DOUBLE SCALING LIMITS AND
AIRY FUNCTIONS FROM QUASIHOMOGENEOUS
SINGULARITIES OF VECTOR SIGMA MODELS

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Abstract

A sufficiently large class of vector sigma models admitting r variable singularities is defined. It is shown how quasihomogeneity induces filtrations on algebras of functions and Lie algebras of infinitesimal diffeomorphisms. From the filtrations double scaling limits and Airy functions are derived.

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1. Introduction

In our previous work [1,2] we have shown that in the case of elementary singularities and the unimodular singularity X_9 appearing in one and two vector sigma models one can define double scaling limits and Airy functions to describe the large N limit for the partition functions of these models. Renormalization group equations and beta functions can then be derived a posteriori.

In this work we generalize this approach to quite a large class of singularities namely the quasihomogeneous nondegenerate ones that include the elementary singularities as trivial and X_9 as almost trivial examples. New features show up in this generalization. In the Lie algebra of infinitesimal diffeomorphisms there is a finite "triangular" Lie subalgebra which induces a set of string equations that hold for the Airy function. They can be used to reduce the number of arguments on which this function depends. The concepts of renormalization group equation and beta functions are maintained, the critical indices follow from the type of quasihomogeneity.

We have not constructed explicit examples. But from the experience with such constructions collected in [2] we know that the main algebraic task in the construction is the elimination of the angular variables. We are sure that any quasihomogeneous nondegenerate singularity can be found in the class of models we propose in Section 2 of this work.

The stronger algebraization of the approach presented here as compared with [1,2] brings us closer to our final aim: applying these ideas to matrix models.

2. The sigma models

We take over the notations of [2]. We assume that our models have r vector fields

$$\phi_1, \phi_2, \dots, \phi_r \in \mathbb{R}_N \quad (2.1)$$

from which we derive radial

$$x_i = \phi_i^2 \quad (2.2)$$

and angular variables

$$t_{ij} = \frac{(\phi_i \cdot \phi_j)^2}{x_i x_j} \quad (2.3)$$

We define the action to be Ng

$$g(\phi_1, \phi_2, \dots, \phi_r) = \frac{1}{2} \left\{ \sum_{i=1}^r g_i(x_i) + \sum_{i < j} s_{ij}(x_i x_j t_{ij}) \right\} \quad (2.4)$$

and the partition function

$$Z_N^{(r)} = \int \prod_{i=1}^r d\phi_i \exp \{-Ng(\phi_1, \phi_2, \dots, \phi_r)\} \quad (2.5)$$

Here $d\phi$ is the Lebesgue measure. We are interested only in large N .

We can ascribe a graph to such an action, representing the functions g_i by vertices and the s_{ij} by links between vertices i and j (Fig. 1).

Fig. 1: Graph for the action. A link is not drawn if the function s_{ij} is zero.

All integrations over variables that are not fixed by $\{x_i, t_{ij}\}$ are performed. Let

$$e_i = \frac{\phi_i}{\sqrt{x_i}} \quad (2.6)$$

Then we evaluate

$$F(\{t_{ij}\}) = \int \prod_{i=1}^r d\Omega(e_i) \prod_{j < k} \delta((e_j \cdot e_k)^2 - t_{jk}) \quad (2.7)$$

where the product $\prod_{j < k}$ goes over all (j, k) for which s_{ij} is not zero. The integrations (2.7) are elementary. In the sequel Ω_N denotes the area of the unit sphere in $\mathbb{R}_N$. We give some examples.

