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**GR via Characteristic Surfaces**

by

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**Abstract**

We reformulate the Einstein equations as equations for \textit{families of surfaces} on a four-manifold. These surfaces eventually become characteristic surfaces for an Einstein metric (with or without sources). In particular they are formulated in terms of two functions on $\mathbb{R}^4 \times S^2$, i.e. the sphere bundle over space-time, - one of the functions playing the role of a conformal factor for a family of associated conformal metrics, the other function describing an $S^2$'s worth of surfaces at each space-time point. It is from these families of surfaces themselves that the conformal metric - conformal to an Einstein metric - is constructed; the conformal factor turns them into Einstein metrics. The surfaces are null surfaces with respect to this metric.

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I Introduction

The general theory of relativity (GR), a theory of the geometry of physical space-time, is usually considered as a classical field theory with the basic field being the metric tensor along with its associated connection field and curvature tensor. In this work we will present an alternate point of view to the identical theory - but now the metric tensor, etc., become derived concepts, the fundamental variables being, instead, families of surfaces and a scalar function. These surfaces, which are described by partial differential equations, become the characteristic surfaces of a conformal metric which is obtained directly from the surfaces themselves. The scalar function is a conformal factor turning the conformal metric into an Einstein metric. We emphasize that though, neither the presentation nor the final equations resemble the standard version of GR, the final results are identical to GR.

Though there has been a considerable amount of technical material written on this program already, recent developments considerably simplify the earlier work and suggest that a new complete presentation would be worthwhile. The discussion here is intended to be essentially self-contained.

As we just mentioned the basic idea or goal of the program is the reformulation of classical General Relativity in terms of concepts and variables that are new and can, perhaps, be considered as more fundamental than space-time itself; i.e., so that, in some sense, space-time arises as a derived concept from a more "primary structure" and a metric tensor, satisfying the Einstein equations, appears automatically but as a secondary idea.

The formulation is in terms of two functions, $\Omega(x, S^2)$ and $Z(x, S^2)$, of the space-time points, $x^a$, and parametrized by points on the sphere, i.e., by functions on the sphere-bundle over a space-time manifold, functions on $\mathbb{R}^4 \times S^2$. The first of the functions $\Omega$, is to be considered as a conformal factor for a sphere's worth of conformal metrics - the factor turning the conformal metrics into Einstein metrics - while the second function, $Z$, describes, at each space-time point, a sphere's worth of characteristic surfaces. The conformal metrics are constructed from the characteristic surfaces.
Though, in the end, we obtain a set of four (coupled) equations, three complex and one real, for the two functions, we will begin with a standard Lorentzian manifold $M$, with an Einstein metric $g^{ab}(x^a)$ in some local coordinates and derive from this structure our new set of equations.

In Sec.II, we begin with a discussion of null coordinate conditions, i.e., the use of a special set of coordinates $\theta^i$ associated with a one parameter family of characteristic surfaces, $u = \text{const.} = Z(x^a)$, and a related (almost unique) decomposition of the metric tensor into a conformal factor, $\Omega^2$, and conformal metric, $g^{ij}$, so that $g^{ij} = \Omega^2 g^{ij}$. The next idea is to introduce on the same manifold with the same metric, an $S^2$'s worth of these families of characteristic surfaces, i.e., $u = \text{const.} = Z(x^a, \zeta, \bar{\zeta})$, with $\zeta, \bar{\zeta}$ as stereographic coordinates on $S^2$, parametrizing the family of characteristic surfaces. (An alternate way to describe these families of null surfaces is to say that there are a sphere's worth of null surfaces through each space-time point.) For each fixed value of $\zeta, \bar{\zeta}$ (or for each family of surfaces) we introduce the same type of null coordinate system and conformal decomposition of the metric as before, i.e., we obtain an $S^2$'s worth of metrics, $g^{ij}(\theta^i, \zeta, \bar{\zeta})$ all diffeomorphically equivalent but each associated with a different null coordinate system, $\theta^i(\zeta, \bar{\zeta})$.

Two, rather attractive and, to us, surprising results are the following:

1. The complete null coordinate system that we have been using can be obtained (or defined) solely from the null surfaces, $u = Z(x^a, \zeta, \bar{\zeta})$, themselves, by taking $\zeta, \bar{\zeta}$ derivatives of $Z$, evaluated at a fixed $\zeta, \bar{\zeta}$; more specifically, we define (dropping the * from the $\theta^i$ in this case)

$$
\theta^i = \theta^i(x^a, \zeta, \bar{\zeta}) = (\theta^0, \theta^1, \theta^+, \theta^-) \bar{E} (u, R, \omega, \bar{\omega}) \quad (1.1)
$$

with

$$
\begin{align*}
\quad u &= Z(x^a, \zeta, \bar{\zeta}), \quad R = \partial x^a(Z(x^a, \zeta, \bar{\zeta}), \\
\quad \omega &= Z(x^a, \zeta, \bar{\zeta}), \quad \bar{\omega} = \partial x^a(Z(x^a, \zeta, \bar{\zeta}),
\end{align*}
\quad (1.2)
$$

where the $'$ and $\partial x^a(\cdot)$ are essentially the $\zeta, \bar{\zeta}$ derivatives respectively. See Appendix B. The result is that the $\theta^i = \theta^i$. We will refer to the $\theta^i$ as the intrinsic coordinate system associated with the parametrized families of characteristic surfaces, parametrized by $u = Z(x^a, \zeta, \bar{\zeta})$. 

2. The second result is that the conformal metric $g^{ij}$ can be completely reconstructed from knowledge of the surfaces, $u = Z(x^a, \zeta, \bar{\zeta})$. More specifically, if we take the second derivative of $Z$ and eliminate the $x^a$ through (1.2), i.e., by $x^a = x^a(\theta^i, \zeta, \bar{\zeta})$, we define $\Lambda(\theta^i, \zeta, \bar{\zeta})$ via

$$\frac{d^2 Z}{\delta Z} \equiv \lambda(x^a, \zeta, \bar{\zeta}) = \Lambda(\theta^i, \zeta, \bar{\zeta}).$$

The conformal metric is then an explicit function of $\Lambda(\theta^i, \zeta, \bar{\zeta})$ and its derivatives.

The basic idea of the program is to replace the metric as the basic variable for GR, by the factor $\Omega$ and, either, the $Z(x^a, \zeta, \bar{\zeta})$ or the $\Lambda(\theta^i, \zeta, \bar{\zeta})$, i.e., to find equations for $\Omega$ and $\Lambda(\theta^i, \zeta, \bar{\zeta})$ that are equivalent to the Einstein equations. This is discussed in Sec.III. There will be one real equation that is equivalent to the Einstein equations - the remaining three complex equations are the requirement that the sphere's worth of metrics obtained from the $Z$ are diffeomorphically equivalent.

In Sec.IV we study the structure of these equations and show how they can be "easily" solved perturbatively in the case of asymptotically flat vacuum space-times. In Sec. V we give some examples of solutions.

II. Characteristic Coordinate Systems

We begin by pointing out a set of four related observations concerning sets of characteristic coordinate systems - beginning with a relatively simple one and ending with a basic relationship for us.

We start from a manifold $M$ with Lorentzian metric, $g^{ab}(x^a)$ and a choice of a one-parameter family of characteristic surfaces given by

$$u = \text{constant} = Z(x^a). \quad (2.1)$$

#1. Associated with this family one can choose a preferred coordinate system in the following way: Let $u = Z(x^a)$ be one coordinate, a second could be an affine parameter $r$ along the null geodesics of the surface and the last two, $\omega^\pm$, would label the geodesics on the $u = \text{const.}$ surface. By definition, the null covector $Z_{\cdot a}$ in this coordinate system has the form, $Z_{\cdot a} = \delta_a^0$ and $L^a = g^{ab}Z_{\cdot b} = \delta^a_1$; from this it follows that the metric takes the form
We modify this by introducing, instead of the affine length, a new variable along the geodesics by the transformation

$$u^* = u, \quad R^* = -\int g^{*+}dr, \quad \omega^{*\pm} = \omega^{\pm}.$$ 

This puts the metric into the form (with $g^{01} = \Omega^2$)
\[ g_{ij} = \begin{bmatrix} 0 & g^{01} & 0 & 0 \\ g^{01} & x & x & x \\ 0 & x & x & -g^{01} \\ 0 & x & -g^{01} & x \end{bmatrix} = \Omega^2 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & x & x & x \\ 0 & x & x & -1 \\ 0 & x & -1 & x \end{bmatrix} = \Omega^2 \{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \} = \Omega^2 \{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{11} & h^{1+} & h^{1-} \\ 0 & h^{1+} & h^{++} & 0 \\ 0 & h^{1-} & 0 & h^{--} \end{bmatrix} \} \]

or

\[ g_{ij} = \Omega^2 \{ \eta_{ij} + h_{ij} \} = \Omega^2 g_{ij} \]

We denote these special null coordinates by

\[ \theta^* = \{ \theta^0, \theta^1, \theta^+, \theta^- \} = \{ u^*, R^*, \omega^+, \omega^- \} \]

and use \( x^a \) for any *arbitrary* coordinate system.

