Exact Black-Hole Solution
With Self-Interacting Scalar Field

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Abstract

Einstein gravity minimally coupled to a self-interacting scalar field is investigated in the static and isotropic situation. We explicitly construct in partially closed form a new black-hole solution with exponentially decaying scalar hair. The symmetric interaction potential has a triple-well shape with a smooth but non-analytic minimum at vanishing field. We present numerical data as well as double series expansions around spatial infinity.
It is known for more than 20 years that isotropic and static solutions of Einstein’s equations are very rigid in nature. In vacuo, with $T_{\mu\nu}=0$, where isotropy already implies time-independence, the Schwarzschild metric $g^{(s)}_{\mu\nu}(r)$ is in fact the unique asymptotically flat solution, depending on the two parameters $r_0$ (location of the singularity) and $r_s$ (location of the event horizon). The situation is less clear in the presence of matter, although partial results exist for gravity coupled to Maxwell, Yang-Mills, and/or scalar fields of dilaton, axion or Higgs type. For a review see Ref. [1]. The so-called “no-hair” theorems severely restrict the static field configurations outside the horizon, completely classifying regular and asymptotically flat black-hole solutions by a few conserved charges such as total mass, angular momentum, electric and magnetic charges [2].

The most simple example is that of a minimally gravitationally coupled real scalar field $\phi$ enjoying some self-interaction $V(\phi)$. We are interested in spherically symmetric and static field configurations $(g_{\mu\nu}(r), \phi(r))$, where $r$ denotes the radial coordinate. The Minkowski metric $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ determines our sign convention. As $r \to \infty$, the scalar function $\phi(r)$ approaches a constant which we may normalize to zero. Asymptotic flatness then demands that $V(0) = V'(0) = 0$, i.e. a vanishing cosmological constant and a local extremum of the interaction potential at the origin $\phi=0$.

For this case a scalar no-hair theorem can be demonstrated by a simple argument due to Bekenstein [3]. The scalar field equation, \footnote{We abbreviate $\sqrt{g} = \sqrt{-\det g_{\mu\nu}}$.}

$$\partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \phi = \sqrt{g} V'(\phi) \quad ,$$

receives only contributions from $\mu = \nu = 0$. After multiplying with $\phi(r)$ and integrating from the horizon to infinity, a partial integration yields

$$\left[ \phi g^{rr} \sqrt{g} \partial_r \phi \right]_{r_s}^{r_0} - \int_{r_s}^{r_0} dr \sqrt{g} g^{rr} (\partial_r \phi)^2 = \int_{r_s}^{r_0} dr \sqrt{g} \phi V'(\phi) \quad .$$

The horizon $r_s$ is defined by the largest zero of $g^{rr}$, so that $g^{rr} \geq 0$ in the integration domain. Assuming regularity of $\sqrt{g}$ and $\phi$ at the horizon as well as a fall-off of $\phi(r) = \alpha r^{-1/2}$ for $r \to \infty$, we can drop the boundary terms in eq. (2). The integrands on both sides of the equation are clearly non-negative, provided that

$$\phi(r) V'(\phi(r)) \geq 0 \quad \text{for} \quad r \geq r_s \quad .$$

Since the left-hand integral of eq. (2) comes with a negative sign, both integrals must be zero and, hence, both integrands vanish identically. It follows that the scalar field sits at its asymptotic value, $\phi(r) \equiv 0$, so that merely the Schwarzschild metric results. This eliminates the possibility of non-trivial scalar deformations of the Schwarzschild black
hole. Besides the reasonable regularity and fall-off assumptions above, the non-trivial condition going into this no-hair argument is eq. (3) which means that the potential function must not have a local maximum in the $\phi$-range probed outside the horizon.

In this paper we shall look at the situation where the condition of eq. (3) is violated. Is it possible to improve on the scalar no-hair theorem by weakening or dropping this condition? Or can one find a non-trivial solution for a potential with a local maximum, showing that eq. (3) is indeed essential? We shall provide conclusive evidence for the second choice by explicitly constructing such a solution of the coupled Einstein-scalar equations.