$$r = 2: \quad \begin{array}{c} \bullet \text{---} \bullet \\ 1 \quad s_{12} \quad 2 \end{array} \quad (\text{see [2]})$$

$$F(t_{12}) = \frac{1}{2} \Omega_{N-1} t_{12}^{-\frac{1}{2}} (1-t_{12})^{-\frac{3}{2}} (\Gamma^{(2)}(e_1, e_2))^{\frac{N}{2}} \quad (2.8)$$

where $\Gamma^{(r)}(e_1, e_2, \dots, e_r)$ is the Gram determinant of $\{e_i\}$

$$\Gamma^{(2)}(e_1, e_2) = \begin{vmatrix} e_1^2 & e_1 e_2 \\ e_1 e_2 & e_2^2 \end{vmatrix} = (1 - t_{12}) \quad (2.9)$$

$$F(t_{12}, t_{23}, t_{13}) = \frac{1}{8} \Omega_N \Omega_{N-1} \Omega_{N-2} (t_{12} t_{23} t_{13})^{-\frac{1}{2}} \cdot \Gamma^{(3)}(e_1, e_2, e_3)^{\frac{N}{2}-2} \quad (2.10)$$

All signs of square roots of t_{ij} have to be admitted and summed over.

In this fashion the partition function can be represented in the form

$$Z_N^{(r)} = \Omega_N^{a_0} \Omega_{N-1}^{a_1} \Omega_{N-2}^{a_2} \dots \Omega_{N-S}^{a_S} \sum_{\text{signs}} \int \prod_{i < j} dt_{ij} \cdot f(\{t_{ij}\}) \int_0^\infty \prod_{i=1}^r \frac{dx_i}{x_i} \exp \left\{ \frac{N}{2} \psi(\{x_i\}, \{t_{ij}\}) \right\} \quad (2.11)$$

where in the r -simplex case (graph with r vertices and $\frac{1}{2} r(r-1)$ links) the "reduced action" ψ is

$$\psi = \log \left\{ \left(\prod_{i=1}^r x_i \right) \Gamma^{(r)}(e_1, \dots, e_r) \right\} - \prod_{i=1}^r g_i(x_i) - \sum_{i < j} s_{ij} (x_i x_j t_{ij}) \quad (2.12)$$

The domain $\mathcal{D}$ is obtained as follows. It contains the point where all the $\{e_i\}$ are pairwise orthogonal. Since $e_i e_j = 0$ is a branch point for $\sqrt{t_{ij}}$, we continue from here along different branches until we reach a zero of the Gram determinant belonging to this combination of signs. Finally we have to sum over all combinations of signs.

3. Singularities

At a singularity S all first derivatives of ψ vanish

$$S: \frac{\partial \psi}{\partial x_i} = \frac{\partial \psi}{\partial t_{ij}} = 0 \quad \text{all } i, (ij). \quad (3.1)$$

Such singularities denominate the large N behaviour of $Z_N^{(r)}$ (r fixed). With the shorthand

$$\zeta_{ij} = x_i x_j t_{ij} \quad (3.2)$$

we obtain from (3.1) ($s_{ij} = s_{ji}$, $t_{ij} = t_{ji}$)

$$x_i g'_i(x_i) + \sum_j \zeta_{ij} s'_{ij}(\zeta_{ij}) = 1 \quad (3.3)$$

$$\frac{t_{ij}}{\Gamma^{(r)}} \frac{\partial \Gamma^{(r)}}{\partial t_{ij}} - \zeta_{ij} s'_{ij}(\zeta_{ij}) = 0 \quad (3.4)$$

In the generic case (3.3), (3.4) are enough constraints to fix all coordinates of S

$$\dim \{\text{algebraic set } S\} = 0 \quad (3.5)$$

We are interested in nongeneric cases where functions $\{g_i, s_{ij}\}$ are chosen in a specific way to allow for a finite number of degrees of freedom for S . In the generic case S is of Morse type (Gaussian), there is simple scaling and a standard $\frac{1}{N}$ expansion which is of no interest for us here.