(As an aside we point out that \( \eta_{ij} \) can be described as a flat space metric defined (almost uniquely) by the requirement that \( Z_{,i} \) is both null and covariantly constant with respect to \( \eta_{ij} \) and the form of \( h_{ij} \) can be described as the tensor that annihilates \( Z_{,i} \), i.e., \( h_{ij} Z_{,i} = 0 \) and is trace-free with respect to \( \eta_{ij} \), i.e., \( \eta_{ij} h_{ij} = 0 \). Note that though (2.3b) resembles the form of the metric used in linear theory there is no implication of linearization. Of the ten components of the metric, four have been determined by the coordinate conditions, one goes into the \( \Omega \) and the remaining five into the \( h_{ij} \).

#2. The next observation we want to explore is the consideration of a set (a sphere's worth) of these one-parameter families of null surfaces. They are given in the form

\[ u = \text{constant} = Z(x^a, \zeta, \bar{\zeta}) \]

(2.4)

where for each fixed \( \zeta, \bar{\zeta} \) (a point on \( S^2 \) in complex stereographic coordinates) we have a one parameter family of characteristic surfaces.

(The prototype of the situation we have in mind is to consider a fiducial past light cone \( F \), with each of the sphere's worth of null generators labeled by \( \zeta, \bar{\zeta} \) and the points on each
generator labeled by $u$. The null surfaces we are considering are simply the past light-cones of each point of $F$, labeled by $u, \zeta, \bar{\zeta}$. A particular example$^{5,6}$ of the fiducial surface is the null cone at infinity, i.e., $l$. See Fig.1.)

Using the same coordinate construction as in #1, we obtain an $S^2$'s worth of null coordinate systems and metrics. In other words, if we had begun with the metric, $g^{ab}(x)$, in some local coordinates $x^a$, we, in this manner, would have constructed a sphere's worth of coordinate transformations

$$\theta^* \equiv \{ u, R^*, \omega^*+, \omega^*- \} = \theta^i(x, \zeta, \bar{\zeta})$$

with

$$u = Z(x, \zeta, \bar{\zeta}), \quad R^* = R^*(x, \zeta, \bar{\zeta}), \quad \omega^*+ = \omega^*+(x, \zeta, \bar{\zeta}), \quad \omega^- = \omega^--(x, \zeta, \bar{\zeta})$$

and a sphere's worth of metrics

$$g^{ij} = \Omega^2 \{ \eta^{ij} + h^{ij} \} = \theta^i, a \theta^j, b g^{ab}(x)$$

with $\Omega = \Omega(\theta^i, \zeta, \bar{\zeta})$ and $h^{ij} = h^{ij}(\theta^i, \zeta, \bar{\zeta})$

Note that the sphere's worth of metrics are all isometric, i.e. are diffeomorphically equivalent; all being equivalent to the $g^{ab}(x)$.

Remark 1: An obvious, but very important point that we will return to shortly is that a small change in the value of $\zeta$ and/or $\bar{\zeta}$ generates an infinitesimal coordinate transformation between two neighboring $\theta^i$ coordinate systems.

#3. Our next observation is slightly surprising. If we begin with the $S^2$'s worth of null surfaces $u = Z(x, \zeta, \bar{\zeta})$ we can define three new functions of $(x, \zeta, \bar{\zeta})$ by taking the first derivatives of $Z$ in the $\zeta, \bar{\zeta}$ directions and the mixed second derivative, i.e., by

$$\omega = Z(x^a, \zeta, \bar{\zeta}), \quad \bar{\omega} = \\nabla' x \to Z(x^a, \zeta, \bar{\zeta}), \quad R = \\nabla' x \to Z(x^a, \zeta, \bar{\zeta}),$$

where the operators $'$ and $\\nabla' x \to$ are essentially the $\zeta, \bar{\zeta}$ derivatives respect- ively$^{7,8}$. The observation is that the set of four functions

$$\theta^i = (u, R, \omega, \bar{\omega})$$

$$u = Z(x, \zeta, \bar{\zeta}), \quad \omega = Z(x^a, \zeta, \bar{\zeta}), \quad \bar{\omega} = \\nabla' x \to Z(x^a, \zeta, \bar{\zeta}), \quad R = \\nabla' x \to Z(x^a, \zeta, \bar{\zeta}),$$

$$(2.7)$$
constitutes a special case of the coordinate transformation (2.5), i.e., \( \theta^i = \theta^{*i} \), and the transformation (2.7) for each value of \( \zeta \), \( \bar{\zeta} \) yields the same coordinate conditions that led to the metric form (2.3). Knowledge of just the \( Z(x, \zeta, \bar{\zeta}) \) defines uniquely our special characteristic coordinate system which we will refer to as intrinsic coordinates.

Though the proof of this result is quite easy we will postpone it for later.

Remark 2: It is easy to show that there are families of null surfaces in Minkowski space so that the Jacobian of (2.7) is non-vanishing. From this follows the existence, in arbitrary space-times, of (local) families of null surfaces, \( u = Z(x, \zeta, \bar{\zeta}) \), with some finite \( (\zeta, \bar{\zeta}) \) range, also with non-vanishing Jacobian. We assume that \( Z(x, \zeta, \bar{\zeta}) \) has this property.

Since, by assumption, the transformation to the intrinsic coordinates is well defined, the four gradients

\[
\theta^i_\alpha = \theta^i_\alpha(x, \zeta, \bar{\zeta}) = (Z_\alpha, Z_\alpha, X^{to}(\zeta)Z_\alpha, X^{to}(\bar{\zeta})Z_\alpha) \tag{2.8}
\]

form an independent set of covectors.

The transformation is thus invertible, so that

\[
x^\alpha = x^\alpha (\theta^i, \zeta, \bar{\zeta}) \tag{2.9}
\]

Remark 3: For later use we define

\[
^{\nu^2}Z \equiv \lambda (x^\alpha, \zeta, \bar{\zeta}) \tag{2.10}
\]

and by eliminating the \( x^\alpha \) via (2.9) we have

\[
^{\nu^2}Z \equiv \Lambda(\theta^i, \zeta, \bar{\zeta}) \tag{2.11}
\]

\( \Lambda(\theta^i, \zeta, \bar{\zeta}) \) will play a basic role for us. We emphasize the important idea of inverting (2.11), namely, if the \( \Lambda(\theta^i, \zeta, \bar{\zeta}) \) and its conjugate are given (with \( \theta^i \) as in 2.7) and (2.11) is considered as a differential equation for \( Z \), then it is possible to show that there is a four parameter family of solutions, \( u = Z(x^\alpha, \zeta, \bar{\zeta}) \) with \( x^\alpha \) as the four parameters. From (2.11) we also have the important integrability condition on \( \Lambda; \)

\[
^{\nu^2} \Lambda = \text{\textit{X}}^{to}(\theta^i)^2 \Lambda \text{ that must be satisfied.}
\]

#4. Our last observation is that the terms \( h^{ij} \) (or the conformal metric \( g^{ij} \)) in the metric (2.3) or (2.6), are completely determined by the knowledge of \( Z(x^\alpha, \zeta, \bar{\zeta}); h^{ij} \) is expressible as an
algebraic function of derivatives of $Z$. More specifically $h_{ij}$ is an explicit function of $\Lambda(\theta^i, \zeta, \bar{\zeta})$ (see Remark 3) and its $\theta^i$ and $\zeta, \bar{\zeta}$ derivatives, i.e.,
\[
h_{ij} = h_{ij}(\Lambda(\theta^i, \zeta, \bar{\zeta})) \quad \text{or} \quad g_{ij} = g_{ij}(\Lambda(\theta^i, \zeta, \bar{\zeta})) \tag{2.12}
\]
The proof of this will be given shortly.

Remark 4: This result is not surprising since, from the $S^2$ set of null surfaces, $Z(x^a, \zeta, \bar{\zeta})$ at each point, there should be enough information to construct the conformal metric. The main point is that there is a relatively simple explicit means of constructing the conformal metric from the $Z(x^a, \zeta, \bar{\zeta})$. See below.

