Critical Behavior of
Dimensionally Continued Black Holes

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Abstract

The critical behavior of black holes in even and odd dimensional spacetime is studied based on Bañados-Teitelboim-Zanelli (BTZ) dimensionally continued black holes. Second order phase transitions are found in even dimensional spacetime. In this case critical values for the mass, temperature and entropy for charged BTZ black holes are calculated; however Schwarzschild black holes for zero cosmological constant $\Lambda$ do not present critical behavior. For a given spacetime dimension $D$ and finite $\Lambda$, we find that the electric charge $Q$ controls the appearance of one or two critical transitions. As a special case we refer to the situation of $D = 4$ for any value of the cosmological constant, and for $\Lambda = 0$ we recover results previously found in the literature. It is shown that there are no critical transitions in odd dimensions, and this result prevails for any value of the dimensionality.

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1. Introduction

The possibility of critical behavior of classical objects such as black holes in general relativity is an interesting and open question. This behavior has been discovered by Choptuik [1] in connection with the numerical study of gravitational collapse of massless scalar fields. In that paper, a universal behavior of the black-hole mass described by a critical exponent $\beta \sim 0.37$ independent of the initial shape of the collapsing scalar field was found. Since then, critical behavior in other collapsing systems have been reported [2].

To study the thermodynamics of black holes, and in particular their heat capacity and critical behavior, it is assumed that there is an existing analogy between the laws of thermodynamics and the laws that govern black hole mechanics derived from general relativity. This was first established by Bardeen, Carter and Hawking [3]. To guarantee the analogy between the latter systems, one needs to make the formal replacements $E \rightarrow M$, $T \rightarrow C\kappa$ and $S \rightarrow A/8\pi C$, where $A$ is the area, $\kappa$ de surface gravity and $C$ is a constant [4]. With these substitutions the four laws of black hole thermodynamics can be enunciated and the study of critical behavior seems to be a plausible natural extension of this early idea. In this direction, Lousto [5] has argued in favor of critical behavior in a transition previously found by Davies [6]. He further calculated the critical exponents of black holes in $D = 4$ dimensions and showed the validity of the scaling laws. However, the relationship between the results found in [1, 2] and [5] are yet to be understood.

Bañados, Teitelboim and Zanelli (BTZ) [7] have reported black hole solutions for even and odd dimensional spacetime as a particular dimensional continuation of general relativity with non-vanishing cosmological constant $\Lambda$. They deal with a negative value of $\Lambda$, which is related to a length scale $l$ by $\Lambda = -a/l^2$, $a$ being a positive number. By a suitable choice of coefficients in the Lovelock action they manage to obtain a unique solution for the metric with dressed singularity, only for positive masses. The entropy is then a monotonically increasing function of $r_+$, and therefore they give meaning to the second law of thermodynamics for black holes.

In this paper we study the occurrence of phase transitions and the possibility of critical behavior in dimensionally continued BTZ black holes. We review some of the results found in [7] which we derive by use of implicit calculations, and particularize them for the different cases where $Q$ and $\Lambda$ are
zero or non-vanishing. In our study, as in the BTZ paper, we only consider non-rotating \((J = 0)\) black holes.

In sections 2 and 3 we study the critical behavior of black holes in even and odd dimensions respectively, and the critical values for the relevant thermodynamic variables are calculated. We then evaluate the full (not per unit mass) thermal capacity at constant \(Q\) as a way to study the possible phase transitions. We will look for those values that make the thermal heat become divergent. We find that these transitions are possible in even dimensions – not for the case of Schwarzschild black holes with zero cosmological constant – but no phase transitions are present in odd dimensions. We also study possible discontinuities in the derivatives of the thermal capacity to assure that there are no phase transitions of any order, for \(D\) odd. Section 4 is dedicated to give the final conclusions.