To next order in the Taylor series of ψ at S we obtain the Hessian $\mathcal{H}$. We choose

$$\xi_i = \frac{x_i - x_i^{(0)}}{x_i^{(0)}} \quad (3.6)$$

$$\tau_{ij} = \frac{t_{ij} - t_{ij}^{(0)}}{t_{ij}^{(0)}} \quad (3.7)$$

as variables for the Taylor series which translates S to zero. For the case (2.10) ($r = 3$) we obtain

$$-X = \begin{bmatrix} U_1 & \sigma_{12} & \sigma_{13} & \sigma_{12} & 0 & \sigma_{13} \\ & U_2 & \sigma_{23} & \sigma_{12} & \sigma_{23} & 0 \\ & & U_3 & 0 & \sigma_{23} & \sigma_{13} \\ & & & W_{12} & \tau_{2,13} & \tau_{1,23} \\ \text{symm.} & & & & W_{23} & \tau_{3,12} \\ & & & & & W_{31} \end{bmatrix} \quad (3.8)$$

where the parameters are ordered as

$$\xi_1, \xi_2, \xi_3, \tau_{12}, \tau_{23}, \tau_{31}$$

and

$$U_i = 1 + x_i^2 g_i''(x_i) + \sum_j \zeta_{ij} s_{ij}''(\zeta_{ij}) \quad (3.9)$$

$$\sigma_{ij} = \zeta_{ij} s_{ij}'(\zeta_{ij}) + \zeta_{ij}^2 s_{ij}''(\zeta_{ij}) \quad (3.10)$$

$$\tau_{i,jk} = \Gamma^{(3)-2} \left\{ t_{ij} t_{ik} - \frac{1}{2} (t_{12} t_{23} t_{13})^{\frac{1}{2}} (1 + t_{ij} + t_{ik} - t_{jk}) \right\} \quad (3.11)$$

$$W_{ij} = \zeta_{ij}^2 s_{ij}''(\zeta_{ij}) + \left[\frac{t_{ij}}{\Gamma} \frac{\partial \Gamma}{\partial t_{ij}} \right]^2 - \frac{t_{ij}^2}{\Gamma} \frac{\partial^2 \Gamma}{\partial t_{ij}^2} \quad (3.12)$$

where all arguments are evaluated at S.

The structure of the matrix X is similar in the general case. If

$$\sigma_{ij}|_S = U_i|_S = 0 \quad \text{all } i, (i,j) \quad (3.13)$$

then

$$\text{corank } X \geq r \quad (3.14)$$

We will be interested in cases where the equality sign holds.

The asymptotic evaluation of $Z_N^{(r)}$ for large N can be started in such a case by integration over the angle variables $\{t_{ij}\}$. This is a Gaussian integral and leads to a $\frac{1}{N}$ expansion with coefficient functions depending on $\{\xi_i\}$. The procedure is canonical and was performed explicitly in [2]. We concentrate now on the leading term of the $\frac{1}{N}$ expansion, we have to deal with a Morse reduced action

$$\exp \left\{ -\frac{N}{2} f(\xi) \right\} \quad (3.15)$$

We assume that this action function $f(\xi)$ lies in the neighborhood of a critical function

$$f_c(\xi) = f_\delta(\xi) + f_{>\delta}(\xi) \quad (3.16)$$

where f_δ is quasihomogeneous of type $\vec{\alpha} \in \mathbb{Q}_+^r$ and degree δ .

Let $\mathbb{K} \in \mathbb{N}_0^r$ and denote a monomial of ξ by

$$\xi^{\mathbb{K}} = \xi_1^{k_1} \xi_2^{k_2} \dots \xi_r^{k_r} \quad (3.17)$$

If we apply the multiplicative group $\mathbb{R}_+$ to ξ by

$$\xi_i \rightarrow \lambda^{\alpha_i} \xi_i \quad (3.18)$$

then

$$\xi^{\mathbb{K}} \rightarrow \lambda^{\mathbb{K} \cdot \vec{\alpha}} \xi^{\mathbb{K}} \quad (3.19)$$