Proof of the Coordinate Conditions and Derivation of the Conformal Metric

Beginning with the coordinate transformation (2.7) (scalar functions), $\theta^i = \theta^i(x, \zeta, \bar{\zeta})$, a gradient basis $\theta^i_a$ and dual vectors $\theta^i_a$ can be formed with $\theta^i_a = \{\theta^0_a, \theta^+_a, \theta^-_a, \theta^1_a\} = \{Z_a, \nabla\theta^0()Z_a, \nabla\theta^+(())Z_a\}$. From the transformation of the metric $g^{ab}(x)$ we have
\[
g_{ij}(\theta^i, \zeta, \bar{\zeta}) = g^{ab}(x^a)\theta^i_a\theta^j_b \tag{2.13}
\]
and the inversion
\[
g^{ab}(x^a) = g_{ij}(\theta^i, \zeta, \bar{\zeta})\theta^a_i\theta^j_b. \tag{2.14}
\]
From (2.12) and (2.13) we have, for example,
\[
g^{00} = g^{ab}(x^a)Z_aZ_b, \quad g^{0+} = g^{ab}(x^a)Z_a\nabla\theta^0()Z_b, \quad g^{0-} = g^{ab}(x^a)Z_a\nabla\theta^+()Z_b \tag{2.15}
\]
One sees, from the assumption that $u = Z(x, \zeta, \bar{\zeta})$ is a characteristic surface, and $Z_a$ is a null covector that
\[
g^{00} = g^{ab}(x^a)Z_aZ_b = 0 \tag{2.16}
\]
and, by taking '$' and $\nabla\theta()$ derivatives of (2.16) (and the fact that $g^{ab}$ depends only on $x^a$) we have that
\[
g^{0+} = \overset{\prime}{g^{00}} = 0, \quad g^{0-} = \overset{\prime}{\nabla\theta()}g^{00} = 0, \quad g^{01} + g^{+-} = \overset{\prime}{\nabla\theta()}'\overset{\prime}{g^{00}} = 0. \tag{2.17}
\]
This result is equivalent to the coordinate conditions that was used in (2.3) proving our contention in observation #3.
Our contention in observation #4, is easily proved in a similar manner. By repeated \( ^\text{'} \) and \( \partial \times \text{to} ('') \) operations on \( g^{00} = 0 \) it is possible [See Fig.2.] to express all the components of \( h_{ij} \) or \( g_{ij} \) in terms of the \( \Lambda_{ia} \), i.e., one obtains explicitly the conformal metric, with \( g^{01}(\theta^i, \zeta, \bar{\zeta}) \) as the conformal factor:

\[
g_{ij} = g^{01} g^{ij}(\Lambda_{ia}). \tag{2.18}
\]

For example, we have that

\[
^{\text{'}}_{\Lambda^2} g^{00} = g^{ab} \partial \times Z_{a} \times Z_{b} + g^{ab} Z_{a} \times Z_{b} = g^{++} + g^{ab} Z_{a} \Lambda_{ab} = 0. \tag{2.19}
\]

\[
^{\text{'}} \partial \times \text{to} ('') \partial \times g^{00} = g^{ab} \partial \times \text{to} ('') Z_{a} \times \text{to} ('') Z_{b} + g^{ab} Z_{a} \partial \times \text{to} ('') \times Z_{b} = g^{--} + g^{ab} Z_{a} \Lambda_{ab} = 0.
\]

Using the decomposition of the gradient of \( \Lambda \) in the \( \theta^i_{ia} \) basis, i.e.,

\[
\Lambda_{ia} = \Lambda_{ia} \theta^i_{ia} = \Lambda_{a} + \Lambda_{1} \times \partial \times \text{to} ('') Z_{a} + \Lambda_{+} \times \partial \times \text{to} ('') Z_{a} + \Lambda_{-} \times \partial \times \text{to} ('') Z_{a} \quad (2.20)
\]

in (2.19), we have that

\[
g^{++} = -g^{01} \Lambda_{1} \quad \text{and} \quad g^{--} = -g^{01} \Lambda_{1}. \tag{2.21a}
\]

or

\[
g^{++} = -\Lambda_{1} \quad \text{and} \quad g^{--} = -\Lambda_{1}. \tag{2.21b}
\]
From \( \nabla^2 g^{00} = 0 \), \( \nabla^2 g^{00} = 0 \) and \( \nabla^2 g^{00} = 0 \),

the remaining components of the conformal metric, namely \( g^{+1} \), \( g^{-1} \), and \( g^{11} \), can be obtained in the form

\[
g^{+1} = g^{++}, \quad g^{-1} = g^{--}, \quad g^{11} = -2g^{01} + \nabla^2 g^{+1} + \nabla^2 g^{ab} \Lambda^a \Lambda_b \quad (2.22)
\]

where each component is a function of \( \Lambda \) with an overall factor of \( g^{01} \). The metric is described via equations (2.16), (2.17), (2.21) and (2.22).

(The detailed expressions are given in Appendix A.)

**Remark 5:** We now must address a subtle but very important point. If we had continued to take further \( \nabla \) and \( \nabla^2 \) derivatives of \( g^{00} = 0 \), as for example, \( \nabla^3 g^{00} = 0 \) or \( \nabla^3 g^{00} = 0 \), [See Fig.2.] we would have found a set of identities. If, however, we adopt the alternative point of view, that we want to find the function \( u = Z(x, \zeta, \bar{\zeta}) \) or preferably \( \Lambda(\theta^i, \zeta, \bar{\zeta}) \), so that the sphere's worth of metrics obtained from (2.18) are, in fact, diffeomorphically equivalent, then these "identities" become the conditions for the existence of a single metric. They will be referred to as the "metricity conditions".

To understand and find these conditions it is useful to use the inversion of (2.13), namely

\[
g^{ab}(x^a) = g^{ij}(\theta^i, \zeta, \bar{\zeta}) \theta^i_a \theta^j_b \quad (2.14*)
\]

There are, to begin with, twenty (ten complex) metricity conditions, obtained from the independence of \( g^{ab} \) on \( (\zeta, \bar{\zeta}) \), namely

\[
\nabla^a g^{ab}(x^a) = 0 \quad (2.23)
\]

For technical reasons it turns out that it is more convenient to work with (2.23) in the gradient basis, i.e., with

\[
\theta^i_a \theta^j_b g^{ab}(x^a) = 0 \quad \text{and} \quad \theta^i_a \theta^j_b \nabla^a g^{ab}(x^a) = 0. \quad (2.24)
\]

By applying (2.24) to (2.14*), we obtain the twenty equations (ten complex)

\[
\nabla^a g^{ij}(\theta^i, \zeta, \bar{\zeta}) + g^{km}(\theta^i, \zeta, \bar{\zeta}) \nabla^a(\theta^k m^b) \theta^j_\Lambda \theta^j_\Lambda = 0
\]

\[
\nabla^a g^{ij}(\theta^i, \zeta, \bar{\zeta}) + g^{km}(\theta^i, \zeta, \bar{\zeta}) \nabla^a(\theta^k m^b) \theta^j_\Lambda \theta^j_\Lambda = 0. \quad (2.25)
\]

It is relatively easy to show that of these equations, eight determine the metric components (all except the condition \( g^{00} = 0 \) and the \( g^{01} \) which is the unknown conformal factor), four are
already identically satisfied by virtue of the detailed expressions for the conformal metric in terms of the $\Lambda(\theta^i, \zeta, \bar{\zeta})$ (or via the detailed expression for the T's: see Appendix B). There are actually two more identities; see below. The remaining eight are the final metricity conditions;

$$'g^{01}(\theta^i, \zeta, \bar{\zeta}) + g^{km}(\theta^i, \zeta, \bar{\zeta})' (\theta^a_k \theta^b_m) \theta^0 \theta^1 = 0 \quad (2.26)$$

and

$$'g^{++}(\theta^i, \zeta, \bar{\zeta}) + g^{km}(\theta^i, \zeta, \bar{\zeta})' (\theta^a_k \theta^b_m) \theta^+ \theta^+ = 0 \quad (2.27a)$$

$$'g^{+1}(\theta^i, \zeta, \bar{\zeta}) + g^{km}(\theta^i, \zeta, \bar{\zeta})' (\theta^a_k \theta^b_m) \theta^a+ \theta^b+ = 0 \quad (2.27b)$$

$$'g^{11}(\theta^i, \zeta, \bar{\zeta}) + g^{km}(\theta^i, \zeta, \bar{\zeta})' (\theta^a_k \theta^b_m) \theta^a \theta^b = 0 \quad (2.27c)$$

and their complex conjugates. Aside from (2.26) which involves both $\Omega$ (linearly) and $\Lambda(\theta^i, \zeta, \bar{\zeta})$, the remaining three equations (equivalent to $\gamma^{00} = 0$, $\nabla \times \Theta^0(\gamma^0 = 0$ and $\nabla \times \Theta^0(\gamma^0 = 0$) are functions of just the $\Lambda_{ij}$ and its derivatives.

Though it is far from obvious and took considerable effort to prove, it turns out that (2.27b) is an identity (and carries no information) by virtue of the previous equations. (See Appendix B.) The metricity conditions are thus the six (three complex) equations (2.26) and (2.27a&c) - rather than the original eight - and there are really six identities, rather than the four of the previous paragraph. It should be pointed out here that only (2.27a) is really a new condition on $Z$ and/or $\Lambda$. The other conditions, with (2.27a), turn out to be equivalent to the integrability conditions, $\gamma^{2} \Lambda = \nabla \times \Theta^2 \Lambda$.

