Abstract. We derive the exact action for a damped mechanical system (and the special case of the linear oscillator) from the path integral formulation of the quantum Brownian motion problem developed by Schwinger and by Feynman and Vernon. The doubling of the phase-space degrees of freedom for dissipative systems and thermal field theories is discussed and the initial values of the doubled variables are related to quantum noise effects.

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1. Introduction

A dissipative system is physically incomplete and a microscopic theory must include the details of processes responsible for dissipation, including quantum effects. One then would start from the beginning with a Hamiltonian that describes a complete system, the bath and the system-bath interaction. Subsequently, the description of the original dissipative system is recovered by the reduced density matrix obtained by eliminating the bath variables which originate the damping and the fluctuations. The problem with dissipative systems in quantum mechanics is indeed that canonical commutation relations (ccr) are not preserved by time evolution due to damping terms. The role of fluctuating forces is in fact the one of preserving the canonical structure.

The above strategy, however, may not always be viable since it requires the knowledge of the details of the processes inducing the dissipation; these details may not be explicitly known and the dissipation mechanisms are sometime globally described by such parameters as friction, resistance, viscosity etc.. In such a case a different route must be followed. On the other hand, the attempt to derive, from a variational principle, the equations of motion defining the dissipative system requires the introduction of additional complementary equations\[1\].

The latter approach has been pursued in refs. 2 and 3 where the quantization of the one-dimensional damped linear harmonic oscillator (dho) has been studied by doubling the phase-space degrees of freedom. The new degrees of freedom thus introduced play the role of the bath degrees of freedom and make it possible to derive the dho equation from a variational principle.

The quantum hamiltonian is obtained as

\[ H = H_o + H_I, \]

\[ H_o = \hbar \Omega (A^\dagger A - B^\dagger B), \quad H_I = i\hbar \Gamma (A^\dagger B^\dagger - AB), \]  

(1)

where \( \Gamma \) is the decay constant, \( \Omega \) the frequency and \( A^\dagger, B^\dagger, A \) and \( B \) are creation and annihilation operators with usual ccr\[2,3\].

As pointed out in refs. 2 and 3, only if one restricts the Hilbert space to the subspace of those states, \( |\psi > \), which are annihilated by \( B \): \( B|\psi >= 0 \), the eigenstates of \( H \) merge into those of the simple undamped harmonic oscillator when \( \Gamma \to 0 \). Therefore the states generated by \( B^\dagger \) represent the sink where the energy dissipated by the quantum damped oscillator flows: the \( B \)-oscillator thus represents the reservoir or heat bath coupled to the \( A \)-oscillator.

The dynamical group structure is that of \( SU(1,1) \) and it has also been shown\[3\] that the canonical quantization of the dho can be achieved in a consistent
way in the infinite volume (or thermodynamic) limit of Quantum Mechanics, i.e. in Quantum Field Theory (QFT) where infinitely many unitarily inequivalent representations of the ccr are allowed. The reason for this is that the set of states of the dho splits into unitarily inequivalent representations (i.e. into disjoint folia, in the $C^*$-algebra formalism) each one representing the states of the oscillator at time $t$ and therefore QFT provides the proper setting to discuss dho: the irreversible, non-unitary time evolution is thus described as tunneling across unitarily inequivalent representations (breakdown of time reversal invariance or arrow of time). In such a description a central role is played by the degeneracy among the vacua of the inequivalent representations, which is formally expressed by the fact that the free Hamiltonian $H_o$ is the Casimir operator of the $SU(1,1)$ group and therefore a constant of motion, $[H_o, H_I] = 0$. Quantum processes changing separately $n_A$ and $n_B$ are allowed provided their difference is left constant and therefore $H_o$ may have (infinitely) many degenerate zero-energy states (vacua). The positiveness of $H_o$ is also ensured by its constancy in time, once its initial (positive) value has been assigned.

In ref. 3 it has been shown that the dho states are time dependent thermal states, as expected due to the statistical nature of dissipation. In particular the formalism for the dho turns out to be similar to the one of real time QFT at finite temperature, also called thermo-field dynamics (TFD)[4], which is indeed based on the doubling of the degrees of freedom.