The function f_δ is quasihomogeneous of type $\vec{\alpha}$ and degree δ if under substitution of (3.19)

$$f_\delta(\xi) \rightarrow \lambda^{\delta} f_\delta(\xi) \quad (3.20)$$

which means that f_δ is a linear combination of monomials (3.17) such that

$$\vec{k} \cdot \vec{\alpha} = \delta \quad (3.21)$$

The second term in (3.16) is a formal series of monomials with degree

$$\delta' > \delta.$$

Applying a diffeomorphism to f_c one can bring it to a form which is canonical. An algorithm has been developed to achieve this [3]. For example, for the bimodular singularity $W_{1,0}$ ($r = 2$) this canonical form is

$$\xi_1^4 + (a_0 + a_1 \xi_2) \xi_1^2 \xi_2^3 + \xi_2^6 \quad (3.22)$$

$(a_0 \neq \pm 2)$

The quasihomogeneous principal part is

$$\xi_1^4 + a_0 \xi_1^2 \xi_2^3 + \xi_2^6 \quad (3.23)$$

so that

$$\vec{\alpha} = \left[\frac{1}{4}, \frac{1}{6} \right], \quad \delta = 1 \quad (3.24)$$

Of course we can also choose

$$\vec{\alpha} = \left[\frac{1}{2}, \frac{1}{3} \right], \quad \delta = 2 \quad (3.25)$$

but we will usually normalize the degree δ to one. The remaining monomial in (3.22) has degree

$$\delta' = \frac{7}{6} \quad (3.26)$$

with respect to the type $\vec{\alpha}$ (3.24).

The form of the singularity given in (3.22) exemplifies a theorem [3, vol. 1, p.194] saying that singularities with quasi-homogeneous principal term f_δ can always be brought by a diffeomorphism to the form of a finite sum

$$f_c(\xi) = f_\delta(\xi) + \sum_{\vec{k}} c_{\vec{k}} \xi^{\vec{k}}$$

$$\vec{\alpha} \cdot \vec{k} > \delta \quad (3.27)$$

We assume that f in (3.15) can by an adjustment of the "coupling constants", i.e. the functional form of $\{g_{i,s_{ij}}\}$ be brought to the form f_c (3.27). In turn a slight deformation of f_c , i.e. a small deviation of the coupling constants from their critical values, gives us back f .

4. Algebras and filtrations

Consider the algebra of formal power series in r variables ξ_i

$$A = \left\{ a(\xi) = \sum_{\vec{m} \in \mathbb{N}_0^r} a_{\vec{m}} \xi^{\vec{m}} \right\} \quad (4.1)$$

$$a(\xi) b(\xi) = \sum_{\vec{m} \in \mathbb{N}_0^r} \left[\sum_{0 \leq \vec{n} \leq \vec{m}} a_{\vec{n}} b_{\vec{m}-\vec{n}} \right] \xi^{\vec{m}} \quad (4.2)$$

where

$$\vec{m} \geq \vec{n} \text{ iff } \vec{m} - \vec{n} \in \mathbb{N}_0^r \quad (4.3)$$

If

$$a_{\vec{m}} = 0 \quad \text{for } \vec{m} \cdot \vec{\alpha} < \delta \quad (4.4)$$

then $a(\xi)$ is an element of the ideal A_δ which is defined with respect to the type $\vec{\alpha}$. Obviously

$$A_\delta \supset A_{\delta'}, \quad \text{if } \delta' > \delta \quad (4.5)$$

$$A_0 = A \quad (4.6)$$

The quotient A/A_δ is called the algebra of "quasi-jets" of degree smaller δ . In this fashion the type $\vec{\alpha}$ defines a filtration of the algebra A .

Functions f generate ideals in A by their formal Taylor expansion at zero. Let

$$\left[\frac{\partial f}{\partial \xi_1}, \frac{\partial f}{\partial \xi_2}, \dots, \frac{\partial f}{\partial \xi_r} \right]$$

denote the ideal in $\mathbf{A}$ generated by the first derivatives of f . Then

$$Q_f = \mathbf{A} / \left[\frac{\partial f}{\partial \xi_1}, \dots, \frac{\partial f}{\partial \xi_r} \right] \quad (4.7)$$

is denoted the local algebra of f . It plays an important role in singularity theory [3] and

$$\mu(f) = \dim Q_f \quad (4.8)$$

is the multiplicity of the singularity represented by f . If

$$\mu(f) < \infty \quad (4.9)$$

the singularity f is called nondegenerate. We assume that the singularity of f_c is nondegenerate and quasihomogeneous.