2. Even dimensional black holes

To study spherically symmetric solutions of black holes we start from the metric
\[
ds^2 = -g^2(r)dt^2 + g^{-2}(r)dr^2 + r^2 d\Omega^2,
\]
where the coefficient \(g^2(r)\) can be expressed as a function of \(r, M,\) and \(Q\) \([7]\) as
\[
g^2(r) = 1 + \frac{r^2}{l^2} - \left[ \frac{2M}{r} - \frac{Q^2}{(D - 3)r^D - 2} \right]^{\frac{1}{D-2}}, \quad \text{with} \quad D = 2n.
\]
The equation \(g^2(r) = 0\) has zero, one or two real solutions, depending on the values of \(M\) and \(Q\). We take the greater root \(r_+\) as the black hole horizon. However the explicit expression for \(r_+\) in terms of \(T, Q,\) and the parameters \(D\) and \(l\) is rather complicated, so we will use an implicit approach. From \(g^2(r) = 0\) we find
\[
1 + \frac{r_+^2}{l^2} = \left[ \frac{2M}{r_+} - \frac{Q^2}{(D - 3)r_+^D - 2} \right]^{\frac{1}{D-2}},
\]
a constraint relation which we will use in following expressions to write the relevant thermodynamical quantities. For \(l \to \infty\), that is \(\Lambda = 0\), equation (3) gives us \(r_+ = 2M\) for the case of Schwarzschild black holes, whereas the
finite cosmological constant situation, gives \( r_+ < 2M \). Therefore the largest Schwarzschild horizon happens for the \( \Lambda = 0 \) scenario.

From the standard expression of the temperature

\[
T = \frac{1}{4\pi} \left( \frac{dg^2(r)}{dr} \right)_{r=r_+}
\]

and relation (3), we find

\[
T(r_+, Q) = \frac{1}{4\pi} \left[ \frac{2r_+}{l^2} + \frac{1}{n-1} \left( 1 + \frac{r_+^2}{l^2} \right) \frac{1}{r_+} - \frac{1}{n-1} \left( 1 + \frac{r_+^2}{l^2} \right)^{2-n} \frac{Q^2}{r_+^{D-2}} \right],
\]

where in the above expression \( r_+ = r_+(T, Q) \).

In the case of \( Q = 0 \) and a non-vanishing \( \Lambda \), it is easy to see that the above expression for the temperature reduces to the result obtained in [7]

\[
T_{Q=0} = \frac{1 + (2n - 1)(r_+/l)^2}{4\pi(n-1)r_+}.
\]

In the instance of \( \Lambda = 0 \) \((l \to \infty)\) and for charged black holes, the expression for the temperature reduces to

\[
T^\infty = \frac{1}{4\pi(n-1)} \left[ \frac{1}{r_+} - \frac{Q^2}{r_+^{D-1}} \right]
\]

and

\[
T^\infty_{Q=0} = \frac{1}{4\pi(n-1)r_+} = \frac{1}{8\pi(n-1)M}.
\]

These results reproduce the standard relations found in the literature, for \( D = 4 \).

The expression of the entropy of the black hole in even dimensions can be computed in a closed form, obtaining [7]

\[
S(r_+) = \pi l^2 \left[ (1 + \frac{r_+^2}{l^2})^{n-1} - 1 \right],
\]

which for \( l \to \infty \) reduces to

\[
S^\infty(r_+) = \pi (n-1)r_+^2.
\]
Phase transitions

For the study of the phase transitions we will assume that the system is held in equilibrium at some temperature $T$ with a surrounding heat bath. If we consider a small, reversible transfer of energy between the hole and its environment, this absorption will be isotropic, and will occur in such a way that the angular momentum $J$ and the electric charge $Q$ remain unchanged [5]. The heat capacity we will calculate is related to this transfer of energy. In our study we are only concerned with $J = 0$ cases.

Let us now use the results from (5) and (8) to evaluate the full thermal capacity at constant electric charge. We obtain

$$C_Q = T \frac{\partial S}{\partial T} \bigg|_Q$$

$$= T \frac{8\pi^2(n-1)^2(1 + \frac{r_+^2}{\rho^2})^{n-2}r_+}{\frac{2n-1}{\rho^2} - \frac{1}{r_+^2} - Q^2[1 - 2n(1 + \frac{r_+^2}{\rho^2})^{1-n}\frac{1}{r_+^2} - (1 + \frac{r_+^2}{\rho^2})^{2-n} \frac{D-1}{r_+^{D-1}}]} \tag{10}$$

Since we are interested in critical behavior, we look for solutions of $r_+ = r_c$ that make the denominator (that we will denote by $\Delta$) vanish in the above expression. We shall divide our study in different cases:

1. The case of $l \to \infty$ and $Q = 0$ does not present any transition, since there is no value of $r_+$ that makes $\Delta = 0$. Taking the limit $l \to \infty$ in eq.(10) we obtain $C_Q = -2\pi(n-1)r_+^2$, which is always negative for any value of $n$. For a Schwarzschild black hole $r_+ = 2M = 1/4\pi(n-1)T$ leading to $C_Q = -M/T$, which agrees with the result found in [5], for $D = 4$. It is worth noting that this negativity of the full thermal capacity of Schwarzschild black holes persists, independently of the dimension $D$.

