\end{array} \right]
\]
where the \( \langle \ldots | \ldots \rangle \) are SU\(_3\) Wigner coefficients and the \((\Lambda)\) quantum numbers determine a canonical enumeration of the multiplicity problem, as discussed in considerable detail in Ref. 17).

The result given above is rather formidable in appearance, but the meaning is clear by analogy to the SU\(_2\) result given earlier. The essential feature of this direct product, for us, is that the SU\(_3\) Wigner coefficients factor; the SU\(_2\) factor is the usual Wigner coefficient. Thus it follows that we have the particular coefficient \( \Phi^{ff'f}_{\Omega000} \) appearing once again. This coefficient then determines the rule for the conservation of the quantum number \( f \). Since there are now several possibilities for \( f \) (unlike the SU\(_2\) example where \( f = \ell \) was determined by the representation labels) one may consider several possible rules for this quantum number:

(a) the weakest assignment is to assign only an "\( f \) parity", \((-)^{f}\), to the various states. Then this parity is multiplicatively conserved. Such an assignment is not exhaustive since there will exist in general several states of the same \( f \) parity belonging to a given representation.

(b) A stronger assignment is to consider \( f \) itself as a quantum number, with an \( f \) parity \((-)^{f}\). Then the rule for \( f \) is given by the familiar rules of orbital angular momenta, \((f\) is a constant for a given representation and is the isospin value of one arbitrary \( I_z = Y = 0 \) state contained in the given representation).

(Note that in both (a) and (b) the possibility of an intrinsic \( f \) parity has been disregarded.)

To summarize: one may assign to the states \( \left[ \begin{array}{c} 2k+3 \cr \ell \cr \ell' \cr \ell+k \cr \ell'+k \cr \ell+1 \cr \ell+1 \cr \ell+1 \cr \end{array} \right] \) of the octet model an additional "internal" \(^*)\) quantum number \( f \) where \( f = \ell, \ell + 1, \ldots, \ell + k \) and an \( f \) parity \((-)^{f}\), and this internal quantum number obeys the composition rules of orbital angular momenta and parity. (For conjugate states

\(^*)\) It might help in understanding this quantum number to note the close similarity between our construction and the symmetric top (rotator) problem in angular momentum. We have, in effect, introduced an SU\(_3\) system with "internal" isospin.
\[
\begin{bmatrix}
2k+3 & k+3 \\
\ell & 0
\end{bmatrix}
\]
the \( f \) quantum number also ranges over the same values \( f = \ell \ldots \ell + k \). All states of the octet model are of one of these two forms.)

Whether or not this possibility for an additional quantum number in the octet model is of value or not depends on the assignment to experimentally observed systems. Tentatively one might consider the following assignments:

(a) Singlet \([[00]]\): \( f = 0 \) only; parity +.

(b) Octet \([[21]]\): \( f = 0, 1 \)
assign baryons \( f = 1 \) parity -.
mesons \( f = 0 \) parity +.

(c) Decuplet \([[30]]\): \( f = 1 \) only, parity -.
Decuplet \([[33]]\): \( f = 1 \) only, parity -.

From the octet assignments alone two conclusions can already be drawn.

(1) The singlet cannot be produced from a baryon octet-meson octet interaction. It seems reasonable to take the singlet to be a meson, in accord with the SU\(_6\) assignment (except if in pairs).

(2) The decuplet \([[30]]\) necessarily is a baryon, since it cannot be produced from meson octets alone (except if in pairs).

Since our purpose was only to illustrate another possibility for the introduction of discrete, parity-like, quantum numbers we shall not consider the physical implication of the above octet example further.
ACKNOWLEDGEMENTS

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APPENDIX

We shall prove here the validity of the "weak G parity" conservation rules only for the $SU_3$ case, since this is typical of the considerations that enter.

One needs only to consider explicitly the direct product of the group element $(e,r)$ belonging to $\mathcal{J}_{\text{ext}}$. For the representations having $G' = \mathcal{E} = \mathcal{E}_1$ (which requires that $\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \mathcal{A} \end{bmatrix}$) the element $(e,r)$ corresponds to a matrix whose trace is:

$$\text{tr} \left( \mathcal{D}^{(A,\varepsilon)}(e,r) \right) = \varepsilon_1 \cdot \text{tr} \left( C^{-1} \right)$$

(The matrix $C$ has been explicitly given in Section (III).)