Our starting point is the generalized Einstein-Hilbert action

$$ S[g, \phi] = \int d^4x \sqrt{g} \left[ R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right] . \quad (4) $$

It is extremized by eq. (1) and

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} \left( \frac{1}{2} \partial_\phi \cdot \partial_\phi + V \right) \equiv -T_{\mu\nu} \quad (5) $$

which may be simplified to

$$ R_{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} V(\phi) = 0 . \quad (6) $$

In the isotropic and static case all field degrees of freedom are functions of the radial coordinate $r$ only, and the metric can be reduced to two functions by residual coordinate transformations. The field configuration is then given by

$$ \phi = \phi(r) \quad \text{and} \quad ds^2 = -G(r) dt^2 + G(r)^{-1} dr^2 + S(r)^2 d\Omega^2 \quad (7) $$

where we have chosen a somewhat non-standard gauge [4]. We often use

$$ \sigma(r) = -\ln S(r) \quad \Leftrightarrow \quad S(r) = e^{-\sigma(r)} \quad (8) $$

instead of $S(r)$. Eqs. (1) and (6) reduce to four coupled ordinary second-order differential equations, three of which are independent. We chose

$$ \sigma'' - \sigma'^2 + \frac{1}{4} \phi'^2 = 0 $$

$$ G'' - 2G'\sigma' - V(\phi) = 0 $$

$$ G'' - 2G(2\sigma'^2 - \sigma'') + 2e^{\sigma} = 0 , \quad (9) $$

where the prime denotes a derivative with respect to $r$. Obviously, we must have $V(-\phi) = V(\phi)$. Given the potential $V(\phi)$, Eqs. (9) determine the three functions $\phi(r)$, $G(r)$ and $\sigma(r)$ (or $S(r)$). Their asymptotical behavior for $r \to \infty$ is

$$ \phi(r) \to 0 , \quad G(r) \sim 1 - \frac{h}{r} , \quad \sigma(r) \sim -\ln r \quad (10) $$
and we fix the freedom of shifting $r$ by taking the black-hole singularity at the origin, i.e. putting $G(0) = 0$.

The solution to Eqs. (9) is known analytically only when the scalar field is free and massless. Putting $V(\phi) = 0$, we can solve the second and third of Eqs. (9) for $G(r)$ in terms of $\sigma(r)$. Eliminating $G(r)$ one arrives at an equation for $\sigma(r)$ only,

\[(r-k)\sigma''(r) - (2r-h)\sigma'(r) = \sigma'(r) = 0 , \tag{11}\]

with integration constants $k$ and $h$ satisfying $h < 2k$. This homogeneous Ricatti equation for $\sigma'(r)$ is easily solved by

\[
\sigma(r) = -\frac{k}{h} \ln r - (1 - \frac{k}{h}) \ln(r - h),
\]

\[
= -\frac{k}{h} \ln r - (1 - \frac{k}{h}) \ln(1 - \frac{r}{h}) , \tag{12}\]

where two further integration constants are fixed by the $r \to \infty$ asymptotics and taking the left-most singularity to sit at $r=0$. Eqs. (9) and (10) then directly yield $^2$

\[
G(r) = (1 - \frac{k}{h})^{\frac{k}{h} - 1}
\]

\[
\phi(r) = \pm 2\sqrt{\frac{2}{\frac{k}{h} - 1}} \ln(1 - \frac{k}{h}) . \tag{13}\]

This two-parameter family of solutions was found by Buchdahl [5] already in 1959, though by a completely different route. At $r = h$, however, the scalar field and the metric develop physical singularities, so that the no-hair theorem above is avoided. The only exception obtains for $k = h$ and is, as expected, nothing but the Schwarzschild metric with horizon at $r_s = h$ and $\phi(r) \equiv 0$.