2. We now study the situation of $l \to \infty$ and $Q \neq 0$ and show, that for the special case of $D = 4$ with zero cosmological constant we reproduce the results for the critical thermodynamic quantities found in ref.[5]. Making $\Delta = 0$ we find

$$r_+^c = \left[Q^c(D - 1)\right]^{1/D-2}, \tag{11}$$

and substituting this into eq.(7) we find the value for the critical temperature to be

$$T^c = \frac{1}{2\pi(D-1)r_+^c}. \tag{12}$$
From eq.(3) we have
\[ M^e = \frac{r_+^e}{2} \left[ 1 + \frac{1}{(D-1)(D-3)} \right], \]  
(13)
so we can write the critical temperature in terms of the critical mass as
\[ T^c = \frac{1}{4\pi M^e} \left[ \frac{1}{D-1} + \frac{1}{(D-1)^2(D-3)} \right]. \]  
(14)

For the case of \( D = 4 \) black holes, the critical values reduce to \( T^c = 1/6\pi r_+^e = 1/9\pi M^c \), since \( M^c = 2r_+^e/3 \). As we said, our result coincides with that of [5] where
\[ T^c = (2\pi M[3 + \sqrt{3 - q_J}])^{-1} \]  
(15)
and \( q_J \) satisfies the following constraint
\[ j^2_J + 6j_J + 4q_J = 3. \]

For our case where \( J = 0 \) then \( q = 3/4 \), and therefore \( T^c = (9\pi M^c)^{-1} \).

The following figures correspond to some of the significant graphs for \( D = 4 \), \( Q = 1 \) and zero cosmological constant.

In Fig.1 above we plot the temperature versus \( r_+ \). One can see that there is a maximum value for the temperature, which is the critical one, \( T^c \). The value for \( T = 0 \) corresponds to \( r_+ = Q^{2/D-2} \). As the black hole horizon goes
over $r_+^c$ given by eq.(11), the black hole starts to cool down, reaching zero temperature for infinite horizon.

In Fig.2 we have the mass as a function of $r_+$. The mass behaves as a monotonically increasing function of $r_+$ for those values that make the temperature positive. The region where the mass is monotonically decreasing corresponds to negative values of $T$, thus it does not have physical meaning.

![Fig. 2.](image)

Fig.3 represents the temperature as a function of the mass. For values $M > M^c$ ($r_+ > r_+^c$), this graph reminds us of the behavior $T \sim 1/M$ found in eq.(6) for Schwarzschild black holes, whereas for $M < M^c$, the temperature increases up to the critical value.

![Fig. 3.](image)
In Fig. 4 we represent the heat capacity as a function of temperature. There are two branches, depending on the value of the event horizon being greater or smaller than \( r_+ \). \( C_Q \) is positive for \( r_+ < r_+ \) and negative otherwise. On the right part of the plot below we see the divergent behavior of \( C_Q \) as the horizon reaches its critical value.

As we cross over \( r_+ \), \( C_Q < 0 \), and we then move along the lower branch of the graph.

The expression for the heat capacity found in [5], for \( l \to \infty \), \( D = 4 \), and \( Q \neq 0 \) is given as a function of \( T \), \( S \) and \( M \), and the behavior near the critical temperature \( T^c \) is

\[
C_{JQ} = \frac{4\pi T S M}{1 - 8\pi T M - 4\pi S T^2} \sim \frac{1}{T - T^c},
\]

so that the critical exponent \( \alpha \) is one. To determine \( \alpha \) in our case of \( D \) dimensions, we need to find the multiplicity of \( r_+^c \) in \( \Delta = 0 \). The denominator in relation (10) reduces to

\[
\Delta = \frac{\partial T}{\partial r_+} = \frac{Q^2(D - 1)}{r_+^D} - \frac{1}{r_+^2}
\]
By studying the derivatives of $\Delta$ we find $\Delta'(r_+^c) = \partial^2 T/r_+^2 < 0$ leading to a maximum of the temperature for $r_+ = r_+^c$ (see Fig.1). Therefore we see that there is a single root for $T_c$ and claim that $C_Q \sim (T - T_c)^{-1}$, so $\alpha = 1$ for any even value of $D$. Redefining $\tilde{\Delta} = r_+^D\Delta$ (this does not change the critical values of $C_Q$), we can find the solutions of $\tilde{\Delta} = 0$ through the transformation
\[
r_+ = [Q^2(D - 1)]^{1/2\pi} e^{i\phi},
\]
where $\phi = 2\pi k/(D - 2) = 2\pi$, to keep the solution within the principal value. Therefore $k = D - 2$ gives us the number of complex and real valued roots of $\tilde{\Delta} = 0$. Out of these solutions two of them will be real, with only one ($k = 0$) being positive. Hence the root is of single multiplicity giving us $\alpha = 1$, for any value of $D$.