Using the fact that every self-conjugate $SU_3$ representation has the form: $[A] = [2kk0]$, we find that:

$$\text{tr} \left( C^{-1} \right) = (-)^k (k + 1)$$

(This result can be interpreted — aside from sign — as the number of self-conjugate states belonging to the representation.)

Consider next the direct product ($[A] = [2kk0]$ and $[B] = [\ell \ell 0]$):

$$\begin{align*}
\text{tr} & \left[ \mathcal{D}^{(A,\varepsilon)}(e,r) \otimes \mathcal{D}^{(B,\varepsilon)}(e,r) \right] \\
& = \left( \text{tr} \mathcal{D}^{(A,\varepsilon)}(e,r) \right) \cdot \left( \text{tr} \mathcal{D}^{(B,\varepsilon)}(e,r) \right) \\
& = \varepsilon_1 (-)^k (k + 1) \cdot \varepsilon_2 (-)^\ell (\ell + 1)
\end{align*}$$
To proceed further we must recall that for $SU_3$, the direct product $[2kk0] \otimes [2\ell \ell 0]$ contains the self-conjugate representations $[2mm0]$ with $m = k+\ell, k+\ell-1, \ldots, k-\ell$ with the corresponding multiplicities $1, 2, 3, \ldots, \ell+1, \ell, \ldots, 2, 1$.

Since the non-self-conjugate representations contained in the direct product have zero trace for the element $(e, x)^{\ell}$, we must satisfy the equation:

$$\epsilon_1 \epsilon_2 (-)^{k+\ell} (k+1)(\ell+1) = \sum_{m = k-\ell}^{k+\ell} \epsilon_m (-)^m (m+1)$$

where the repetitions correspond to the multiplicity given above. By induction, one shows that the solution is given by $\epsilon_m (-)^m = \epsilon_1 \epsilon_2 (-)^{k+\ell}$ for the first occurrence of the representation $[2mm0]$ with the remaining occurrences taking the signs $-\epsilon_m, +\epsilon_m, \ldots$.

Thus the terms in the sum cancel in pairs, and only the representations with odd multiplicity survive, each once. The result is the identity, familiar from the addition of angular momenta, that:

$$\epsilon_1 \epsilon_2 (-)^{k+\ell} (k+1)(\ell+1) =$$

$$= \epsilon_{k+\ell} (-)^{k+\ell} (k+\ell+1) + \epsilon_{k+\ell-1} (-)^{k+\ell-1} (k+\ell-1) + \ldots + \epsilon_{k-\ell} (-)^{k-\ell} (k-\ell+1)$$

$$= (-)^{k+\ell} \sum_{i=0}^{2\ell} (k-\ell+1+2i) = (-)^{k+\ell} (k+1)(\ell+1)$$

Thus $\epsilon_f = \epsilon_1 \epsilon_2$, i.e., the first occurrence of the representations $[2mm0]$ have the same $G$ parity, which is conserved multiplicatively for these representations. The remaining representations have either $G$ parity $= 0$ or contain equal numbers of $G$ parities $\pm \epsilon_1 \epsilon_2$.

This establishes rule (b) of Section (VI) for the $SU_3$ case, and thus completes the proof of the weak $G$ parity rules for the extension $\mathcal{U}$ of $SU_3$. 1002
<table>
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<tr>
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<th>$n = \frac{p^{k-1} \beta}{(\alpha \beta)}$</th>
<th>$SU_n$</th>
<th>$R_{4k-1} \cdot (\alpha &gt; 0)$</th>
<th>$R_{4k} \cdot (\ell &gt; c)$</th>
<th>$R_6$</th>
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<tr>
<td>Centres</td>
<td>$C_n = {A}$</td>
<td>$C_2 = {A}$</td>
<td>$C_4 = {A}$</td>
<td>$V_4 = C_2 \times C_2 = {A} \times {B}$</td>
<td>$C_2 = {A}$</td>
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<td>Charge conjugations belong to</td>
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<td>$Z$</td>
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<tr>
<td>Number of inequivalent parities of each type</td>
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<td>$3$</td>
<td>$4$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

Involuntary Automorphisms $R$, Centres, Admissible elements $Z$ for $R^2 = Z$ - Types of Parities
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