The purpose of this paper is the following. In the approach of refs.1-3 the system is perturbed in a manner that its time development is governed by different dynamics depending on the positive and negative sense of time. In this work we want to show that there is consistency between the variational principle in the Lagrangian formalism and such different dynamical behaviors relative to a different sense of time. Or, in other words, on a closed time path the variational principle is sufficient to determine the expectation values of any physical observable for a given initial condition. We will obtain the exact action for the dho in the path integral formalism. In particular we will reach some conclusion on the initial values of the doubled variables and relate them to the probability of quantum fluctuations in the ground state, a result which is interesting also in the more general case of thermal field theories, such as TFD, with the doubled variable formalism.

We will cast the canonical quantization scheme of the dho of ref. 3 in the formalism developed by Schwinger[5] and Feynman and Vernon[6] since it is particularly suited to our task.

2. Lagrangian formalism
Our purpose is to explore the manner in which the Lagrangian model for quantum dissipation of refs. 2-3 arises from the formulation of the quantum Brownian motion problem as described by Schwinger[5] and by Feynman and Vernon[6]. Let us first recall (and generalize) the Lagrangian of dho[2,3]. The Langrangian for a particle of mass \( \mu \), damped by a mechanical resistance \( R \) moving in a potential \( V \), reads

\[
L(\dot{x}, \dot{y}, x, y) = \mu \dot{x}\dot{y} - V(x + (1/2)y) + V(x - (1/2)y) + (R/2)(x\dot{y} - y\dot{x}).
\] (2)

The classical Lagrangian equations of motion implied by Eq.(2) read

\[
\mu \ddot{x} + R\dot{x} + (1/2)[V'(x + (1/2)y) + V'(x - (1/2)y)] = 0,
\] (3a)

\[
\mu \ddot{y} - R\dot{y} + [V'(x + (1/2)y) - V'(x - (1/2)y)] = 0.
\] (3b)

It is easy to see that for \( V(x \pm (1/2)y) = (1/2)k(x \pm (1/2)y)^2 \) Eqs. (3) give the dho equation and its complementary equation of refs. 2 and 3.

If from the manifold of solutions to Eqs.(3) we choose those for which the \( y \) coordinate is constrained to be zero, then Eqs.(3) simplify to

\[
y = 0; \quad \mu \ddot{x} + R\dot{x} + V'(x) = 0.
\] (3c)

Thus we obtain a classical damped equation of motion from a Lagrangian theory at the expense of introducing an “extra” coordinate \( y \), later constrained to vanish. (Note that \( y(t) = 0 \) is a true solution to Eqs.(2) so that the constraint is not in violation of the equations of motion.)

The question to which we address ourselves is the following. Does the introduction of an “extra coordinate” make any sense in the context of conventional quantum mechanics? Consider the special case of zero mechanical resistance. For that case, one should begin with the Hamiltonian for an isolated particle,

\[
H = -(\hbar^2/2\mu)(\partial/\partial Q)^2 + V(Q).
\] (4)

For an isolated particle one obtains in quantum mechanical theory the density matrix equation of motion

\[
i\hbar(\partial\rho/\partial t) = [H, \rho],
\] (5)

which indeed requires two coordinates (say \( Q_+ \) and \( Q_- \)). In the coordinate representation, Eqs.(4) and (5) read

\[
i\hbar(\partial/\partial t) < Q_+|\rho(t)|Q_- > =
\]
\[-\left(\frac{\hbar^2}{2\mu}\right)[(\partial/\partial Q_+)^2 - (\partial/\partial Q_-)^2] + [V(Q_+) - V(Q_-)]\right) < Q_+ |\rho(t)| Q_+ >. \quad (6)\]

Employing the coordinates \(x\) and \(y\),

\[
Q_\pm = x \pm (1/2)y, \quad \text{(7)}
\]

and the associated density matrix function

\[
W(x, y, t) = < x + (1/2)y |\rho(t)| x - (1/2)y >, \quad \text{(8)}
\]

simplifies Eq.(6) which now reads

\[
i\hbar(\partial/\partial t)W = \mathcal{H}_o W \quad \text{(9a)}
\]

\[
\mathcal{H}_o = (p_x p_y /\mu) + V(x + (1/2)y) - V(x - (1/2)y), \quad \text{(9b)}
\]

\[
p_x = -i\hbar(\partial/\partial x), \quad p_y = -i\hbar(\partial/\partial y). \quad \text{(9c)}
\]

Of course the “Hamiltonian” Eq.(9b) may be constructed from the “Lagrangian”

\[
\mathcal{L}_o = \mu x y - V(x + (1/2)y) + V(x - (1/2)y). \quad \text{(10)}
\]

We have then the justification for introducing Eq.(2) at least for the case \(R = 0\).