As the value of $D$ goes to infinity, it is easy to see that $r_+^c \to 1$. For any dimension equal or greater than four we find that the value of the critical event horizon becomes smaller as the dimension of spacetime increases. However, this critical horizon will never become zero.

(3) Let us now look into the case of $l$ finite and $Q = 0$. The critical value we obtain for the Schwarzschild horizon is
\[
r_+^c = \frac{l}{\sqrt{2n - 1}}.
\]
From relation (3) the critical mass is
\[
M^c = \frac{l}{2(2n - 1)^{1/2}},
\]
and eq.(6) gives us
\[
T_{Q=0}^c = \frac{1}{2\pi(n - 1)r_+^c} = \frac{\sqrt{2n - 1}}{2\pi(n - 1)^{1/2}}.
\]

The critical entropy at the horizon is given by
\[
S^c(r_+) = \pi l^2[1 + \left(\frac{1}{2n - 1}\right)^{n-1} - 1].
\]
For the particular case of four dimensional spacetime \((n = 2)\), we find the following critical results: \(r_+^c = l/\sqrt{3}\), \(M^c = 2l/\sqrt{3}\), \(T^c = \sqrt{3}/2\pi l\) and \(S^c(r_+) = \pi l^2/3\).

As the dimensionality goes to infinity the critical temperature goes to zero and the critical entropy becomes

\[
\lim_{n \to \infty} S^c(r_+) = \pi l^2 \left[ e^{\frac{1}{2}} - 1 \right].
\]  

(4) In the most general case of \(l\) finite, \(Q \neq 0\) and any value of \(D\), it is difficult to find the explicit critical expressions of the thermodynamic variables. Let us look at the \(D = 4\) scenario. The roots of the denominator in \((10)\) are given by

\[
r_+^c 4 - \frac{l^2}{4} r_+^c 2 + l^2 Q^2 = 0.
\]

For \(Q > l/6\) there are no real valued solutions and hence no phase transitions, while for \(Q = l/6\) there is one transition, and \(Q < l/6\) presents two. It is possible to show numerically that for a given \(D\), different than four, there are up to two critical transitions, depending again on the ratio between \(Q\) and \(l\).
In Fig. 5 above we plot the full thermal capacity as a function of $r_+$ for the case of $D = 4$. The situation with two phase transitions (b), $0 < Q < l/6$, presents three branches. As the electric charge approaches zero, the asymptote on the left part of the figure drifts towards the left, until $r_{+\text{min}}$ becomes zero. Therefore, Schwarzschild black holes with finite $\Lambda$ [case (3)], as the limiting case where $Q = 0$ [case (4)], present only one critical transition. As $Q$ increases both asymptotes in (b) get together, until $Q$ reaches the threshold value of $l/6$, at which point there is no negative branch for the heat capacity. For greater values of $Q$, (b) becomes (a), and there are no transitions at all.

In Fig. 6 above, we show how the heat capacity looks as a function of the temperature. If we start with $r_+ = 0$ and increase the horizon of the black hole, the temperature increases up to $T_{\text{max}}^c$, and we see that the heat capacity (positive branch of $C_Q$ starting at $C_Q = 0$) increases as well. As the horizon goes over $r_{+\text{min}}^c$, the temperature decreases (negative branch values of $C_Q$) down to a minimum value $T_{\text{min}}^c$. Should the horizon continue to increase over $r_{+\text{max}}^c$, the temperature increases to infinity (second positive branch of $C_Q$).

The steepest shape of the heat capacity around the critical values of the temperature is related to the fact that the critical exponents should be small.
3. Odd dimensional black holes

Just as in the previous section, to study odd dimensional black holes we start from the expression of the metric (1), where

$$g^2(r) = 1 + \frac{r^2}{l^2} - [M + 1 - \frac{Q^2}{2(D - 3) r^{D-3}}]^{\frac{1}{D-1}}, \text{ with } D = 2n - 1. \quad (23)$$

As in the previous part, we find the value of the horizon $r_+$ from $g^2(r) = 0$, so

$$1 + \frac{r_+^2}{l^2} = [M + 1 - \frac{Q^2}{2(D - 3) r^{D-3}_+}]^{\frac{1}{D-1}}. \quad (24)$$

The relation above tells us that there are no Schwarzschild black holes in odd dimensions when we restrict ourselves to the case of zero cosmological constant. The reason being that $M$ becomes zero.