Denote the unit vectors

$$\vec{e}_i = (0, 0, \dots, \underset{\substack{\uparrow \\ \text{position } i}}{1}, \dots, 0) \quad (4.10)$$

Then a diffeomorphism in form of a formal power series is given by r functions $\{g_1, \dots, g_r\}$

$$g_j = \xi_j + \sum_{\vec{m} > \vec{e}_j} g_{\vec{m},j} \xi^{\vec{m}} \quad (4.11)$$

g defines an action g^* on $\mathbf{A}$ by

$$f \in \mathbf{A} : g^* f = f \circ g \quad (4.12)$$

The filtration on $\mathbf{A}$ induces a filtration of diffeomorphisms. If

$$(g^* - 1)\mathbf{A}_\lambda \subset \mathbf{A}_{\lambda+d} \quad (4.13)$$

for all λ (w.r.t. $\vec{\alpha}$), then g is said to be of order d .

Let g be an infinitesimal diffeomorphism,

$$g_i(\xi) = \xi_i \text{ for all } i \neq j$$

$$g_j(\xi) = \xi_j + \epsilon \xi^{\vec{m}}, \quad \vec{m} > \vec{e}_j$$

$$(\epsilon \searrow 0) \quad (4.14)$$

then

$$f = \sum_{\vec{k} \in \mathbb{N}_0^r} a_{\vec{k}} \xi^{\vec{k}} \quad (4.15)$$

is mapped into

$$(g^* - 1)f = \epsilon \sum_{\vec{k}} a_{\vec{k}} k_j \xi^{\vec{k} - \vec{e}_j + \vec{m}} + o(\epsilon^2) \quad (4.16)$$

Derivation with respect to ϵ at $\epsilon = 0$ yields

$$\ell_{\vec{m},j} f \text{ with } \ell_{\vec{m},j} = \xi^{\vec{m}} \frac{\partial}{\partial \xi_j} \quad (4.17)$$

These operators commute such as

$$[\ell_{\vec{m},i}, \ell_{\vec{n},j}] = n_i \ell_{\vec{m} + \vec{n} - \vec{e}_i, j} - m_j \ell_{\vec{m} + \vec{n} - \vec{e}_j, i} \quad (4.18)$$

and span a Lie algebra denoted w_∞ -algebra. Whereas the elements of $\mathbf{A}$ are by their series represented as functions on $\mathbb{N}_0^r$ or the positive orthant of the cubic lattice $\mathbb{Z}^r$, the basis elements $\ell_{\vec{m},i}$ are ascribed to the links of this lattice.

The filtration on the diffeomorphisms induces a grading on the w_∞ -algebra. Let

$$\deg(\ell_{\vec{m},i}) = \vec{\alpha} \cdot (\vec{m} - \vec{e}_i) \quad (4.19)$$

so that

$$\deg \left[[\ell_{\vec{m},i}, \ell_{\vec{n},j}] \right] = \deg \ell_{\vec{m},i} + \deg \ell_{\vec{n},j} \quad (4.20)$$

This grading is with respect to the type $\vec{\alpha}$.