When the potential $V(\phi)$ is not constant, an analytic solution to Eqs. (9) seems hard to come by, even for simple cases of $V$. To keep the choice of potential open, it is useful to translate the $V$-dependence in Eqs. (9) to a fourth function

\[
U(r) := V(\phi(r)) \tag{14}\]

so that a solution is given by the quartet $(\phi, G, \sigma, U)$. Now it turns out to be fruitful to reverse the roles of the potential and the scalar field in our problem. In other words, we are going to first choose some fixed scalar field configuration $\phi(r)$ and then seek to determine the corresponding metric and potential function $U(r)$ from which the field potential $V(\phi)$ can be reconstructed. $^3$ The advantage of this approach is that the first of Eqs. (9) is the simplest and can actually be solved analytically for a certain function $\phi(r)$.

$^2$The location $h$ of the second singularity is then positive. The sign of the scalar field is undetermined.

$^3$in the region where the chosen $\phi$ takes values.
To be more precise, the inhomogeneous Ricatti equation

$$\sigma''(r) - \sigma'(r)^2 + \frac{1}{4}\phi'(r)^2 = 0 \tag{15}$$

is analytically solvable for the ansatz

$$\phi(r) = \phi_0 e^{-mr} \quad \text{with} \quad m > 0 \ . \tag{16}$$

One finds

$$m \ S(r) \equiv m \ e^{-\sigma(r)} = K_0(\frac{1}{2}\phi(r)) + (\ln \frac{\phi}{2} + \gamma) I_0(\frac{1}{2}\phi(r)) \tag{17}$$

where $K_0$ and $I_0$ are modified Bessel functions, and the integration constants are fixed by demanding $S(r \to \infty) \sim r$.

The associated functions $G(r)$ and $U(r)$ can in principle be computed by going into the remaining two of Eqs. (9), which can be brought to the integral form

$$G(r) = S(r)^2 \int_r^\infty \frac{dr'}{S(r')^4} (2r' - 3h) \tag{18}$$

$$U(r) = \frac{1}{2}G(r) \left(\frac{S(r)^4}{S(r)^2}\right)' + (2r - 3h) \left(\frac{1}{S(r)^2}\right)' - \frac{2}{S(r)^2} \ ,$$

with one integration constant $h$. Expanding the Bessel functions for small argument (around $r=\infty$), Eq. (17) becomes

$$S(r) = r + \sum_{k=1}^{\infty} \frac{1}{k!^2} (\frac{1}{2}\phi_0)^{2k} \left[r + \frac{1}{m} \sum_{j=1}^{k} \frac{1}{j} \right] e^{-2kmr} \ . \tag{19}$$

Eq. (18) then produces series expansions in the form

$$\sum_{k=1}^{\infty} r f_k(\frac{1}{r}) e^{-2kmr} + \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} r^2 f_{k\ell}(\frac{1}{r}) \ Ei(-2\ell mr) e^{-2kmr} \tag{20}$$

for $G(r) - (1 + \frac{1}{r})$ and $U(r)$, with polynomials $f_k$ and $f_{k\ell}$. The integration constant $h$ is related to the event horizon $r_s$ by

$$h = \frac{2 \int_{r_s}^{\infty} dr r S(r)^{-4}}{3 \int_{r_s}^{\infty} dr \ S(r)^{-4}} \ , \tag{21}$$

where the latter is defined by $G(r_s) = 0$. It is convenient to set $r_s = 1$ which amounts to measuring distance in units of horizon lengths. The remaining parameters of our solution are the scalar field amplitude $\phi_0$ and its mass parameter $m$.

We have evaluated the expansions à la Eq. (20) up to order $e^{-10mr}$ and tested them in the original Eqs. (9). The numerical accuracy outside the horizon is generally of $O(10^{-10})$ and decreases near the horizon to $O(10^{-7})$. It is instructive to plot the deviation of $S(r)$
and $G(r)$ from the Schwarzschild case. This is done in Figs. 1 and 2, for $r_s=1$, $\phi_0=1$, and three different values of $m$. It is evident that the metric is regular except for the standard Schwarzschild coordinate singularity at $r=1$.  

![Fig. 1: Deviation of $S(r)$ from Schwarzschild case](image1)

![Fig. 2: Deviation of $G(r)$ from Schwarzschild case](image2)

The most interesting object is, of course, the potential. Fig. 3 shows the function $U(r)$ for $\phi_0=1$ and three values of $m$. A closer look reveals that $U(r)$ has a local maximum and for $r \to \infty$ approaches zero from above.