In the following they will be written out up to linear terms, - the detailed versions are given in Appendix A.

$$'\omega - \gamma \nabla \omega = 0, \quad W = \Lambda, + + \gamma \nabla \times \Theta^0(\Lambda_{+1} + O(\Lambda^2)) \quad (2.26^*)$$

$$\Lambda_{-} = \gamma^0 \Lambda_{1} + O(\Lambda^2) \quad (2.27a^*)$$

$$\gamma^{3}\Lambda_{1} - 2' \nabla \times \Theta^0(\Lambda_{+4} + \gamma \nabla \times \Theta^0(\Lambda_{0 - 4}) + \gamma \nabla \times \Theta^0(\Lambda_{1} = O(\Lambda^2)) \quad (2.27c^*)$$

As the equations (2.26*) and (2.27*) are fundamental to us, we emphasize that if we had started with the set of null surfaces $u = Z(x, \zeta, \bar{\zeta})$ they would be identically satisfied but if we are looking for the null surfaces, i.e., the $Z$ or $\Lambda$, then they become differential conditions to be imposed.
Summary. We now adopt a new point of view towards geometry on a Lorentzian manifolds. Instead of a metric $g^{ab}(x)$ on $\mathcal{M}$, as the fundamental variable, we consider as the basic variable(s) a family of surfaces on $\mathcal{M}$ given by $u = \text{const.} = Z(x, \zeta, \bar{\zeta})$ -- (or preferably its second derivative, $\Lambda(\theta^i, \zeta, \bar{\zeta}) = \partial^2 Z$) and a "scalar" field $\Omega = \Omega(\theta^i, \zeta, \bar{\zeta})$. When these functions $\{\Lambda(\theta^i, \zeta, \bar{\zeta})$ and $\Omega(\theta^i, \zeta, \bar{\zeta})\}$ are given and satisfy Eqs. (2.26) and (2.27) they define a Lorentzian metric with the "surfaces" being characteristic surfaces of this metric - from this "new" point of view the surfaces are basic and the metric is a "derived" concept. Though any Lorentzian metric can be constructed in this manner, we have not yet said a word about the Einstein equations. How they are to be incorporated into this scheme will be discussed in the next section.

III. The Einstein Equations

The Einstein equations $G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R = \kappa T^{ab}$ can be imposed on our new variables in the following fashion: Since, in our construction, $Z_a(x^a, \zeta, \bar{\zeta})$ is a null one-form at $x^a$, that for all $\zeta, \bar{\zeta}$ spans (or almost spans) the null cone, the Einstein equations can be written

$$G^{ab} Z_a Z_b = \kappa T^{ab} Z_a Z_b.$$  \hspace{1cm} (3.1)

That (3.1) really is equivalent (with a minor caveat; see below) to the Einstein equations is easily seen by applying $'$ and \(\nabla_r T \) several times to it, obtaining in this manner nine of the ten components of $G^{ab}$ in the $\theta_i^a$ basis. (See Remark 6 for a discussion of the tenth component, namely the trace of $G^{ab}$ and the related caveat.) If we now substitute the metric from Eq.(2.3) into (3.1) we obtain a single remarkably simple relationship between $\Omega$ and $\Lambda$, namely

$$D^2 \Omega - Q \Omega = \kappa T^{ab} Z_a Z_b \Omega^3$$ \hspace{1cm} (3.2)

or in the vacuum case

$$D^2 \Omega - Q \Omega = 0$$ \hspace{1cm} (3.3)

where $D = \partial / \partial \theta^1 = \partial / \partial R$ and $Q = Q(\Lambda, 1)$ is a relatively simple function of $\Lambda, 1$ having the structure $Q = O(\Lambda^2)$. The complete expression for $Q$ is given in Appendix A.

The vacuum Einstein equations have become (formally) a single, second order, linear ordinary differential equation for $\Omega$ assuming that the $\Lambda$ is known. In actuality Eq. (3.3) must be viewed as an equation for both $\Lambda$ and $\Omega$ which is to be solved by coupling it to the metricity
conditions. (We will see later that in a perturbation expansion (3.3) can however be looked on as an equation for $\Omega$ with $\Lambda$ known. This arises because of the fact that $Q = O(\Lambda^2)$.)

**Remark 6:** Though at first appearance it might seem surprising that the single Eq. (3.2) or (3.3) is equivalent to the full set of Einstein equations, it in fact is quite natural, since as $\zeta$, $\bar{\zeta}$ vary over the light-cone, we are really obtaining many (an infinite number) of components. What has been lost is the trace term since $g^{ab}Z_a Z_b = 0$ - it however returns as a "constant" of integration in the solution to (3.2) or (3.3). There is however a subtlety that we should mention. If the Einstein equations $G_{ab} = \kappa T_{ab}$, are decomposed into a trace-free part $G_{ab} - \Phi g_{ab}G = \kappa(T_{ab} - \Phi g_{ab}T)$, [equivalent to our (3.2) or (3.3)], and $G = \kappa T$, then, via the Bianchi Identities, $a G^{ab} \equiv 0 = \kappa a T^{ab}$, the *trace-free part* determines the trace $G$ uniquely up to a constant, i.e., $b G = \kappa a (T^{ab} - \Phi \delta^{ab} T)$. If we had solved only the nine trace-free equations, then this latter equation would be identically satisfied. In our version, our Eq. (3.3) though equivalent to the *nine* trace-free Einstein equations, *does not contain* the tenth one until the metricity conditions are included. The reason for this is that we do not yet have a unique metric (until the metricity conditions are applied) and therefore until then we do not have the Bianchi Identities which are needed to determine the trace. We have in this sense a strong "mixing" of (3.3) with the metricity conditions. What we have done here is trade off the ten Einstein equations as equations for the ten components of the metric as functions of the four space-time coordinates $x^a$, with the new version where the Einstein *equation* is one equation for the $\Omega$ and $\Lambda$ in the six variables, $\theta^i$ and $\zeta$, $\bar{\zeta}$ but *augmented by the three complex metricity conditions*.

**IV. The Structure of the New Equations**

Our version of the vacuum Einstein equations consists of the (dynamic) linear second order o.d.e., (3.3) which we will refer to as (E) and the three (kinematic) *metricity* conditions, (2.26*), (2.27a*) and (2.27c*) which for brevity we will refer to as (*)& (a*) and (c*), i.e.,

$$D^2 \Omega - Q \Omega = 0, \quad Q = Q(\Lambda, 1) = O(\Lambda^2) \quad (E)$$

$$\Omega - \theta W \Omega = 0, \quad W = \Lambda + \theta X\lambda(\Lambda_{1} + O(\Lambda^2) \quad (*)&$$

$$\Lambda_{1} = \theta' \Lambda_{1} + O(\Lambda^2) \quad (a*)$$
These equations are a set of explicit differential equations for the dependent variables \( \{ \Omega, \Lambda, \bar{\Lambda} \} \) in terms of the six independent variables \( \{ \theta^i, \zeta, \bar{\zeta} \} \). The two equations that contain \( \Omega \), (E) and (*) are both linear in \( \Omega \) and have fairly simple expressions in the \( \Lambda, \bar{\Lambda} \). The last two, (a*) and (c*), contain only the \( \Lambda \) and \( \bar{\Lambda} \) and, aside from the linear terms, are reasonably complicated.

With the exception of special solutions, e.g., Minkowski space, half conformally-flat spaces, special symmetries, etc. [See Sec. V.], it is highly unlikely that general classes of solutions to these equations can be found. The basic idea is to fall back on a perturbation scheme - but even this has certain difficult issues of practicality and perhaps principle. The main issue is what variables to use -- is it "better" to use the pair \( \Lambda \) and \( \bar{\Lambda} \) or is it "better" or more natural to use the more fundamental variable \( Z(x, \zeta, \bar{\zeta}) \).

Though we have an (aesthetic?) preference for the use of the \( \Lambda \) and \( \bar{\Lambda} \) pair, it turns out (at least at the present time) to be simpler and clearer to use \( Z \). (There is however some evidence that the use of \( \Lambda \) might be as simple.) Perhaps with more experience we will see that it makes no difference - or that the different variables should be used to study different situations.

We conclude this section with a "point of view" towards these equations and with it an attractive and simple version of a perturbation scheme.

The starting point is to consider (E) as a linear second order equation for \( \Omega \) with some unknown \( \Lambda \) in the \( Q \). Since \( Q = O(\Lambda^2) \), we can write the solution to (E) as

\[
\Omega = 1 + O(\Lambda^2)
\]  

where we have used asymptotic flatness (i.e., \( \Omega \Rightarrow 1 \) as \( R \Rightarrow \) ) to determine the two "constants" of integration. When (4.1) is substituted into (*), we obtain

\[
\Lambda, + + \theta \\chi \text{t}(') \Lambda, = O(\Lambda^2)
\]  

while (a*) remains

\[
\Lambda, - - \theta' \Lambda, = O(\Lambda^2)
\]  

and (c*) becomes, using (4.2) and (4.3)
Remark 7: Note that the general solution of the linearized vacuum Einstein equation (E) is \( \Omega = 1 \). The main content of the linearized equations has been transferred to the metricity conditions (4.2), (4.3a) and (4.3c).

Since we are integrating only up to the linear terms we consider (4.2) and (4.3a & c) without the order symbol. Even so it is not immediately clear what to do with them. It, however, turns out that they have a very attractive structure. By studying the integrability conditions between Eqs. (4.2) and (4.3a), on both the (+,-) derivatives and the (', \( \partial \times \partial \)) derivatives, two important equations can be derived. See Appendix C. They are

\[
\mathcal{E} \Lambda \equiv \eta^{ij} \partial_i \partial_j \Lambda \equiv 2(\Lambda,_{01} - \Lambda,_{11} - \Lambda,_{+-}) = O(\Lambda^2) \tag{4.4}
\]

and

\[
\Lambda,_{01} + \mathcal{E}' (' \partial \times \partial) \Lambda,_{1} = 2 \Lambda,_{1} = O(\Lambda^2). \tag{4.5}
\]

Remark 8: It is at this point (though it is not at all obvious) there appears to be a break in the two methods of solving the equations - i.e., using either the \( \Lambda \) or the \( \zeta \) as basic variables. We will take the later direction, concentrating on (4.5), leaving (4.4) for a future paper. (We however point out that (4.4) is really a rather strange version of the wave equation (neglecting the non-linear terms), involving not just one solution - but a sphere's worth of solutions parametrized by \( \zeta, \bar{\zeta} \), all matched with each other so that the remaining \( \Lambda \) equations (metricity conditions) are satisfied.)