5. String equations

Consider a partition function

$$Z_{\text{sing}} = \int_C d^T \xi \exp [-f(\xi)] \quad (5.1)$$

where C is some chain on which the integral converges exponentially. With the transpose operators

$$\ell_{\vec{m},i}^T = -\frac{\partial}{\partial \xi_i} \xi^{\vec{m}} \quad (5.2)$$

we obtain

$$\int_C d^T \xi \ell_{\vec{m},i}^T \exp [-f(\xi)] = 0 \quad (5.3)$$

If

$$f = \sum_{\vec{m} \in \mathbb{N}_0^T} a_{\vec{m}} \xi^{\vec{m}} \quad (5.4)$$

then by differentiation with respect to the expansion parameters $a_{\vec{m}}$ we get

$$\ell_{\vec{m},i}^T Z_{\text{sing}} = 0 \quad (5.5)$$

where

$$\ell_{\vec{m},i}^T = -m_i \frac{\partial}{\partial a_{\vec{m}-\vec{e}_i}} + \sum_{\vec{k} \geq \vec{e}_i} k_i a_{\vec{k}} \frac{\partial}{\partial a_{\vec{m}+\vec{k}-\vec{e}_i}} \quad (5.6)$$

Thus we obtain a faithful representation of the positive subalgebra of the w_∞ -algebra and Z_{sing} represents a ground state functional for this.

6. Double scaling limits

Now we consider the integral (3.15)

$$\int d^T \xi g(\xi) \exp \left[-\frac{N}{2} f(\xi) \right] \quad (6.1)$$

with an arbitrary amplitude function $g(\xi)$. f is assumed to lie in a neighborhood of a nondegenerate quasihomogeneous singularity of type $\vec{\alpha}$: $f_c(\xi)$. We aim at an evaluation of (6.1) at large N .

The filtration allows us to decompose

$$f(\xi) = \sum_{\delta=0}^1 f_\delta(\xi) + f_{>1}(\xi) \quad (6.2)$$

where the degree of the principal part of $f_c(\xi)$ has been normalized to one by an appropriate choice of $\vec{\alpha}$. Consider

$$f_1(\xi) = \sum_{\substack{\vec{k} \\ \vec{\alpha} \cdot \vec{k}=1}} a_{\vec{k}} \xi^{\vec{k}} \quad (6.3)$$

A scale transformation applied to ξ (3.18)

$$\xi \rightarrow \lambda^{\vec{\alpha}} \xi \quad (6.4)$$

compensates for the factor $\frac{N}{2}$ if

$$\lambda^{\frac{N}{2}} = 1 \quad (6.5)$$

At the same time we want to let all coupling constants of a minimal set tend to their critical values

$$\{\gamma_n\} : \gamma_n \rightarrow \gamma_n^c \quad (6.6)$$

in such a fashion that

$$f_\delta(\xi) \rightarrow 0 \text{ for all } 0 < \delta < 1 \quad (6.7)$$

Explicitly we write

$$f_\delta(\xi) = \sum_{\substack{\vec{k} \\ (\vec{\alpha} \cdot \vec{k} = \delta)}} t_{\delta, \vec{k}} \xi^{\vec{k}} \quad (6.8)$$

$$\gamma_n - \gamma_n^c = \Theta_n \quad (6.9)$$

$$t_{\delta, \vec{k}} = \sum_n \mathcal{M}_{\delta, \vec{k}; n} \Theta_n + O_2(\Theta) \quad (6.10)$$

Now double scaling means that with $N \rightarrow \infty$, the limit

$$\Theta_n \rightarrow 0 \quad \text{for all } n \quad (6.11)$$

is coupled such that ($\delta = 1$ included)

$$\lim \left[\frac{N}{2} \right]^{1-\delta} t_{\delta, \vec{k}} = z_{\delta, \vec{k}} \quad (6.12)$$

($z_{\delta, \vec{k}}$ finite)

We want only a "minimal" set of coupling constants as coordinates in the set of deformations of a singularity. Thus we should minimize the number of independent coefficients $t_{\delta, \vec{k}}$ in (6.8), in order to make $\mathcal{M}$ in (6.10) a nonsingular linear map at the same time. For this purpose we apply all possible diffeomorphisms.