![Fig. 3a: Potential function $U(r)$](image3)

![Fig. 3b: Detail of $U(r)$](image4)

Composing $U(r)$ with $r(\phi) = -\frac{1}{m}\ln |\phi/\phi_0|$, one arrives at Fig. 4 which displays the interaction potential $V(\phi)$ in the range $\phi \in [0, \phi_0=1]$. The interesting region is blown up to show the local maximum of $V$. Reflection at $\phi=0$ extends the function to negative $\phi$, so that a triple-well potential results. $G(r)$ and $U(r)$ diverge at $r=0$, so that $V(\phi)$ explodes

\[\text{For large enough amplitudes } \phi_0 \text{ the metric function } G(r) \text{ develops a new zero which can then be taken as the event horizon, without qualitatively changing the discussion.}\]
for $\phi \to \phi_0$. Clearly, our solution escapes the consequences of the no-hair theorem by violating the criterion $\phi V'(\phi) > 0$, demonstrating that this condition is a necessary one.

In spherical coordinates, the energy-momentum tensor of our solution reads

$$T_{\mu\nu} = \text{diag} \left( \rho G, \ p_r G^{-1}, \ p_t S^2, \ p_t S^2 \sin^2 \theta \right)$$

with energy density $\rho$, radial pressure $p_r$, and tangential pressure $p_t$ given by

$$\rho = +\frac{1}{4} G \phi'^2 + \frac{1}{2} V$$
$$p_r = +\frac{1}{4} G \phi'^2 - \frac{1}{2} V$$
$$p_t = -\frac{1}{4} G \phi'^2 - \frac{1}{2} V \quad (23)$$

Two different types of pressure occur since black-hole configurations are isotropic but not homogeneous [6]. Energy density and pressures are plotted as functions of $r$ in Fig. 5 and are seen to be regular for $r > 0$. The energy density is negative near the horizon due to the negative potential there.

Fig. 4a: Interaction potential $V(\phi)$

Fig. 4b: Detail of $V(\phi)$

Fig. 5a: Functions $\rho$, $p_r$ and $p_t$ outside the horizon

Fig. 5b: Functions $\rho$, $p_r$ and $p_t$ inside the horizon
In order to verify the consistency of our solution, we finally develop an power series expansion around \( r = \infty \). Let us first discuss the expansion of a general solution to Eqs. (9). The natural expansion parameter is \( \frac{1}{r} \). Naively assuming analyticity of all functions in \( \frac{1}{r} \) we attempt

\[ S(r) = r + \sum_{i=1}^{\infty} s_i r^i \]  

and plug this into

\[ \phi'(r)^2 = 4 S''(r)/S(r) \]  

and Eqs. (18) to obtain power series for \( \phi, G \) and \( U \). The leading terms of these expansions come out to be

\[ \phi(r) = O(r^{-k}) \implies U(r) = O(r^{-2k-2}) \implies V(\phi) = O(|\phi|^{2+k}) \]  

so that only \( k=1 \) may lead to a potential \( V \) with a local analytic minimum at the origin. However, in this case the leading coefficient of \( U(r) \) vanishes and, hence, \( V(\phi) \sim |\phi|^2 \). Clearly, we have to go beyond simple power series and give up analyticity at \( r=\infty \).

On the other hand, Eqs. (16), (19) and (20) show that our solution is essentially “non-perturbative”, since \( e^{-mr} \) is not analytic in the expansion parameter \( \frac{1}{r} \). Hence, a double expansion, in \( \frac{1}{r} \) and \( e^{-mr} \), is needed. Keeping only the leading non-trivial order in \( e^{-mr} \), we write down an improved general ansatz

\[ \phi(r) = e^{-mr} r^a \sum_{i \geq 0} f_i r^{-i} + O(e^{-2mr}) \]

\[ \sigma(r) = -\ln r + e^{-2mr} r^b \sum_{i \geq 0} \sigma_i r^{-i} + O(e^{-4mr}) \]

\[ G(r) = 1 - \frac{h}{r} + e^{-2mr} r^c \sum_{i \geq 0} g_i r^{-i} + O(e^{-4mr}) \]

\[ U(r) = e^{-2mr} r^d \sum_{i \geq 0} u_i r^{-i} + O(e^{-4mr}) \]

and insert into Eqs. (9) to obtain equations for the coefficients \( f_i, \sigma_i, g_i, \) and \( u_i \). The leading powers come out to be

\[ 2a = b = c = d \]  