Eq. (4.5) can be integrated (on the \( R \equiv \theta^1 \) variable) to become

\[
\Lambda,_{0} = \sigma_{,0}(\theta^0, \zeta, \bar{\zeta}) \mathcal{E}' (' \partial \times \partial) \Lambda,_{1} = 2 \Lambda,_{1} = O(\Lambda^2) \tag{4.6}
\]

with \( \sigma_{,0}(u, \zeta, \bar{\zeta}) \), the "constant" of integration, becoming the free data (characteristic). (Though it is not obvious, \( \sigma_{,0}(u, \zeta, \bar{\zeta}) \) is the Bondi news function.)

Our final task is to simplify (4.3c) with the use of (4.6). By applying \( \partial \times \partial \) to (4.6) and substituting it into (4.3c) we obtain (with its complex conjugate)

\[
^3 \bar{\Lambda},_{1} = -4 \partial \times \partial(\sigma_{,0} + O(\Lambda^2)) \quad \text{and} \quad \partial \times \partial(3 \Lambda,_{1} = -4 \mathcal{E}' \sigma_{,0} + O(\Lambda^2) \tag{4.7}
\]
which, in turn, is resubstituted into (4.3c), after applying $\mathbf{\nabla}\mathbf{\nabla}u$ to it. This results in

$$\mathbf{\nabla}\mathbf{\nabla}u \mathbf{\nabla}^2 \Lambda = \mathbf{\nabla}\mathbf{\nabla}^2 \sigma + \mathbf{\nabla}\mathbf{\nabla}^2 \sigma_0 + O(\Lambda^2)$$

(4.8a)

or on integration

$$\mathbf{\nabla}\mathbf{\nabla}u \mathbf{\nabla}^2 \Lambda = \mathbf{\nabla}\mathbf{\nabla}^2 \sigma + \mathbf{\nabla}\mathbf{\nabla}^2 \sigma_0 + O(\Lambda^2)$$ .

(4.8b)

Finally, using the definition, $\Lambda = \mathbf{\nabla}\mathbf{\nabla}Z$, we have our basic equation for iteration, namely

$$\mathbf{\nabla}\mathbf{\nabla}^2 Z = \mathbf{\nabla}\mathbf{\nabla}^2 \sigma + \mathbf{\nabla}\mathbf{\nabla}^2 \sigma_0 + O(\Lambda^2)$$

(4.9)

where the free data $\sigma = \sigma(Z, \zeta, \bar{\zeta})$ is an arbitrary function of three variables.

The terms $O(\Lambda^2)$ are determined iteratively (going back to (E) and are in general quite complicated though the quadratic terms are reasonable. With a similar formalism to ours but using the vanishing of the Bach tensor instead of the Einstein equations, Lionel Mason obtained (4.9).

The iteration procedure is as follows:

Let $Z = Z_M + Z_1 + Z_2 + \ldots$.

with

$$\mathbf{\nabla}\mathbf{\nabla}^2 Z_M = 0$$

(4.9a)

whose solution $Z_M = Z_M(x, \zeta, \bar{\zeta})$ yields Minkowski space (See Sec. V). The next approximation

$$\mathbf{\nabla}\mathbf{\nabla}^2 Z_1 = \mathbf{\nabla}\mathbf{\nabla}^2 \sigma(Z_M, \zeta, \bar{\zeta}) + \mathbf{\nabla}\mathbf{\nabla}^2 \sigma(Z_M, \zeta, \bar{\zeta}) + O(\Lambda^2)$$

(4.9b)

is equivalent to linear theory. Note that the operator $\mathbf{\nabla}\mathbf{\nabla}^2$ is essentially the double Laplacian on the sphere and possesses a simple, unique Green’s function so that the solutions can be given explicitly in integral form.

The first non-linear term is

$$\mathbf{\nabla}\mathbf{\nabla}^2 Z_2 = \mathbf{\nabla}\mathbf{\nabla}^2 \sigma(Z_M + Z_1, \zeta, \bar{\zeta}) + \mathbf{\nabla}\mathbf{\nabla}^2 \sigma(Z_M + Z_1, \zeta, \bar{\zeta}) + O(\Lambda^2),$$

(4.9c)

which again can be integrated via the same Green’s function since the $O(\Lambda^2)$ is known from the first order theory.

Remark 9: Note that in the iteration procedure, the $u = Z(x, \zeta, \bar{\zeta})$ - which are the characteristic surfaces - are substituted into the free data $\sigma(Z, \zeta, \bar{\zeta})$. It begins with the Minkowski surfaces, $Z_M$. 

but gets corrected at each stage of the approximation. This does not happen in the conventional version of the perturbation procedure which uses the same uncorrected Minkowski light-cones at every order of the approximation.

V. Special Solutions

Minkowski space: It is easy to check that the equations \((a^*)\) and \((c^*)\) are identically satisfied by \(\Lambda = 0\). This makes \(h^{ij} = 0\), leading, from (2.3), to a conformally flat space-time,

\[
g^{ij} = \Omega^2 \eta^{ij}. \tag{5.1}
\]

\((E)\) and \((*)\) become, with \(\Lambda = 0\),

\[
D^2 \Omega = 0 \quad \text{and} \quad ' \Omega = \nabla \Omega = 0. \tag{5.2}
\]

The solution

\[
\Omega = 1
\]

is then the Minkowski metric. The vanishing of \(\Lambda\), i.e.,

\[
'2Z = \Lambda = 0 \tag{5.4}
\]

implies that

\[
u = Z = x^a l_a (\zeta, \bar{\zeta}), \tag{5.5}
\]

with \(Z_a = l_a (\zeta, \bar{\zeta})\) a normalized null vector given in Minkowski coordinates by

\[
\Theta_a (\zeta, \bar{\zeta}) = \frac{1}{\sqrt{2(1 + \zeta \bar{\zeta})}} (1 + \zeta \bar{\zeta}, \zeta + \bar{\zeta}, i(\zeta - \bar{\zeta}), -1 + \zeta \bar{\zeta})
\]

and \(x^a\), the Minkowski space coordinates, appearing as constants of integration.

De Sitter Space. Since De Sitter Space is conformally flat, we once again have that \(\Lambda = 0\) and \(\Lambda = 0\) as in the Minkowski case. But now we must solve for \(\Omega\) from (5.1). Since \(\Omega = \nabla \Omega = 0\), then \(\Omega\) is independent of \(\zeta, \bar{\zeta}\) and thus \(\Omega = \Omega (x^a)\) with \(x^a\) the Minkowski coordinates; In addition, since \(D = \partial / \partial R = \theta^a \partial_a = l^a (\zeta, \bar{\zeta}) \partial_a\), we have that \(D^2 \Omega = 0\) implies that \(l^a b \partial_a \partial_b \Omega = 0\). The general solution, using \(\eta_{ab} l^a l^b = 0\), is then

\[
\Omega = \text{constant} + \lambda_a x^a + \lambda \eta_{ab} x^a x^b. \tag{5.6}
\]

The constant can be transformed to one by a rescaling and the \(\lambda_a\) set equal to zero by an origin shift, yielding one form of the De Sitter metric.

Linear Theory.
This was discussed in the previous section under the perturbation scheme.

**Self-Dual Metrics.**

If we drop the requirement that our solutions be real - i.e., *allow complex solutions* (which means that the equations (*), (a*) and (c*) are independent of their conjugate versions) and start with the ansatz that

\[ ^{2}Z = \Lambda = \sigma(u, \zeta, \bar{\zeta}) \quad \text{and} \quad \overline{\sigma}(u, \zeta, \bar{\zeta}) = 0 \]

we have immediately that \( \Lambda,_{1} = \Lambda,_{+} = \Lambda,_{-} = 0 \), (with no implication about \( \bar{\Lambda} \)) which in turn implies that \( Q(\Lambda) = 0 \) so that (E) becomes \( D^{2}\Omega = 0 \). Choosing the solution \( \Omega = 1 \) leads, from (*) and its conjugate, to \( W = \overline{W} = 0 \). The remaining equations, (a*), and (c*) are identically satisfied. Our final field equation for the self-dual space-times is\(^{13,14,15}\) thus

\[ ^{2}Z = \sigma(Z, \zeta, \bar{\zeta}). \]

When a regular solution \( Z = Z(x^{a}, \zeta, \bar{\zeta}) \) of (5.7) is found then the \( \bar{\Lambda} = \Lambda \) and conformal factor \( \Omega \). Unfortunately they are fairly complicated and it is not clear how to obtain them directly from our equations.

**Schwarzschild, Kerr, and Charged Kerr.**

By starting with these metrics and integrating the null geodesics, it has been possible\(^{16,17}\) to find the characteristic surfaces \( u = Z(x, \zeta, \bar{\zeta}) \), \( \Lambda \), and conformal factor \( \Omega \). Unfortunately they are fairly complicated and it is not clear how to obtain them directly from our equations.