Finally we observe that for \(V(x \pm (1/2)y) = (1/2)k(x \pm (1/2)y)^2\) it is easy to obtain the Hamiltonian (1) from the Lagrangian Eq.(2)[2,3].

We also notice that \(\mathcal{H}_o\) and \(\mathcal{H}_\mathcal{Z}\) in Eq.(1) are the free Hamiltonian and the generator of Bogolubov transformations, respectively, in TFD[4]. Our present discussion thus includes the doubling of degrees of freedom in finite temperature QFT.

### 3. Damping

Now let us suppose that the particle interacts with a thermal bath at temperature \(T\). The interaction Hamiltonian between the bath and the particle is taken as

\[
H_{int} = -fQ, \quad \text{(11)}
\]

where \(Q\) is the particle coordinate and \(f\) is the random force on the particle due to the bath.
In the Feynman-Vernon\cite{6} quantum Brownian motion scheme, the effective
action for the particle has the form
\[ A[x, y] = \int_{t_i}^{t_f} dt L_o(\dot{x}, \dot{y}, x, y) + I[x, y], \]  
(12)
where \( L_o \) is defined in Eq.(10) and
\[ e^{(i/\bar{\hbar})I[x, y]} = \langle e^{(-i/\bar{\hbar})\int_{t_i}^{t_f} f(t)Q(t)dt} \rangle_+ \langle e^{(i/\bar{\hbar})\int_{t_i}^{t_f} f(t)Q(t)dt} \rangle_+ >. \]  
(13)
In Eq.(13): (i) The average is with respect to the thermal bath; (ii) “(.)_+” denotes
time ordering and “(.,)_{-}” denotes anti-time ordering; (iii) The c-number coordi-
nates \( Q_\pm \) are defined as in Eq.(7). We observe that if the interaction between
the bath and the coordinate \( Q \) (i.e \( H_{\text{int}} = -fQ \)) were turned off, then the operator
\( f \) of the bath would develop in time according to \( f(t) = e^{iH_Rt/\bar{\hbar}}f e^{-iH_Rt/\bar{\hbar}} \) where
\( H_R \) is the Hamiltonian of the isolated bath (decoupled from the coordinate \( Q \)). \( f(t) \) is the force operator of the bath to be used in Eq.(13).

Assuming that the particle first makes contact with the bath at the initial
time \( t_i \), the reduced density matrix function in Eq.(8) obeys at a final time
\[ W(x_f, y_f, t_f) = \int_{-\infty}^{\infty} dx_i \int_{-\infty}^{\infty} dy_i K(x_f, y_f, t_f; x_i, y_i, t_i)W(x_i, y_i, t_i), \]  
(14)
with the path integral representation for the time development
\[ K(x_f, y_f, t_f; x_i, y_i, t_i) = \int_{x(t_f) = x_f}^{x(t_i) = x_i} Dx(t) \int_{y(t_f) = y_f}^{y(t_i) = y_i} Dy(t) e^{(i/\bar{\hbar})A[x, y]}. \]  
(15)

The evaluation of \( I[x, y] \) for a linear passive damping thermal bath involves
several Greens functions all of which have been discussed by Schwinger\cite{5} in his
work on quantum Brownian motion. For completeness of presentation we here
discuss these results.

4. Greens Functions

The fundamental correlation function for the random force on the particle
due to the thermal bath is given by
\[ G(t - s) = (i/\bar{\hbar}) < f(t)f(s) >. \]  
(16)
From this one may construct the time ordered Greens function and the anti-time ordered Greens function

\[ G_+(t-s) = \theta(t-s)G(t-s) + \theta(s-t)G(s-t), \]  
\[ G_-(t-s) = \theta(s-t)G(t-s) + \theta(t-s)G(s-t), \]  

where

\[ \theta(t-s) = 1 \text{ if } t > s, \]  
and

\[ \theta(t-s) = 0 \text{ if } t < s. \]  