The limit $l \to \infty$ is not well defined due to the dependence on $l$ of $r_+$. When $l \to \infty$, $r_+$ also goes to infinity and one is left only with the black hole interior [7]. However, for the case of $l$ finite, there is the possibility for such black holes, since in this case $M$ takes positive values.

As we did in section 2, we will write the general expressions for the temperature and entropy of the black hole as a function of the dimensionality. From (4) and (23), the temperature is found to be

$$T = \frac{1}{4\pi} \left[ \frac{2r_+}{l^2} - \frac{Q^2}{2(n-1)r^{D-2}_+}(1 + \frac{r_+^2}{l^2})^{2-n} \right], \quad (25)$$

where we have used again the implicit approach coming from the constraint equation$^2$ (24).

For the case of Schwarzschild black holes, the relation found in [7] is recovered,

$$T_{Q=0} = \frac{r_+}{4\pi l^2}. \quad (26)$$

$^1$As we did for the even dimensional case we will follow the expressions of the metric found in the paper of Bañados, Teitelboim and Zanelli, reference [7].

$^2$Here one can see that for $l \to \infty$ and $Q \neq 0$ we arrive at a negative value of the temperature, for any value of $n$. 

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In odd dimensions the expression for the entropy of the black hole given in [7] does not have a close form, being

\[ S = 4\pi(n-1) \int_0^{r^+} dr (1 + \frac{r_+^2}{l^2})^{n-2}. \] (27)

For the case of \( D = 4 \) the previous expression reduces to the usual \( S = 4\pi r_+ \).

**Phase transitions**

The full thermal capacity at constant \( Q \) is given by

\[ C_Q = T \frac{16\pi^2(n-1)(1 + \frac{r_+^2}{l^2})^{2n-3} r_+^{D-1}}{C^2 \left( 1 + \frac{r_+^2}{l^2} \right)^n - \frac{r_+^{D-1}}{2(n-1)} \left( 4n-\frac{7}{2}r_+^2 + D - 2 \right)}. \] (28)

As we did in section 2, we will denote the denominator of eq. (28) by \( \Delta \), and we will be interested in the following cases:

(1) For \( l \) finite and \( Q = 0 \), \( \Delta \) reduces to a polynomial expression in \( r_+ \), which is finite and non-zero, for all \( r_+ \). As \( T \) is proportional to \( r_+ \), according to (26), this then implies that \( C_Q \) is finite and regular for any temperature. Therefore there are no critical transitions.

(2) The general case, \( l \) finite and \( Q \neq 0 \), does not present any transitions either. It is easy to see, by inspection of (28), that there is no transition with \( C_Q \) divergent (\( \Delta \neq 0 \)) for any value of \( T, Q \) or \( l \). For finite values of the heat capacity there is no transition either, as the derivatives will be regular to any order. This can be seen from the fact that one can write

\[ \frac{\partial C_Q}{\partial T} = \frac{\partial S}{\partial T} + T \frac{\partial}{\partial T} \left( \frac{\partial S}{\partial T} \right). \] (29)

The first term on the right-hand side is equal to \( C_Q/T \) and it is finite, since it is the ratio of two polynomials in \( r_+ \) with positive coefficients. The second term reduces to

\[ \frac{1}{\Delta} \frac{\partial}{\partial r_+} \left( \frac{C_Q}{T} \right), \]

where the ratio \( C_Q/T \) and its derivative with respect to \( r_+ \) are both finite. Because \( \Delta \) never vanishes in odd dimensions, we can say that the second
of the terms in (29) keeps \( \partial C_Q / \partial T \) finite. Doing a similar analysis for the following derivatives of \( C_Q \) we find them all to be finite. Therefore in odd-dimensionally continued BTZ black holes there are no critical transitions; the full thermal capacity and all its derivatives remain finite for all values of \( T \) and \( Q \). This result is general for any dimension.

4. Conclusions

In this paper we studied the possibility of critical transitions in Bañados, Teitelboim and Zanelli dimensionally continued black holes. By studying the behavior of the full thermal capacity at constant electric charge \( Q \), we found that even dimensional BTZ black holes present critical transitions. These are second order transitions due to the divergency of the heat capacity. However, odd dimensional BTZ black holes do not present criticality. We also found that Schwarzschild black holes do not present transitions for \( \Lambda = 0 \), although they do, for finite \( \Lambda \). The case of \( Q \neq 0 \) was also studied, and we found the possibility of zero, one or two transitions, depending on the values of \( Q, l \) and \( D \). In the case of four dimensional spacetime and vanishing cosmological constant, we recovered the results found in the literature for the critical values of the thermodynamic variables.

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References