**VI. Summary and Conclusions**

We begin by first summarizing our claims; we contend that we have a radically different view (from the conventional version) of GR, where, in addition to a scalar function (conformal factor, \( \Omega \)), the basic geometric objects are a family of surfaces described by a single function \( u = Z(x^{a}, \zeta, \bar{\zeta}) \) (or probably better, its second derivatives with respect to \( \zeta \) and \( \bar{\zeta} \), i.e., the \( \Lambda \) and \( \bar{\Lambda} \), from which the \( Z \) can be reconstructed) chosen to satisfy a set of conditions or differential equations (the metricity conditions). From this it follows that a conformal metric exists so that these surfaces are characteristic surfaces of the metric. It also follows that any Lorentzian metric
can be encoded into an appropriately chosen $Z$ and conformal factor $\Omega$. From this point of view, the (vacuum) Einstein equations become, remarkably, a single simple second order, linear, o.d.e. [Eq. (3.3)] for the conformal factor. What has happened is that in this version, the Einstein equations have become quite simple, with the difficult non-linear problems being shifted to the metricity conditions. One should not make the mistake of thinking that it is possible to either 1) first solve (3.3) and then the metricity conditions or 2) first solve the metricity conditions and then (3.3). The two sets are inextricably related and must be solved simultaneously. However if one goes to the iteration scheme it is easily seen from the non-linear structure that it is possible to "toggle" back and forth between the two sets, solving first (3.3) then the metricity equations and back to (3.3), etc.

From another point of view, we want to emphasize how different this formulation is from the usual one. As we have already stressed it uses characteristic surfaces as its basic variable - but when thought of as differential equations for $\Lambda(\theta^i, \zeta, \bar{\zeta})$ and $\Omega(\theta^i, \zeta, \bar{\zeta})$ the equations are, in a sense, very strange; - though they are explicit differential equations for the $\Lambda$ and $\Omega$, with the six independent variables, $(\theta^i, \zeta, \bar{\zeta})$, nevertheless, the four $\theta^i$ are themselves $\zeta$ and $\bar{\zeta}$ derivatives of a single function $Z(\zeta, \bar{\zeta})$ with no mention of space-time. [See Eq. (2.7) and (2.11).] Evolution, in the conventional sense, does not appear - there being no "time" variable in the formulation. The space-time $M$ itself arises when $Z(x, \zeta, \bar{\zeta})$ is constructed from the $\Lambda(\theta^i, \zeta, \bar{\zeta})$ with the $x^a$ originating as "constants of integration", i.e., - the manifold $M$ and the characteristic surfaces simply appear as solutions of $\nabla^2 Z = \Lambda(\theta^i, \zeta, \bar{\zeta})$. One then calculates, by simply differential and algebraic operations, the associated metric which is automatically Lorentzian and vacuum. It appears to us highly unlikely that other field theories of physical interest can be formulated in a like manner.

At this early stage of our understanding of this new view, we find it difficult to assess how potent or effective this approach will be to the problem of obtaining new solutions of the equations - it seems that, at least for certain problems, e.g., pure radiation studied perturbatively, it is very well suited - but for others, perhaps problems with sources and/or singularities, poorly
suited. It certainly appears well suited for global linear and second order theory. The advantage is that the approach is based on characteristic surfaces - and perturbatively, these surfaces are corrected at each stage of the approximation.

But perhaps for us, the greatest advantage of looking at GR in this manner, is conceptual - it permits one to focus on different issues. One can think of possible generalizations of GR - we mention one, without necessarily taking it seriously. Consider our equations, but omit one or more of the metricity conditions. The remaining equations lead to some sort of a Finslerian metric.

Another, and more important, conceptual issue that can be raised is: how does one think of the quantization of GR from this point of view. One of the most important approaches to the study of Quantum GR is via the canonical formulation. This is based on the foliation of space-time by space-like hyper surfaces. The variables (to be turned into operators of the quantum theory) have conventionally been the three-metric and the canonical conjugate second fundamental form given on the three-surfaces - (or the new variables of Ashtekar, a tetrad and a complex connection). In any case, they are conventional fields on a three manifold. From our view, we have no fields at all, (except perhaps the conformal factor). Our variables are the geometric surfaces themselves. It is not at all clear what the idea of quantization would mean in this context. It might well imply that this formulation of GR is simply inappropriate for the study of quantum GR or conversely, it might imply that GR is not "quantizable" in any conventional sense. The observation that suggests that this later point of view should be taken seriously is the fact that there is a formulation of Maxwell theory\textsuperscript{18} based on characteristic surfaces, that closely mirrors the present version of GR - but in that version there is no trouble (at least at a formal level) in quantizing the Maxwell field. In that case it is the fact that the characteristic surfaces are there and fixed that allows the (formal) quantization -- in our case the characteristic surfaces are the variables themselves and the whole procedure (apparently) breaks down.

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References:


Appendix A

The Linearized Metric

\[ g^{00} = g^{0+} = g^{0-} = 0, \]
\[ g^{01} = 1, \quad g^{+-} = -1 \]
\[ g^{++} = -\Lambda, \quad g^{--} = -\Lambda \]
\[ g^{1+} = -\Lambda + \Lambda, \quad g^{1-} = -\Lambda + \Lambda \]
\[ g^{11} = -2 - \Lambda + \Lambda + \Lambda + \Lambda \]

(A.1)

or using the vacuum field equations

\[ g^{11} = -2 - \Lambda + \Lambda \]

The full Metric

\[ g^{ij} = g^{ab} \theta^{i}_{a} \theta^{j}_{b} \]
\[ g^{00} = g^{0+} = g^{0-} = 0, \]
\[ g^{01} = \Omega^{2}, \quad g^{+-} = -\Omega^{2} \]
\[ g^{++} = -\Omega^{2} \Lambda, \quad g^{--} = -\Omega^{2} \Lambda \]
\[ g^{1+} = -\Lambda - \Lambda + \Lambda + \Lambda + \Lambda \]
\[ g^{1-} = -\Lambda - \Lambda + \Lambda + \Lambda \]
\[ g^{11} = -2 - \Lambda - \Lambda + \Lambda + \Lambda \]

with W defined in (A.4). From B.6, by choosing \( g^{ij} \) as \( g^{+1} \), and using \( T_{1}^{+} = \delta^{1}_{1} \), we have

\[ g^{11} = \Lambda + \Lambda + \Lambda + \Lambda \]

(A.2)

Eq. (A.2), with \( W \) from (A.4) and the \( T_{1}^{1} \) and \( \bar{T}_{1}^{1} \) from (B.4), yields the last component of \( g^{ij} \) as \( \Omega^{2} \) times an expression in \( \Lambda \) and \( \bar{\Lambda} \), and we finally have \( g^{ij} = \Omega^{2} g^{ij}(\Lambda, \bar{\Lambda}) \).

Comment

There are several alternative paths that could be taken for the calculations that lead to the expressions for \( g^{+1}, g^{-1}, * , a^{*}, c^{*}, W \) and \( \bar{W} \). They lead to different but equivalent expressions for the metricity conditions as well as for \( W \). We give two of them below.
The Metricity Conditions

\[ \star \Omega = 8W\Omega, \tag{A.3} \]

with

\[ W(1 - \mathcal{A}_1 \overline{\mathcal{A}}_1) \equiv (T^{1}_{1} - \Lambda_1 \overline{T}^{1}_{1}) - ^{\mathcal{A}}(\mathcal{X}t^{(*)}\mathcal{A}_1 - ^{8}\Lambda_1 \overline{\mathcal{A}}_1) \tag{A.4} \]

or

\[ W = \Lambda_{+} + ^{8}\mathcal{X}t^{(*)}\Lambda_1 + O(\Lambda^2) \]

with

\[ q = 1 - \Lambda_1 \overline{\Lambda}_1. \]

(a*) \[ \overline{\Lambda}_1 - 2\Lambda_1 = (W + ^{1}\mathcal{X}t)\Lambda_1. \]

(b*) \[ 2g^+ + g^+i(T_i^1 - ^{\mathcal{A}}(\mathcal{X}t^{(*)})T^i_j + ^{\mathcal{A}}(\mathcal{X}t^{(*)})T^i_j - T^i_jT^i_j) \]

\[ - g^{ii}(\mathcal{X}t^{(*)}\Lambda_{ii} + \Lambda_{ij} \overline{\mathcal{X}}_j + \Lambda_{ij} \overline{\mathcal{A}}_1) = 0 \tag{A.6} \]

or

\[ ^{13}\overline{\mathcal{A}}_1 - 2\mathcal{X}t^{(*)}\Lambda_{+} + 4\mathcal{X}t^{(*)}\Lambda_{0} - 4\Lambda_{+} - 2\mathcal{X}t^{(*)}\Lambda_1 = O(\Lambda^2). \]

Eq. (A.6) can be explicitly expressed in terms of the \( \Lambda_{ii} \) and their \( \star \) and \( \mathcal{X}t^{(*)} \) derivatives by replacing all the \( g^{ij} \) by their known expressions in terms of the \( \Lambda_{ii} \). The \( T^i_j \) are given in Eqs. (B.4a&b).

The Vacuum Einstein Equation

\( E \)

\[ D^2 \Omega = Q(\Lambda)\Omega, \tag{A.7} \]

with

\[ Q = - \frac{1}{4q} D\overline{\mathcal{A}}_1 DA_1 - \frac{3}{8q^2}(Dq)^2 + \frac{1}{4q} D^2 q, \quad q = 1 - \Lambda_1 \overline{\Lambda}_1 \tag{A.8} \]
Appendix B

The Definition of $\Phi^s$.