The retarded and advanced Greens functions are defined by

\[ G_{ret}(t-s) = \theta(t-s)[G(t-s) - G(s-t)], \]  
and

\[ G_{adv}(t-s) = \theta(s-t)[G(s-t) - G(t-s)]. \]  

From a causal engineering view point, the mechanical impedance \( Z(\zeta) \) (analytic in the upper half complex frequency plane \( \text{Im} \, \zeta > 0 \)) is determined by the retarded Greens function

\[ -i\zeta Z(\zeta) = \int_0^\infty dt G_{ret}(t)e^{i\zeta t}. \]  

The time domain quantum noise in the fluctuating random force

\[ N(t-s) = (1/2) < f(t)f(s) + f(s)f(t) >, \]  

is distributed in the frequency domain

\[ N(t-s) = \int_0^\infty d\omega S_f(\omega)\cos[\omega(t-s)], \]  

in accordance with the Nyquist theorem

\[ S_f(\omega) = (\hbar\omega/\pi)cosh(\hbar\omega/2kT)\text{Re}Z(\omega + i0^+). \]  

The mechanical resistance is defined

\[ R = \lim_{\omega \to 0} \text{Re}Z(\omega + i0^+). \]
Finally, the time ordered and anti-time ordered Greens functions describe both the retarded and advanced Greens functions as well as the quantum noise,

\[ G_{\pm}(t-s) = \pm(1/2)[G_{\text{ret}}(t-s) + G_{\text{adv}}(t-s)] + (i/\hbar)N(t-s). \] (25)

5. Bath Interactions

For the particle interacting with the bath describing a linear passive mechanical impedance, Eq.(13) may be evaluated following Feynman and Vernon as,

\[
\mathcal{I}[x,y] = (1/2) \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds [G_{+}(t-s)Q_{+}(t)Q_{+}(s) + G_{-}(t-s)Q_{-}(t)Q_{-}(s) - 2G(t-s)Q_{-}(t)Q_{+}(s)].
\] (26)

Eq.(26) is more simple when expressed in terms of coordinates \(x\) and \(y\),

\[
\mathcal{I}[x,y] = (1/2) \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds [G_{\text{ret}}(t-s) + G_{\text{adv}}(t-s)][x(t)y(s) + x(s)y(t)] + (i/2\hbar) \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t-s)y(t)y(s).
\] (27)

Further simplification results if one defines the retarded force on \(y\)

\[
F_{y}^{\text{ret}}(t) = \int_{t_i}^{t_f} ds G_{\text{ret}}(t-s)y(s),
\] (28a)

and the advanced force on \(x\),

\[
F_{x}^{\text{adv}}(t) = \int_{t_i}^{t_f} ds G_{\text{adv}}(t-s)x(s).
\] (28b)

The interaction between the bath and the particle is then

\[
\mathcal{I}[x,y] = (1/2) \int_{t_i}^{t_f} dt [x(t)F_{y}^{\text{ret}}(t) + y(t)F_{x}^{\text{adv}}(t)] + (i/2\hbar) \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t-s)y(t)y(s).
\] (29)
6. The Final Action

Putting all the pieces together, we have from Eqs.(12) and (29) the real part of the action

\[ \mathcal{R}e A[x, y] = \int_{t_i}^{t_f} dt \mathcal{L}, \quad (30a) \]

\[ \mathcal{L} = \mu \dot{x} \dot{y} - [V(x + (1/2)y) - V(x - (1/2)y)] + (1/2)[xF_y^{ret} + yF_x^{adv}], \quad (30b) \]

and the imaginary part of the action

\[ \mathcal{I}m A[x, y] = (1/2\hbar) \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t - s)y(t)y(s). \quad (30c) \]

The central Eqs.(30) are rigorously exact for linear passive damping due to the bath when the path integral Eq.(15) is employed for the time development of the density matrix.

The lagrangian Eq.(2) can now be viewed as the approximation to Eq.(30b) with \( F_y^{ret} = R\dot{y} \) and \( F_x^{adv} = -R\dot{x} \). The classical constraint \( y = 0 \) occurs because nonzero \( y \) yields an “unlikely process” in view of the large imaginary part of the action (in the classical \( \hbar \to 0 \) limit) implicit in Eq.(30c). On the contrary, at quantum level nonzero \( y \) may allow quantum noise effects arising from the imaginary part of the action.

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References

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