Our solution was obtained by taking

\[ a = 0 \quad \text{and} \quad f_i = \phi_0 \delta_{i0} \quad \forall i \]  

\[ A \text{ constant} (s_0) \text{ term contradicts } G(r) = 1 - \frac{h}{r} + O(r^{-2}) \text{ with } h \text{ appearing in Eqs. (18).} \]
which yields
\[ \sigma_0 = -\frac{1}{10} \phi_0^2 \quad , \quad \sigma_1 = -\frac{1}{100} \phi_0^2 \quad , \quad \sigma_i = 0 \quad \forall i \geq 2 \quad (30) \]
and
\[
\begin{align*}
\sigma_{0} &= \frac{1}{8} \phi_0^2 \\
\sigma_{1} &= -\frac{1}{8} m^{-1} (1 + 2h) \phi_0^2 \\
\sigma_{2} &= \frac{1}{8} m^{-2} (1 + 2h) \phi_0^2 \\
\sigma_{3} &= -\frac{1}{8} m^{-3} (2 + 3h) \phi_0^2 \\
&\vdots
\end{align*}
\quad (31)
\]
This result agrees perfectly with the series in Eqs. (19) and (20),
\[
\begin{align*}
\sigma(r) &= -\ln r - \frac{1}{10} \phi_0^2 e^{-2mr} \left[ 1 + \frac{1}{mr} \right] + O(e^{-4mr}) \\
G(r) &= 1 - \frac{h}{r} + \frac{1}{5} \phi_0^2 r^2 Ei(-2mr) m^2(2 + 3h) \\
&\quad + \frac{1}{45} \phi_0^2 e^{-2mr} \left[ 4m(2 + 3h) r + (2 - 6h) - \frac{2}{m} \frac{1}{r} + \frac{3h}{m} + \frac{1}{r^2} \right] + O(e^{-4mr}) \\
U(r) &= \phi_0^2 Ei(-2mr) m^2(2 + 3h) \\
&\quad + \frac{1}{5} \phi_0^2 e^{-2mr} m^2 \left[ 1 + \frac{2h}{m} + O\left(\frac{1}{r^2}\right) \right] + O(e^{-4mr}) \quad ,
\end{align*}
\]
after expanding the exponential-integrals.

Let us take a look at the potential \( V(\phi) \). The non-analyticities of \( U(r) \sim e^{-2mr} \) and \( r(\phi) = -\frac{1}{m} \ln |\phi/\phi_0| \) tend to compensate each other so that \( V(\phi) \sim \frac{1}{2} m \phi^2 \). However, the additional power series \( \sum_i u_i r^i \) introduces logarithmic terms into the potential
\[
V(\phi) = \frac{1}{2} m \phi^2 \left[ 1 + \frac{2 + 3h}{2 \ln |\phi/\phi_0|} + \frac{2 + 3h}{2 \ln^2 |\phi/\phi_0|} + \ldots \right] + O(\phi^4) \quad (33)
\]
which are necessary to match the expansions. It should be noted that \( V(\phi) \) is nevertheless \( C^\infty \) smooth at \( \phi = 0 \). This exemplifies how non-trivial solutions require a non-analyticity at \( r = \infty \).

To summarize, we have constructed an explicit static and isotropic black-hole solution to Einstein’s equations minimally coupled to a self-interacting scalar field. It is regular outside the central singularity, with the standard coordinate singularity signalling a regular event horizon. An exponentially decaying scalar field configuration belongs to an interaction potential which, firstly, possesses a local maximum and, secondly, is perfectly smooth but not analytic at an (asymptotically attained) local minimum. The first feature circumvents the no-hair theorem while the second one seems to be generic. It would be interesting and physically relevant to investigate the stability properties of this solution. Work in this direction is in progress.
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