The differential operator $\Phi^s$, [referred to as edh] (essentially the derivative with respect to $\zeta$) has been used extensively throughout this work. It is a version of the covariant derivative operator on the sphere. Instead of acting on tensors it acts on spin-weighted objects.

Its definition is $\Phi^s = P^{1-s}\partial/(P^s\partial \zeta)$ where $s$ is the spin weight and $P = 1 + \zeta \bar{\zeta}$ with a similar definition for $\Lambda_{\text{to}(\cdot)}$. Thus since $Z(x, \zeta, \bar{\zeta})$ is a spin-weight zero object, we have from (2.7)

$$\omega = 'Z(x^a, \zeta, \bar{\zeta}) = P \partial Z/\partial \zeta, \quad \bar{\omega} = \Lambda_{\text{to}(\cdot)}Z(x^a, \zeta, \bar{\zeta}) = P \partial Z/\partial \bar{\zeta},$$

$$R = \Lambda_{\text{to}(\cdot)}Z(x^a, \zeta, \bar{\zeta}) = P^2 \partial^2 Z/\partial \zeta \partial \bar{\zeta}.$$ 

In these operations the derivatives have been taken at constant $x^a$. The problem arises as to the meaning of the operator when the functions are of $\theta^i$ and $\zeta, \bar{\zeta}$. It turns out that $'\Phi(\theta^i, \zeta, \bar{\zeta})$ is a perfectly well defined differential operator (though complicated) involving ordinary partial derivatives with respect to both the $\theta^i$ and $\zeta$. To see this, we have (where the prime on the $'$ indicates the operation holding the $\theta^i$ constant) that

$$'\Phi(\theta^i, \zeta, \bar{\zeta}) = '\Phi(\theta^i, \zeta, \bar{\zeta}) + \Phi_{,i} '\theta^i$$

$$= '\Phi(\theta^i, \zeta, \bar{\zeta}) + P_{,0} 'Z + \Phi_{,+} 'Z + \Phi_{,-} \Lambda_{\text{to}(\cdot)}Z + \Phi_{,1} \Lambda_{\text{to}(\cdot)}Z$$

or

$$'\Phi(\theta^i, \zeta, \bar{\zeta}) = '\Phi(\theta^i, \zeta, \bar{\zeta}) + \omega \Phi_{,0} + \Lambda \Phi_{,+} + R \Phi_{,-} + (\Lambda_{\text{to}(\cdot)}\Lambda - 2\omega)\Phi_{,1}.$$ 

This definition is not good enough since it involves $\Lambda_{\text{to}(\cdot)}$, which has not yet been defined, acting on $\Lambda$. It, however, can be "fixed" in the following fashion. By setting $\Phi = \Lambda$ in (A.8) and considering the complex conjugate equation to (A.8) with now $\Phi = \Lambda$, we obtain a pair of linear algebraic equations for $\Lambda_{\text{to}(\cdot)}\Lambda$ and $\bar{\Lambda}$ in terms of the ordinary partial derivatives of $\Lambda$ and $\bar{\Lambda}$, which when solved and replaced in (B.1) yield a complicated but explicit differential operator (depending on $\Lambda$) for $'\Phi(\theta^i, \zeta, \bar{\zeta})$, namely (B.1), with
\( \Lambda \to (') \Lambda - 2 \omega = \frac{\Lambda_{i1} \tilde{J} + J}{1 - \Lambda_{i1} \Lambda_{i1}} \) with \( J = -2\omega + \Omega \Lambda_{i0} + \Lambda \to (') \Lambda + R \Lambda_{+} + \Lambda \Lambda_{\ldots} \).

**Calculations with \( ' \theta_{i,a}^\dagger \) and \( \Lambda \to (') \theta_{i,a} \).**

Throughout this work (See e.g., Eqs.2.25, 2.26 or 2.27) use has been made of \( ' \) and \( \Lambda \to (') \) applied to \( \theta_{a,i} \) or its inverse \( \theta_{a,i}^{-1} \). We will, in this section, organize these calculations into a unified set of relations which then allows a great simplification in many of the manipulations.

Since \( ' \theta_{a,i}^\dagger \) and \( \Lambda \to (') \theta_{i,a} \) are one forms, they can be expressed as a linear combination of the basis set \( \theta_{a,i} \), i.e., we have

\[ ' \theta_{a,i}^\dagger = T_{j}^i \theta_{a,j} \], \quad \Lambda \to (') \theta_{i,a} = \tilde{T}_{j}^i \theta_{a,j} \] (B.2)

All the \( T_{j}^i \), with the exception of \( T_{1}^i \), are simple expressions in terms of \( \Lambda_{i} \) and can be calculated from \( ' \theta_{i,a}^\dagger = T_{j}^i \theta_{j,a} \) and from the definitions, i.e.,

\[ ' \theta_{0,a}^\dagger \equiv ' Z_{a} = \omega_{a} = \theta_{a}^{+} \], \quad (B.3a) \]

\[ ' \theta^{+},a \equiv ' \omega_{a} = \Lambda_{a} = \Lambda_{i} ' \theta_{i,a} \], \quad ' \theta^{-},a \equiv ' \bar{\omega}_{a} \equiv \Lambda \to (') Z_{a} = R_{a} = \theta_{a}^1 \], \quad (B.3b) \]

\[ ' \theta_{1,a} \equiv ' R_{a} \equiv ' 2 \Lambda \to (') Z_{a} \equiv \Lambda \to (') \Lambda_{a} - 2 \omega_{a} \]
\[ = \Lambda \to (') (\Lambda_{i} ' \theta_{i,a}) - 2 \omega_{a} = \Lambda \to (') (\Lambda_{i} ' \theta_{i,a}) - 2 \theta_{a}^{+} \]

(B.3c)

thus

\[ T_{0}^i = \delta_{i}^{+} \], \quad \Lambda_{a} = \Lambda_{i} \]
\[ qT_{1}^i = (\Lambda_{i} \bar{\Lambda}_{a} + ' \bar{\Lambda}_{i} + \bar{\Lambda}_{i} \delta_{1} - 2 \delta_{1} ') \Lambda_{i1} \]
\[ + \Lambda \to (') \Lambda_{i} + \Lambda_{a} ' \bar{\Lambda}_{i} + \Lambda_{0} \delta_{1} + \Lambda_{a} + \delta_{1} - 2 \delta_{1} \] (B.4)

\( T_{1}^i \) (see B.3c), the most complicated of the \( T \)'s, needs the conjugates of (B.3) for its evaluation and is obtained, after a lengthy calculation using (B.2) on \( \Lambda \to (') (\Lambda_{i} ' \theta_{i,a}) \) and its conjugate, repeatedly, from the following relationships:
which gives the decomposition of \( \Lambda_{\alpha} \) and \( \Lambda_{\alpha} \) into the \( \theta_i^\alpha \) basis.

(An alternate method to obtain the \( T^1_i \), not as straightforward as the the above, but actually a much shorter calculation, is to look at the integrability conditions on the pair (B.2) applied to the component \( \theta^+_{\alpha \alpha} \) and its complex conjugate. See below. This integrability condition is an algebraic expression involving \( T^1_i \) and the other \( T \)s, (functions of the \( \Lambda_{\alpha i} \), which with its complex conjugate, is easily solvable for the \( T^1_i \)).

An immediate use for (B.2) is in the simplification of (2.25). Using (B.2), the Eqs. (2.25) takes the form

\[
\mathcal{g}_{ij}(\theta^i, \zeta, \bar{\zeta}) - \mathcal{g}_{ik} T^k_j - \mathcal{g}_{jk} T^k_i = 0 \quad \text{(B.6)}
\]

The integrability conditions for (B.6) are

\[
2s(ij)g_{ij} = \mathcal{g}^{ik}(\mathcal{X}t^i_{\alpha \alpha})T^k_j - \mathcal{T}^k_i - \mathcal{T}^m_j + \mathcal{T}^m_i T^j_m - T^m_i T^j_m
\]

(2.25b)

where \( s(ij) \) is the spin-weight of the component \( g_{ij} \). They are equivalent (though it is not obvious) to the integrability conditions on (B.2), namely

\[
2s(i)\theta^i = \theta^k_{\alpha \alpha}(\mathcal{X}t^i_{\alpha \alpha})T^k_i - \mathcal{T}^k_i + \mathcal{T}^m_i T^j_m - T^m_i T^j_m\quad \text{(B.7)}
\]

where \( s(i) \) is the spin-weight of \( \theta^i_{\alpha \alpha} \).