Usually one assumes f to lie in Q_{fc} but this is not enough in general. In fact we need the maximal Lie-subalgebra of the w_∞ -algebra of nonpositive degree with respect to $\vec{\alpha}$. We denote the Lie-algebra spanned by $\{\ell_{\vec{m}, i}^{\vec{\alpha}}\}$ with

$$\deg \ell_{\vec{m}, i}^{\vec{\alpha}} \leq 0 \quad (6.13)$$

i.e.

$$\vec{\alpha} \cdot (\vec{m} - \vec{e}_i) \leq 0 \quad (6.14)$$

the "triangular algebra" $t(\vec{\alpha})$. It contains the translations ($\vec{m} = 0$) and the dilations $\{\ell_{\vec{e}_i, i}^{\vec{\alpha}}\}_{i=1}^r$ at least.

Let $\vec{\alpha}$ belong to a homogeneous singularity. Then $t(\vec{\alpha})$ contains $gl(r)$ spanned by all $\{\ell_{\vec{e}_i, j}^{\vec{\alpha}}\}$ and nothing more than the translations and $gl(r)$. If $\vec{\alpha}$ belongs to a proper quasi-homogeneous singularity, then $\vec{\alpha}$ can be ordered

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r \quad (6.15)$$

with one proper inequality at least. Then $t(\vec{\alpha})$ contains the triangular subalgebra of $gl(r)$ spanned by

$$\{\ell_{\vec{e}_i, j}^{\vec{\alpha}}\}_{i \leq j} \quad (6.16)$$

But this may not be all of $t(\vec{\alpha})$. It may, namely, be possible to refine the ordering (6.15) at the step $i, i+1$ as follows

$$\alpha_i < 2\alpha_i < 3\alpha_i < \dots < s_i \alpha_i < \alpha_{i+1} \quad (6.17)$$

which has obvious consequences we do not want to elaborate in detail.

Now we assume that the minimalization has been performed. As the result of (6.4), (6.5), (6.12) we obtain as most important factor from (6.1) the Airy function

$$\Phi[z] = \int_C d^r \xi \exp [-z(\xi)] \quad (6.18)$$

$$z(\xi) = \sum_{0 < \delta \leq 1} z_{\delta, \vec{k}}(\xi) \quad (6.19)$$

$$z_{\delta, \vec{k}}(\xi) = \sum_{\vec{k}} z_{\delta, \vec{k}} \xi^{\vec{k}} \quad (6.20)$$

($\vec{\alpha} \cdot \vec{k} = \delta$)

The integration contour C^* is a rational linear combination of chains in $\mathbb{C}^r$ which "run from ∞ to ∞ " but their actual construction is an unsolved problem. Of course the convergence along C^* ought to be exponential.

The arguments $z_{\delta, \vec{k}}$ with $\delta < 1$ in this Airy function can be expressed as "scale invariant functions" of N and $\{\Theta_n\}$ by (6.12), (6.10). Substituting these expressions into the Airy function, one can see that it satisfies the renormalization group equation expressing consistency in the double scaling limit

$$\left[N \frac{\partial}{\partial N} - \sum_n \beta_n(\Theta) \frac{\partial}{\partial \gamma_n} \right] \Phi[z] = 0 \quad (6.21)$$

where one can derive the linear terms of β_n from

$$\beta_n(\Theta) = \sum_{n'} S_{nn'} \Theta_{n'} + O_2(\Theta) \quad (6.22)$$

and the susceptibility matrix S is

$$S_{nn'} = \sum_{\vec{k}, \delta} \mathcal{M}_{n, \delta, \vec{k}}^{-1} (1-\delta) \mathcal{M}_{\delta, \vec{k}; n'} \quad (6.23)$$

7. Some comments

The approach presented here can be generalized to singularities that are not quasi-homogeneous. But the classification of such singularities is quite incomplete and e.g. constructive methods for canonical forms are almost missing at all. The singularities treated in [1,2] are all quasi-homogeneous in turn. The critical indices ([2], Table 2) can be derived from the degrees δ of quasihomogeneous monomials used in this work. On the other hand the class of quasihomogeneous singularities is well suited for a study of sigma models and sufficiently general.

References

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