These integrability conditions, (B.8) and/or(B.7), can be used to give relatively simple explicit forms to the metricity conditions, (2.27b&c). Specifically, if in (B.6) we choose \( g_{ij} \) to be \( g^{++} \) (and respectively \( g^{+1} \)), we have the metricity condition, (2.27b) (and respectively, (2.27c)). If we now subtract \( \mathcal{X}t^i_{\alpha \alpha} \) applied to the metricity condition (2.27a) from (2.27b) the result is
precisely the integrability conditions (B.7) with $g^{ij}$ equal to $g^{++}$. This, as well as the analogous one with $g^{+1}$, are (equivalent) to the full versions of our equations (2.27$b^*$) and (2.27$c^*$), the final two metricity conditions. This new version of (2.27$b^*$) follows from (B.8) with $\theta^i_a$ equal to $\theta^+_a$. But (B.8) with $\theta^i_a$ equal to $\theta^+_a$ turns out to be identically satisfied from the definition of the $T$'s. (See the parenthetic remark above.). We thus have the important result that metricity condition b* is an identity. The explicit version of c* is given in Appendix A.

The meaning and content of the metricity conditions and their relationship to the integrability conditions, $\overset{\wedge}{\Lambda} = \mathcal{X}\Theta$, is more fully explored in [19].

**Commutation Relations between ′ and the $\theta^i$ Derivatives**

A major source of computational difficulty is the fact that the ′ and $\mathcal{X}\Theta$ operators do not commute with the $\theta^i$ derivatives, i.e., with $\theta^i_\ a = i$. These commutation relations can be obtained in the following manner:

Consider

\[
'(\Phi, i) \equiv (\theta^i_\ a \Phi) = (\theta^i_\ a^) \ a \Phi + (\theta^i_\ i^) \ a \Phi = -T^i_j \Phi, j + ('\Phi)_{, i}
\]

or

\[
'(\Phi, i) - ('\Phi)_{, i} = -T^i_j \Phi, j \quad (B.9)
\]

where we have used $\theta^i_\ a = -T^i_j \theta^j_\ a$, which follows from $\theta^i_\ a = T^i_j \theta^j_\ a$ and $\theta^i_\ a \theta^j_\ a = \delta^i_\ j$. 


Appendix C.

Two of, what appear to be, the most important equations that arise from our new (surface) point of view, are Eqs. (4.4) and (4.5), the "wave" equation for $\Lambda$ and the data equation, namely

$$\mathcal{L} \Lambda \equiv \eta^{ij} \partial_i \partial_j \Lambda \equiv 2(\Lambda,_{01} - \Lambda,_{11} - \Lambda,_{++}) = O(\Lambda^2) \quad (C.1)$$

$$\Lambda,_{01} + c(\partial \Lambda,_{11})_{,1} - 3\Lambda,_{11} = O(\Lambda^2). \quad (C.2)$$

They arise from the integrability conditions between equations (4.2) and (4.3a). We give a sketch of the derivation which relies heavily on the commutation relations between the $'$ derivatives and the $\theta^i$ derivatives. [See Appendix B.] For simplicity we will, in the remainder of this section, drop the $O(\Lambda^2)$ symbol and work with just the linear terms. Since the variables we are working with are already first order we only need the commutation relations, Eq. B.2 to zeroth order, namely

$$'\Phi,_{00} - ('\Phi),_{00} = 0 \quad (C.3a)$$

$$'\Phi,_{11} - ('\Phi),_{11} = - \Phi,_{--}$$

$$'\Phi,_{++} - ('\Phi),_{++} = - \Phi,_{00} - 2 \Phi,_{1}$$

$$'\Phi,_{--} - ('\Phi),_{--} = 0$$

(For future use in second order calculations, we give the first order commutation relations:

$$'\Phi,_{00} - ('\Phi),_{00} = - \partial \Lambda,_{00} \Phi,_{1} - \Lambda,_{0} \Phi,_{+} \quad (C.3b)$$

$$'\Phi,_{11} - ('\Phi),_{11} = - (\partial \Lambda,_{11} + \Lambda,_{+})\Phi,_{1} - \Lambda,_{1} \Phi,_{+} - \Phi,_{--}$$

$$'\Phi,_{++} - ('\Phi),_{++} = - \Phi,_{00} - (2 - \partial \Lambda,_{11} + \Lambda,_{+})\Phi,_{1} - \Lambda,_{+} \Phi,_{+}$$

$$'\Phi,_{--} - ('\Phi),_{--} = - (\partial \Lambda,_{11} + \Lambda,_{0} - 2\Lambda,_{1})\Phi,_{1} - \Lambda,_{-} \Phi,_{+}$$

(For future use in second order calculations, we give the first order commutation relations:

We begin with (4.2) and (4.3a),

$$\Lambda,_{+-} = - \partial \Lambda,_{1} \quad \text{and} \quad \Lambda,_{-} = \partial \Lambda,_{1} \quad (C.4)$$

Applying the (-) derivative to the first and the (+) derivative to the second and subtracting, using the commutation relations (C.3), we obtain

$$\partial \Lambda,_{1} +'\Lambda,_{1+} + 2\Lambda,_{10} - 4\Lambda,_{11} = 0. \quad (C.5)$$
Also from (C.4) applying the (1) derivative to each, we obtain, again using the commutation relationships (C.3),

\[ \Lambda_{+,1} = - \chi \rt X \to (') \Lambda_{+,11} \quad \text{and} \quad \Lambda_{-,1} = ' \Lambda_{-,11}. \]  

(C.6)

Applying ' and \( \rt X \to (') \), respectively to each of (C.6) leads to,

\[ ' \Lambda_{+,1} = \chi ' \rt X \to (') \Lambda_{+,11} \quad \text{and} \quad \rt X \to (') \Lambda_{-,1} = ' \rt X \to (') \Lambda_{-,11} + 4 \Lambda_{-,11}. \]  

(C.7)

The commutation relations between ' and \( \rt X \to (') \) have been used several times. We now substitute both equations of (C.7) into (C.5), noting that \( \theta^i \) derivatives commute with each other. This leads, after some simplification, to

\[ \Lambda_{10} = - \chi ' \rt X \to (') \Lambda_{11}. \]  

(C.8)

From the commutation relations (C.3), after a lengthy calculation, we find

\[ 5 ' \rt X \to (') \Lambda_{11} = - 6 \Lambda_{-,11} - 3 \Lambda_{-,10} + 3 (' \rt X \to (') \Lambda_{11})_{-1}. \]  

(C.9)

Finally, using (C.9) in (C.8) we have (C.2), namely

\[ \Lambda_{01} + \phi (' \rt X \to (') \Lambda_{11})_{-1} - \theta \Lambda_{-,11} = 0. \]  

(C.1) can be derived by applying the (-) operator to the first of (C.4), commuting it through the \( \rt X \to (') \) on the right side and then eliminating the \( \Lambda_{-,1} \) via (C.6). The resulting equation when used with (C.8) finally yields (C.1),

\[ \chi \Lambda \equiv \eta^{\hat{u}} \partial_\hat{u} \partial_j \Lambda \equiv 2 (\Lambda_{01} - \Lambda_{-,11} - \Lambda_{+,1}) = 0. \]
Fig. 1: A Fiducial Null Cone with Families of Null Surfaces Parametrized by the Points $u$ on each of the Null Generators $(\zeta, \bar{\zeta})$. 
\[ \begin{align*}
g_{ij} &= g^{ab} \theta_i^a \theta_j^b \equiv \theta^i \theta^j \\
(Z_{a}, \partial Z_{a}^{*}, \partial \partial Z_{a}^{*}, \partial \partial \partial Z_{a}^{*}) &= (\theta^{0}_a, \theta^{+}_a, \theta^{-}_a, \theta^{1}_a) \\
g^{ab} Z_{a} Z_{b} &= 0 \\
g^{00} &= Z \cdot Z = 0
\end{align*} \]

\[ \begin{align*}
\tilde{\partial}(g^{ab} Z_{a} Z_{b}) &= 0 \\
g^{ab} \partial Z_{a} Z_{b} &= g^{0-} = 0 \\
g^{ab} \partial \partial Z_{a} Z_{b} &= g^{0+} = 0
\end{align*} \]

\[ \begin{align*}
\tilde{\partial}^2(g^{ab} Z_{a} Z_{b}) &= 0 \\
g^{-+} + \Lambda \cdot Z &= 0 \\
g^{++} + g^{01} &= 0 \\
g^{++} + \Lambda \cdot Z &= 0
\end{align*} \]

\[ \begin{align*}
\tilde{\partial}^3(g^{ab} Z_{a} Z_{b}) &= 0 \\
\tilde{\partial}[\Lambda \cdot Z] + 2[\Lambda \cdot \partial Z] &= 0
\end{align*} \]

\[ \begin{align*}
\tilde{\partial}^2 \partial \partial^3(g^{ab} Z_{a} Z_{b}) &= 0 \\
\tilde{\partial} \partial \partial^3(g^{ab} Z_{a} Z_{b}) &= 0 \\
\tilde{\partial} \partial \partial^3(g^{ab} Z_{a} Z_{b}) &= 0 \\
\tilde{\partial} \partial \partial^3(g^{ab} Z_{a} Z_{b}) &= 0
\end{align*} \]

\[ \begin{align*}
\partial \partial \partial \partial^3(g^{ab} Z_{a} Z_{b}) &= 0 \\
g^{11} & \text{ determined}
\end{align*} \]

\[ \begin{align*}
\tilde{\partial} \partial \partial \partial g^{ab} Z_{a} Z_{b} &= 0 \\
\tilde{\partial} \partial \partial \partial g^{ab} Z_{a} Z_{b} &= 0
\end{align*} \]

Fig. 2: The metric diamond with the metricity conditions A, B, and C.